MATH 8254: ALGEBRAIC GEOMETRY
WEIL DIVISORS, RATIONAL FUNCTIONS AND LINE BUNDLES

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1. Global and local rational functions

The matter is that rational functions near a point of a scheme X are defined as quotients of regular functions, while global rational functions on X are defined as just as regular functions on dense open subsets. For some reason, Vakil and Hartshorne sweep the following statement under the rug.

Let X be a **reduced** Noetherian scheme of pure dimension n, regular in codimension 1 and

$$K(X) = \varinjlim_{\text{Dense open } U \subset X} \mathcal{O}_X(U)$$

be the ring of *rational functions on* X. In other words, a rational function is a regular function on a dense open $U \subset X$, and two rational functions are equivalent, if they agree on the intersection of their domains. A scheme X is *reduced* if for each open $U \subset X$, $\mathcal{O}_X(U)$ is a reduced ring, i.e., contains no nonzero nilpotents. I did not require this in class, which is okay, but things will become more cumbersome without it. For example, instead of saying φ is not the zero function on each irreducible component of X in the lemma below, we would have to say there is at least one point p in each irreducible component at which φ is not zero in the fiber $\mathcal{O}_X|_p = \mathcal{O}_{X,p}/p\mathcal{O}_{X,p}$.

Proposition 1. *Let $p \in X$ be a point of codimension 1. Then there is a natural ring homomorphism $K(X) \to K(\mathcal{O}_{X,p})$.*

Proof. By choosing an irreducible component of X containing the point p, we may assume WLOG that X is **irreducible, which is something I missed in class on Friday.** In fact, the regularity of X in codimension 1 implies that p lies in a unique irreducible component of X: this is because the stalk $\mathcal{O}_{X,p}$ is an integral domain.

Choose an affine open neighborhood $U_p \cong \operatorname{Spec} A \subset X$ of p.

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Suppose we are given $\varphi \in K(X)$ defined as a regular function $\varphi \in \mathcal{O}_X(U)$ on an open dense subset $U \subset X$. Since $U \cap U_p \neq \varnothing$, we may find a point $q \in U \cap U_p$ and an open neighborhood $U_q \subset U \cap U_p$ of q on which $\varphi = f/g$, where $f, g \in A$ and $g \notin x$ for each $x \in U_q \subset \operatorname{Spec} A$. Note that, while f/g is defined only on U_q, the regular functions f and g are defined everywhere on U_p. We claim that f/g is defined in the fraction field $K(A_p)$ as f_p/g_p. For that, we just need to know that $g_p \neq 0$ in the stalk $\mathcal{O}_{X,p} = A_p$, which is an integral domain, because of the regularity in codimension 1 assumption. Suppose $g_p = 0$. Since g_p is the germ of $g \in \mathcal{O}(U_p) = A$, we have $g = 0$ in an open neighborhood V_p of p. Since X is **irreducible**, V_p is dense in it and $V_p \cap U_q \neq \varnothing$. Thus, $g = 0$ there, which contradicts the fact that $g \notin x$ for each $x \in U_q$.

So, we define the homomorphism $K(X) \to K(\mathcal{O}_{X,p})$ as

$$\varphi \mapsto f_p/g_p.$$

It is not hard to see that it will be a homomorphism, provided we show it is independent of the choice of a point q and presentation $\varphi = f/g$ near it we have chosen. If we choose another point q', neighborhood $U_{q'}$, and presentation $\varphi = f'/g'$, we again know that $U_q \cap U_{q'} \neq \varnothing$ and there $f'/g' = f/g$. Therefore $f'_p/g'_p = f_p/g_p$, and we are done. □

2. Principal divisors and generic points

The above statement allows us to define the valuation $v_p(\varphi)$ for a codimension-one point $p \in X$ and a rational function φ which is not the zero function on any irreducible component of X by taking the image of φ in $K(\mathcal{O}_{X,p})$ and then using the valuation $v_p : K(\mathcal{O}_{X,p})^\times \to \mathbb{Z}$. We want to connect such $p \in X$ with a closed subset of $Y \subset X$ of codimension one. It is going to be $Y = \overline{\{p\}}$, but we'd rather need the converse of this construction.

A primer on the generic point of an irreducible closed $Y \subset X$: For $X = \operatorname{Spec} A$, every point $p \in X$ (i.e., a prime ideal of A) gives rise to an irreducible closed subset,

$$\{q \in \operatorname{Spec} A \mid q \supset p\} = V(p) = V(\bigcup_{S \subset p} S) = \bigcap_{S \subset p} V(S) = \overline{\{p\}}.$$

Thus every point of an affine scheme is present in two incarnations: as a point p and as an irreducible closed subset $V(p)$, related by closure: $\overline{\{p\}} = V(p)$. Conversely, every irreducible closed subset is of the form $V(p)$, where p is a prime ideal of A, i.e., a point of $\operatorname{Spec} A$. The idea of a generic point is that in this case we say that p is the *generic point* of $V(p)$. This construction generalizes to arbitrary schemes, as follows.

We claim that every irreducible closed subset Y of a scheme X has a unique point $\eta \in Y$ dense in Y: $\overline{\{\eta\}} = Y$, called the *generic point of* Y. Pick an affine open $U \subset X$ such that $U \cap Y \neq \varnothing$ and let η be the generic point of $U \cap Y$. Then, since $U \cap Y$ is dense in Y, *eta* will be a generic point of Y as well. On the other hand, any other generic point η' of Y will also be dense in $U \cap Y$ and thereby equal to η, as $V(\eta') = V(\eta)$ implies $\eta' = \eta$ by the Nullstellensatz for affine schemes.

Then for an irreducible closed subset $Y \subset X$ of codimension 1, we define

$$v_Y(\varphi) := v_\eta(\varphi).$$

Now, a *principal* divisor is the *divisor of a rational function*

$$\operatorname{div} \varphi := \sum_{\substack{\text{irred. } Y \subset X \\ \operatorname{codim} Y = 1}} v_Y(\varphi)[Y].$$

Principal divisors form a subgroup $\operatorname{Prin} X$ of the group $\operatorname{Weil} X$ of Weil divisiors.

Lemma 2. *This is a finite sum.*

Proof. It is enough to show finiteness for each irreducible component of X, which means we can assume X is irreducible.

Let $U = \operatorname{Spec} A$ be an affine open subset on which φ is regular and therefore $\varphi \in A$, $\varphi \neq 0$. Then $Z := X \setminus U$ is a proper closed subset of X. Therefore, it contains a finite number of irreducible divisors. This is because if Z is of codimension 0, it must be the whole X, which could not be the case when Z is proper. If $\operatorname{codim} Z = 1$, then an irreducible divisor of X contained in Z must be an irreducible component of Z, of which there are finitely many. If $\operatorname{codim} Z > 1$, then it is too small to contain an irreducible divisor of codimension 1.

Thus, every other divisor will intersect $U = \operatorname{Spec} A$ nontrivially, and it is enough to show that there are finitely many divisors $Y \subset U$ of the Noetherian scheme U for which $v_Y(\varphi) \neq 0$. This actually means $v_Y(\varphi) > 0$, because φ is regular on U. So, we are again talking about irreducible divisors contained in the proper closed set $V(\varphi) \subsetneq U$, as A is reduced and $\varphi \neq 0$ cannot be contained in all prime ideals. The same argument as for Z above shows that U contains finitely many irreducible divisors. □

3. Line bundles and Weil divisors

Now, suppose s is a rational section of a line bundle $\mathcal{L}$ on X (i.e., s is a regular section on a dense open subset, with a similar equivalence relation as for rational functions) and s is **not the zero section on any irreducible component of** X, which is something I forgot to mention in class. Then, for each irreducible closed $Y \subset X$, $\operatorname{codim} Y = 1$, we can trivialize $\mathcal{L}$ in a neighborhood U of the generic point of Y: $\mathcal{L}|_U \cong \mathcal{O}_X|_U$. Then regular sections of $\mathcal{L}$ on open subsets of U will be identified with regular functions with same domain, and hence, our rational section s will be identified with a rational function on U. This allows us to define the valuation of s at Y as above. It is independent of the choice of trivialization, because any two trivializations differ by an invertible function, whose valuation is zero.

Then the *divisor of the rational section* s is

$$\operatorname{div} s := \sum_{\substack{\text{irred. } Y \subset X \\ \operatorname{codim} Y = 1}} v_Y(s)[Y].$$

The proof of the lemma above generalizes to show that this sum is finite.

The set of isomorphism classes of such pairs $(\mathcal{L}, s)$ forms a group with respect to $\otimes$ and $(\mathcal{L}, s)^{-1} = (\mathcal{L}^\vee, s^{-1})$, s^{-1} being a rational section of $\mathcal{L}^\vee$ such that $s^{-1}(s) = 1$ under evaluation $\mathcal{L}^\vee \otimes \mathcal{L} \xrightarrow{\sim} \mathcal{O}_X$. We have a group homomorphism

$$\{(\mathcal{L}, s)\}/\text{isomorphism} \to \operatorname{Weil} X,$$
$$(\mathcal{L}, s) \mapsto \operatorname{div} s,$$

where

$$\operatorname{Weil} X = \{ \sum_{\substack{\text{irred. } Y \\ \operatorname{codim} Y = 1}} n_Y[Y] \mid n_y \in \mathbb{Z}, n_Y = 0 \text{ for almost all } Y\}$$

is the group of Weil divisors with respect to addition. Since two rational sections s and s' of $\mathcal{L}$ differ by a rational function factor: $s' = \varphi s$ (since $\mathcal{L} \otimes \mathcal{L}^{-1} = \mathcal{O}_X$), and $v_Y(s') = v_Y(\varphi) + v_Y(s)$, we also have a homomorphism

$$\{\mathcal{L}\}/\text{isomorphism} \to \operatorname{Cl} X,$$

where $\operatorname{Cl} X := \operatorname{Weil} X / \operatorname{Prin} X$.

The point of these is that these homomorphisms are isomorphisms in good enough situations.