Math 8254: Algebraic Geometry
Locally Free Sheaves and Vector Bundles; Differentials

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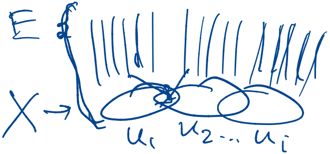
Motivation

(Fiber) bundles in all sorts of geometry and topology have a local triviality property:



$E \to X$ fiber bundle with fiber F

if over an open cover $\{U_i\}$ of X

$E|_{U_i} \to U_i$ is isom-c to $F \times U_i \to U_i$

If F is a v. space, then fiber bundle becomes a v. bdle

Quasi-coherent sheaves are like vector bundles (the fiber over a closed point P is a vector space M/PM over the residue field k_P), but are not locally trivial: think of. . . the skyscraper sheaf

"R/P

k_P on $X \ni P$, P closed pt
scheme

$$(\text{fiber of } k_P \text{ at } x \in X \text{ (closed pt)}) \cong \begin{cases} k_P & \text{if } x = P \\ 0 & \text{if } x \neq P \end{cases}$$

Locally free sheaves

Definition: X a ringed space, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. $\mathcal{F}$ is

- *free* if it's isom-c to $\bigoplus_{i\in I} \mathcal{O}_X$: $\mathcal{F} \cong \bigoplus_{i\in I} \mathcal{O}_X$
- *locally free* (or a *vector bundle*) if $\exists$ open cover $\{U_i\}$ of X, s.t. $\forall i$ $\mathcal{F}|_{U_i}$ is free on U_i.

  $\oplus \mathcal{O}_{U_i}$ $\oplus \mathcal{O}_{U_j}$

  
- *rank* (if X is connected): $\operatorname{rank} \mathcal{F} = |I|$
- *invertible* (or a *line bundle*) if $\mathcal{F}$ is loc. free of rank 1

Remark: On a scheme X locally free sheaves are automatically quasi-coherent: $\mathcal{F}|_{U_i} \cong \bigoplus_{\alpha\in I} \mathcal{O}_{U_i} = \widetilde{\bigoplus_{\alpha\in I} R_i}$ $\quad \operatorname{Spec} R_i = U_i$

Example

$\mathcal{O}_{\mathbb{P}^n}(d)$ on $\mathbb{P}^n$ but not k_P $\qquad \mathcal{O}_{\mathbb{P}^n}(d)|_{U_i} \cong \mathcal{O}_{U_i}$ invertible

$P \in \mathbb{P}^1$ $\quad k_P$ skyscraper is not loc. free

The Picard group

$\mathcal{L}$ line bdle (invertible sheaf)

X ringed space

$\mathcal{L}^\vee \otimes \mathcal{L} \longrightarrow \mathcal{O}_X$

morphism of sheaves

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes \mathcal{L} \longrightarrow \mathcal{O}_X$

$\varphi \otimes s \longmapsto \varphi(s)$

$\mathcal{L}^\vee|_U = \mathcal{H}om(\mathcal{O}_U, \mathcal{O}_U) \simeq \mathcal{O}_U$

$\Rightarrow \mathcal{L}^\vee|_U \otimes \mathcal{L}|_U \to \mathcal{O}_U$ isom-m

$\Rightarrow \mathcal{L}^\vee \otimes \mathcal{L} \to \mathcal{O}_X$ isom-m; $\mathcal{L}^{-1} := \mathcal{L}^\vee$

$\mathrm{Pic}\, X = \{$group of iso classes of invertible sheaves on $X\}$ $\otimes$

Constructions of locally free sheaves on a scheme

Motto Constructions of Comm. Alg. preserving free modules will work in AG as constructions of locally free sheaves. Here X is a scheme and the sheaves are locally free.

$\mathcal{F}, \mathcal{G}$ loc. free sheaves

- *direct sum* $\mathcal{F} \oplus \mathcal{G}$ $\qquad \widetilde{M \oplus N}$
- *tensor product* $\mathcal{F} \otimes \mathcal{G}$ $\qquad \widetilde{M \otimes N}$
- *dual* $\mathcal{F}^\vee$ $\qquad \widetilde{M^\vee}$
- *pullback* $f: Y \to X$, ($\operatorname{Spec} S \to \operatorname{Spec} R$) $\qquad f^* \mathcal{F} \;\widetilde{\;}\; S \otimes_R M$ locally associated (of same rank)

Thus the *twisting sheaf* $\mathcal{O}_X(d)$ on a projective variety (closed subscheme of $\mathbb{P}^n$) is locally free: $i: X \hookrightarrow \mathbb{P}^n$ (closed subsch)

$\mathcal{O}_X(d) := i^* \mathcal{O}_{\mathbb{P}^n}(d)$ invertible sheaf on X