Finite Element Exterior Calculus and Applications

Part II

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The fundamental theorem of numerical analysis

Convergence, consistency, and stability of discretizations

The basic idea:

Continuous problem: $L : X \to Y$ bounded linear operator between Banach spaces

$$\text{Given } f \in Y \text{ find } u \in X \text{ such that } Lu = f.$$

Discrete problem: $L_h : X_h \to Y_h$ operator on finite dimensional spaces.

$$\text{Given } f_h \in Y_h \text{ find } u_h \in X_h \text{ such that } L_h u_h = f_h$$

- The discretization is *convergent* if u_h is sufficiently near u.
- The discretization is *consistent* if L_h and f_h are sufficiently near L and f.
- The discretization is *stable* if the discrete problem is well-posed.

Theorem (Fundamental theorem of numerical analysis)

A discretization which is consistent and stable is convergent.

Measuring convergence, consistency, and stability

To quantify convergence we use

1. A norm in the space X_h.
2. A representative U_h in X_h.

The *discretization error* is then $\|U_h - u_h\|_{X_h}$. The method is convergent if it tends to 0 as $h \to 0$.

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In this context the fundamental theorem is easy:

$$L_h u_h = f_h \implies L_h U_h - L_h u_h = L_h U_h - f_h \implies U_h - u_h = L_h^{-1}(L_h U_h - f_h)$$

$$\|U_h - u_h\|_{X_h} \le \|L_h^{-1}\|_{\mathcal{L}(Y_h, X_h)} \|L_h U_h - f_h\|_{Y_h}$$

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Discretization of Hilbert complexes

Motivation: why mixed formulation?

$H^1 \xrightarrow{\text{grad}} H(\text{rot}) \xrightarrow{\text{rot}} L^2 \qquad (dd^* + d^*d)u = (-\operatorname{grad}\operatorname{rot} + \operatorname{curl}\operatorname{rot})u = \lambda u$

Primal formulation: Find u such that $(du, dv) + (d^*u, d^*v) = \lambda(u, v) \qquad \forall v$

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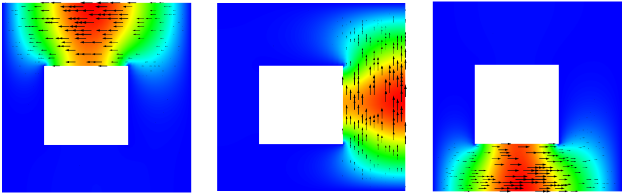
$\mathcal{P}_1 \otimes \mathbb{V}$
4K elts



$\lambda_1 = 1.94$ $\qquad \lambda_2 = 2.02$ $\qquad \lambda_3 = 2.26$

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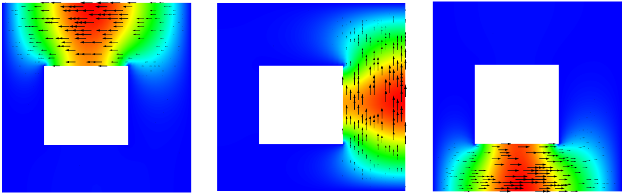

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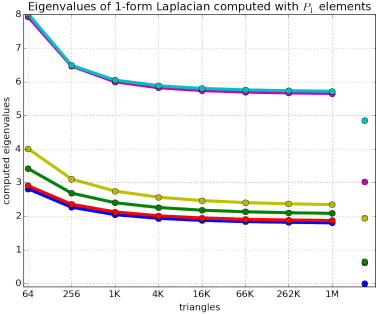
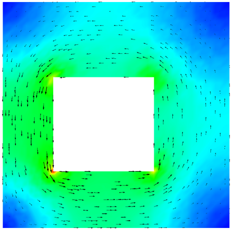


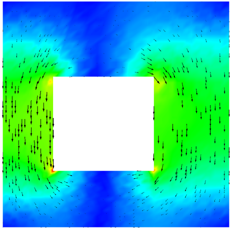
Eigenvalues of the 1-form Hodge Laplacian (FEEC)

$\mathcal{P}_1^- \Lambda^0 \times$
$\mathcal{P}_1^- \Lambda^1$

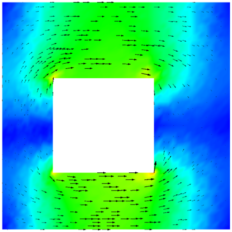
4K elts



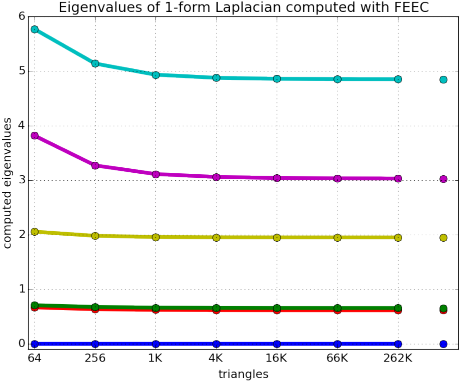
$\lambda_1 = 0$



$\lambda_2 = 0.617$



$\lambda_3 = 0.658$

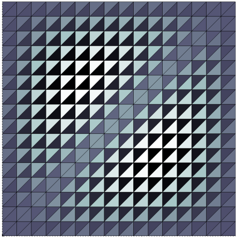


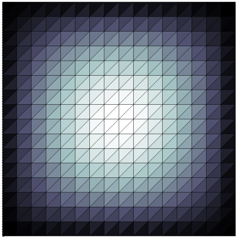
Motivation: why do we need special elements?

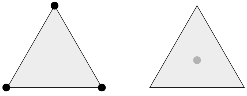
Mixed Poisson eq: $H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \to 0$.

The obvious choice of Lagrange $\mathcal{P}_1$ for V^0 and P_0 for V^1 is unstable.

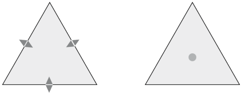
RT, BDM families are stable.







$\mathcal{P}_1 \otimes \mathbb{V}$ (Lagrange) $\quad$ $\mathcal{P}_0$

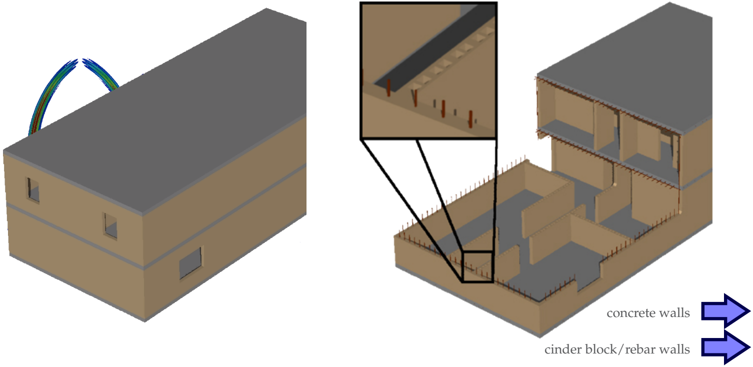


$\mathcal{P}_1^- \Lambda^1$ (RT) $\quad$ $\mathcal{P}_0 \Lambda^2$

Motivation: a real computation

Stowell–Fassenfass–White, IEEE Trans. Ant. & Prop. 2008

- Solved time-dependent Maxwell equations using $\mathcal{Q}_1^- \Lambda^1$ for E and $\mathcal{Q}_1^- \Lambda^2$ for B (Nédélec elements of the first kind on bricks)
- 10,114,695,855 brick elements ($\approx$ 1 cm resolution)
- $\approx 60,000,000,000$ unknowns
- $\approx 12,000$ time steps of 14 picoseconds



Discretizing the mixed formulation

We therefore consider finite element discretizations of the mixed form:

Given $f \in W^k$, find $\sigma \in V^{k-1}$, $u \in V^k$, and $p \in \mathfrak{H}^k$ s.t.

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, d\tau \rangle &= 0, & \tau &\in V^{k-1}, \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, & v &\in V^k, \\ \langle u, q \rangle &= 0, & q &\in \mathfrak{H}^k. \end{aligned}$$

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- Choose f.d. subspaces $V_h^j \subset V^j$
- Define $\mathfrak{Z}_h^j$, $\mathfrak{B}_h^j$, $\mathfrak{H}_h^j = \{v \in \mathfrak{Z}_h^j \,|\, v \perp \mathfrak{B}_h^j\}$

Given $f \in W^k$, find $\sigma_h \in V_h^{k-1}$, $u_h \in V_h^k$, and $p_h \in \mathfrak{H}_h^k$ s.t.

$$\begin{aligned} \langle \sigma_h, \tau \rangle - \langle u_h, d\tau \rangle &= 0, && \tau \in V_h^{k-1}, \\ \langle d\sigma_h, v \rangle + \langle du_h, dv \rangle + \langle p_h, v \rangle &= \langle f, v \rangle, && v \in V_h^k, \\ \langle u_h, q \rangle &= 0, && q \in \mathfrak{H}_h^k. \end{aligned}$$

Choice of subspaces

For *any* choice of the V_h^j there exists a unique solution.

However, the consistency and stability of the numerical method depends vitally on the choice of subspaces.

Remark:
Note that $\mathfrak{Z}_h \subset \mathfrak{Z}, \quad \mathfrak{B}_h \subset \mathfrak{B}$, but in general $\mathfrak{H}_h \not\subseteq \mathfrak{H}$. So the mixed method is slightly *nonconforming*, and this also contributes to the consistency error.

Key assumptions

We need the spaces $V_h^j \subset V^j$ (at least for $j = k-1, k, k+1$) to satisfy three properties:

1. Approximation property: Of course V_h^j must afford good approximation of elements of V^j. This can be formalized with respect to a family of subspaces parametrized by h by requiring

$$\lim_{h\to 0} \inf_{v \in V_h^j} \|w - v\|_V = 0, \quad w \in V^j$$

   (or $= O(h^r)$ for w in some dense subspace, or ...)

2. Subcomplex property: $dV_h^{k-1} \subset V_h^k$ and $dV_h^k \subset V_h^{k+1}$, so

$$V_h^{k-1} \xrightarrow{d} V_h^k \xrightarrow{d} V_h^{k+1}$$

   is a subcomplex.

Bounded cochain projection

3. Bounded cochain projection: Most important, we assume that there exists a *cochain map* from the H-complex to the subcomplex which is a *projection* and is *bounded*.

$$\begin{array}{ccccc} V^{k-1} & \xrightarrow{d} & V^k & \xrightarrow{d} & V^{k+1} \\ \pi_h^{k-1}\big\downarrow & & \pi_h^k\big\downarrow & & \pi_h^{k+1}\big\downarrow \\ V_h^{k-1} & \xrightarrow{d} & V_h^k & \xrightarrow{d} & V_h^{k+1} \end{array}$$

- For now, boundedness is in V-norm: $\|\pi_h v\|_V \le c\|v\|_V$. But later we will need W-boundedness, which is a stronger requirement.
- A bounded projection is *quasioptimal*:

$$\|v - \pi_h v\|_V \le c \inf_{w \in V_h^j} \|v - w\|_V, \quad v \in V^j$$

First consequences from the assumptions

From the subcomplex property

$$V_h^{k-1} \xrightarrow{d} V_h^k \xrightarrow{d} V_h^{k+1}$$

is itself a closed H-complex. (We take $W_h^k = V_h^k$ but with the W-norm.)

Therefore there is a discrete adjoint operator d_h^* (its domain is all of W_h^k), a discrete Hodge decomposition

$$V_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{B}_{kh}^*.$$

and a discrete Poincaré inequality

$$\|z\|_V \le c_h^P \|dz\|, \quad z \in \mathfrak{Z}_h^{k\perp_{V_h}}.$$

Preservation of cohomology

THEOREM

Given: a closed H-complex, and a choice of f.d. subspaces satisfying the subcomplex property and admitting a V-bdd cochain projection π_h. Assume also the (very weak) approximation property

$$\|q - \pi_h q\| < \|q\|, \quad 0 \neq q \in \mathfrak{H}^k.$$

Then π_h induces an isomorphism from $\mathfrak{H}^k$ onto $\mathfrak{H}_h^k$. Moreover,

$$\operatorname{gap}(\mathfrak{H}, \mathfrak{H}_h) \leq \sup_{\substack{q \in \mathfrak{H} \\ \|q\|=1}} \|q - \pi_h q\|_V.$$

$$\operatorname{gap}(\mathfrak{H}, \mathfrak{H}_h) := \max\left(\sup_{\substack{u \in \mathfrak{H} \\ \|u\|=1}} \inf_{v \in \mathfrak{H}_h} \|u - v\|_V, \sup_{\substack{v \in \mathfrak{H}_h \\ \|v\|=1}} \inf_{u \in \mathfrak{H}} \|u - v\|_V \right).$$

Uniform Poincaré inequality and stability

THEOREM

Given: a closed H-complex, and a choice of f.d. subspaces satisfying the subcomplex property and admitting a V-bdd cochain projection π_h. Then

$$\|v\|_V \le c^P \|\pi_h\| \|dv\|_V, \quad v \in \mathfrak{Z}_h^{k\perp} \cap V_h^k.$$

COROLLARY (STABILITY AND QUASIOPTIMALITY OF THE MIXED METHOD)

The mixed method is stable (uniform inf-sup condition) and satisfies

$$\begin{aligned} &\|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\| \\ &\qquad \le C\Big(\inf_{\tau \in V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v \in V_h^k} \|u - v\|_V + \inf_{q \in V_h^k} \|p - q\|_V \\ &\qquad\qquad + \mu \inf_{v \in V_h^k} \|P_{\mathfrak{B}} u - v\|_V \Big), \end{aligned}$$

where $\mu = \mu_h = \sup_{r \in \mathfrak{H}^k, \|r\|=1} \|(I - \pi_h) r\|.$

Improved error estimates

In addition to $\mu = \|(I - \pi_h)P_{\mathfrak{H}}\|$, define $\delta, \eta = o(1)$ by

$$\delta = \|(I - \pi_h)K\|_{\mathrm{Lin}(W,W)}, \quad \eta = \|(I - \pi_h)d^{[*]}K\|_{\mathrm{Lin}(W,W)}.$$

$$\text{When } V_h^k \supset \mathcal{P}_r, \quad \mu = O(h^{r+1}), \quad \eta = O(h), \quad \delta = \begin{cases} O(h^2), & r > 0, \\ O(h), & r = 0, \end{cases}$$

Theorem

Given: an H-complex satisfying the compactness property, and a choice of f.d. subspaces satisfying the subcomplex property and admitting a W-bdd cochain projection π_h. Then

$$\|d(\sigma - \sigma_h)\| \le cE(d\sigma), \qquad \|\sigma - \sigma_h\| \le c[E(\sigma) + \eta E(d\sigma)],$$

$$\|d(u - u_h)\| \le c\{E(du) + \eta[E(d\sigma) + E(p)]\},$$

$$\|u - u_h\| \le c\{E(u) + \eta[E(du) + E(\sigma)] + (\eta^2 + \delta)[E(d\sigma) + E(p)] + \mu E(P_{\mathfrak{B}}u)\}.$$

Numerical tests

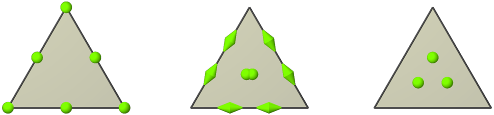
$$-\operatorname{grad}\operatorname{div} u + \operatorname{curl}\operatorname{rot} u = f \text{ in } \Omega \text{ (unit square)}, \quad u \cdot n = \operatorname{rot} u = 0 \text{ on } \partial\Omega$$

(magnetic BC)

$$0 \to H^1 \xrightarrow{\text{grad}} H(\text{rot}) \xrightarrow{\text{rot}} L^2 \to 0$$

$\sigma_h \in V_h^0 \subset H^1, \quad u_h \in V_h^1 \subset H(\text{rot})$

$$\begin{aligned} \langle \sigma_h, \tau \rangle - \langle u_h, \operatorname{grad} \tau \rangle &= 0, & \tau \in V_h^{k-1}, \\ \langle \operatorname{grad} \sigma_h, v \rangle + \langle \operatorname{rot} u_h, \operatorname{rot} v \rangle &= \langle f, v \rangle, & v \in V_h^k. \end{aligned}$$

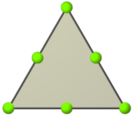


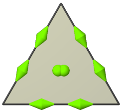
V_h^0 Lagrange $\quad$ V_h^1 R-T $\quad$ V_h^2 DG

All hypotheses are met…

Numerical solution of vector Laplacian, magnetic BC

$\|\sigma - \sigma_h\|$	rate	$\|\nabla(\sigma - \sigma_h)\|$	rate	$\|u - u_h\|$	rate	$\|\mathrm{rot}(u - u_h)\|$	rate
2.16e-04	3.03	2.63e-02	1.98	2.14e-03	1.99	1.17e-02	1.99
2.70e-05	3.00	6.60e-03	1.99	5.37e-04	1.99	2.93e-03	2.00
3.37e-06	3.00	1.65e-03	2.00	1.34e-04	2.00	7.33e-04	2.00
4.16e-07	3.02	4.14e-04	2.00	3.36e-05	2.00	1.83e-04	2.00
	3		2		2		2





Numerical solution of vector Laplacian, Dirichlet BC

For Dirichlet boundary conditions, $\sigma = -\operatorname{div} u$ is sought in H^1, u is sought in $\mathring{H}(\mathrm{rot})$ (the BC $u \cdot t = 0$ is essential, $u \cdot n = 0$ is natural).

There is no complex, so our theory does not apply.

$\|\sigma - \sigma_h\|$	rate	$\|\nabla(\sigma - \sigma_h)\|$	rate	$\|u - u_h\|$	rate	$\|\mathrm{rot}(u - u_h)\|$	rate
1.90e-02	1.62	2.53e+00	0.63	1.22e-03	2.01	1.55e-02	1.58
6.36e-03	1.58	1.68e+00	0.60	3.05e-04 r	2.00	5.33e-03	1.54
2.18e-03	1.54	1.14e+00	0.56	7.63e-05	2.00	1.85e-03	1.52
7.58e-04	1.52	7.89e-01	0.53	1.91e-05	2.00	6.49e-04	1.51
	1.5		0.5		2		1.5

DNA-Falk-Gopalakrishnan M3AS 2011

Eigenvalue problems

$$\text{Find } \lambda \in \mathbb{R}, 0 \neq u \in D(L) \text{ s.t. } Lu = \lambda u, u \perp \mathfrak{H}$$

$$\lambda \|u\|^2 = \|du\|^2 + \|d^* u\|^2 > 0 \qquad \text{so} \qquad \lambda > 0 \text{ and } Ku = \lambda^{-1} u.$$

By the compactness property, $K : W^k \to W^k$ is compact and self-adjoint, so $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$.
Denote by v_i corresponding orthonormal eigenvalues, $E_i = \mathbb{R} v_i$.

Mixed discretization:

$$\text{Find} \quad \lambda_h \in \mathbb{R}, \quad 0 \neq (\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k \quad \text{s.t.}$$

$$\begin{aligned} \langle \sigma_h, \tau \rangle - \langle u_h, d\tau \rangle &= 0, & \tau &\in V_h^{k-1}, \\ \langle d\sigma_h, v \rangle + \langle du_h, dv \rangle + \langle p_h, v \rangle &= \lambda_h \langle u_h, v \rangle, & v &\in V_h^k, \\ \langle u_h, q \rangle &= 0, & q &\in \mathfrak{H}_h^k. \end{aligned}$$

$$0 < \lambda_{1h} \leq \lambda_{2h} \leq \ldots \leq \lambda_{N_h h}, \quad v_{ih} \text{ orthonormal}, \quad E_{ih} = \mathbb{R} v_{ih}$$

Convergence of eigenvalue problems

Let $\sum_{i=1}^{m(j)} E_i$ be the span of the eigenspaces of the first j *distinct* eigenvalues. The method *converges* if $\forall\, j, \epsilon > 0$, $\exists\, h_0 > 0$ s.t.

$$\max_{1\le i\le m(j)} |\lambda_i - \lambda_{ih}| \le \epsilon \quad \text{and} \quad \text{gap}\left(\sum_{i=1}^{m(j)} E_i, \sum_{i=1}^{m(j)} E_{i,h}\right) \le \epsilon \qquad \text{if } h \le h_0.$$

A sufficient (and necessary) condition for eigenvalue convergence is operator norm convergence of the discrete solution operator $K_h P_h$ to K (Kato, Babuska–Osborn, Boffi–Brezzi–Gastaldi):

$W \to W_h$ orthog.

The mixed discretization of the eigenvalue problem converges if

$$\lim_{h\to 0} \|K_h P_h - K\|_{\mathcal{L}(W,W)} = 0.$$

Eigenvalue convergence follows from improved estimates

$$\|u-u_h\| \le c\{E(u)+\eta[E(du)+E(\sigma)]+(\eta^2+\delta)[E(d\sigma)+E(p)]+\mu E(P_{\mathfrak{B}}u)\}$$

$$E(d\sigma)+E(p)+E(P_{\mathfrak{B}}u) \le \|d\sigma\|+\|p\|+\|u\| \le \|f\|$$

$$E(u) \le \delta\|f\|, \qquad E(du)+E(\sigma) \le \eta\|f\|$$

Therefore

$$\|(K-K_hP_h)f\| \le \delta+\eta^2+\mu \to 0$$

Rates of convergence also follow, included doubled convergence rates for eigenvalues…