FM 5012
6th class

Finite difference methods for solving the Black-Scholes equation

Here we are going to study finite difference methods for numerically solving the Black-Scholes equation. We are going to proceed as follows. First, we consider European options and we describe the problem with which the value of the option is modeled, that is, the Black-Scholes partial differential equation with the terminal and boundary conditions corresponding to a call or put option. We then show how to transform the Black-Scholes equation into a "heat" equation and apply a finite difference technique to numerically solve it.

1. The model for European options.

We know that if "S" denotes the price of the underlying asset and "f" denotes the value of the option, we have

(1.1a)
$$\frac{\partial}{\partial t} f + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} f + rS\frac{\partial}{\partial S} f = rf$$

in $(0, S_M) \times (0, T)$, where "σ" is the volatility, and "r" the risk-free interest rate. The above equation is the Black-Scholes equation. We must complete it with terminal and boundary conditions in order to determine the solution we are interested in.

So, for a "call", the terminal condition is

(1.1b)$_{call}$ $$f(S,T) = \max\{S-K, 0\} \qquad \forall\, S \in (0, S_M),$$

where "K" is the strike price, and the boundary conditions are

(1.1c)$_{call}$ $$f(0,t) = 0 \qquad \forall\, t \in (0,T),$$
(1.1d)$_{call}$ $$f(S_M,t) = S_M - K e^{-r(T-t)} \qquad \forall\, t \in (0,T).$$

If we are considering a "put", we have instead

(1.1b)$_{put}$ $$f(S,T) = \max\{K-S, 0\} \qquad \forall\, S \in (0, S_M),$$

and

(1.1c)$_{put}$ $$f(0,t) = K e^{-r(T-t)} \qquad \forall\, t \in (0,T),$$
(1.1d)$_{put}$ $$f(S_M,t) = 0 \qquad \forall\, t \in (0,T),$$

where in this case $S_M = \infty$.

The equation (1.1a) can be transformed into a (backward) "heat" equation in two steps. First, we change variables as follows:

$$S = e^x$$

and obtain that, for $\phi(x,t) := f(S,t)$, the equation (1.1a) becomes

$$\frac{\partial}{\partial t}\phi + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\phi + (r - \tfrac{1}{2}\sigma^2)\frac{\partial}{\partial x}\phi - r\phi = 0$$

We then set

$$\phi(x,t) = e^{\alpha x + \beta t} F(x,t)$$

and obtain that the equation for F is

$$\frac{\partial}{\partial t}F + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}F + (\sigma^2\alpha + r - \tfrac{1}{2}\sigma^2)\frac{\partial}{\partial x}F$$
$$+ \left(\beta + \frac{\alpha^2\sigma^2}{2} + (r - \tfrac{\sigma^2}{2})\alpha - r\right)F = 0.$$

So taking α and β such that the last two terms are equal to zero, we obtain the equation we sought, namely, the backward heat equation

$$\frac{\partial}{\partial t}F + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}F = 0.$$

Note that we can do this provided $\sigma^2 > 0$, otherwise we cannot eliminate the third term. In the case in which $\sigma = 0$, we have

$$\frac{\partial}{\partial t}F + r\frac{\partial F}{\partial x} + (\beta + r\alpha - r)F = 0$$

and taking, for example, $\beta = 0$ and $\alpha = 1$, we obtain the so-called "transport" equation

$$\frac{\partial}{\partial t} F + r \frac{\partial F}{\partial x} = 0.$$

Thus, when the volatility is negligible, the Black-Scholes equation is equivalent not to a heat equation but to a transport equation. These two types of equations are of rather different character and this is why we are going to study them separately.

2. Approximating the solution of the heat equation.

Let us set $\sigma^2 = 2$ and consider the solution f of

(2.1 a) $$\frac{\partial}{\partial t} f + \frac{\partial^2}{\partial S^2} f = 0 \qquad \text{in } (0, S_M) \times (0, T)$$

with terminal condition

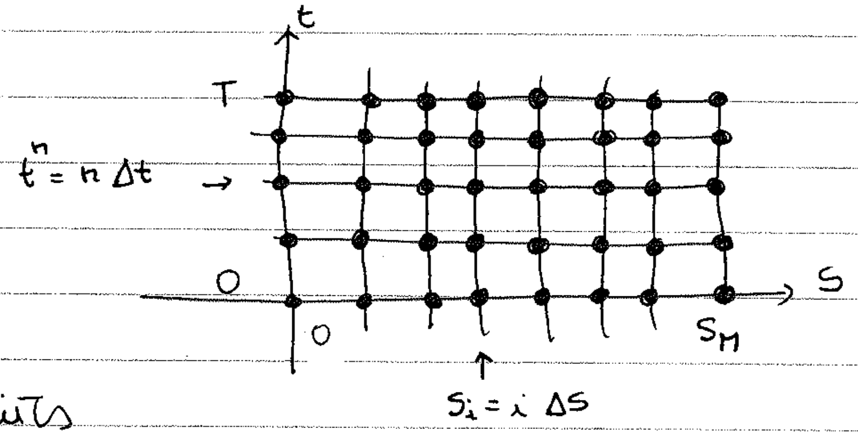
(2.1 b) $$f(S, T) = f_T(S) \qquad \text{for } S \in (0, S_M)$$

and boundary conditions

(2.1 c) $$f(0, t) = f_0(t) \qquad \text{for } t \in (0, T)$$

(2.1 d) $$f(S_M, t) = f_{S_M}(t) \qquad \text{for}$$

To obtain an approximation to "f by using finite difference methods, we proceed as follows. First, we discretize the domain $(0, S_M) \times (0, T)$ by replacing it with the set



of points

$$\{ (S_i, t^n) \quad i = 0, \dots, I \text{ and } n = 0, \dots, N \}$$

as illustrated above. For simplicity we take (S_i, t^n) to be of the form $(i\Delta S, n\Delta t)$, where ΔS and Δt are "stepsizes". We are thus going to find an approximation to f at each point (S_i, t^n) which will be denoted by f_i^n.

Next, we discretize the terminal condition and boundary conditions as follows:

(2.2b) $$f_i^N = f_T(S_i) \qquad i = 1, \dots, I-1,$$

(2.2c) $$f_0^n = f_0(t^n) \qquad n = 0, \dots, N$$

(2.2d) $$f_I^n = f_{S_M}(t^n) \qquad n = 0, \dots, N$$

Finally, we discretize the heat equation. We take the following simple discretization:

(2.2a)
$$\frac{1}{\Delta t}\left(f_i^n - f_i^{n-1}\right) + \frac{1}{\Delta S^2}\left(f_{i+1}^n - 2 f_i^n + f_{i-1}^n\right) = 0$$

for $i = 1, \dots, I-1$ and $n = 1, \dots, N$.

Note that we have to compute the values $\{f_i^{n-1}\}_{i=0}^{I}$ given the values $\{f_i^n\}_{i=0}^{I}$ and that we can do this without inverting any matrix. Indeed, we have

$$f_i^{n-1} = \frac{\Delta t}{\Delta S^2} f_{i+1}^n + \left(1 - 2\frac{\Delta t}{\Delta S^2}\right) f_i^n + \frac{\Delta t}{\Delta S^2} f_{i-1}^n .$$

Because of this, we say that this method is <u>explicit</u>. Next we show that the method is consistent, stable under the condition

$$\frac{\Delta t}{\Delta S^2} \leq \frac{1}{2}$$

and convergent. Moreover, we show that if the ratio $\Delta t / \Delta S^2$ is held constant, the error is no bigger than a constant times Δt.

2.1 <u>Consistency</u>

We say that the method is consistent if the quantity

$$TE_i^n = \frac{1}{\Delta t}\left(f(S_i, t^n) - f(S_i, t^{n-1})\right) + \frac{1}{\Delta S^2}\left(f(S_{i+1}, t^n) - 2 f(S_i, t^n) + f(S_{i-1}, t^n)\right)$$

tends to zero as Δt and ΔS^2 go to zero and (S_i, t^n) go to some point in $(0, S_M) \times (0, T)$, whenever f is an exact solution of the heat equation.

To see that our method is actually consistent, we only have to use Taylor expansions around the point (S_i, t^n). To deal with the first term of TE, we use

$$f(S_i, t^{n-1}) = f(S_i, t^n) + (t^{n-1} - t^n) \frac{\partial}{\partial t} f(S_i, t^n) + \int_{t^n}^{t^{n-1}} (t^{n-1} - \tau) \frac{\partial^2}{\partial t^2} f(S_i, \tau)\, d\tau$$

to obtain that

$$\frac{1}{\Delta t}\left(f(S_i, t^n) - f(S_i, t^{n-1})\right) = \frac{\partial}{\partial t} f(S_i, t^n) - \int_{t^{n-1}}^{t^n} \left(\frac{\tau - t^{n-1}}{\Delta t}\right) \frac{\partial^2}{\partial t^2} f(S_i, \tau)\, d\tau .$$

Note that, assuming that $\frac{\partial^2}{\partial t^2} f(S_i, \tau)$ is continuous for $\tau \in (t^{n-1}, t^n)$, a simple application of the mean value theorem gives that

(2.3a) $$\frac{1}{\Delta t}\left(f(S_i, t^n) - f(S_i, t^{n-1})\right) = \frac{\partial}{\partial t} f(S_i, t^n) - \frac{\Delta t}{2} \frac{\partial^2}{\partial t^2} f(S_i, \tau^n)$$

where τ^n is some number in (t^{n-1}, t^n). As a consequence we have that if Δt goes to zero and t^n, t^{n-1} go to t, the expression

$$\frac{1}{\Delta t}\left(f(S_i, t^n) - f(S_i, t^{n-1})\right) \text{ goes to } \frac{\partial}{\partial t} f(S_i, t),$$

as expected.

Let us now deal with the other term of TE. To do that, we need the following Taylor expansion

$$f(S,t^n) = f(S_i,t^n) + (S-S_i)\frac{\partial}{\partial S} f(S_i,t^n) + \frac{(S-S_i)^2}{2}\frac{\partial^2}{\partial S^2} f(S_i,t^n) + \frac{(S-S_i)^3}{6}\frac{\partial^3}{\partial S^3} f(S_i,t^n) + \int_{S_i}^{S} \frac{(S-\sigma)^3}{6}\frac{\partial^4}{\partial S^4} f(\sigma,t^n)\, d\sigma$$

We then obtain

$$\frac{1}{\Delta S^2}\left(f(S_{i+1},t^n) - 2f(S_i,t^n) + f(S_{i-1},t)\right) = \frac{\partial^2}{\partial S^2} f(S_i,t^n) + \int_{S_{i-1}}^{S_{i+1}} K_i(\sigma)\frac{\partial^4}{\partial S^4} f(\sigma,t^n)\, d\sigma$$

where

$$K_i(\sigma) = \begin{cases} (S_{i+1}-\sigma)^3/6\Delta S^2 & \text{for } \sigma \in (S_i, S_{i+1}), \\ -(S_{i-1}-\sigma)^3/6\Delta S^2 & \text{for } \sigma \in (S_{i-1}, S_i). \end{cases}$$

Again, if we assume that $\frac{\partial^4}{\partial S^4} f(\sigma,t^n)$ is continuous for $\sigma \in (S_{i-1}, S_{i+1})$, we can apply the mean value theorem since $K_i > 0$ for $\sigma \in (S_{i-1}, S_{i+1})$ to obtain

(2.3b)
$$\frac{1}{\Delta s^2}\left(f(s_{i+1},t^n) - 2f(s_i,t^n) + f(s_{i-1},t^n)\right) = \frac{\partial^2}{\partial s^2} f(s_i,t^n) + \frac{\Delta s^2}{12}\frac{\partial^4}{\partial s^4} f(\sigma_i,t^n)$$

for some value σ_i in (s_{i-1}, s_{i+1}). As a consequence, if Δs goes to zero and s_{i+1}, s_i, s_{i-1} go to some point s, then

$$\frac{1}{\Delta s}\left(f(s_{i+1},t^n) - 2f(s_i,t^n) + f(s_{i-1},t^n)\right) \text{ goes to } \frac{\partial^2}{\partial s^2} f(s,t^n),$$

as expected.

If we now insert our expressions (2.3a) and (2.3b) in the definition of TE, we get that

(2.4a)
$$TE_i^n = \frac{\partial f}{\partial t}(s_i,t^n) + \frac{\partial^2}{\partial s^2} f(s_i,t^n) + R_i^n$$

where

(2.4b)
$$R_i^n = -\frac{\Delta t}{2}\frac{\partial^2}{\partial t^2} f(s_i,\tau^n) + \frac{\Delta s^2}{12}\frac{\partial^4}{\partial s^4} f(\sigma_i,t^n).$$

thus, if f is a solution of the heat equation, we have that

(2.5)
$$TE_i^n = R_i^n = -\frac{\Delta t}{2}\frac{\partial^4}{\partial s^4} f(s_i,\tau^n) + \frac{\Delta s^2}{12}\frac{\partial^4}{\partial s^4} f(\sigma_i,t^n)$$

and we see that TE_i^n goes to zero as Δt and ΔS go to zero and (S_i, t^n) go to some point (S, t). this shows that the method is consistent.

2.2 Stability (with respect to the terminal data).

Suppose that we have two solutions of our finite difference method (2.2) which only differ in the terminal condition f_T. Let us denote them by f_1 and f_2. We want to know if a small difference between f_{T1} and f_{T2} induce a small difference between f_1 and f_2.

To do this, we set $\delta = f_1 - f_2$, and note that, by (2.2), it must satisfy the following equations:

(2.6a) $$\frac{1}{\Delta t}(\delta_i^n - \delta_i^{n-1}) + \frac{1}{\Delta S^2}(\delta_{i+1}^n - 2\delta_i^n + \delta_{i-1}^n) = 0 \quad \text{for } i = 1, \dots, I-1 \text{ and } n = 1, \dots, N,$$

(2.6b) $$\delta_i^N = f_{T1}(S_i) - f_{T2}(S_i) \qquad \text{for } i = 1, \dots, I-1$$

and

(2.6c) $$\delta_0^n = 0 \qquad \text{for } n = 0, \dots, N,$$

(2.6d) $$\delta_I^n = 0 \qquad \text{for } n = 0, \dots, N.$$

From these equations, we must deduce our stability result. However, working with a continuous version of these equations is much easier and usually provides us with a "map" that will allow us not to get lost when dealing with the more complicated case of discretized equations.

So, let us consider the following "continuous" version of equations (2.6):

$$\frac{\partial}{\partial t}\delta + \frac{\partial^2}{\partial s^2}\delta = 0 \qquad \text{in } (0, S_M)\times(0,T),$$

$$\delta(s,T) = f_{1T}(s) - f_{2T}(s) \qquad \text{for } s \in (0, S_M),$$

$$\delta(0,t) = 0 \qquad \text{for } t \in (0,T),$$

$$\delta(S_M, t) = 0 \qquad \text{for } t \in (0,T).$$

We are going to show that the L^2 norm of δ at any time $t < T$, namely,

$$\|\delta(t)\| := \left\{\int_0^{S_M} \delta^2(s,t)\, ds\right\}^{1/2}$$

is not bigger than the L^2-norm of the terminal condition. To obtain this simple stability result, we proceed in several steps.

Step 1. We multiply the heat equation by δ to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\delta^2 + \delta \cdot \frac{\partial^2}{\partial s^2}\delta = 0$$

<u>Step 2</u>. We integrate in s from 0 to s_M to obtain

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 + \int_0^{s_M} \delta \cdot \frac{\partial^2}{\partial s^2}\delta = 0.$$

<u>Step 3.</u> We integrate by parts and use the boundary conditions on δ to get

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 - \int_0^{s_M}\left(\frac{\partial \delta}{\partial s}\right)^2 = 0$$

<u>Step 4</u> We integrate in time from t to T to get

$$\frac{1}{2}\|\delta(T)\|^2 - \frac{1}{2}\|\delta(t)\|^2 - \int_t^T\int_0^{s_M}\left(\frac{\partial}{\partial s}\delta(\sigma,\tau)\right)^2 d\sigma\, d\tau = 0$$

<u>Step 5</u> We conclude that

$$\|\delta(t)\| \leq \|\delta(T)\| = \|f_{T1} - f_{T2}\|$$

for all $t \in [0,T]$.

Armed with this "map", we are now ready to deal with our finite difference scheme. We follow each of the steps of the continuous case.

<u>Step 1.</u> We multiply equation (2.6a) by δ_i^n. We get

$$\delta_i^n \frac{1}{\Delta t}\left(\delta_i^n - \delta_i^{n-1}\right) + \delta_i \frac{1}{\Delta s^2}\left(\delta_{i+1}^n - 2\delta_i^n + \delta_{i-1}^n\right) = 0$$

We are not going to touch the second term, but we need to work on the first one to render it a discrete version of $\frac{1}{2}\frac{\partial}{\partial t}\delta^2$. It is not difficult to do this. Indeed, we can write

$$\delta_i^n \frac{1}{\Delta t}\left(\delta_i^n - \delta_i^{n-1}\right) = \frac{1}{2\Delta t}\left({\delta_i^n}^2 - {\delta_i^{n-1}}^2\right) + \frac{1}{2\Delta t}\left(\delta_i^n - \delta_i^{n-1}\right)^2$$

Hence,

$$\frac{1}{2\Delta t}\left({\delta_i^n}^2 - {\delta_i^{n-1}}^2\right) + \frac{1}{2\Delta t}\left(\delta_i^n - \delta_i^{n-1}\right)^2 + \delta_i \frac{1}{\Delta s^2}\left(\delta_{i+1}^n - 2\delta_i^n + \delta_{i-1}^n\right) = 0$$

for $i = 1, \dots, I-1$ and $n = 1, \dots, N$.

<u>Step 2</u>. Now we multiply by Δs and sum over i from 1 to $I-1$. We obtain

$$\frac{1}{2\Delta t}\left(\sum_{i=1}^{I-1} {\delta_i^n}^2 \Delta s - \sum_{i=1}^{I-1} {\delta_i^{n-1}}^2 \Delta s\right) +$$
$$\frac{1}{2\Delta t}\sum_{i=1}^{I-1}\left(\delta_i^n - \delta_i^{n-1}\right)^2 \Delta s +$$
$$\sum_{i=1}^{I-1} \delta_i^n \frac{1}{\Delta s^2}\left(\delta_{i+1}^n - 2\delta_i^n + \delta_{i-1}^n\right) = 0$$

this suggest to take as a discrete L^2-norm,

$$\|\delta^n\| := \left\{\sum_{i=1}^{I-1} (\delta_i^n)^2 \Delta s\right\}^{1/2},$$

so that we have

$$\frac{1}{2}\left(\frac{\|\delta^n\|^2 - \|\delta^{n-1}\|^2}{\Delta t}\right) +$$

$$\frac{1}{2\Delta t}\sum_{i=1}^{I-1}(\delta_i^n - \delta_i^{n-1})^2\,\Delta s +$$

$$\sum_{i=1}^{I-1}\delta_i^n\frac{1}{\Delta s^2}(\delta_{i+1}^n - 2\delta_i^n + \delta_{i-1}^n)\,\Delta s = 0.$$

Step 3. Here we have to carry out a discrete version of the integration by parts. We work on the third term of the left-hand side as follows:

$$\sum_{i=1}^{I-1}\delta_i^n\frac{1}{\Delta s^2}(\delta_{i+1}^n - 2\delta_i^n + \delta_{i-1}^n) = \sum_{i=1}^{I-1}\delta_i^n\frac{1}{\Delta s}\left(\frac{\delta_{i+1}^n-\delta_i^n}{\Delta s} - \frac{\delta_i^n-\delta_{i-1}^n}{\Delta s}\right)\Delta s$$

$$= \sum_{i=1}^{I-1}\left(\frac{\delta_i^n}{\Delta s}\right)\left(\frac{\delta_{i+1}^n-\delta_i^n}{\Delta s}\right)\Delta s - \sum_{i=1}^{I-1}\left(\frac{\delta_i^n}{\Delta s}\right)\left(\frac{\delta_i^n-\delta_{i-1}^n}{\Delta s}\right)\Delta s$$

$$= \sum_{i=1}^{I-1}\left(\frac{\delta_i^n}{\Delta s}\right)\left(\frac{\delta_{i+1}^n-\delta_i^n}{\Delta s}\right)\Delta s - \sum_{i=0}^{I-2}\left(\frac{\delta_{i+1}^n}{\Delta s}\right)\left(\frac{\delta_{i+1}^n-\delta_i^n}{\Delta s}\right)\Delta s$$

Note that we only changed the index "i" to "$i+1$" in the second term. Now we use the boundary conditions (2.6c) and (2.6d) to write

$$\sum_{i=1}^{I-1}\delta_i^n\frac{1}{\Delta s^2}(\delta_{i+1}^n - 2\delta_i^n + \delta_{i+1}^n)$$

$$= \sum_{i=0}^{I-1}\left(\frac{\delta_i^n}{\Delta s}\right)\left(\frac{\delta_{i+1}^n-\delta_i^n}{\Delta s}\right)\Delta s - \sum_{i=0}^{I-1}\left(\frac{\delta_{i+1}^n}{\Delta s}\right)\left(\frac{\delta_{i+1}^n-\delta_i^n}{\Delta s}\right)\Delta s$$

$$= -\sum_{i=0}^{I-1}\left(\frac{\delta_{i+1}^n-\delta_i^n}{\Delta s}\right)^2\Delta s.$$

As a consequence, we have

$$\frac{1}{2}\left(\frac{\|\delta^n\|^2-\|\delta^{n-1}\|^2}{\Delta t}\right)+Ⓗ^n=0,$$

where

$$Ⓗ^n=\frac{1}{2\Delta t}\sum_{i=1}^{I-1}(\delta_i^n-\delta_i^{n-1})^2\Delta s-\sum_{i=0}^{I-1}\left(\frac{\delta_{i+1}^n-\delta_i^n}{\Delta s}\right)^2\Delta s .$$

Step 3'. Before proceeding, we need to obtain a better expression for $Ⓗ^n$. Indeed, note that, unlike the continuous case, here it is not clear that $Ⓗ^n$ is a nonpositive quantity. If this is not the case, we would not be able to continue the "map" provided by the continuous case. We have thus to try to show that $Ⓗ^n$ is indeed a non-negative quantity.

To do this, we begin by using equation (2.6a) to write

$$Ⓗ^n=\frac{1}{2\Delta t}\sum_{i=1}^{I-1}\Delta t^2\left(\frac{\delta_{i+1}^n-2\delta_i^n+\delta_{i-1}^n}{\Delta s^2}\right)^2\cdot\Delta s$$
$$-\sum_{i=0}^{I-1}\left(\frac{\delta_{i+1}^n-\delta_i^n}{\Delta s}\right)^2\Delta s$$

and setting $z_i=(\delta_{i+1}^n-\delta_i^n)/\Delta s$, we rewrite the above expression as

$$Ⓗ^n=\frac{\Delta t}{2\Delta s^2}\sum_{i=1}^{I-1}(z_i-z_{i-1})^2\Delta s-\sum_{i=0}^{I-1}(z_i)^2\Delta s .$$

Now note that

$$\sum_{i=1}^{I-1} (z_i - z_{i-1})^2 \Delta s = \sum_{i=1}^{I-1} (z_i^2 + z_{i-1}^2 - 2 z_i z_{i-1}) \Delta s$$

and that

$$z_i \, z_{i-1} = \left(\frac{z_i + z_{i-1}}{2}\right)^2 - \left(\frac{z_i - z_{i-1}}{2}\right)^2 .$$

This implies that

$$\sum_{i=1}^{I-1} (z_i - z_{i-1})^2 \Delta s = \sum_{i=1}^{I-1} (z_i^2 + z_{i-1}^2) \Delta s$$

$$- \frac{1}{2} \sum_{i=1}^{I-1} (z_i + z_{i-1})^2 \Delta s$$

$$+ \frac{1}{2} \sum_{i=1}^{I-1} (z_i - z_{i-1})^2 \Delta s$$

and so

$$\sum_{i=1}^{I-1} (z_i - z_{i-1})^2 \Delta s = 2 \sum_{i=1}^{I-1} (z_i^2 + z_{i-1}^2) \Delta s$$

$$- \sum_{i=1}^{I-1} (z_i + z_{i-1})^2 \Delta s .$$

Finally,

$$\sum_{i=1}^{I-1} (z_i - z_{i-1})^2 \Delta s = 4 \sum_{i=0}^{I-1} z_i^2 \Delta s - 2 z_0^2 \Delta s - 2 z_{I-1}^2 \Delta s$$

$$- \sum_{i=1}^{I-1} (z_i + z_{i-1})^2 \Delta s .$$

This implies that

$$\Theta^n = -\left(1 - 2\frac{\Delta t}{\Delta s^2}\right)\sum_{i=0}^{I-1} z_i^2\,\Delta s$$

$$- 2\left(z_0^2 + z_{I-1}^2\right)\Delta s - \sum_{i=1}^{I-1}\left(z_i + z_{i-1}\right)^2 \Delta s$$

and we can see that if $\frac{\Delta t}{\Delta s^2} \leq \frac{1}{2}$, $\Theta^n \leq 0$, just as we wanted. We can now proceed with our "map".

<u>Step 4.</u> We multiply by Δt and sum from $n+1$ to N to obtain

$$\frac{1}{2}\|\delta^N\|^2 - \frac{1}{2}\|\delta^n\|^2 + \sum_{m=n+1}^{N}\Theta^m\,\Delta t = 0$$

<u>Step 5.</u> Since $\Theta^m \leq 0$ for $\frac{\Delta t}{\Delta s^2} \leq 1/2$, we have that

$$\|\delta^n\| \leq \|\delta^N\| = \|f_{T1} - f_{T2}\|$$

for all $n \leq N$.

We have thus shown that the method is stable with respect to the initial data.

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2.3. Convergence. Here we show that the approximation provided by our finite difference method converges to the exact solution as the parameters Δt and Δs go to zero, provided, of course, $\frac{\Delta t}{\Delta s^2} \le \frac{1}{2}$. We are going to do that by estimating the discrete L^2-norm of the error at any given time t^n,

$$\| e^n \| := \left\{ \sum_{i=1}^{I-1} (e_i^n)^2 \, \Delta s \right\}^{1/2}$$

where $\quad e_i^n = f(s_i, t^n) - f_i^n$.

As we did when studying the stability of the numerical scheme with respect to the terminal data, we begin by writing the equations satisfied by the error. They are the following:

(2.7a) $$\frac{1}{\Delta t}(e_i^n - e_i^{n-1}) + \frac{1}{\Delta s^2}(e_{i+1}^n - 2e_i^n + e_{i-1}^n) = R_i^n$$
for $i = 1, \dots, I-1$
and $n = 1, \dots, N$,

(2.7b) $$e_i^N = 0 \qquad \text{for } i = 1, \dots, I-1,$$

and

(2.7c) $$e_0^n = 0 \qquad \text{for } n = 0, \dots, N,$$

(2.7d) $$e_I^n = 0 \qquad \text{for } n = 0, \dots, N.$$

Here R_i^n is nothing but the expression given by (2.4b)!

Now we consider the continuous version of these equations, namely,

$$\frac{\partial}{\partial t} e + \frac{\partial^2}{\partial s^2} e = R \qquad \text{in } (0, S_M) \times (0, T),$$

$$e(s, T) = 0 \qquad \text{for } s \in (0, S_M),$$
$$e(0, t) = 0 \qquad \text{for } t \in (0, T),$$
$$e(S_M, t) = 0 \qquad \text{for } t \in (0, T),$$

and obtain our "map" to follow when dealing with the discrete problem. We proceed as in the study of the stability with respect to the initial data.

<u>Step 1</u>. We multiply the equation by e to obtain

$$\frac{1}{2} \frac{\partial}{\partial t} e^2 + e \cdot \frac{\partial^2}{\partial s^2} e = R \cdot e$$

<u>Step 2</u>. We integrate in s from 0 to S_M:

$$\frac{1}{2} \frac{d}{dt} \| e \|^2 + \int_0^{S_M} e \cdot \frac{\partial^2}{\partial s^2} e = \int_0^{S_M} R \cdot e$$

<u>Step 3</u> We integrate by parts and use the boundary conditions on e to get

$$\frac{1}{2} \frac{d}{dt} \| e \|^2 - \int_0^{S_M} \left(\frac{\partial}{\partial s} e \right)^2 = \int_0^{S_M} R \cdot e$$

Step 4. We integrate in time to get

$$\frac{1}{2}\|e(T)\|^2 - \frac{1}{2}\|e(t)\|^2 - \int_t^T\int_0^{S_M}\left(\frac{\partial}{\partial s}e\right)^2 = \int_t^T\int_0^{S_M} R\cdot e\,.$$

Step 5. This implies that, since $e(T)=0$,

$$\frac{1}{2}\|e(t)\|^2 \leq -\int_t^T\int_0^{S_M} R\cdot e,$$

and by the Cauchy-Schwarz inequality, that

(2.8) $$\|e(t)\|^2 \leq 2\int_t^T \|R(\tau)\|\cdot\|e(\tau)\|\,d\tau$$

for all $t \in (0,T)$.

Step 6. Setting

$$\phi(t) = 2\int_t^T \|R(\tau)\|\cdot\|e(\tau)\|\,d\tau$$

we have that

$$\phi(t) \leq 2\int_t^T \|R(\tau)\|\,d\tau \max_{t\leq\tau\leq T}\|e(\tau)\|$$

$$\leq 2\int_t^T \|R(\tau)\|\,d\tau\cdot \max_{t\leq\tau\leq T}\sqrt{\phi(\tau)}$$

and so

$$\phi(t) \leq 2 \int_t^T \| R(\tau) \| \, d\tau \cdot \sqrt{\phi(t)} .$$

this implies that

$$\sqrt{\phi(t)} \leq 2 \int_t^T \| R(\tau) \| \, d\tau$$

and hence that

$$\| e(t) \| \leq 2 \int_t^T \| R(\tau) \| \, d\tau .$$

We are now ready to deal with the discrete case.

<u>Step 1.</u> We multiply equation (2.7a) by e_i^n to get

$$e_i^n \frac{1}{\Delta t} (e_i^n - e_i^{n-1})$$

$$+ e_i^n \frac{1}{\Delta s^2} (e_{i+1}^n - 2 e_i^n + e_{i-1}^n) = e_i^n R_i^n$$

for $i = 1, \dots, I-1$ and $n = 0, \dots, N-1$.

Now, proceeding as for the previous stability study, we get

$$\frac{1}{2\Delta t}\left({e_i^n}^2 - {e_i^{n-1}}^2\right) + \frac{1}{2\Delta t}\left(e_i^n - e_i^{n-1}\right)^2$$
$$+ e_i^n \cdot \frac{1}{\Delta s^2}\left(e_{i+1}^n - 2e_i^n + e_{i-1}^n\right) = R_i^n e_i^n .$$

Step 2. We multiply by Δs and sum over i to obtain

$$\frac{1}{2\Delta t}\left(\sum_{i=1}^{I-1} {e_i^n}^2 \Delta s - \sum_{i=1}^{I-1} {e_i^{n-1}}^2 \Delta s\right)$$
$$+ \frac{1}{2\Delta t}\sum_{i=1}^{I-1}\left(e_i^n - e_i^{n-1}\right)^2 \Delta s$$
$$+ \sum_{i=1}^{I-1} e_i^n \frac{1}{\Delta s^2}\left(e_{i+1}^n - 2e_i^n + e_{i-1}^n\right)\Delta s = \sum_{i=1}^{I-1} R_i^n e_i^n \Delta s .$$

Step 3. We carry out a discrete integration by parts to obtain

$$\frac{1}{2\Delta t}\left(\|e^n\|^2 - \|e^{n-1}\|^2\right) + \Psi^n = \sum_{i=1}^{I-1} R_i^n e_i^n \Delta s$$

where,

$$\Psi^n = \frac{1}{2\Delta t}\sum_{i=1}^{I-1}\left(e_i^n - e_i^{n-1}\right)^2 \Delta s - \sum_{i=0}^{I-1}\left(\frac{e_{i+1}^n - e_i^n}{\Delta s}\right)^2 \Delta s;$$

see "Step 3" of the stability (with respect to the initial data) estimate.

By equation (2.7a),

$$e_i^n - e_i^{n-1} = -\frac{\Delta t}{\Delta s^2}\left(e_{i+1}^n - 2e_i^n + e_{i-1}^n\right) + R_i^n \Delta t$$

and so

$$\Psi^n = \frac{1}{2\Delta t}\sum_{i=1}^{I-1}\left(\frac{\Delta t}{\Delta s^2}\left(e_{i+1}^n - 2e_i^n + e_{i-1}^n\right) - \Delta t R_i^n\right)^2 \Delta s$$

$$- \sum_{i=0}^{I-1}\left(\frac{e_{i+1}^n - e_i^n}{\Delta s}\right)^2 \Delta s.$$

$$= \frac{1}{2\Delta t}\sum_{i=1}^{I} \Delta t^2 \left(\frac{e_{i+1}^n - 2e_i^n + e_{i-1}^n}{\Delta s^2}\right)^2 \Delta s$$

$$- \sum_{i=0}^{I-1}\left(\frac{e_{i+1}^n - e_i^n}{\Delta s}\right)^2 \Delta s$$

$$- \frac{1}{2\Delta t}\sum_{i=1}^{I-1} 2\Delta t^2 \left(\frac{e_{i+1}^n - 2e_i^n + e_{i-1}^n}{\Delta s^2}\right) R_i^n \Delta s$$

$$+ \frac{1}{2\Delta t}\sum_{i=1}^{I-1} \Delta t^2 (R_i^n)^2 \Delta s.$$

Setting $z_i = (e_{i+1}^n - e_i^n)/\Delta s$, we get

$$\Psi^n = \frac{\Delta t}{2\Delta s^2}\sum_{i=1}^{I-1}(z_i - z_{i-1})^2 \Delta s - \sum_{i=0}^{I-1} z_i^2 \Delta s$$

$$- \frac{\Delta t}{\Delta s}\sum_{i=1}^{I-1}(z_i - z_{i-1}) R_i^n \Delta s$$

$$+ \frac{\Delta t}{2}\sum_{i=1}^{I-1}(R_i^n)^2 \Delta s$$

From step 3' of our stability study (see pages 15, 16 and 17), we get that

$$\Psi^n = -\left(1 - 2\frac{\Delta t}{\Delta s^2}\right)\sum_{i=0}^{I-1} z_i^2\,\Delta s - 2\left(z_0^2 + z_{I-1}^2\right)\Delta s$$

$$-\sum_{i=1}^{I-1}\left(z_i + z_{i-1}\right)^2\Delta s$$

$$-\frac{\Delta t}{\Delta s}\sum_{i=1}^{I-1}\left(z_i - z_{i-1}\right)R_i^n\,\Delta s + \frac{\Delta t}{2}\sum_{i=1}^{I-1}\left(R_i^n\right)^2\Delta s.$$

Let us now use the fact that, by equation (2.7a),

$$\frac{1}{\Delta s}\left(z_i - z_{i-1}\right) = R_i^n - \frac{e_i^n - e_i^{n-1}}{\Delta t},$$

to write

$$-\frac{\Delta t}{\Delta s}\sum_{i=1}^{I-1}\left(z_i - z_{i-1}\right)R_i^n\,\Delta s = -\Delta t\sum_{i=1}^{I-1}\left(R_i^n\right)^2\Delta s$$

$$+\sum_{i=1}^{I-1} R_i^n e_i^n\,\Delta s$$

$$-\sum_{i=1}^{I-1} R_i^n e_i^{n-1}\,\Delta s.$$

This implies that

$$\Psi^n = -\left(1 - 2\frac{\Delta t}{\Delta s^2}\right)\sum_{i=0}^{I-1} z_i^2\,\Delta s - 2\left(z_0^2 + z_{I-1}^2\right)\Delta s$$

$$-\sum_{i=1}^{I-1}\left(z_i + z_{i-1}\right)^2\Delta s$$

$$+\sum_{i=1}^{I-1} R_i^n e_i^n\,\Delta s - \sum_{i=1}^{I-1} R_i^n e_i^{n-1}\,\Delta s - \frac{\Delta t}{2}\sum_{i=1}^{I-1}\left(R_i^n\right)^2\Delta s,$$

and so

$$\frac{1}{2\Delta t}\left(\|e^n\|^2 - \|e^{n-1}\|^2\right) + \circledast^n = \sum_{i=1}^{I-1} R_i^n e_i^{n-1} \Delta s$$

where

$$\begin{aligned}\circledast^n = & -\left(1 - 2\frac{\Delta t}{\Delta s^2}\right) \sum_{i=0}^{I-1} z_i^2 \Delta s \\ & - 2(z_0^2 + z_{I-1}^2)\,\Delta s \\ & - \sum_{i=1}^{I-1} (z_i + z_{i-1})\,\Delta s \\ & - \frac{\Delta t}{2} \sum_{i=1}^{I-1} (R_i^n)^2 \Delta s .\end{aligned}$$

Note that $\circledast^n \leq 0$ if $\frac{\Delta t}{\Delta s^2} \leq \frac{1}{2}$.

Step 4. Multiplying by Δt and adding on n, we get

$$\frac{1}{2}\|e^N\|^2 - \frac{1}{2}\|e^n\|^2 + \sum_{m=n+1}^{N} \circledast^m \Delta t = \sum_{m=n+1}^{N} \sum_{i=1}^{I} R_i^m e_i^{m-1} \Delta s\, \Delta t.$$

Step 5. Since by the terminal condition (2.75), $e^N = 0$, and since $\circledast^n \leq 0$ if $\frac{\Delta t}{\Delta s^2} \leq \frac{1}{2}$, we get that

$$\frac{1}{2}\|e^n\|^2 \leq -\sum_{m=n+1}^{N} \sum_{i=1}^{I} R_i^m e_i^{m-1} \Delta s\, \Delta t$$

for $n = 0, \ldots, N-1$. Applying the Cauchy-Schwarz inequality, we get

$$\frac{1}{2}\|e^n\|^2 \le \sum_{m=n+1}^{N} \|R^m\| \, \|e^{m-1}\| \, \Delta t$$

for $n=0,\dots,N-1$. This implies that

$$\|e^n\|^2 \le \phi^n \qquad \text{for } n=0,\dots,N-1.$$

where $\phi^n = 2\sum_{m=n+1}^{N} \|R^m\| \, \|e^{m-1}\| \, \Delta t$.

Step 6. We have

$$\phi^n \le 2\sum_{m=n+1}^{N} \|R^m\| \Delta t \cdot \max_{n\le m\le N-1} \|e^m\|$$

and so

$$\phi^n \le 2\sum_{m=n+1}^{N} \|R^m\| \Delta t \cdot \max_{n\le m\le N-1} \sqrt{\phi^m}.$$

This implies that

$$\phi^n \le 2\sum_{m=n+1}^{N} \|R^m\| \Delta t \cdot \sqrt{\phi^n}$$

and so

$$\sqrt{\phi^n} \le 2\sum_{m=n+1}^{N} \|R^m\| \Delta t.$$

we can conclude that

$$\| e^n \| \leq 2 \sum_{m=n}^{N-1} \| R^m \| \Delta t$$

for $n = 0, \ldots, N-1$.

If we maintain the ratio $\frac{\Delta t}{\Delta s^2} = \lambda$ fixed as Δt and Δs go to zero, for $\lambda \leq 1/2$, we have, by (2.5), that

$$R_i^n = \Delta t \left(-\frac{1}{2} \frac{\partial^4}{\partial s^4} f(s_i, \tau^n) + \frac{1}{12\lambda} \frac{\partial^4}{\partial s^4} f(\sigma_i, t^n) \right),$$

and so

$$\| e^n \| \leq \Delta t \cdot C \qquad \text{for} \quad n = 0, \ldots, N-1,$$

where

$$C = 2 \sum_{m=n}^{N-1} \left\{ \sum_{i=1}^{I-1} \left(-\frac{1}{2} \frac{\partial^4}{\partial s^4} f(s_i, \tau^n) + \frac{1}{12\lambda} \frac{\partial^4}{\partial s^4} f(\sigma_i, t^n) \right)^2 \Delta s \right\}^{1/2} \Delta t.$$

Since,

$$C \approx 2 \int_{t^n}^{T} \left| \frac{1}{2} - \frac{1}{12\lambda} \right| \left\| \frac{\partial^4}{\partial s^4} f(t) \right\| dt,$$

we have that the error is of order Δt !

Let us end by estimating the number of operations of the method. Since the value f_i^{n-1} can be obtained by using three multiplications and two sums, that is, by using five operations, we need a total of

$$5 \cdot (I-1) \cdot N \sim 5 I N$$

operations. Since $I = S_M / \Delta S$ and $N = T / \Delta t$, the number of operations is

$$(5 \cdot S_M \cdot T) / (\Delta S \cdot \Delta t)$$

or, since the ratio $\lambda = \Delta t / \Delta S^2$ is kept fixed,

$$(5 \cdot S_M \cdot T) / (\lambda \, \Delta S^3).$$

This means that each time we double the number of points I, or, equivalently, divide ΔS by 2, our number of operations is multiplied by 8.

Next, we show that by using an implicit scheme, the number of operations can be significantly reduced.

3. An implicit scheme for the heat equation.

Here we consider the following scheme for the problem (2.1):

(3.1a) $$\frac{1}{\Delta t}\left(f_i^n - f_i^{n-1}\right) + \frac{1}{\Delta S^2}\left(f_{i+1}^{n-1} - 2f_i^{n-1} + f_{i-1}^{n-1}\right) = 0$$

for $i = 1, \dots, I-1$ and $n = 1, \dots, N$,

(3.1b) $$f_i^N = f_T(S_i)$$ for $i = 1, \dots, I-1$,

and

(3.1c) $$f_0^n = f_0(t^n)$$ for $n = 0, \dots, N$,

(3.1d) $$f_I^n = f_{S_M}(t^n)$$ for $n = 0, \dots, N$.

Note that, unlike our explicit scheme (2.2), this scheme requires the inversion of a matrix in order to compute the values $\{f_i^{n-1}\}_{i=1}^{I-1}$ from the already known values $\{f_i^n\}_{i=0}^{I}$. Indeed, we can rewrite the equation (3.1a) as

$$-\frac{\Delta t}{\Delta S^2} f_{i-1}^{n-1} + \left(1 + 2\frac{\Delta t}{\Delta S^2}\right) f_i^{n-1} - \frac{\Delta t}{\Delta S^2} f_{i+1}^{n-1} = f_i^n$$

If we set $\lambda = \Delta t / \Delta S^2$, this simply becomes

$$-\lambda f_{i-1}^{n-1} + (1 + 2\lambda) f_i^{n-1} - \lambda f_{i+1}^{n-1} = f_i^n$$

for $i = 1, \dots, I-1$.

or, in matrix notation

$$
\begin{array}{c}
\text{I-1 rows} \updownarrow
\end{array}
\begin{bmatrix}
-\lambda & (1+2\lambda) & -\lambda & 0 & \cdots & 0 \\
0 & -\lambda & (1+2\lambda) & -\lambda & \cdots & 0 \\
0 & 0 & -\lambda & (1-2\lambda) & \cdots & 0 \\
 & & \cdots & & & \\
0 & 0 & 0 & 0 & \cdots & -\lambda
\end{bmatrix}
\begin{bmatrix}
f_0^{n-1} \\ f_1^{n-1} \\ f_2^{n-1} \\ f_3^{n-1} \\ \vdots \\ f_I^{n-1}
\end{bmatrix}
=
\begin{bmatrix}
f_1^n \\ f_2^n \\ f_3^n \\ \vdots \\ f_{I-1}^n
\end{bmatrix}.
$$

$\longleftrightarrow$ I+1 columns

Since f_0^{n-1} and f_I^{n-1} are given by the boundary conditions (3.1c) and (3.1d) respectively, we can rewrite further the above matrix equation as follows:

$$
\begin{array}{c}
\text{I-1 rows} \updownarrow
\end{array}
\begin{bmatrix}
(1+2\lambda) & -\lambda & 0 & \cdots & 0 \\
-\lambda & (1+2\lambda) & -\lambda & \cdots & 0 \\
0 & -\lambda & (1+2\lambda) & \cdots & 0 \\
 & & \cdots & & \\
0 & 0 & 0 & \cdots & (1+2\lambda)
\end{bmatrix}
\begin{bmatrix}
f_1^{n-1} \\ f_2^{n-1} \\ f_3^{n-1} \\ \vdots \\ f_{I-1}^{n-1}
\end{bmatrix}
=
\begin{bmatrix}
f_1^n + \lambda f_0^{n-1} \\ f_2^n \\ f_n^3 \\ \vdots \\ f_{I-1}^{n-1} + \lambda f_I^{n-1}
\end{bmatrix}
$$

$\longleftrightarrow$ I-1 columns

This is the matrix equation we must solve at each iteration in time. Since it is tridiagonal, its LU-decomposition can be easily computed at the very beginning. In this way, the number of operations needed at each time iteration is $2\cdot(I-1)$ times a sum, a multiplication and a division, that is, $6(I-1)$ operations. This is comparable with the number of operations needed for the explicit scheme. However, we are going to see that we do not need to carry out that many time steps.

Of course, we still need to investigate if the matrix under consideration is invertible. This issue can be deduced from our stability study, but also from a simple argument. If a matrix A is a square matrix, it is known that it is invertible if and only if the only solution of the equation $Ax=0$ is $x=0$. If A is the matrix we are considering, a simple computation gives us that

$$
\begin{aligned}
x^T A x &= (1+2\lambda)\, x_1^2 - 2\lambda\, x_1 x_2 \\
&\quad + (1+2\lambda)\, x_2^2 - 2\lambda\, x_2 x_3 \\
&\quad + \cdots \\
&\quad + (1+2\lambda)\, x_{I-1}^2 \\
&= x_1^2 + \lambda\,(x_1-x_2)^2 + \lambda\,(x_2-x_3)^2 + \cdots \\
&\quad + \lambda\,(x_{I-2}-x_{I-1})^2 + x_{I-1}^2
\end{aligned}
$$

and if $Ax=0$, we obtain immediately that $x=0$. Our matrix is thus invertible.

Let us show that this scheme is consistent, stable and hence convergent.

3.1 <u>Consistency.</u>

In this case, we have that

$$TE_i^n = \frac{1}{\Delta t}\left(f(S_i,t^n) - f(S_i,t^{n-1})\right) + \frac{1}{\Delta S^2}\left(f(S_{i+1},t^{n-1}) - 2f(S_i,t^{n-1}) + f(S_{i-1},t^{n-1})\right)$$

If we carry out Taylor expansions around the point (S_i, t^{n-1}), we obtain that

$$\frac{1}{\Delta t}\left(f(S_i,t^n) - f(S_i,t^{n-1})\right) = \frac{\partial}{\partial t} f(S_i,t^{n-1}) + \int_{t^{n-1}}^{t^n} \left(\frac{t^n-\tau}{\Delta t}\right) \frac{\partial^2}{\partial t^2} f(S_i,\tau)\,d\tau$$

and, as we saw in section 2.2 for the explicit scheme,

$$\frac{1}{\Delta S^2}\left(f(S_{i+1},t^{n-1}) - 2f(S_i,t^{n-1}) + f(S_{i-1},t^{n-1})\right) = \frac{\partial^2}{\partial S^2} f(S_i,t^{n-1}) + \int_{S_{i-1}}^{S_i} K_i(\sigma) \frac{\partial^4}{\partial S^4} f(\sigma,t^{n-1})\,d\sigma$$

where K_i is the same function obtained for the explicit scheme; see page 8. Hence

$$TE_i^n = \frac{\partial f}{\partial t}(S_i,t^{n-1}) + \frac{\partial^2}{\partial S^2} f(S_i,t^{n-1}) + R_i^{n-1}$$

where

(3.2) $$R_i^{n-1} = + \frac{\Delta t}{2} \frac{\partial^2}{\partial t^2} f(s_i, \tau^n) + \frac{\Delta S^2}{12} \frac{\partial^4}{\partial S^4} f(\sigma_i, t^{n-1})$$

for some τ^n in (t^{n-1}, t^n) and some σ_i in (S_{i-1}, S_{i+1}). If "f" is a solution of the heat equation, we get that

$$TE_i^n = R_i^{n-1} = \frac{\Delta t}{2} \frac{\partial^4}{\partial S^4} f(S_i, \tau^n) + \frac{\Delta S^2}{12} \frac{\partial^4}{\partial S^4} f(\sigma_i, t^{n-1})$$

and we see that TE_i^n goes to zero as the parameters Δt and ΔS go to zero and (S_i, t^{n-1}) goes to some point in $(0, S_M) \times (0, T)$. this shows that the method is consistent.

3.2 Stability (with respect to the initial data).
In this case our $\delta = f_1 - f_2$ satisfies the following equation:

(3.3a) $$\frac{1}{\Delta t}(\delta_i^n - \delta_i^{n-1}) + \frac{1}{\Delta S^2}(\delta_{i+1}^{n-1} - 2\delta_i^n + \delta_{i-1}^n) = 0 \quad \text{for } i = 1, \dots, I-1 \text{ and } n = 1, \dots, N,$$

(3.3b) $$\delta_i^N = f_{T1}(S_i) - f_{T2}(S_i) \quad \text{for } i = 1, \dots, I-1,$$

and

(3.3c) $$\delta_0^n = 0 \quad \text{for } n = 0, \dots, N,$$

(3.3d) $$\delta_I^n = 0 \quad \text{for } n = 0, \dots, N.$$

We follow the same steps followed in section (2.2).

Step 1. We multiply equation (3.3a) by δ_i^{n-1}, not by δ_i^n as in the previous case. We get

$$\delta_i^{n-1} \cdot \frac{1}{\Delta t}(\delta_i^n - \delta_i^{n-1}) + \delta_i^{n-1} \frac{1}{\Delta S^2}(\delta_{i+1}^{n-1} - 2\delta_i^{n-1} + \delta_{i-1}^{n-1}) = 0,$$

and since

$$\delta_i^{n-1} \frac{1}{\Delta t}(\delta_i^n - \delta_i^{n-1}) = \frac{1}{\Delta t}(\delta_i^{n^2} - \delta_i^{n-1^2}) - \frac{1}{2\Delta t}(\delta_i^n - \delta_i^{n-1})^2,$$

we readily obtain

$$\frac{1}{2\Delta t}(\delta_i^{n^2} - \delta_i^{n-1^2}) - \frac{1}{2\Delta t}(\delta_i^n - \delta_i^{n-1})^2 + \delta_i^{n-1} \frac{1}{\Delta S^2}(\delta_{i+1}^{n-1} - 2\delta_i^{n-1} + \delta_{i-1}^{n-1}) = 0.$$

for $i = 1, \dots, I-1$ and $n = 1, \dots, N$.

Note that the sign of the second term is now negative, unlike the case of the explicit scheme!

Step 2. Multiplying by ΔS and adding over i from 1 to $I-1$, we get

$$\frac{1}{2\Delta t}(\|\delta^n\|^2 - \|\delta^{n-1}\|^2) -$$
$$\frac{1}{2\Delta t} \sum_{i=1}^{I-1} (\delta_i^n - \delta_i^{n-1})^2 \Delta S +$$
$$\sum_{i=1}^{I-1} \delta_i^{n-1} \frac{1}{\Delta S^2}(\delta_{i+1}^{n-1} - 2\delta_i^{n-1} + \delta_{i-1}^{n-1}) \Delta S = 0$$

Step 3. Here we carry out the same discrete integration by parts we did for the explicit scheme to get

$$\frac{1}{2}\left(\frac{\|\delta^n\|^2-\|\delta^{n-1}\|^2}{\Delta t}\right)+\Theta^n=0,$$

where

$$\Theta^n=-\frac{1}{2\Delta t}\sum_{i=1}^{I}(\delta_i^n-\delta_i^{n-1})^2\Delta s-\sum_{i=0}^{I-1}\left(\frac{\delta_{i+1}^{n-1}-\delta_i^{n-1}}{\Delta s}\right)^2\Delta s.$$

Step 4. Multiplying by Δt and summing in n, we get

$$\frac{1}{2}\|\delta^N\|^2-\frac{1}{2}\|\delta^n\|^2+\sum_{m=n+1}^{N}\Theta^m\Delta t=0.$$

Step 5. Since $\Theta^m\le 0$, we obtain that

$$\|\delta^n\|\le\|\delta^N\|=\|f_{1T}-f_{2T}\|$$

for all $n\le N$.

We have thus shown the stability of the method. Note that this time there is no restriction on the ratio $\Delta t/\Delta s^2$, as for the explicit scheme!

3.3 <u>Convergence</u>. In this case, the equations for the error are the following:

(3.4a) $$\frac{1}{\Delta t}(e_i^n - e_i^{n-1}) + \frac{1}{\Delta S^2}(e_{i+1}^{n-1} - 2e_i^{n-1} + e_{i-1}^{n-1}) = R_i^{n-1}$$
for $i = 1, \dots, I-1$
and $n = 1, \dots, N$,

(3.4b) $$e_i^N = 0$$ for $i = 1, \dots, I-1$,

and

(3.4c) $$e_0^n = 0$$ for $n = 0, \dots, N$,

(3.4d) $$e_I^n = 0$$ for $n = 0, \dots, N$.

Here, R_i^n is given by (3.2)!

We follow the steps of section 2.3.

<u>Step 1</u>. We multiply (3.4a) by e_i^{n-1} and get

$$\frac{1}{2\Delta t}({e_i^n}^2 - {e_i^{n-1}}^2) - \frac{1}{2\Delta t}(e_i^n - e_i^{n-1})^2$$
$$+ e_i^{n-1} \cdot \frac{1}{\Delta S^2}(e_{i+1}^{n-1} - 2e_i^{n-1} + e_{i-1}^{n-1}) = R_i^{n-1} e_i^{n-1}.$$

<u>Step 2</u>. We multiply by ΔS and sum over i to get

$$\frac{1}{2\Delta t}(\|e^n\|^2 - \|e^{n-1}\|^2) - \frac{1}{2\Delta t}\sum_{i=1}^{I-1}(e_i^n - e_i^{n-1})^2$$
$$+ \sum_{i=1}^{I-1} e_i^{n-1} \frac{1}{\Delta S^2}(e_{i+1}^{n-1} - 2e_i^{n-1} + e_{i-1}^{n-1})\,\Delta S = \sum_{i=1}^{I-1} R_i^{n-1} e_i^{n-1}\,\Delta S.$$

Step 3. We carry out the discrete integration by parts to obtain

$$\frac{1}{2\Delta t}\left(\|e^n\|^2 - \|e^{n-1}\|^2\right) + \Theta^n = \sum_{i=1}^{I-1} R_i^{n-1} e_i^{n-1} \Delta s$$

where $\Theta^n = -\sum_{i=0}^{I-1} \left((e_{i+1}^{n-1} - e_i^{n-1})/\Delta s\right)^2 \Delta s.$

Step 4. We multiply by Δt and sum over n to get

$$\frac{1}{2}\left(\|e^N\|^2 - \|e^n\|^2\right) + \sum_{m=n+1}^{N} \Theta^m \Delta t$$
$$= \sum_{m=n+1}^{N} \sum_{i=1}^{I-1} R_i^{m-1} e_i^{m-1} \Delta s \cdot \Delta t.$$

Step 5 Since $e^N = 0$ by (3.4b) and $\Theta^m \le 0$, we get

$$\frac{1}{2}\|e^n\|^2 \le -\sum_{m=n+1}^{N} \sum_{i=1}^{I-1} R_i^{m-1} e_i^{m-1} \Delta s \cdot \Delta t,$$

and by the Cauchy-Schwarz inequality,

(3.5) $$\|e^n\|^2 \le 2\sum_{m=n+1}^{N} \|R^{m-1}\| \, \|e^{m-1}\| \, \Delta t$$

for $n = 0, \ldots, N-1$.

Step 6. Setting

$$\phi^n = 2\sum_{m=n}^{N} \|R^{m-1}\| \, \|e^{m-1}\| \, \Delta t,$$

we get that

$$\phi^n \leq 2 \sum_{m=n}^{N} \|R^{m-1}\| \Delta t \cdot \max_{n-1 \leq m \leq N-1} \|e^m\|$$
$$\leq 2 \sum_{m=n}^{N} \|R^{m-1}\| \Delta t \cdot \max_{n-1 \leq m \leq N-1} \sqrt{\phi^{m+1}}$$
$$\leq 2 \sum_{m=n}^{N} \|R^{m-1}\| \Delta t \qquad \sqrt{\phi^n}$$

and so

$$\|e^n\| \leq \sqrt{\phi^n} \leq 2 \sum_{m=n}^{N} \|R^{m-1}\| \Delta t.$$

this implies that

$$\|e^n\| \lessapprox C_1 \Delta t + C_2 \Delta s^2 \qquad \text{for } n = 0, \dots, N-1$$

where

$$C_1 = \sum_{m=n}^{N} \left\{ \sum_{i=1}^{J-1} \left(\frac{\partial^4}{\partial s^4} f(s_i, \tau^{n-1}_i) \right)^2 \Delta s \right\}^{1/2} \Delta t$$

$$C_2 = \sum_{m=n}^{N} \frac{1}{6} \left\{ \sum_{i=1}^{J-1} \left(\frac{\partial^4}{\partial s^4} f(\sigma_i, t^{n-1}) \right)^2 \Delta s \right\}^{1/2} \Delta t .$$

Since

$$C_1 \approx \int_{t^n}^{T} \left\| \frac{\partial^4}{\partial s^4} f(t) \right\| dt$$
$$C_2 \approx \frac{1}{6} \int_{t^n}^{T} \left\| \frac{\partial^4}{\partial s^4} f(t) \right\| dt$$

we have that the error is of order $(\Delta t + \Delta s^2)$. Note that for this scheme Δt and Δs can be taken independent of each other, unlike what happened for the explicit scheme.

4. The Crank-Nicolson scheme.

The Crank-Nicolson scheme is the following:

(4.1a) $$\frac{1}{\Delta t}\left(f_i^n - f_i^{n-1}\right) + \frac{1}{2}\frac{1}{\Delta S^2}\left(f_{i+1}^n - 2f_i^n + f_{i-1}^n\right) + \frac{1}{2}\frac{1}{\Delta S^2}\left(f_{i+1}^{n-1} - 2f_i^{n-1} + f_{i-1}^{n-1}\right) = 0$$

for $i = 1, \dots, I-1$ and $n = 1, \dots, N$,

(4.1b) $$f_i^N = f_T(S_i) \qquad \text{for } i = 1, \dots, I-1,$$

and

(4.1c) $$f_0^n = f_0(t^n) \qquad \text{for } n = 0, \dots, N,$$

(4.1d) $$f_I^n = f_{S_M}(t^n) \qquad \text{for } n = 0, \dots, N.$$

The first thing to note is that the equation (4.1a) defining the Crank-Nicolson method is obtained by formally averaging the explicit and implicit schemes we have considered before. As we are going to see, this averaging results in a more accurate method.

The second thing to note is that this is an implicit scheme. If we set $\lambda = \Delta t / \Delta S^2$, the equations (4.1a) can be rewritten as

$$-\frac{\lambda}{2} f_{i-1}^{n-1} + (1+\lambda) f_i^{n-1} - \frac{\lambda}{2} f_{i+1}^{n-1} = \frac{\lambda}{2} f_{i-1}^n - (1-\lambda) f_i^n + \frac{\lambda}{2} f_{i+1}^n$$

or, in matrix form

$I-1$ rows

$$\begin{bmatrix} -\frac{\lambda}{2} & (1+\lambda) & -\frac{\lambda}{2} & 0 & \cdots & 0 \\ 0 & -\frac{\lambda}{2} & (1+\lambda) & -\frac{\lambda}{2} & \cdots & 0 \\ 0 & 0 & -\frac{\lambda}{2} & (1+\lambda) & \cdots & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & -\frac{\lambda}{2} \end{bmatrix} \begin{bmatrix} f_0^{n-1} \\ f_1^{n-1} \\ f_2^{n-2} \\ \vdots \\ f_I^{n-1} \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{2} f_0^n + (1-\lambda) f_1^n + \frac{\lambda}{2} f_2^n \\ \frac{\lambda}{2} f_1^n + (1-\lambda) f_2^n + \frac{\lambda}{2} f_3^n \\ \frac{\lambda}{2} f_2^n + (1-\lambda) f_3^n + \frac{\lambda}{2} f_4^n \\ \vdots \\ \frac{\lambda}{2} f_{I-2}^n + (1-\lambda) f_{I-1}^n + \frac{\lambda}{2} f_I^n \end{bmatrix}$$

$I+1$ columns

Since f_0^{n-1} and f_I^{n-1} are given by the boundary conditions (4.1c) and (4.1d) respectively, the matrix equation we need to solve is

$$\begin{bmatrix} 1+\lambda & -\frac{\lambda}{2} & 0 & \cdots & 0 \\ -\frac{\lambda}{2} & (1+\lambda) & -\frac{\lambda}{2} & \cdots & 0 \\ 0 & -\frac{\lambda}{2} & (1+\lambda) & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & (1+\lambda) \end{bmatrix} \begin{bmatrix} f_1^{n-1} \\ f_2^{n-1} \\ f_3^{n-1} \\ \vdots \\ f_{I-1}^{n-1} \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \\ \quad \end{bmatrix} + \begin{bmatrix} \frac{\lambda}{2} f_0^{n-1} \\ 0 \\ 0 \\ \vdots \\ \frac{\lambda}{2} f_I^{n-1} \end{bmatrix}.$$

Once the LU decomposition of the matrix on the left-hand side is computed, we need $6(I-1)$ operations to obtain the solution provided we already have the right-hand side, which needs $5(I-1)$ operations. So the Crank-Nicolson needs $11(I-1)$ operations per time step.

Let us show that this scheme is consistent, stable and convergent.

4.1 <u>Consistency</u>

We have

$$TE_i^n = \frac{1}{\Delta t}\left(f(s_i,t^n) - f(s_i,t^{n-1})\right)$$
$$+ \frac{1}{2}\frac{1}{\Delta s^2}\left(f(s_{i+1},t^n) - 2f(s_i,t^n) + f(s_{i-1},t^n)\right)$$
$$+ \frac{1}{2}\frac{1}{\Delta s^2}\left(f(s_{i+1},t^{n-1}) - 2f(s_i,t^{n-1}) + f(s_{i-1},t^{n-1})\right)$$

to explore the consistency of this method, we are going to carry out Taylor expansions around the point $(s_i, \frac{1}{2}(t^n+t^{n-1})) =: (s_i, t^{n-1/2})$. So, we have

$$\frac{1}{\Delta t}\left(f(s_i,t^n) - f(s_i,t^{n-1})\right) = \frac{\partial}{\partial t} f(s_i, t^{n-1/2})$$
$$+ \int_{t^{n-1}}^{t^n} K^n(\tau)\frac{\partial^3}{\partial t^3} f(s_i,\tau)\, d\tau$$

where

$$K^n(\tau) = \begin{cases} (t^n-\tau)^2/2\Delta t & \text{for } \tau \in (t^{n-1/2}, t^n), \\ (\tau - t^{n-1})^2/2\Delta t & \text{for } \tau \in (t^{n-1}, t^{n-1/2}). \end{cases}$$

Now, from our previous analyses of consistency, we know that

$$\frac{1}{\Delta s^2}\left(f(s_{i+1},t^m) - 2f(s_i,t^m) + f(s_{i-1},t^m)\right) = \frac{\partial^2}{\partial s^2} f(s_i,t^m) + \int_{s_{i-1}}^{s_{i+1}} K_i(\sigma)\frac{\partial^4}{\partial s^4} f(\sigma,t^m)\,d\sigma$$

Hence

$$\Theta := \begin{cases} \frac{1}{2}\frac{1}{\Delta s^2}\left(f(s_{i+1},t^n) - 2f(s_i,t^n) + f(s_{i-1},t^n)\right) \\ + \frac{1}{2}\frac{1}{\Delta s^2}\left(f(s_{i+1},t^{n-1}) - 2f(s_i,t^{n-1}) + f(s_{i-1},t^{n-1})\right) \end{cases}$$

$$= \frac{1}{2}\left(\frac{\partial^2}{\partial s^2} f(s_i,t^n) + \frac{\partial^2}{\partial s^2} f(s_i,t^{n-1})\right) + \int_{s_{i-1}}^{s_{i+1}} K_i(\sigma)\,\frac{1}{2}\left(\frac{\partial^4}{\partial s^4} f(\sigma,t^n) + \frac{\partial^4}{\partial s^4} f(\sigma,t^{n-1})\right) d\sigma.$$

Since

$$\frac{1}{2}\left(\phi(t^n) + \phi(t^{n-1})\right) = \phi(t^{n-1/2}) + \int_{t^{n-1}}^{t^n} \mathbb{K}^n(\tau)\,\phi''(\tau)\,d\tau$$

where

$$\mathbb{K}^n(\tau) = \begin{cases} (t^n - \tau)/2 & \text{for } \tau \in (t^{n-1/2}, t^n), \\ (\tau - t^{n-1})/2 & \text{for } \tau \in (t^{n-1}, t^{n-1/2}), \end{cases}$$

we get that

$$\Theta = \frac{\partial^2}{\partial s^2} f(s_i, t^{n-1/2}) + \int_{t^{n-1}}^{t^n} \mathbb{K}^n(\tau) \frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial s^2} f(s_i,\tau)\, d\tau$$

$$+ \int_{s_{i-1}}^{s_{i+1}} K_i(\sigma) \frac{1}{2}\left(\frac{\partial^4}{\partial s^4} f(\sigma, t^n) + \frac{\partial^4}{\partial s^4} f(\sigma, t^{n-1})\right) d\sigma$$

As a consequence

$$TE_i^n = \frac{\partial}{\partial t} f(s_i, t^{n-1/2}) + \frac{\partial^2}{\partial s^2} f(s_i, t^{n-1/2}) + R_i^{n-1/2}$$

where

$$R_i^{n-1/2} = \int_{t^{n-1}}^{t^n} K^n(\tau) \frac{\partial^3}{\partial t^3} f(s_i,\tau)\, d\tau$$

$$+ \int_{t^{n-1}}^{t^n} \mathbb{K}^n(\tau) \frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial s^2} f(s_i,\tau)\, d\tau$$

$$+ \int_{s_{i-1}}^{s_{i+1}} K_i(\sigma) \frac{1}{2}\left(\frac{\partial^4}{\partial s^4} f(\sigma, t^n) + \frac{\partial^4}{\partial s^4} f(\sigma, t^{n-1})\right) d\sigma .$$

Applying the mean-value theorem, we get

$$R_i^{n-1/2} = \frac{1}{24} \Delta t^2 \frac{\partial^3}{\partial t^3} f(s_i, \tau_1^n)$$

$$+ \frac{1}{4} \Delta t^2 \frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial s^2} f(s_i, \tau_2^n)$$

$$+ \frac{1}{12} \Delta s^2 \frac{\partial^4}{\partial s^4} f(\sigma_i, t_3^n)$$

for some values T_1^n, T_2^n and T_3^n in (t^{n-1}, t^n) and σ_i in (S_{i-1}, S_{i+1}).

If f is a solution of the heat equation,

(4.2) $$TE_i^n = R_i^{n-1/2} = \frac{1}{24}\Delta t^2 \frac{\partial^6}{\partial S^6} f(S_i, T_1^n) + \frac{1}{4}\Delta t^2 \frac{\partial^6}{\partial S^6} f(S_i, T_2^n) + \frac{1}{12}\Delta S^2 \frac{\partial^4}{\partial S^4} f(\sigma_i, T_3^n).$$

4.2 <u>Stability.</u>

In this case, we have

(4.3a) $$\frac{1}{\Delta t}(\delta_i^n - \delta_i^{n-1}) + \frac{1}{2\Delta S^2}(\delta_{i+1}^n - 2\delta_i^n + \delta_{i-1}^n) + \frac{1}{2\Delta S^2}(\delta_{i+1}^{n-1} - 2\delta_i^{n-1} + \delta_{i-1}^{n-1}) = 0$$

for $i = 1, \dots, I-1$ and $n = 1, \dots, N$,

(4.3b) $$\delta_i^N = f_{T1}(S_i) - f_{T2}(S_i) \qquad \text{for } i = 1, \dots, I-1,$$

and

(4.4c) $$\delta_0^n = 0 \qquad \text{for } n = 0, \dots, N,$$

(4.4d) $$\delta_I^n = 0 \qquad \text{for } n = 0, \dots, N.$$

We proceed by following (2.2).

<u>Step 1</u>. We multiply the equation (4.3a) by $(\delta_i^n + \delta_i^{n-1})/2$ to get

$$\frac{1}{2\Delta t}\left(\delta_i^{n^2} - \delta_i^{n-1^2}\right) + \left(\frac{\delta_i^n + \delta_i^{n-1}}{2}\right)\cdot\left[\frac{1}{2}\frac{1}{\Delta s^2}\left(\delta_{i+1}^n - 2\delta_i^n + \delta_{i-1}^n\right)\right.$$
$$\left. + \frac{1}{2}\frac{1}{\Delta s^2}\left(\delta_{i+1}^{n-1} - 2\delta_i^{n-1} + \delta_{i-1}^{n}\right)\right] = 0.$$

If we set $\delta_i^{n-1/2} := (\delta_i^n + \delta_i^{n-1})/2$, we can rewrite the above equation in the following way

$$\frac{1}{2\Delta t}\left(\delta_i^{n^2} - \delta_i^{n-1^2}\right) + \delta_i^{n-1/2}\frac{1}{\Delta s^2}\left(\delta_{i+1}^{n-1/2} - 2\delta_i^{n-1/2} + \delta_{i+1}^{n-1/2}\right) = 0$$

for $i = 1, \dots, I-1$ and $n = 1, \dots, N$.

<u>Step 2</u>. Multiplying by Δs and adding over i, we get

$$\frac{1}{2\Delta t}\left(\|\delta^n\|^2 - \|\delta^{n-1}\|^2\right) + \sum_{i=1}^{I-1}\delta_i^{n-1/2}\frac{1}{\Delta s^2}\left(\delta_{i+1}^{n-1/2} - 2\delta_i^{n-1/2} + \delta_{i-1}^{n-1/2}\right)\Delta s = 0$$

<u>Step 3</u>. We can now integrate by parts exactly as we did it before to obtain

$$\frac{1}{2\Delta t}\left(\|\delta^n\|^2 - \|\delta^{n-1}\|^2\right) + \Theta^n = 0$$

where $$\Theta^n = -\sum_{i=0}^{I-1}\left(\frac{\delta_{i+1}^{n-1/2} - \delta_i^{n-1/2}}{\Delta s}\right)^2\Delta s.$$

<u>Step 4</u>. Multiplying by Δt and summing on n, we get

$$\frac{1}{2}\|\delta^N\|^2 - \frac{1}{2}\|\delta^n\|^2 + \sum_{m=n+1}^{N} \Theta^m \Delta t = 0 .$$

<u>Step 5</u>. Since $\Theta^m \le 0$, we conclude that

$$\|\delta^n\| \le \|\delta^N\| = \|f_{1T} - f_{2T}\|$$

for all $n \le N$.

4.3 <u>Convergence</u>.

The equations for the error are

(4.1a)
$$\frac{1}{\Delta t}(e_i^n - e_i^{n-1}) + \frac{1}{2}\frac{1}{\Delta S^2}(e_{i+1}^n - 2e_i^n + e_{i-1}^n) + \frac{1}{2}\frac{1}{\Delta S^2}(e_{i+1}^{n-1} - 2e_i^{n-1} + e_{i-1}^{n-1}) = R_i^{n-1/2}$$

for $i = 1, \dots, I-1$ and $n = 1, \dots, N$,

(4.1b) $e_i^N = 0$ for $i = 1, \dots, I-1$,

and

(4.1c) $e_0^n = 0$ for $n = 0, \dots, N$,

(4.1d) $e_I^n = 0$ for $n = 0, \dots, N$.

Again, we follow section 2.3.

<u>Step 1.</u> We multiply by $e_i^{n-1/2} := (e_i^n + e_i^{n-1})/2$ to get

$$\frac{1}{2\Delta t}\left({e_i^n}^2 - {e_i^{n-1}}^2\right) + e_i^{n-1/2}\frac{1}{\Delta s^2}\left(e_{i+1}^{n-1/2} - 2e_i^{n-1/2} + e_{i-1}^{n-1/2}\right) = R_i^{n-1/2}\, e_i^{n-1/2}.$$

<u>Step 2</u> We multiply by Δs and add over i to get

$$\frac{1}{2\Delta t}\left(\|e^n\|^2 - \|e^{n-1}\|^2\right) + \sum_{i=1}^{I-1} e_i^{n-1/2}\frac{1}{\Delta s^2}\left(e_{i-1}^{n-1/2} - 2e_i^{n-1/2} + e_{i+1}^{n-1/2}\right)\Delta s$$
$$= \sum_{i=1}^{I-1} R_i^{n-1/2}\, e_i^{n-1/2}\,\Delta s.$$

<u>Step 3</u> We carry out the discrete integration by parts:

$$\frac{1}{2\Delta t}\left(\|e^n\|^2 - \|e^{n-1}\|^2\right) + \Theta^n = \sum_{i=1}^{I-1} R_i^{n-1/2}\, e_i^{n-1/2}\,\Delta s$$

where

$$\Theta^n = -\sum_{i=0}^{I-1}\left(\frac{e_{i+1}^{n-1/2} - e_i^{n-1/2}}{\Delta s}\right)^2 \Delta s.$$

<u>Step 4.</u> We multiply by Δt and add on n to get

$$\frac{1}{2}\|e^N\|^2 - \frac{1}{2}\|e^n\|^2 + \sum_{m=n+1}^{N} \Theta^m\,\Delta t$$
$$= \sum_{m=n+1}^{N}\sum_{i=1}^{I-1} R_i^{n-1/2}\, e_i^{n-1/2}\,\Delta s\cdot\Delta t$$

<u>Step 5</u> Since $e^N = 0$ by (4.4b) and $\Theta^m \leq 0$,

$$\tfrac{1}{2}\|e^n\|^2 \le -\sum_{m=n+1}^{N}\sum_{i=1}^{I-1} R_i^{m-1/2} e_i^{m-1/2}\,\Delta s\cdot\Delta t\,,$$

and by the Cauchy-Schwarz inequality

$$\tfrac{1}{2}\|e^n\|^2 \le \sum_{m=n+1}^{N} \|R^{m-1/2}\|\;\|e^{m-1/2}\|\,\Delta t,$$

for $n=0,\dots,N-1$.

Step 6. Setting

$$\phi^n = \sum_{m=n}^{N} \|R^{m-1/2}\|\;\|e^{m-1/2}\|\,\Delta t$$

we obtain

$$\begin{aligned}\frac{1}{\Delta t}(\phi^{n+1}-\phi^n) &= -\|R^{n-1/2}\|\;\|e^{n-1/2}\| \\ &\geqslant -\|R^{n-1/2}\|\,\tfrac{1}{2}\left(\|e^n\|+\|e^{n-1}\|\right)\end{aligned}$$

and since

$$\|e^n\| \le \sqrt{2}\,\sqrt{\phi^{n+1}}$$

we get

$$\frac{1}{\Delta t}(\phi^{n+1}-\phi^n) \geqslant -\frac{1}{\sqrt{2}}\|R^{n-1/2}\|\left(\sqrt{\phi^{n+1}}+\sqrt{\phi^n}\right).$$

This implies that

$$\frac{1}{\Delta t}\,\frac{(\phi^{n+1}-\phi^{n})}{\sqrt{\phi^{n+1}}+\sqrt{\phi^{n}}} \geq -\frac{1}{\sqrt{2}}\,\| R^{n-1/2} \|,$$

and since

$$\frac{\phi^{n+1}-\phi^{n}}{\sqrt{\phi^{n+1}}+\sqrt{\phi^{n}}} = \sqrt{\phi^{n+1}}-\sqrt{\phi^{n}},$$

that

$$\frac{1}{\Delta t}\left(\sqrt{\phi^{n+1}}-\sqrt{\phi^{n}}\right) \geq -\frac{1}{\sqrt{2}}\,\| R^{n-1/2} \|$$

for $n = 1, \dots, N-1$. As a consequence

$$\sqrt{\phi^{N}} - \sqrt{\phi^{n}} \geq -\frac{1}{\sqrt{2}} \sum_{m=n}^{N-1} \| R^{m-1/2} \| \,\Delta t$$

Hence

$$\sqrt{\phi^{n}} \leq \sqrt{\phi^{N}} + \frac{1}{\sqrt{2}} \sum_{m=n}^{N-1} \| R^{m-1/2} \| \,\Delta t.$$

Now,

$$\frac{1}{2}\,\| e^{N-1} \|^{2} \leq \| R^{N-1/2} \| \,\frac{1}{2}\,\| e^{N-1} \| \,\Delta t$$

since $e^{N}=0$. this implies

$$\|e^{N-1}\| \leq \|R^{N-1/2}\| \, \Delta t$$

and

$$\phi^N = \|R^{N-1/2}\| \, \|e^{N-1}\| \, \Delta t \leq \|R^{N-1/2}\|^2 \, \Delta t^2.$$

As a consequence

$$\sqrt{\phi^n} \leq \|R^{N-1/2}\| \, \Delta t + \frac{1}{\sqrt{2}} \sum_{m=n}^{N-1} \|R^{m-1/2}\| \, \Delta t$$

and so

$$\|e^n\| \leq \|R^{N-1/2}\| \sqrt{2} \, \Delta t + \sum_{m=n+1}^{N-1} \|R^{m-1/2}\| \, \Delta t$$

$$= (1-\sqrt{2}) \, \|R^{N-1/2}\| \, \Delta t + \sum_{m=n+1}^{N} \|R^{m-1/2}\| \, \Delta t.$$

This shows that

$$\|e^n\| \overset{\sim}{\leq} C_1 \, \Delta t^2 + C_2 \, \Delta S^2.$$

The Crank-Nicolson scheme has then second order in both "t" and "S".

5. <u>The upwinding scheme for the transport equation</u>

In the extreme (and unlikely!) case in which the volatility σ is equal to zero, we have seen that the Black-Scholes equation can be transformed into the transport equation

$$\frac{\partial F}{\partial t} + r\frac{\partial F}{\partial x} = 0 \qquad \text{in} \quad (-\infty, \ln S_M) \times (0,T)$$

We are going to consider the following problem

(5.1a) $$\frac{\partial F}{\partial t} + r\frac{\partial F}{\partial x} = 0 \qquad \text{in} \quad (0,L) \times (0,T),$$

with terminal condition

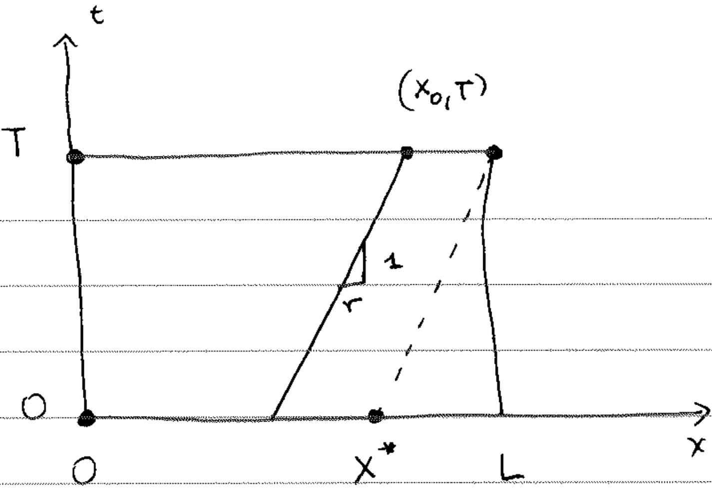
(5.1b) $$F(x,T) = F_T(x) \qquad \forall\, x \in (0,L)$$

and boundary condition

(5.1c) $$F(L,t) = F_L(t) \qquad \forall\, t \in (0,T),$$

in order to illustrate the application of finite difference techniques to this kind of equations.

Before doing this, we need to make some important remarks about the exact solution of (5.1). The first is that, unlike the heat equation, here we only impose a boundary condition at $x = L$ and <u>not</u> at $x = 0$. Let us see why this must be the case.



Consider the curve

$$(5.2) \qquad \begin{cases} \dfrac{d}{dt} x = r & \forall\, t \in (0,T), \\ x(T) = x_0 \end{cases}$$

This curve is nothing but the straight line $(x(t), t)$ where

$$x(t) = x_0 + r\,(t-T);$$

see the figure above. Now, if we restrict the solution of the transport equation (5.1a) to such a line, we see that its value is constant. Indeed

$$\frac{d}{dt} F(x(t), t) = \frac{\partial}{\partial t} F(x(t), t) + \frac{d x(t)}{dt} \frac{\partial}{\partial x} F(x(t), t)$$

$$= 0$$

Hence

$$(5.3) \qquad F(x(t), t) = F(x_0, T) = F_T(x_0) \qquad \forall\, t \in (0,T).$$

In this way, we see that the terminal condition (5.1b) determines the value of F in the trapezoid determined by the points $(0,0)$, $(x^*, 0)$, (L,T) and $(0,T)$. The values of F in the triangle of vertices $(x^*, 0)$, $(L,0)$ and (L,T) can be determined in a similar way by using the boundary condition (5.1c). We thus see that there is no need of a boundary condition at $x=0$, and that if such a condition were imposed the solution of (5.1) might simply not exist.

Note also that equation (5.3) can be interpreted as a reflection that the information "stored" in the terminal condition travels without deformation at the constant speed of "r" to the left as time diminishes. We must take this fact into account when deriving our finite difference scheme.

The simplest finite difference scheme for the problem (5.1) is the following:

(5.4a) $$\frac{1}{\Delta t}\left(F_i^n - F_i^{n-1}\right) + \frac{r}{\Delta x}\left(F_{i+1}^n - F_i^n\right) = 0 \qquad \text{for } i = 0, \dots, I-1 \text{ and } n = 1, \dots, N,$$

(5.4b) $$F_i^N = F_T(i\,\Delta x) \qquad \text{for } i = 0, \dots, I-1,$$

(5.4c) $$F_I^n = F_L(n\,\Delta t) \qquad \text{for } n = 0, \dots, N.$$

Note that this scheme is explicit since (5.4a) can be rewritten as follows:

$$F_i^{n-1} = \left(1 - r\frac{\Delta t}{\Delta x}\right) F_i^n + \left(r\frac{\Delta t}{\Delta x}\right) F_{i+1}^n$$

Note also that if $r\Delta t/\Delta x = 1$,

$$F_i^{n-1} = F_{i+1}^n$$

and we obtain the exact solution. this relation is the discrete version of (5.3). So by approximating $\frac{\partial F}{\partial x}$ at the point $(\underbrace{i\Delta x}_{x_i}, \underbrace{n\Delta t}_{t^n})$ by the difference

$$\frac{F_{i+1}^n - F_i^n}{\Delta x}$$

we are taking into account that the information travels (since $r>0$) from right to left as t decreases. For this reason the scheme is called "upwinding".

Next we show that this scheme is consistent, stable under the condition

$$r\frac{\Delta t}{\Delta x} \leq 1$$

and convergent. Moreover, we show that it is a first-order accurate method.

5.1 Consistency.

We say that the method is consistent if

$$TE_i^n = \frac{1}{\Delta t}\left(F(x_i,t^n) - F(x_i,t^{n-1})\right) + \frac{r}{\Delta x}\left(F(x_{i+1},t^n) - F(x_i,t^n)\right),$$

goes to zero as Δt and Δx go to zero and (x_i, t^n) goes to some point in $(0, L) \times (0, T)$. whenever F is an exact solution of the transport equation (5.1a).

Since

$$F(x_i, t^{n-1}) = F(x_i, t^n) - \Delta t \frac{\partial}{\partial t} F(x_i, t^n) + \int_{t^n}^{t^{n-1}} (t^{n-1} - \tau) \frac{\partial^2}{\partial t^2} F(x_i, \tau)\, d\tau,$$

and since

$$F(x_{i+1}, t^n) = F(x_i, t^n) + \Delta x \frac{\partial}{\partial x} F(x_i, t^n) + \int_{x_i}^{x_{i+1}} (x_{i+1} - s) \frac{\partial^2}{\partial x^2} F(s, t^n)\, ds$$

we readily obtain that

$$TE_i^n = \frac{\partial F}{\partial t}(x_i, t^n) - \int_{t^{n-1}}^{t^n} \left(\frac{\tau - t^{n-1}}{\Delta t} \right) \frac{\partial^2}{\partial t^2} F(x_i, \tau)\, d\tau$$

$$+ r \frac{\partial F}{\partial x}(x_i, t^n) + r \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - s}{\Delta x} \right) \frac{\partial^2}{\partial x^2} F(s, t^n)\, ds,$$

and if F satisfies (5.1a),

(5.5a) $$TE_i^n = R_i^n$$

where

(5.5b) $$R_i^n = -\frac{\Delta t}{2} \cdot r^2 \cdot \frac{\partial^2 F}{\partial x^2}(x_i, \tau^n) + \frac{\Delta x}{2} \cdot r \cdot \frac{\partial^2 F}{\partial x^2}(s_i, t^n)$$

for some $\tau^n \in (t^{n-1}, t^n)$ and some $s_i \in (x_i, x_{i+1})$.

We thus see that TE_i^n goes to zero as Δt and Δx go to zero and (x_i, t^n) goes to some point (x,t) in $(0,L)\times(0,T)$. the scheme is thus consistent.

5.2 Stability (with respect to the terminal data).

We proceed as usual and set $\delta = F_1 - F_2$ where F_1 and F_2 solve (5.4) with terminal conditions F_{T1} and F_{T2} respectively. We want to see under what conditions on Δt and Δx, if $F_{T1} - F_{T2}$ is "small" implies that $F_1 - F_2$ is also "small".

the equations satisfied by δ are

(5.6a) $$\frac{1}{\Delta t}\left(\delta_i^n - \delta_i^{n-1}\right) + \frac{r}{\Delta x}\left(\delta_{i+1}^n - \delta_i^n\right) = 0 \qquad \text{for } i = 0, \dots, I-1 \text{ and } n = 1, \dots, N,$$

(5.6b) $$\delta_i^N = F_{T1}(x_i) - F_{T2}(x_i) \qquad \text{for } i = 0, \dots, I-1,$$

(5.6c) $$\delta_I^n = 0 \qquad \text{for } n = 0, \dots, N.$$

Let us carry out our stability analysis on the following "continuous" version of the above equations, namely,

$$\frac{\partial}{\partial t}\delta + r\frac{\partial}{\partial x}\delta = 0 \qquad \text{in } (0,L)\times(0,T),$$

$$\delta(x,T) = F_{T1}(x) - F_{T2}(x) \qquad \forall x \in (0,L),$$

$$\delta(L,t) = 0 \qquad \forall t \in (0,T).$$

We proceed as follows.

Step 1. Multiplying the transport equation by δ, we get

$$\frac{1}{2}\frac{\partial}{\partial t}\delta^2 + \frac{1}{2} r \frac{\partial}{\partial x}\delta^2 = 0 .$$

Step 2. Integrating in x from 0 to L, we get

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 + \frac{1}{2} r\, \delta^2 \Big|_0^L = 0$$

and using the boundary condition, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 - \frac{1}{2} r\, \delta^2(x=0) = 0 .$$

Step 3. Integrating in time from t to T,

$$\frac{1}{2}\|\delta(T)\|^2 - \frac{1}{2}\|\delta(t)\|^2 - \frac{1}{2} r \int_t^T \delta^2(0,\tau)\, d\tau = 0 .$$

Step 4 We conclude that

$$\|\delta(t)\|^2 + r\int_t^T \delta^2(0,\tau)\, d\tau = \|\delta(T)\|^2$$

and hence

$$\|\delta(t)\| \leq \|\delta(T)\| \qquad \forall\, t \in (0,T).$$

Let us now go back to our discrete problem. We proceed as suggested by the above "map".

Step 1. We multiply (5.6a) by δ_i^n to obtain

$$\frac{1}{2}\frac{1}{\Delta t}\left((\delta_i^n)^2 - (\delta_i^{n-1})^2\right) + \frac{1}{2\Delta t}\left(\delta_i^n - \delta_i^{n-1}\right)^2$$
$$+ \frac{r}{2}\frac{1}{\Delta x}\left((\delta_{i+1}^n)^2 - (\delta_i^n)^2\right) - \frac{r}{2\Delta x}\left(\delta_{i+1}^n - \delta_i^n\right)^2 = 0$$

for $i = 0, \dots, I-1$ and $n = 1, \dots, N$.

Step 2. We multiply the equation by Δx and add in i from 0 to $I-1$. We obtain

$$\frac{1}{2\Delta t}\left(\sum_{i=0}^{I-1} (\delta_i^n)^2 \Delta x - \sum_{i=0}^{I-1} (\delta_i^{n-1})^2 \Delta x \right)$$

$$+ \frac{1}{2\Delta t} \sum_{i=0}^{I-1} (\delta_i^n - \delta_i^{n-1})^2 \Delta x$$

$$+ \frac{r}{2}\left((\delta_I^n)^2 - (\delta_0^n)^2 \right)$$

$$- \frac{r}{2} \sum_{i=0}^{I-1} (\delta_{i+1}^n - \delta_i^n)^2 = 0,$$

and if we set

$$\| \delta^n \| := \left\{ \sum_{i=0}^{I-1} (\delta_i^n)^2 \Delta x \right\}^{1/2},$$

we get

$$\frac{1}{2\Delta t}\left(\| \delta^n \|^2 - \| \delta^{n-1} \|^2 \right) - \frac{r}{2} (\delta_0^n)^2 - H^n = 0,$$

where

$$H^n = -\frac{1}{2\Delta t} \sum_{i=0}^{I-1} (\delta_i^n - \delta_i^{n-1})^2 \Delta x$$

$$+ \frac{r}{2} \sum_{i=0}^{I-1} (\delta_{i+1}^n - \delta_i^n)^2$$

$$- \frac{r}{2} (\delta_I^n)^2.$$

Step 2'. Since by the boundary condition $\delta_I^n = 0$, and using the fact that

$$\left(\delta_i^n - \delta_i^{n-1}\right) = -r\frac{\Delta t}{\Delta x}\left(\delta_{i+1}^n - \delta_i^n\right)$$

for $i = 0, \dots, I-1$ and $n = 1, \dots, N$, we get that

$$\begin{aligned} (H)^n &= \left[-\frac{\Delta x}{2\Delta t}\cdot\left(r\frac{\Delta t}{\Delta x}\right)^2 + \frac{r}{2}\right]\sum_{i=0}^{I-1}\left(\delta_{i+1}^n - \delta_i^n\right)^2 \\ &= \frac{1}{2}\frac{\Delta x}{\Delta t}\left[\left(\frac{r\Delta t}{\Delta x}\right) - \left(\frac{r\Delta t}{\Delta x}\right)^2\right]\sum_{i=0}^{I-1}\left(\delta_{i+1}^n - \delta_i^n\right)^2 \\ &= \frac{1}{2}\frac{\Delta x}{\Delta t}\left(\frac{r\Delta t}{\Delta x}\right)\left[1 - \left(r\frac{\Delta t}{\Delta x}\right)\right]\sum_{i=0}^{I-1}\left(\delta_{i+1}^n - \delta_i^n\right)^2 \\ &\geq 0, \end{aligned}$$

if $\dfrac{r\Delta t}{\Delta x} \leq 1$.

Step 3. Multiplying by Δt and adding on n from $m+1$ to N, we get

$$\frac{1}{2}\|\delta^N\|^2 - \frac{1}{2}\|\delta^m\|^2 - \frac{r}{2}\sum_{n=m+1}^{N}\left(\delta_0^n\right)^2\Delta t - \sum_{n=m+1}^{N}(H)^n\,\Delta t = 0$$

Step 4. We conclude then that if $r\frac{\Delta t}{\Delta x} \leq 1$,

$$\|\delta^m\| \leq \|\delta^N\| \qquad \text{for } m = 0, \dots, N-1.$$

5.3 Convergence. Now that we have established that the method is stable with respect with the initial condition, let us show that it converges. So, we assume that $r\frac{\Delta t}{\Delta x} \le 1$ and show that

$$\|e^n\| = \left\{ \sum_{i=0}^{I-1} (e_i^n)^2 \Delta x \right\}^{1/2},$$

where $e_i^n = F(x_i, t^n) - F_i^n$, goes to zero as Δt and Δx go to zero.

The equations satisfied by the error are

(5.7a) $$\frac{1}{\Delta t}(e_i^n - e_i^{n-1}) + \frac{r}{\Delta x}(e_{i+1}^n - e_i^n) = R_i^n \quad \text{for } i=0,\dots,I-1 \text{ and } n=1,\dots,N,$$

(5.7b) $$e_i^N = 0 \quad \text{for } i=0,\dots,I-1,$$

(5.7c) $$e_I^n = 0 \quad \text{for } n=0,\dots,N,$$

where R_i^n is given by (5.5b)!

Now, consider the continuous version of these equations

$$\frac{\partial}{\partial t}e + r\frac{\partial e}{\partial x} = R \quad \text{in } (0,L)\times(0,T),$$

$$e(x,T) = 0 \quad \forall\, x \in (0,L),$$

$$e(L,t) = 0 \quad \forall\, t \in (0,T).$$

We can obtain an estimate of e in terms of R as follows:

Step 1. Multiplying the equation by e, we get

$$\frac{1}{2}\frac{\partial}{\partial t}e^2 + \frac{r}{2}\frac{\partial}{\partial x}e^2 = Re.$$

Step 2. Integrating in x from 0 to L, we get

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 - \frac{r}{2}e^2(x=0) = \int_0^L Re$$

Step 3. Integrating in t from t to T, we get

$$\frac{1}{2}\|e(T)\|^2 - \frac{1}{2}\|e(t)\|^2 - \frac{r}{2}\int_t^T e^2(0,\tau)\,d\tau = \int_t^T\int_0^L R\cdot e$$

and since by the terminal condition $e(T)=0$, we obtain

$$\|e(t)\|^2 + r\int_t^T e^2(0,\tau)\,d\tau = -2\int_t^T\int_0^L R\cdot e.$$

Step 4. Since $r \geq 0$, applying the Cauchy-Schwarz inequality, we get

$$\|e(t)\|^2 \leq 2\int_t^T \|R(\tau)\|\;\|e(\tau)\|\,d\tau. \qquad \forall\, t\in(0,T).$$

Step 5. We can now proceed as in "Step 6" in pages 20 and 21, to get that

$$\|e(t)\| \leq 2\int_t^T \|R(\tau)\|\,d\tau \qquad \forall\, t\in(0,T).$$

Let us now deal with the discrete case.

Step 1. Multiplying (5.7a) by e_i^n, we get

$$\frac{1}{2\Delta t}\left((e_i^n)^2 - (e_i^{n-1})^2\right) + \frac{1}{2\Delta t}\left(e_i^n - e_i^{n-1}\right)^2$$

$$+ \frac{r}{2\Delta x}\left((e_{i+1}^n)^2 - (e_i^n)^2\right) - \frac{1}{2\Delta x}\left(e_{i+1}^n - e_i^n\right)^2 = R_i^n e_i^n$$

for $i = 0, \dots, I-1$ and $n = 1, \dots, N$.

Step 2. Multiplying by Δx and adding over i from 0 to $I-1$, we get

$$\frac{1}{2\Delta t}\left(\|e^n\|^2 - \|e^{n-1}\|^2\right) + \frac{1}{2\Delta t}\sum_{i=0}^{I-1}\left(e_i^n - e_i^{n-1}\right)^2 \Delta x$$

$$+ \frac{r}{2}\left((e_I^n)^2 - (e_0^n)^2\right) - \frac{r}{2\Delta x}\sum_{i=0}^{I-1}\left(e_{i+1}^n - e_i^n\right)^2 \Delta x = \sum_{i=0}^{I-1} R_i^n e_i^n \Delta x$$

By the boundary condition (5.7c) $e_I^n = 0$ and so

$$\frac{1}{2\Delta t}\left(\|e^n\|^2 - \|e^{n-1}\|^2\right) + \Psi^n = \sum_{i=0}^{I-1} R_i^n e_i^n \Delta x$$

where

$$\Psi^n = \frac{1}{2\Delta t}\sum_{i=0}^{I-1}\left(e_i^n - e_i^{n-1}\right)^2 \Delta x - \frac{r}{2}(e_0^n)^2$$

$$- \frac{r}{2\Delta x}\sum_{i=0}^{I-1}\left(e_{i+1}^n - e_i^n\right)^2 \Delta x.$$

Let us work on the expression for Ψ^n. We have, by equations (5.7a),

$$\Psi^n = -\frac{r}{2}(e_0^n)^2 + \frac{1}{2\Delta t}\sum_{i=0}^{I-1}\left(-\frac{\Delta t}{\Delta x} r\,(e_{i+1}^n - e_i^n) + \Delta t\, R_i^n\right)^2 \Delta x - \frac{r}{2\Delta x}\sum_{i=0}^{I-1}(e_{i+1}^n - e_i^n)^2 \Delta x$$

$$= -\frac{r}{2}(e_0^n)^2 - \sum_{i=0}^{I-1}(e_{i+1}^n - e_i^n)^2 \Delta x \left[-\frac{r}{2\Delta x} + \frac{1}{2\Delta t}\left(\frac{\Delta t}{\Delta x} r\right)^2\right] - \sum_{i=0}^{I-1}\left(r\frac{\Delta t}{\Delta x}\right)(e_{i+1}^n - e_i^n) R_i^n \Delta x + \frac{\Delta t}{2}\sum_{i=0}^{I-1}(R_i^n)^2 \Delta x.$$

Using again equation (5.7a) to write

$$\frac{r\Delta t}{\Delta x}(e_{i+1}^n - e_i^n) = -(e_i^n - e_i^{n-1}) + \Delta t\, R_i^n,$$

we get

$$\Psi^n = -\frac{r}{2}(e_0^n)^2 - \left[\frac{\Delta t}{2}\right]\left(r\frac{\Delta t}{\Delta x}\right)\left(1 - r\frac{\Delta t}{\Delta x}\right)\sum_{i=0}^{I-1}(e_{i+1}^n - e_i^n)^2 \Delta x + \sum_{i=0}^{I-1} e_i^n R_i^n \Delta x - \sum_{i=0}^{I-1} e_i^{n-1} R_i^n \Delta x - \frac{\Delta t}{2}\sum_{i=0}^{I-1}(R_i^n)^2 \Delta x.$$

This implies that

$$\frac{1}{2\Delta t}\left(\|e^n\|^2 - \|e^{n-1}\|^2\right) - \frac{r}{2}(e_0^n)^2 + \Theta^n = \sum_{i=0}^{I-1} R_i^n e_i^{n-1} \Delta x$$

where

$$\Theta^n = -\frac{\Delta t}{2}\left(r\frac{\Delta t}{\Delta x}\right)\left(1 - r\frac{\Delta t}{\Delta x}\right)\sum_{i=0}^{I-1}(e_{i+1}^n - e_i^n)^2 \Delta x$$
$$- \frac{\Delta t}{2}\sum_{i=0}^{I-1}(R_i^n)^2 \Delta x.$$

Step 3. Multiplying by Δt and adding on n,

$$\frac{1}{2}\|e^N\|^2 - \frac{1}{2}\|e^n\|^2 - \frac{r}{2}\sum_{m=n+1}^{N}(e_0^m)^2 \Delta t$$
$$+ \sum_{m=n+1}^{N}\Theta^n \Delta t = \sum_{m=n+1}^{N}\sum_{i=0}^{I-1} R_i^m e_i^{m-1} \Delta x\, \Delta t.$$

Since $\Theta^n \le 0$ for $r\frac{\Delta t}{\Delta x} \le 1$, and since $e^N = 0$ by the Terminal condition (5.75), we get that

$$\|e^n\|^2 \le -2\sum_{m=n+1}^{N}\sum_{i=0}^{I-1} R_i^n e_i^{m-1} \Delta x\, \Delta t$$

and, after a simple application of the Cauchy-Schwarz inequality,

$$\|e^n\|^2 \le 2\sum_{m=n+1}^{N}\|R^m\| \cdot \|e^{m-1}\| \Delta t$$

for $n = 0, \ldots, N-1$.

<u>Step 4.</u> Proceeding as in "step 6" (in pages 26 and 27) we get that

$$\|e^n\| \leq 2 \sum_{m=n+1}^{N} \|R^m\| \, \Delta t \qquad \text{for } n=0,\dots,N-1.$$

Since, by (5.5b),

$$2 \sum_{m=n+1}^{N} \|R^m\| \, \Delta t \approx r \cdot \Delta x \left(1 - r \frac{\Delta t}{\Delta x}\right) \int_t^T \left\| \frac{\partial^2 F}{\partial x}(\tau) \right\| d\tau$$

we see that the method is first-order accurate in both Δt and Δx.

5.4. <u>The stability condition $r \frac{\Delta t}{\Delta x} \leq 1$.</u>

Here we want to argue that the stability condition $r \frac{\Delta t}{\Delta x} \leq 1$ reflects the fact that the numerical scheme can "catch information" faster than the transport equation can "propagate" it.

Note that r is a speed and, by (5.2) and (5.3), it is the speed at which the solution of the transport equation "travels". Also $\frac{\Delta t}{\Delta x}$ has the dimension of a velocity and can be thought of as being the speed at which information can be propagated by the numerical scheme; see (5.4a).