# Math 8254: Algebraic Geometry
# Sheaf Cohomology, Continued

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# Further properties of cohomology

**Standing assumptions for the cohomology topic**: $X$ a separated, compact scheme over a field $k$; $\mathcal{F}$ a quasi-coherent sheaf on $X$

$U \subset X \xrightarrow{p} \operatorname{Spec} k \quad p^{-1}(\mathcal{O}_{\operatorname{Spec} k})(X) = k \xrightarrow{p^*} \mathcal{O}_X(X)$ $k$-v.sp.

### Theorem

1. $H^0(X, \mathcal{F}) = \mathcal{F}(X)$;
2. *If $X$ is affine, then* $H^p(X, \mathcal{F}) = 0 \quad \forall p > 0$;
3. *$X$ is projective* variety $/k = \bar{k}$ *of dim $n$, then* $H^p(X, \mathcal{F}) = 0 \quad \forall p > n$;

   Grothendieck's vanishing theorem: $X$ Noetherian top. space of dim $n$, $\mathcal{F}$ sheaf of ab. gps on $X \Rightarrow H^p(X, \mathcal{F}) = 0 \quad \forall p > n$.
4. *If $i : X \hookrightarrow Y$ is a closed embedding, then* $H^p(Y, i_*\mathcal{F}) = H^p(X, \mathcal{F}) \quad \forall p \geq 0$.

**Proof**: (2) $\{X\}$ is an affine open cover of $X$

$C^p(X, \mathcal{F}) = 0 \quad \forall p > 0 \Rightarrow H^p(X, \mathcal{F}) = 0$ as subquotient

# Proof of theorem, continued

(3) $X$ proj. of dim $n$, $X^n \subset \mathbb{P}^N$ closed ~~subscheme~~ subvariety

$f \neq$ const hom. polynomial in $S(X) = k[X_0, .., X_N] / I_p(X)$

$$\dim V_p(f) = \dim X - 1$$

(true for projective $X$ as well as for affine

$V_p(f_0) \subset X \subset \mathbb{P}^N$

b/c $\dim V_p(f) = \max_{i=0}^{N} \dim V_p(f) \cap U_i$, $= \dim X - 1$ )

$V_p(f_0) \cap V_p(f_1) \cap .. \cap V(f_n)$

$$\dim V_p(f_0, .., f_n) = n - (n+1) < 0$$

for some $f_0, f_1, .., f_n \in S(X)$ $\Rightarrow$ $V_p(f_0, .., f_n) = \emptyset$

$j = 0, .., n$ $\quad U_j := X \setminus V_p(f_j)$ affine open

$\forall j$ $\quad f_j$ becomes linear after Veronese $X \subset \mathbb{P}^N \subset_{\text{veronese}} \mathbb{P}^M$

# Proof of theorem, continued

$$C^p(X,\mathcal{F}) = \bigoplus_{i_0<\dots<i_p} \mathcal{F}(V_{i_0\dots i_p}) = 0$$

for $p>n$, b/c cover ~~has~~ only $n+1$ elements $V_0, \dots, V_n$

$$\Rightarrow H^p(X,\mathcal{F}) = 0 \quad \forall p>n.$$

(4) $C^p(Y, i_*\mathcal{F}) = \bigoplus_{i_0<\dots<i_p} i_*\mathcal{F}(U_{i_0\dots i_p})$

$X \xrightarrow{i} Y$, $\{U_i\}$ affine open cover of $Y$

$$= \bigoplus \mathcal{F}(i^{-1}(U_{i_0\dots i_p})) = \bigoplus \mathcal{F}(X\cap U_{i_0\dots i_p})$$

$V_i := X\cap U_i \subset X \Rightarrow$

$$= \bigoplus \mathcal{F}(V_{i_0\dots i_p})$$

$$= C^p(X,\mathcal{F})$$

These identifications are compatible with $d$'s $\Rightarrow$ Čech cxs are isom-c $\Rightarrow$ so is cohom.

# The long exact sequence (LES)

**Theorem**

*The cohomology functor turns naturally a short exact sequence of sheaves on a scheme $X$:*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0,$$

*into a long exact sequence (LES)*

$$\begin{aligned} 0 &\to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \\ &\to \dots \\ &\to H^p(X, \mathcal{F}_1) \to H^p(X, \mathcal{F}_2) \to H^p(X, \mathcal{F}_3) \\ &\to H^{p+1}(X, \mathcal{F}_1) \to H^{p+1}(X, \mathcal{F}_2) \to H^{p+1}(X, \mathcal{F}_3) \\ &\to \dots \end{aligned}$$

Pick affine open cover of X

$$0 \to C^\bullet(\mathcal{F}_1) \to C^\bullet(\mathcal{F}_2) \to C^\bullet(\mathcal{F}_3) \to 0$$

a SES of complexes, i.e.

$$\begin{tikzcd}
0 \arrow[r] & C^p(\mathcal{F}_1) \arrow[r] \arrow[d, "d_1"] & C^p(\mathcal{F}_2) \arrow[r] \arrow[d, "d_2"] & C^p(\mathcal{F}_3) \arrow[r] \arrow[d, "d_3"] & 0 \\
0 \arrow[r] & C^{p+1}(\mathcal{F}_1) \arrow[r] & C^{p+1}(\mathcal{F}_2) \arrow[r] & C^{p+1}(\mathcal{F}_3) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Exact, b/c $U_{i_0 \dots i_p}$ $\forall p \geq 0$ are affine, $\mathcal{F}_j$ q-coh

$\mathcal{F}_1|_{U_{i_0 \dots i_p}}$

$$0 \to \mathcal{F}_1(U_{i_0 \dots i_p}) \to \mathcal{F}_2(U_{i_0 \dots i_p}) \to \mathcal{F}_3(U_{i_0 \dots i_p}) \to 0$$

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

# Proof of theorem, continued

Snake Lemma:

$$0 \to \ker d_1 \to \ker d_2 \to \ker d_3 \to \operatorname{Coker} d_1 \to \operatorname{Coker} d_2 \to \operatorname{Coker} d_3 \to 0$$

is exact

$$\operatorname{Coker} d_3 = C^p(F_3)/\operatorname{Im} d_3$$

$$\begin{tikzcd}
 & \operatorname{Coker} d_1^p \arrow[r] \arrow[d, "d_1^{p+1}"] & \operatorname{Coker} d_2^p \arrow[r] \arrow[d, "d_2^{p+1}"] & \operatorname{Coker} d_3^p \arrow[r] \arrow[d, "d_2^{p+1}"] & 0 \\
0 \arrow[r] & \ker d_1^{p+1} \arrow[r] & \ker d_2^{p+1} \arrow[r] & \ker d_3^{p+1} &
\end{tikzcd}$$

Snake Lemma

$$H^p(F_1) \to H^p(F_2) \to H^p(F_3) \to H^{p+1}(F_1) \to H^{p+1}(F_2) \to H^{p+1}(F_3) \quad \square$$