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A simple introduction to error estimation for nonlinear hyperbolic conservation laws

Some ideas, techniques, and promising results

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Preface

In these notes, we present a simple introduction to the topic of *a posteriori* error estimation for nonlinear hyperbolic conservation laws. This is a topic of great practical interest which has been receiving increasing attention from many researchers in the last several years. On the other hand, the highly complex character of its mathematics often obscures the main ideas behind the technical manipulations. Aware of this unfortunate situation, we have written these notes in an attempt to emphasize the *ideas* and simplify, as much as possible, the presentation of the *techniques*.

The reader has to be warned, however, that there is still much to be researched in this area. As a consequence, these are notes about *ongoing* research– not about fully solved problems. Thus, the purpose of these notes is not to tell the story of a long ago fought battle, but to entice the reader to participate in one that is currently taking place!

We have many people to thank. Let us start with the three organizers. Thanks to Mark Ainsworth for the invitation to give a series of lectures at the EPSRC VIII-th Summer School in Numerical Analysis, University of Leicester, UK, July 13–17, 1998, the material of which is contained in these notes; to Marco Marletta for his excellent editorial work; and, last but not least, to Jeremy Levesly for making sure everybody had a good time. Next, we must thank John Barrett, Charlie Elliot, David Sylvester, and my good friend Endre Süli for their questions and remarks. Finally, we must thank all the participants to the EPSRC VIII-th Summer School for being a patient and engaging audience.

1. Introduction

In these notes, we present a simple introduction to the topic of *a posteriori* error estimation for nonlinear hyperbolic conservation laws. The interest in this topic stems from two facts:

- The first is that many relevant physical phenomena are modeled by conservation laws that when *formally* modified become nonlinear hyperbolic conservation laws. This modification takes place when the terms associated with some physical phenomenon that is *not dominant* are dropped from the equations. An instance of this situation occurs in secondary oil recovery when the capillarity effects are neglected; another occurs in compressible fluid flow when the viscosity and heat transfer effects are dropped.
- The second is that a good error estimate is essential for devising mathematically sound adaptive algorithms. *A posteriori* error estimates are important because they tell us how good a *given* approximation u is without having to know the exact solution v. The *a posteriori* error estimates we are interested in are of the form

  $$\| v - u \| \leq \Psi(u).$$

  With such an estimate, we can ensure that the error is smaller than a given tolerance τ, by simply finding an approximation u such that

  $$\Psi(u) \leq \tau.$$

  The problem is now how to achieve this goal with minimal computational effort. The devising of algorithms to solve this problem is probably the most important topic in modern computational mathematics for partial differential equations.

The organization and content of these notes strongly reflect the fact that this is a rapidly expanding field in which there are more questions than answers. Thus, the purpose of these notes is to bring the reader as fast as possible face to face with what we think are some of the most important current issues in this field. The reader will find (i) some *ideas* about how to approach the problem of *a posteriori* error estimation for nonlinear hyperbolic conservation laws, (ii) some *techniques* that allow us to implement those ideas, and some (iii) *promising* results which reflect the current state of the development of the approach we consider:

- The *ideas* that we develop are:
  1. That nonlinear hyperbolic conservation laws are (in many important cases) approximations to equations of different mathematical type,
  2. That *a posteriori* error estimates are nothing but continuous dependence results,

3. That in order to obtain *a posteriori* error estimates for nonlinear hyperbolic conservation laws, we must first obtain them for the original equations the hyperbolic conservation laws are an approximation of.

- The *techniques* we study are *duality* techniques for *a posteriori* error estimates. There are many adaptive algorithms for nonlinear hyperbolic conservation laws and most, if not all, are based in heuristic arguments. We will not concern ourselves with those heuristic arguments; instead, we consider techniques for obtaining rigorously proven *a posteriori* error estimates. The *techniques* we consider allow us:
  1. Not to have to actually solve an adjoint equation,
  2. Not to have to explicitly deal with the nonlinear convective terms,
  3. To obtain continuous dependence results that hold when the singular limit process that gives rise to the hyperbolic conservation law takes place.
- The *promising* results we consider are partial results in which we numerically test the *a posteriori* error estimates obtained in this approach.

To develop the program just sketched, we adopt as our model problem the Cauchy problem for the nonlinear scalar convection-diffusion equation

$$v_t + \nabla \cdot \mathbf{f}(\mathbf{v}) - \nu \Delta \mathbf{v} = \mathbf{0},$$

and take the following nonlinear hyperbolic conservation law

$$v_t + \nabla \cdot \mathbf{f}(\mathbf{v}) = \mathbf{0},$$

to be its *formal* modification when the diffusion effects modeled by the term $\nu \Delta v$ become 'negligible.' Thus, in chapter 3, we construct continuous dependence results for the convection-diffusion equation that will also hold for the nonlinear hyperbolic conservation law; this chapter constitutes the core of these notes. In chapter 4, we show how these continuous dependence results single out in a natural way a unique solution of the hyperbolic conservation law, the so-called *entropy* solution. We also show how the hyperbolic conservation law *inherits* effortlessly all the continuous dependence results constructed for the original model. Then, we apply these results to obtain a *a posteriori* error estimate for the Engquist-Osher scheme. In chapter 5, we treat the case of a *continuous* approximate solution and in chapter 6, the case of a *discontinuous* approximate solution.

We have chosen the above model problem, not only because it is a simple and relevant case but also because it is the *only* case for which the approach to error estimation we present in these notes has been developed. This means that the extension of this approach to systems is an exciting challenge and a rich, wide-open field of research. In chapter 7, we end these notes with some concluding remarks.

2. Some convection-diffusion problems

In this chapter, we consider two conservation laws modeling physical phenomena of practical interest, traffic flow and phase propagation in solids, and illustrate on them the typical modification of the equations that gives rise to *ill posed* problems for nonlinear hyperbolic conservation laws.

When some of the physical phenomena are considered to be *non-dominant*, it is a wide spread practice to *formally* drop the terms in the equations modeling these phenomena. Thus, in traffic flow, when the driver's awareness of the conditions ahead is negligible, second-order terms of the equations are dropped. In phase propagation in solids, when transport is dominant, the high-order terms modeling the phase transition are dropped.

We will see that the main problem with this *formal* modification is that the resulting problem becomes *ill-posed.* We argue that this happens because, although the neglected physical phenomena can be correctly considered to be not important in most parts of the domain, *they are still crucial in small, key parts of the domain.* For example, in traffic flow, the driver's awareness of the conditions ahead is essential near a strong variation of the density of cars; as we all know, this can only take place in a small set of the space domain. Also, in phase propagation in solids, the viscosity and capillarity effects, which contain all the information of the physics of the phase transition, are crucial only near the phase transition; the phase transition occurs in small subsets of the domain. The *formal* modification of the equations is thus equivalent to the removal of essential physical information and this, not surprisingly, induces a loss of *well-posedness* of the resulting problem.

In this chapter, we show how this happens for traffic flow and phase transitions which are nonlinear conservation laws. We note that in the linear case, this phenomenon does not occur. We end this chapter by some concluding remarks.

2.1 Traffic flow

2.1.1 The model. If ρ represents the density of cars in a highway and v represents the flow velocity, the fact that cars do not appear out of the blue or vanish spontaneously in the middle of the highway can be written mathematically as follows:

$$\rho_t + (\,\rho\, v\,)_x = 0.$$

In a first approximation, the flow velocity v can be thought to be a function of the density of cars ρ only, say $V(\rho)$. In this case, the above conservation law becomes a nonlinear hyperbolic conservation law,

$$\rho_t + (\,f(\rho)\,)_x = 0,$$

where $f(\rho) = \rho\, V(\rho)$ is the so-called density flow. It is reasonable to assume that $\rho \mapsto V(\rho)$ is a decreasing mapping and that for a given density, say $\rho^\star$, the velocity V is equal to zero; this corresponds to the situation in which the cars are bumper to bumper. The simplest case is the following:

$$V(\rho) = v_{max}\,(1 - \frac{\rho}{\rho^\star}),$$

where v_{max} represents the maximum velocity, and it corresponds to a quadratic density flow,

$$f(\rho) = (\rho^\star\, v_{max})\,(\frac{\rho}{\rho^\star})\,(1 - \frac{\rho}{\rho^\star}).$$

A better model for the flow velocity v takes into account our tendency of avoid those crazy nuts that get too close to our car and our tendency to not get too close to a high concentration of cars. A simple way to model of this tendency is to take

$$\rho\, v = f(\rho) - \nu\, \rho_x.$$

Witham [27] claims that the term our $\nu\,\rho_x$ models our 'awareness of conditions ahead,' since when we perceive a high density of cars ahead, we try to suitably decrease our speed to avoid a potentially dangerous situation; of course, for this to happen, the coefficient ν must be positive. With this choice of flow velocity, our original conservation law becomes

$$\rho_t + (\,f(\rho)\,)_x - \nu\,\rho_{xx} = 0,$$

and has the good taste of giving rise to mathematically *well posed* initial value problems.

2.1.2 Traveling waves and the diffusion coefficient ν. It is reasonable to expect that when the convection is dominant, that is, when the number

$$\frac{\rho^\star\, v_{max}}{\nu},$$

is big, the effects of the term $\nu\,\rho_{xx}$ are negligible. This can happen in those parts of the domain in which the quantity ρ_{xx} remains not too big, but cannot happen where ρ_x changes rapidly in a small part of the space domain.

It is well known that if the density flow is linear in ρ the size of those parts of the domain are of order $\sqrt{\nu}$. In the nonlinear case we have in our hands, it turns out that the size of those parts is even smaller: It is of order ν only. The simplest way to illustrate this fact is to look for solutions of our conservation law of the form

$$\rho(x,t) = \phi(\frac{x - c\,t}{\epsilon}).$$

These are called traveling wave solutions.

If we insert this expression for ρ in the conservation law and set

$$\epsilon = \nu,$$

we obtain a simple equation for ϕ, namely,

$$-c\,\phi' + (f(\phi))' - \phi'' = 0.$$

Next, if we assume that

$$\lim_{z\mapsto\infty^\pm} \phi(z) = \rho^\pm,$$

and that

$$\lim_{z\mapsto\infty^\pm} \phi'(z) = 0,$$

we can integrate once to get that ϕ must satisfy the following simple first-order ordinary differential equation:

$$\phi' = f(\phi) - \{\, f(\rho^+) - c\,(\rho^+ - \phi)\,\},$$

where the *speed of propagation* of the traveling wave is

$$c = \frac{f(\rho^+) - f(\rho^-)}{\rho^+ - \rho^-}.$$

There is a unique solution of this equation if and only if

$$\phi'\, sign(\rho^+ - \rho^-) > 0,$$

that is, if

$$\big(\, f(\phi) - \{\, f(\rho^+) - c\,(\rho^+ - \phi)\,\}\,\big)\, sign(\rho^+ - \rho^-) > 0.$$

A simple geometric interpretation of this condition can be obtained as follows. First, we note that the function

$$\phi \mapsto f(\rho^+) - c\,(\phi - \rho^+),$$

is nothing but the linear function that coincides with f at $\phi = \rho^\pm$. Thus, we have the following result.

Theorem 2.1 (Existence of a traveling waves). *A traveling wave solution exists if and only if the graph of f on the interval (ρ^-, ρ^+) (resp., (ρ^+, ρ^-)) lies above (resp., below) the straight line joining the points $(\rho^\pm, f(\rho^\pm))$.*

In the case we are considering, f is a concave function and so the condition for the existence of a traveling wave is satisfied if

$$\rho^- < \rho^+.$$

If there is a traveling wave solution of the form

$$\phi(\frac{x - c\,t}{\nu}),$$

it is clear that strong variations of its derivative must be contained in a set of measure of order ν.

Next, we consider what happens when we let the diffusion coefficient ν go to zero.

2.1.3 Loss of well-posedness when ν goes to zero. It is very easy to see that when we let the diffusion coefficient to zero in the traveling wave solution, we obtain the following limit

$$\rho(x,t) = \lim_{\nu \downarrow 0} \phi\left(\frac{x - c\,t}{\nu}\right) = \begin{cases} \rho^+ & \text{if } x - c\,t > 0, \\ \rho^- & \text{if } x - c\,t < 0. \end{cases}$$

Since this limit can be proven to be a *weak* solution of the following Cauchy problem, also called a Riemann problem,

$$\rho_t + (\, f(\rho)\,)_x = 0,$$
$$\rho(x,0) = \begin{cases} \rho^+ & \text{if } x > 0, \\ \rho^- & \text{if } x < 0, \end{cases}$$

this fact could be thought to be an indication that to *formally* drop the second-order term

$$\nu\, \rho_{xx},$$

from the equation could be mathematically justified. Indeed, it is a well-known fact that piecewise-smooth *weak* solutions of the equation

$$\rho_t + (\, f(\rho)\,)_x = 0,$$

are strong solutions in the parts of the domain they are smooth and satisfy the so-called jump-condition at the discontinuity curves $(x(t), t)$:

$$\frac{d}{dt} x(t) = \frac{f(\rho^+) - f(\rho^-)}{\rho^+ - \rho^-}.$$

Thus, it is clear that the limit of traveling wave solutions that we computed above is a *weak* solution of the Cauchy problem under consideration. However, it is easy to construct infinitely many *weak* solutions for the same Cauchy problem.

To do that, let us fix ideas and set

$$f(\rho) = \rho\,(1 - \rho),$$

and

$$\rho^- = 1/4, \qquad \rho^+ = 3/4.$$

Note that this gives $c = 0$. In this case, the limit of the traveling wave solutions is

$$\rho(x,t) = \lim_{\nu \downarrow 0} \phi\left(\frac{x - c\,t}{\nu}\right) = \begin{cases} 3/4 & \text{if } x > 0, \\ 1/4 & \text{if } x < 0. \end{cases}$$

However, the following functions are also *weak* solutions of the same Cauchy problem, for *all* positive values of the parameter δ:

$$\rho(x,t) = \begin{cases} 3/4 & \text{if } c_1 < x/t, \\ 1/4 - \delta & \text{if } c_2 < x/t < c_1, \\ 3/4 + \delta & \text{if } c_3 < x/t < c_2, \\ 1/4 & \text{if } x/t < c_3, \end{cases}$$

where

$$c_1 = \delta, \qquad c_2 = 0, \qquad c3 = -\delta.$$

Note that the discontinuities $x/t = c_1$ and $x/t = c_3$ do satisfy the condition for the existence of traveling waves of the corresponding parabolic regularization. However, this is *not* true for the discontinuity $x/t = c_2$; in other words, this discontinuity does not 'remember' anything about the physics contained in the modeling of the 'awareness of the conditions lying ahead.' Because of the loss of this crucial information, to *formally* drop the second-order term

$$\nu\,\rho_{xx},$$

from the equations results in the *loss of the well-posedness* of the problem. Moreover, this unfortunate situation is present only if the density flow f is a nonlinear function; indeed, if the density flow f is linear, there is a unique *weak* solution.

2.2 Propagation of phase transitions

2.2.1 The model. The so-called viscosity-capillarity model for phase transitions in van der Waal fluids was proposed independently by Truskinovsky [23] in 1982 and Slemrod [19] in 1984. In the framework of phase transitions in solids, a similar model exists which can be written as follows:

$$\begin{aligned} \gamma_t &= v_x, \\ v_t &= (\sigma(\gamma))_x + \nu\, v_{xx} - \lambda\, \gamma_{xxx}, \end{aligned}$$

where v is the velocity, γ the strain, and σ the stress; the parameter ν is the viscosity and the parameter λ is the capillarity. A very well studied case of strain-stress relations is displayed in Figure 2.1.

The material under consideration has three phases, or crystals, associated with the strain-stress curve. Phase 1 is associated with the strain γ lying of the interval $(-1, \gamma_{1,2})$, phase 2 with the interval $(\gamma_{1,2}, \gamma_{2,3})$, and phase 3 with the interval $(\gamma_{2,3}, \infty)$.

2.2.2 The size of the sets containing the phase transitions. Just as in the case of traffic flow, it can be proven that the width of the boundary layer around a phase boundary of the solution of the system under consideration is of the order of the viscosity coefficient ν; see [20], [1], and [3]. As a consequence, the measure of the sets containing strong variations of the derivatives of both the velocity and the strain must be of the order of the viscosity coefficient ν.

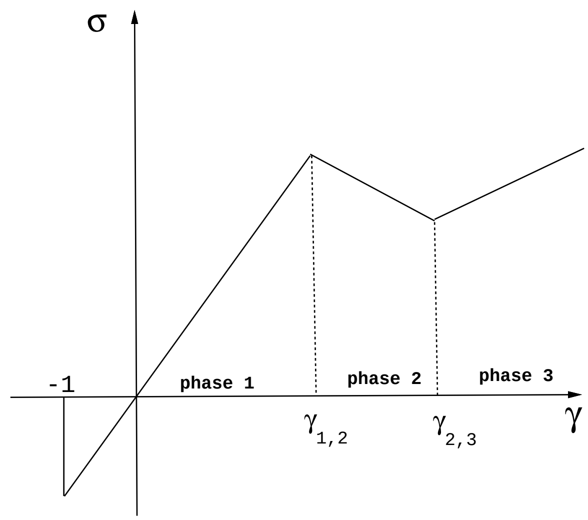

Fig. 2.1. Piecewise-linear strain-stress relation.

A simple way to see this is to construct traveling wave solutions for the system of equations under consideration. It is very simple to see that those traveling waves must depend on the parameter

$$\omega = \frac{\lambda}{\nu^2},$$

which seem to be a measure of the relative importance of viscous and capillarity effects.

2.2.3 The loss of well-posedness. If we let the viscosity and capillarity coefficients ν and λ go to zero while keeping fixed the parameter ω, it is easy to see that the corresponding traveling wave solution will converge to a *weak* solution of a Riemman problem for the system

$$\begin{aligned} \gamma_t &= v_x, \\ v_t &= (\sigma(\gamma))_x, \end{aligned}$$

which is a system of mixed type, since it is well-known that when $\sigma' > 0$ the system is hyperbolic and that when $\sigma' < 0$ the system is elliptic. Since the initial condition of the Riemann problem does not depend on the parameter ω, we see immediately that for each value of this parameter, we have a *weak* solution of the Riemann problem.

Once again, we see that the uniqueness of the *weak* solution of Riemann problem has been lost. This happened because when the terms modeling the viscosity and capillarity effects are dropped from the equations, the physics of the phase transition is lost and this results, once again, in the *loss of the well-posedness* of the corresponding Cauchy problem.

Finally, let us point out that this does not occur if the problem is linear, that is, when σ is a linear function.

2.3 Concluding remarks

We hope to have convinced the reader that to happily drop high-order terms from the equations of a well-posed problem and end up with a seemingly simpler nonlinear conservation law might be a quite dangerous sport. Indeed, even though the effects of those high-order terms can be felt on very small subsets of the domain, they contain essential information of physical processes that cannot be tossed away ... without loosing the well-posedness properties of the problem.

In what follows, we take as our model problem the following Cauchy problem for a convection-diffusion scalar conservation law:

$$\begin{aligned} &v_t + \nabla \cdot \mathbf{f}(v) - \nu \Delta v = r, &&\text{in } \mathbb{R}^d \times (0,T), \\ &v(t=0) = v_0, &&\text{on } \mathbb{R}^d, \end{aligned}$$

which is an extension of the Cauchy problem for the traffic flow model. In chapter 3, we obtain continuous dependence results for this problem and *only then* we pass to the limit in the diffusion coefficient; this is done in section 4.

3. Continuous dependence for nonlinear convection-diffusion

In this chapter, we consider the initial-value problem

$$v_t + \nabla \cdot \mathbf{f}(v) - \nu \Delta v = r, \qquad \text{in } \mathbb{R}^d \times (0,T), \tag{3.1}$$

$$v(t=0) = v_0, \qquad \text{on } \mathbb{R}^d, \tag{3.2}$$

and study how to estimate the effect in the solution v induced by variations in the data. In other words, we are interested in estimating the distance between v_1 and v_2 where v_ℓ denotes the solution of the above initial-boundary value problem with $v_0 = v_{0,\ell}$, $\mathbf{f} = \mathbf{f}_\ell$, $\nu = \nu_\ell$, and $r = r_\ell$, for $\ell = 1,2$.

This chapter is the main chapter of these notes; all the remaining results in this chapter are based on the estimates constructed in this chapter. We construct continuous dependence results that do not break down when the diffusion coefficients go to zero. As a consequence, these continuous dependence results are naturally *inherited* by the *entropy* solution of the corresponding nonlinear hyperbolic conservation law. To construct these estimates, we proceed in several steps; we start from the well-known *duality* technique and successively modify it to fit our interest.

Thus, we start by introducing the main technique to obtain continuous dependence results: the *duality* technique. In this very well-known technique, an expression of the error is obtained in terms of the data of the problem and in terms of the solution of the so-called *adjoint* problem. Since this problem is usually very difficult to study because of the *nonlinear* character of the convective term and because of the smallness of the diffusion coefficients, other techniques have been developed for which no *adjoint* problem has to be solved. The price to pay for this advantage, however, is that w no longer can freely choose the norms in which we measure the distance between two solutions.

The classical example is, of course, the standard energy technique for parabolic equations which gives rise to continuous dependence results in L^2-norms. However, the use of L^2-norms does not allow us to deal with the nonlinearities of the convective terms in a simple way. We show that the use of L^1-like norms does allow us to treat those terms in a simple and elegant way.

The next step is to modify the above technique to obtain continuous dependence results that (i) do not break down when the diffusion coefficients go to zero, and that (ii) can be used to compare the exact solution v with an arbitrary function u of low regularity. We show that these goals can be achieved by introducing the *doubling of the variables* technique. We show that if we *double* the space variables, we can achieve the first goals and that if we further *double* the time variable, we can achieve the second goal.

3.1 The standard *duality* technique and the *adjoint* problem

The main technique to obtain continuous dependence results is the so-called *duality* technique. Let us introduce this technique. We start with the trivial identity for the error $e(t) = v_1(t) - v_2(t)$,

$$\int_{\mathbb{R}^d} e(T)\zeta(T)\,dx = \int_{\mathbb{R}^d} e(0)\zeta(0)\,dx + \int_0^T\!\!\int_{\mathbb{R}^d} (\,e_t\,\zeta + e\,\zeta_t)\,dx\,dt,$$

where the *test* function ζ is to be determined. For simplicity, let us consider the case in which $\mathbf{f}_1 = \mathbf{f}_2 = \mathbf{f}$ and $\nu_1 = \nu_2 = \nu$. In this case, we have that

$$e_t = -\nabla\cdot(\mathbf{f}(v_1) - \mathbf{f}(v_2)) + \nu(\,\Delta v_1 - \Delta v_2\,) + (r_1 - r_2),$$

and so,

$$\int_{\mathbb{R}^d} e(T)\zeta(T)\,dx = \int_{\mathbb{R}^d} e(0)\zeta(0)\,dx + \int_0^T\!\!\int_{\mathbb{R}^d} (e\,A(\zeta) + (r_1 - r_2)\zeta)\,dx\,dt,$$

where the operator A is given by

$$A(\zeta) = \zeta_t + \frac{\mathbf{f}(v_1) - \mathbf{f}(v_2)}{v_1 - v_2}\cdot\nabla\zeta + \nu\,\Delta\zeta.$$

Thus, if we take ζ as the solution of the following initial-value problem

$$A(\zeta) = 0 \qquad \text{in } \mathbb{R}^d \times (0,T), \tag{3.3}$$

$$\zeta(T) = \psi \qquad \text{on } \mathbb{R}^d, \tag{3.4}$$

we will have that

$$\int_{\mathbb{R}^d} e(T)\psi\,dx = \int_{\mathbb{R}^d} e(0)\zeta(0)\,dx + \int_0^T\!\!\int_{\mathbb{R}^d} (r_1 - r_2)\,\zeta\,dx\,dt.$$

Now, a simple application of Hölder's inequality gives us the following continuous dependence result in the L^p-norm, for $1 \le p < \infty$:

$$\|\,e(T)\|_{L^p(\mathbb{R}^d)} \le C\,\{\|\,e(0)\|_{L^p(\mathbb{R}^d)} + \int_0^T \|\,r_1(t) - r_2(t)\,\|_{L^p(\mathbb{R}^d)}\,dt\},$$

where

$$C = \sup_{\psi\in L^q(\mathbb{R}^d)} \frac{\|\,\zeta\,\|_{L^\infty(0,T;L^q(\mathbb{R}^d))}}{\|\,\psi\,\|_{L^q(\mathbb{R}^d)}}, \qquad 1/p + 1/q = 1.$$

In this way, the study of the continuous dependence properties of the solutions of the equation (3.1) is reduced to the study of the solutions of the *adjoint* problem (3.3) and (3.4). This, however, is not a simple matter, not only because of the presence of the nonlinearity $\mathbf{f}$ and the non-divergence form of the equation, but also because of the smallness of the diffusion coefficient ν.

Next, we show that it is possible to obtain continuous dependence results *without* having to solve an *adjoint* problem.

3.2 A technique to bypass the resolution of the *adjoint* problem

The main idea is to try to obtain a continuous dependence result by assuming that the *test* function ζ is of the form

$$\zeta = H(e),$$

where H has to be chosen in such a way that the resolution of the *adjoint* problem could be bypassed. Once again, for the sake of simplicity, we present this technique for the case in which $\mathbf{f}_1 = \mathbf{f}_2 = \mathbf{f}$ and $\nu_1 = \nu_2 = \nu$. In this case, we have the following equation for the error e:

$$\int_{\mathbb{R}^d} e(T)\, H(e(T))\, dx = \int_{\mathbb{R}^d} e(0)\, H(e(0))\, dx + \Gamma(H) + \int_0^T \int_{\mathbb{R}^d} (r_1 - r_2)\, H(e)\, dx\, dt, \tag{3.5}$$

where

$$\Gamma(H) = \int_0^T \int_{\mathbb{R}^d} e\, A(H(e))\, dx\, dt.$$

Note that in the standard *duality* argument, the term $\Gamma(H)$ does not appear in the equation for the error since ζ is chosen to satisfy $A(\zeta) = 0$. This is not the case in this approach; in fact, we have

$$e\, A(H(e)) = e\, H'(e)\, e_t + H'(e)\, (\mathbf{f}(v_1) - \mathbf{f}(v_2)) \cdot \nabla e + \nu\, e \Delta H(e),$$

and so

$$\Gamma(H) = \int_0^T \int_{\mathbb{R}^d} e\, H'(e)\, I\, dx\, dt - \nu \int_0^T \int_{\mathbb{R}^d} |\nabla e|^2\, H'(e)\, dx\, dt,$$

where

$$I = e_t + \frac{\mathbf{f}(v_1) - \mathbf{f}(v_2)}{v_1 - v_2} \cdot \nabla e.$$

Finally, noting that

$$\int_0^T e\, H'(e)\, e_t = \left\{ e\, H(e) - \int_0^e H(s)\, ds \right\}_{t=0}^{t=T},$$

and setting

$$U(e) = \int_0^e H(s)\, ds,$$

we rewrite the equation for the error as follows:

$$\int_{\mathbb{R}^d} U(e(T))\, dx = \int_{\mathbb{R}^d} U(e(0))\, dx + Z(U) + \int_0^T \int_{\mathbb{R}^d} (r_1 - r_2)\, U'(e)\, dx\, dt, \tag{3.6}$$

where

$$Z(U) = -\int_0^T \int_{\mathbb{R}^d} R(e)\, S\, dx\, dt - \nu \int_0^T \int_{\mathbb{R}^d} |\nabla e|^2\, U''(e)\, dx\, dt,$$

where

$$R(e) = \int_0^e s\, U''(s)\, ds \quad \text{and} \quad S = \nabla \cdot \frac{\mathbf{f}(v_1) - \mathbf{f}(v_2)}{v_1 - v_2}.$$

In the classical energy technique for parabolic equations, the *test* function ζ is taken to be

$$\zeta = e \equiv H(e).$$

This means that the classical energy technique corresponds to the technique we are presenting with the following choice of the function U:

$$U(e) = \frac{1}{2}\, e^2.$$

Indeed, in this case, we have

$$Z(U) = -\frac{1}{2} \int_0^T \int_{\mathbb{R}^d} e^2\, \nabla \cdot \big(\frac{\mathbf{f}(v_1) - \mathbf{f}(v_2)}{v_1 - v_2}\big)\, dx\, dt - \nu \int_0^T \int_{\mathbb{R}^d} |\, \nabla e\,|^2\, dx\, dt, \quad (3.7)$$

and (3.6) becomes

$$\begin{aligned} \|\!| e \|\!|_{2,T,\nu}^2 = \quad & \tfrac{1}{2} \|\, e(0)\, \|_{L^2(\mathbb{R}^d)}^2 + \textstyle\int_0^T \int_{\mathbb{R}^d} (r_1 - r_2)\, e\, dx\, dt \\ & -\tfrac{1}{2} \textstyle\int_0^T \int_{\mathbb{R}^d} e^2\, \nabla \cdot \big(\tfrac{\mathbf{f}(v_1) - \mathbf{f}(v_2)}{v_1 - v_2}\big)\, dx\, dt, \end{aligned}$$

where

$$\|\!| e \|\!|_{2,T,\nu}^2 = \frac{1}{2} \|\, e(T)\, \|_{L^2(\mathbb{R}^d)}^2 + \nu \int_0^T \int_{\mathbb{R}^d} |\, \nabla e\,|^2\, dx\, dt.$$

From this equality, it is possible to obtain continuous dependence results with respect to the initial data and with respect to the right-hand side by using standard manipulations. Thus, we have shown that it is possible to compare two solutions of (3.1) and (3.2) *without* having to solve an *adjoint* problem. The price we must pay for this is that we do not have much control on the norm in which we measure the error. In these notes, we assume that this is a reasonable price to pay for not solving the *adjoint* equation.

Note that the term

$$\frac{1}{2} \int_0^T \int_{\mathbb{R}^d} e^2\, \nabla \cdot \big(\frac{\mathbf{f}(v_1) - \mathbf{f}(v_2)}{v_1 - v_2}\big)\, dx\, dt,$$

is equal to zero if $\mathbf{f}(v) = \mathbf{a}\, v$ and $\mathbf{a}$ is a constant vector. However, in general, this term is not easy to handle. This is why we would like to explore the possibility of finding another function U for which the term involving the convective nonlinearity $\mathbf{f}$ has a simpler form. We do this next.

3.3 A very simple way of handling the convective nonlinearity f

A simple glance at the expression (3.2) for $Z(U)$ suggests that, in order to get rid of the term that contains the nonlinearity $\mathbf{f}$, we should find a non-trivial function U such that $e\,U''(e) = 0$. Of course, such a function does *not* exist but we can still achieve our goal if we take U'' to be the limit of a a sequence of smooth functions $\{\, U_\epsilon \,\}_{\epsilon > 0}$ such that

$$\lim_{\epsilon \mapsto 0} e\, U''_\epsilon(e) = 0,$$

It is easy to see that U must be a continuous function, with a kink at the origin and linear elsewhere.

Thus, if we want to consider

$$U(e) = e^+ \equiv \max\{0, e\},$$

we can take, for any positive parameter ϵ, we define

$$U_\epsilon(e) = \int_0^{e/\epsilon} (e - \epsilon\, s)\, \mu(s)\, ds$$

where μ is a smooth nonnegative function with support in $[0,1]$ and integral equal to one. Then, for every fixed value of e, it is easy to verify that

$$\begin{aligned} \lim_{\epsilon \mapsto 0} e\, U''_\epsilon(e) &= 0, \\ \lim_{\epsilon \mapsto 0} U'_\epsilon(e) &= sign(e)^+, \\ \lim_{\epsilon \mapsto 0} U''_\epsilon(e) &\equiv sign'(e)^+ \geq 0, \end{aligned}$$

and that

$$Z(U) = -\nu \int_0^T \int_{\mathbb{R}^d} \nabla e \cdot \nabla sign(e)^+ \, dx\, dt,$$

by the Lebesgue dominated convergence theorem. We thus see that the term containing the convective nonlinearity $\mathbf{f}$ has completely disappeared, as we wanted.

The equation (3.6) becomes the following simple estimate:

$$|\!|\!|\, e \,|\!|\!|^+_{1,T,\nu} = \|\, e^+(0) \,\|_{L^1(\mathbb{R}^d)} + \int_0^T \int_{\mathbb{R}^d} sign(v_1 - v_2)^+ \,(r_1 - r_2)\, dx\, dt, \qquad (3.8)$$

where

$$|\!|\!|\, e \,|\!|\!|^+_{1,T,\nu} = \|\, e^+(T) \,\|_{L^1(\mathbb{R}^d)} + \nu \int_0^T \int_{\mathbb{R}^d} \nabla e \cdot \nabla sign(e)^+ \, dx\, dt. \qquad (3.9)$$

Similar estimates with $(-e)^+$ and $|\,e\,|$ instead of e^+ can be obtained if we take μ to have support in $[-1,0]$ and if we take μ to be an even function with support in $[-1,1]$, respectively. This shows that the use of L^1-like norms leads *naturally* to a simple treatment of the nonlinear convective terms.

Note that the term

$$\nu \int_0^T \int_{\mathbb{R}^d} \nabla e \cdot \nabla sign(e)^+ \, dx \, dt \equiv \lim_{\epsilon \mapsto 0} \nu \int_0^T \int_{\mathbb{R}^d} \nabla e \cdot \nabla U'_\epsilon(e) \, dx \, dt,$$

is a nonnegative term. This strange term is an L^1-version of the more familiar term

$$\nu \int_0^T \int_{\mathbb{R}^d} |\nabla e|^2 \, dx \, dt,$$

obtained by working with L^2-norms.

We have thus shown a way to avoid having to solve an *adjoint* problem and obtain a very simple continuous dependence result that can easily handle the convective nonlinearity $\mathbf{f}$. From that estimate, several important results can be obtained, as we show next.

3.4 Continuous dependence results in L^1-like norms

In this section, we obtain three types of results: (i) *A priori* estimates on the exact solution v of (3.1) and (3.2), (ii) continuous dependence results, and (iii) *a posteriori* error estimates.

3.4.1 *A priori* **estimates of** v**.** From (3.8) and (3.9), several properties of the solution v of (3.1) and (3.2) can be obtained. We are only interested in the following two:

$$v(x,t) \in I(v_0, r) = [a,b], \tag{3.10}$$

where

$$a = \inf_{x\in\mathbb{R}^d} v_0(x) - \| (-r)^+ \|_{L^1(0,T;L^\infty(\mathbb{R}^d))},$$
$$b = \sup_{x\in\mathbb{R}^d} v_0(x) + \| r^+ \|_{L^1(0,T;L^\infty(\mathbb{R}^d))},$$

and

$$\| v \|_{L^\infty(0,T;TV(\mathbb{R}^d))} \le \| v_0 \|_{TV(\mathbb{R}^d)} + \| r \|_{L^1(0,T;TV(\mathbb{R}^d))}, \tag{3.11}$$

where

$$| v |_{TV(\mathbb{R}^d)} = \sup_{\| \mathbf{p} \|_{L^\infty(\mathbb{R}^d,\mathbb{R}^d)=1}} \int_{\mathbb{R}} v \, \nabla \cdot \mathbf{p}.$$

Note how these estimates are independent of the size of the diffusion coefficient ν.

The first estimate follows from (3.8) in two steps. First, we take $v_1 = v$ and

$$v_2(x,t) = c(t) = \sup_{x\in\mathbb{R}^d} v_0(x) + \| r^+ \|_{L^1(0,t;L^1(\mathbb{R}^d))}$$

and obtain that $v(x,t) \le c(t)$. Then, we take $v_2 = v$ and

$$v_1(x,t) = c(t) = \inf_{x\in\mathbb{R}^d} v_0(x) - \|\,(-r)^+\,\|_{L^1(0,t;L^1(\mathbb{R}^d))}$$

to obtain that $v(x,t) \geq c(t)$.

To obtain the second estimate, we first obtain from (3.8) that

$$\|\,e(T)\,\|_{L^1(\mathbb{R}^d)} \quad \leq \|\,e(0)\,\|_{L^1(\mathbb{R}^d)} + \int_0^T\!\!\int_{\mathbb{R}^d} |\,r_1 - r_2\,|\,dx\,dt.$$

Then, taking $v_1 = v$ and $v_2(x,t) = v(x + h\,e_i, t)$, where e_i is one of the canonical Euclidean basis vectors, we get that

$$\|\,v_{x_i}(T)\,\|_{L^1(\mathbb{R}^d)} \quad \leq \|\,v_{x_i}(0)\,\|_{L^1(\mathbb{R}^d)} + \int_0^T\!\!\int_{\mathbb{R}^d} |\,r_{x_i}\,|\,dx\,dt.$$

Adding on i from 1 to d, we get the result.

3.4.2 Continuous dependence. To obtain a general continuous dependence result, we simply rewrite the equation for v_2 as follows

$$(v_2)_t + \nabla\cdot\mathbf{f}_1(v_2) - \nu_1\Delta v_2 = r_2 + \nabla\cdot(\mathbf{f}_1(v_2) - \mathbf{f}_2(v_2)) - (\nu_1 - \nu_2)\,\Delta v_2,$$

and apply the estimate (3.8) to get

$$\begin{aligned} \|\!|\,e\,\|\!|^+_{1,T,\nu_1} \quad \leq \quad & \|\,e^+(0)\,\|_{L^1(\mathbb{R}^d)} + \int_0^T\!\!\int_{\mathbb{R}^d} (\,r_1 - r_2\,)^+\,dx\,dt \\ & + \int_0^T\!\!\int_{\mathbb{R}^d} \big(\nabla\cdot(-\mathbf{f}_1(v_2) + \mathbf{f}_2(v_2))\big)^+\,dx\,dt \\ & + \int_0^T\!\!\int_{\mathbb{R}^d} \big((\,\nu_1 - \nu_2\,)\,\Delta v_2\,\big)^+\,dx\,dt. \end{aligned}$$

Note that the term

$$\int_0^T\!\!\int_{\mathbb{R}^d} \big|\,\nabla\cdot(\mathbf{f}_1(v_2) - \mathbf{f}_2(v_2))\,\big|\,dx\,dt,$$

can be bounded uniformly with respect to the diffusion coefficient ν_2 by

$$\|\,\mathbf{f}_1'(w) - \mathbf{f}_2'(w)\,\|_{L^\infty(I(v_{2,0},r_2))}\,\big(\,\|\,v_{2,0}\,\|_{L^\infty(0,T;TV(\mathbb{R}^d))} + \|\,r_2\,\|_{L^1(0,T;TV(\mathbb{R}^d))}\,\big).$$

This means that the above continuous dependence result for the nonlinearities does hold when the diffusion coefficients tend to zero.

Unfortunately, this is not true for the dependence with respect to the diffusion coefficients themselves since the term

$$|\,\nu_1 - \nu_2\,|\int_0^T\!\!\int_{\mathbb{R}^d} |\Delta v_2|\,dx\,dt,$$

does blow up when the diffusion coefficient ν_2 goes to zero.

3.4.3 An *a posteriori* error estimate. To obtain an *a posteriori* error estimate between the exact solution v of (3.1) and (3.2), and any *smooth* approximation u, we first note that u satisfies the equation

$$u_t + \nabla \cdot \mathbf{f}(u) - \nu \Delta u = r_2,$$

where r_2 is trivially defined by the left-hand side of the above equation. The, we set $v_1 = v$ and $v_2 = u$ in (3.8) and obtain the following very simple *a posteriori* error estimate:

$$\| e \|_{1,T,\nu}^{+} = \| e^{+}(0) \|_{L^1(\mathbb{R}^d)} + \int_0^T \int_{\mathbb{R}^d} sign(v-u)^{+} \, (r - \; u_t + \nabla \cdot \mathbf{f}(u) - \nu \Delta u \,) \, dx \, dt,$$

Note that this estimate does hold if we let the diffusion coefficient ν go to zero. However, in this case, the above estimate becomes *useless* for computational purposes since the quantity

$$\| e \|_{1,T,\nu}^{+} = \| e^{+}(T) \|_{L^1(\mathbb{R}^d)} + \nu \int_0^T \int_{\mathbb{R}^d} \nabla e \cdot \nabla sign(e)^{+} \, dx \, dt,$$

might converge to an order one quantity. Indeed, since the approximate solution u is independent of the diffusion coefficient ν, the expression

$$\lim_{\nu \mapsto 0} \nu \nabla e = (\lim_{\nu \mapsto 0} \nu \nabla v),$$

might be different from zero when the function $\mathbf{f}$ is strictly nonlinear; an example of this situation will be considered in the section 5.3.

An interpretation of this unfortunate situation is the following. If the approximate solution u is to be a good approximation of the exact solution v, it must resolve the strong gradients of v well. Typically, the strong gradients of v are concentrated in very small sets whose measure is proportional to a positive power of the diffusion coefficient ν. If we let ν go to zero, it is clear that the approximate solution u will be totally unable to resolve those structures while remaining *smooth*.

In summary, we have been able to obtain (i) *a priori* estimates on the exact solution v of (3.1) and (3.2), (ii) general continuous dependence results, and (iii) *a posteriori* error estimates. However, we run into essential difficulties when the diffusion coefficients go to zero. In the next section, we show how to overcome this difficulty.

3.5 Allowing the diffusion coefficients to go to zero

To deal with this difficult case, we have to modify the technique we have been working with. This modification has two main ingredients. First, the function v_1 is evaluated at the space variable x whereas the function v_2 is evaluated at a *different* space variable x'; this trick is called the *doubling* of the variables and will allow us to properly deal with the second-order terms.

Since it is not possible to compare v_1 and v_2 at the same points anymore, instead of dealing with the quantity

$$\| (v_1(t) - v_2(t))^+ \|_{L^1(\mathbb{R}^d)},$$

we will consider the quantity

$$\rho^+_{\epsilon_x}(v_1(t), v_2(t)) = \int_{\mathbb{Q}} (\, v_1(x,t) - v_2(x',t)\,)^+ \, \varphi_{\epsilon_x}(x - x')\, dx\, dx',$$

where $\mathbb{Q} = \mathbb{R}^d \times \mathbb{R}^d$,

$$\varphi_{\epsilon_x}(z) = \Pi^d_{i=1} \frac{1}{\epsilon_x}\, \omega(\frac{z_i}{\epsilon_x}),$$

and ω is an even, nonnegative function with support $[-1,1]$ and integral equal to one. The choice of the form of $\rho^+_{\epsilon_x}(v_1(t), v_2(t))$ is appropriate since it is easy to prove that

$$\big|\, \rho^+_{\epsilon_x}(v_1(t), v_2(t)) - \| e^+(t) \|_{L^1(\mathbb{R}^d)} \,\big| \leq \epsilon_x \,|\, v_2(t)\,|_{TV(\mathbb{R}^d)}. \tag{3.12}$$

We are now ready to obtain our new continuous dependence results; we proceed in several steps.

Step 1. We start exactly as in the standard duality argument with a trivial identity for $\rho^+_{\epsilon_x}(v_1(t), v_2(t))$:

$$\rho^+_{\epsilon_x}(v_1(T), v_2(T)) = \rho^+_{\epsilon_x}(v_1(0), v_2(0)) + \int_0^T \int_{\mathbb{Q}} e^+_t \, \varphi_{\epsilon_x}\, dx\, dx'\, dt.$$

For the sake of simplicity, we start by considering the case in which $\mathbf{f}_1 = \mathbf{f}_2 = \mathbf{f}$ and $r_1 = r_2 = 0$. In this case

$$\begin{aligned} e^+_t &= sign(v_1 - v_2)^+ \, (-\nabla_x \cdot \mathbf{f}(v_1) + \nu_1 \Delta_x v_1) \\ &\quad + sign(v_1 - v_2)^+ \, (+\nabla_{x'} \cdot \mathbf{f}(v_2) - \nu_2 \Delta_{x'} v_2). \end{aligned}$$

Next, we rewrite the above expression by using the fact that $v_2 = v_2(x', t)$ does not depend on x, and that $v_1 = v_1(x,t)$ does not depend on x'. We obtain that

$$\begin{aligned} sign(v_1 - v_2)^+ \, \nabla_x \cdot \mathbf{f}(v_1) &= \nabla_x \cdot \mathbf{f}^+(v_1, v_2), \\ -sign(v_1 - v_2)^+ \, \nabla_{x'} \cdot \mathbf{f}(v_2) &= \nabla_{x'} \cdot \mathbf{f}^+(v_1, v_2), \end{aligned}$$

where

$$\mathbf{f}^+(a,b) = (\mathbf{f}(a) - \mathbf{f}(b))\, sign^+(a - b),$$

and that

$$\begin{aligned} sign(e)^+ \, \nu_1 \Delta_x v_1 &= \nu_1 \Delta_x \, e^+ - \nu_1 \, sign'(e)^+ \,|\, \nabla_x v_1 \,|^2, \\ -sign(e)^+ \, \nu_2 \Delta_{x'} v_2 &= \nu_2 \Delta_{x'} \, e^+ - \nu_2 \, sign'(e)^+ \,|\, \nabla_{x'} v_2 \,|^2. \end{aligned}$$

As a consequence,

$$e_t^+ = -(\nabla_x + \nabla_{x'}) \cdot F^+(v_1, v_2) \\ +(\nu_1 \Delta_x + \nu_2 \Delta_{x'})\,(v_1 - v_2)^+ \\ -sign'(v_1 - v_2)^+ (\nu_1 \,|\, \nabla_x v_1 \,|^2 + \nu_2 \,|\, \nabla_{x'} v_2 \,|^2).$$

Hence, after simple integrations by parts, we get

$$\begin{aligned} \rho^+_{\epsilon_x}(v_1(T), v_2(T)) &= \rho^+_{\epsilon_x}(v_1(0), v_2(0)) \\ &+ \int_0^T \int_{\mathbb{Q}} F^+(v_1, v_2) \cdot (\nabla_x + \nabla_{x'}) \varphi_{\epsilon_x} \, dx\, dx'\, dt \\ &+ \int_0^T \int_{\mathbb{Q}} (v_1 - v_2)^+ \, (\nu_1 \Delta_x + \nu_2 \Delta_{x'}) \varphi_{\epsilon_x}) \, dx\, dx'\, dt \\ &- \int_0^T \int_{\mathbb{Q}} sign'(v_1 - v_2)^+ (\nu_1 \,|\, \nabla_x v_1 \,|^2 + \nu_2 \,|\, \nabla_{x'} v_2 \,|^2) \varphi_{\epsilon_x} \, dx\, dx'\, dt. \end{aligned}$$

Finally, using the fact that $\varphi_{\epsilon_x} = \varphi_{\epsilon_x}(x - x')$,

$$\begin{aligned} \rho^+_{\epsilon_x}(v_1(T), v_2(T)) &= \rho^+_{\epsilon_x}(v_1(0), v_2(0)) \\ &+(\nu_1 + \nu_2) \int_0^T \int_{\mathbb{Q}} (v_1 - v_2)^+ \, \Delta_{x'} \varphi_{\epsilon_x} \, dx\, dx'\, dt \\ &- \int_0^T \int_{\mathbb{Q}} sign'(v_1 - v_2)^+ (\nu_1 \,|\, \nabla_x v_1 \,|^2 + \nu_2 \,|\, \nabla_{x'} v_2 \,|^2) \varphi_{\epsilon_x} \, dx\, dx'\, dt. \end{aligned}$$

Step 2. Note that when the diffusion coefficients are equal, we do not immediately recover the estimate (3.8) for the case $\nu_1 = \nu_2$. To work toward that goal, we write

$$\nu_1 + \nu_2 = (\sqrt{\nu_1} - \sqrt{\nu_2})^2 + 2\sqrt{\nu_1\,\nu_2},$$

and rewrite the last equality of **Step 1** as follows:

$$\begin{aligned} \rho^+_{\epsilon_x}(v_1(T), v_2(T)) &= \rho^+_{\epsilon_x}(v_1(0), v_2(0)) \\ &+(\sqrt{\nu_1} - \sqrt{\nu_2})^2 \int_0^T \int_{\mathbb{Q}} (v_1 - v_2)^+ \, \Delta_{x'} \varphi_{\epsilon_x} \, dx\, dx'\, dt \\ &- \int_0^T \int_{\mathbb{Q}} sign'(v_1 - v_2)^+ (\nu_1 \,|\, \nabla_x v_1 \,|^2 + \nu_2 \,|\, \nabla_{x'} v_2 \,|^2) \varphi_{\epsilon_x} \, dx\, dx'\, dt \\ &+2\sqrt{\nu_1\,\nu_2} \int_0^T \int_{\mathbb{Q}} (v_1 - v_2)^+ \Delta_{x'} \varphi_{\epsilon_x} \, dx\, dx'\, dt. \end{aligned}$$

Next, we must work on the last term of the right-hand side which we denote by Ψ. Using once again the facts that $v_2 = v_2(x', t)$ does not depend on x, that $v_1 = v_1(x, t)$ does not depend on x', and that $\varphi_{\epsilon_x} = \varphi_{\epsilon_x}(x - x')$, we get

$$\begin{aligned}
\Psi &= 2\sqrt{\nu_1\,\nu_2}\int_0^T\!\!\int_{\mathbb{Q}} (\,v_1 - v_2\,)^+ \Delta_{x'}\varphi_{\epsilon_x}\,dx\,dx'\,dt\\
&= -2\sqrt{\nu_1\,\nu_2}\int_0^T\!\!\int_{\mathbb{Q}} (\,v_1 - v_2\,)^+ \nabla_x\cdot\nabla_{x'}\varphi_{\epsilon_x}\,dx\,dx'\,dt\\
&= 2\sqrt{\nu_1\,\nu_2}\int_0^T\!\!\int_{\mathbb{Q}} sign'(v_1 - v_2)^+ \nabla_x v_1\cdot\nabla_{x'}\varphi_{\epsilon_x}\,dx\,dx'\,dt\\
&= 2\sqrt{\nu_1\,\nu_2}\int_0^T\!\!\int_{\mathbb{Q}} sign'(v_1 - v_2)^+ \nabla_x v_1\cdot\nabla_{x'} v_2\;\varphi_{\epsilon_x}\,dx\,dx'\,dt.
\end{aligned}$$

Hence, we get

$$\rho^+_{\epsilon_x}(v_1(T),v_2(T) + \mathbb{FLT}^+_{\epsilon_x,T}(v_1,v_2) = \rho^+_{\epsilon_x}(v_1(0),v_2(0)) + \Xi^+_{\epsilon_x,T}(v_1,v_2) \quad (3.13)$$

where

$$\mathbb{FLT}^+_{\epsilon_x,T}(v_1,v_2) = \int_0^T\!\!\int_{\mathbb{Q}} sign'(v_1 - v_2)^+\,|\,\sqrt{\nu_1}\,\nabla_x v_1 - \sqrt{\nu_2}\,\nabla_{x'} v_2\,|^2\,\varphi_{\epsilon_x}\,dx\,dx'\,dt, \quad (3.14)$$

and

$$\Xi^+_{\epsilon_x,T}(v_1,v_2) = (\sqrt{\nu_1} - \sqrt{\nu_2})^2 \int_0^T\!\!\int_{\mathbb{Q}} sign(v_1 - v_2)^+\,\nabla_{x'} v_2\cdot\nabla_{x'}\varphi_{\epsilon_x}\,dx\,dx'\,dt. \quad (3.15)$$

If now we set $\nu_1 = \nu_2$ and let ϵ_x go to zero, we immediately do obtain the estimate (3.8).

Step 3. In the general case of different nonlinearities and different right-hand sides, it is simple exercise to see that the above estimate holds with

$$\begin{aligned}
\Xi^+_{\epsilon_x,T}(v_1,v_2) &= \textstyle\int_0^T\!\int_{\mathbb{Q}} sign(v_1 - v_2)^+\,\nabla_{x'}\cdot(-\mathbf{f}_1(v_2) + \mathbf{f}_2(v_2)\varphi_{\epsilon_x}\,dx\,dx'\,dt \quad (3.16)\\
&\quad + (\sqrt{\nu_1} - \sqrt{\nu_2})^2 \textstyle\int_0^T\!\int_{\mathbb{Q}} sign(v_1 - v_2)^+\,\nabla_{x'} v_2\cdot\nabla_{x'}\varphi_{\epsilon_x}\,dx\,dx'\,dt\\
&\quad + \textstyle\int_0^T\!\int_{\mathbb{Q}} sign(v_1 - v_2)^+\,(\,r_1 - r_2\,)\,\varphi_{\epsilon_x}\,dx\,dx'\,dt.
\end{aligned}$$

Step 4. Now, it is very simple to get an error estimate in $L^\infty(0,T;L^1(\mathbb{R}^d))$. Since, by (3.12),

$$\begin{aligned}
\rho^+_{\epsilon_x}(v_1(T),v_2(T)) &\geq \|\,e^+(T)\,\|_{L^1(\mathbb{R}^d)} - \epsilon_x\,|\,v_2(T)\,|_{TV(\mathbb{R}^d)},\\
\rho^+_{\epsilon_x}(v_1(0),v_2(0)) &\leq \|\,e^+(0)\,\|_{L^1(\mathbb{R}^d)} + \epsilon_x\,|\,v_2(0)\,|_{TV(\mathbb{R}^d)},
\end{aligned}$$

we obtain

$$\begin{aligned}
\|\!|\,e\,\|\!|^+_{1,\epsilon_x,\nu_1,\nu_2,T} \leq\;& \|\,e^+(T)\,\|_{L^1(\mathbb{R}^d)} + 2\,\epsilon_x\,|\,v_2\,|_{L^\infty(0,T;TV(\mathbb{R}^d))} \quad (3.17)\\
&+ \Xi^+_{\epsilon_x,T}(v_1,v_2),
\end{aligned}$$

where

$$\| e \|^{+}_{1,\epsilon_x,\nu_1,\nu_2,T} = \| e^{+}(T) \|_{L^1(\mathbb{R}^d)} + \mathbb{FLT}^{+}_{\epsilon_x,\nu_1,\nu_2,T}(v_1, v_2). \tag{3.18}$$

This is the estimate we were looking for. In the next section, we display the new continuous dependence results that can be obtained from the above estimate and show that they do *not* break down when the diffusion coefficients go to zero.

3.6 New continuous dependence results

From the estimate obtained in the previous section, we will deduce (i) a new continuous dependence result with respect to the diffusion coefficients and (ii) a new *a posteriori* error estimate that holds for less smooth approximate solutions.

3.6.1 Continuous dependence results with respect to the diffusion coefficients. To obtain this result, we take $v_{0,1} = v_{0,2} = v_0$, $\mathbf{f}_1 = \mathbf{f}_2$ and $r_1 = r_2 = r$ in (3.17), (3.18), and (3.16), and use the following estimates

$$\int_0^T \int_{\mathbb{Q}} (\, v_1 - v_2 \,)^{+} \, \Delta_{x'} \varphi_{\epsilon_x} \, dx \, dx' \, dt \le \frac{1}{\epsilon_x} \| v_2 \|_{L^1(0,T;TV(\mathbb{R}^d))},$$

and

$$\int_0^T \int_{\mathbb{Q}} sign(\, v_1 - v_2 \,)^{+} (r_1 - r_2) \, \varphi_{\epsilon_x} \, dx \, dx' \, dt \; \le \; \epsilon_x \| r \|_{L^1(0,T;TV(\mathbb{R}^d))},$$

to get that

$$\| e^{+}(T) \|_{L^1(\mathbb{R}^d)} \le \; 2\epsilon_x \| v_2 \|_{L^\infty(0,T;TV(\mathbb{R}^d))} + \frac{|\sqrt{\nu_1} - \sqrt{\nu_2}\,|}{\epsilon_x} \| v_2 \|_{L^1(0,T;TV(\mathbb{R}^d))}.$$

Minimizing with respect to the parameter ϵ_x, we finally get that

$$\| e(T) \|_{L^1(\mathbb{R}^d)} \le C \, |\sqrt{\nu_1} - \sqrt{\nu_2}\,|,$$

where

$$C^2 = 8 \| v_2 \|_{L^\infty(0,T;TV(\mathbb{R}^d))} \| v_2 \|_{L^1(0,T;TV(\mathbb{R}^d))}.$$

This constant C can be bounded in terms of the initial data v_0 and in terms of the right-hand side r, by the property (3.11).

An immediate consequence of the above estimate is that the sequence $\{ v_\nu \}_{\nu > 0}$ of exact solutions of the parabolic initial value problem (3.1) and (3.2), is a Cauchy sequence in $\mathcal{C}'(\imath, \mathcal{T}; \mathcal{L}^\infty(\mathbb{R}^\lceil))$. Hence, it converges to a unique limit $v^\star$ that also belongs to $\mathcal{C}'(\imath, \mathcal{T}; \mathcal{L}^\infty(\mathbb{R}^\lceil))$. This limit is a *weak* solution of the following initial value problem:

$$\begin{aligned} v^\star_t + \nabla \cdot \mathbf{f}(v^\star) &= r, \qquad \text{on } \mathbb{R}^d \times (0,T), \\ v^\star(t=0) &= v_0, \qquad \text{on } \mathbb{R}^d, \end{aligned}$$

and is called the *entropy* solution. We will discuss this point in detail in the next chapter.

3.6.2 A new *a posteriori* error estimate. Here, we are going to obtain an *a posteriori* error estimate between the exact solution v (3.1) and (3.2) and an approximate solution u.

We assume that it is feasible to numerically resolve the strong gradients of v when the viscosity coefficient ν is not smaller than the value $\hat{\nu}$. In other words, we define an approximate solution u that tries to drive to zero the following residual

$$u_t + \nabla_{x'} \cdot \mathbf{f}(u) - \hat{\nu} \Delta_{x'} u - r,$$

instead of the standard residual

$$u_t + \nabla_{x'} \cdot \mathbf{f}(u) - \nu \Delta_{x'} u - r.$$

Thus, to obtain our *a posteriori* error estimate, we write

$$u_t + \nabla_{x'} \cdot \mathbf{f}(u) - \hat{\nu} \Delta_{x'} u = r_2,$$

where r_2 is trivially defined by the left-hand side of the above equation and set $v_1 = v$ and $v_2 = u$ in (3.17) to obtain

$$\begin{aligned} \| e \|^{+}_{1,\epsilon_x,\nu,\hat{\nu},T} \leq \quad & \| e^{+}(T) \|_{L^1(\mathbb{R}^d)} + 2\,\epsilon_x \, | u |_{L^\infty(0,T;TV(\mathbb{R}^d))} \\ & + \tfrac{(\sqrt{\nu}-\sqrt{\hat{\nu}})^2}{\epsilon_x} \, \| u \|_{L^1(0,T;TV(\mathbb{R}^d))} \\ & + \mathbb{RES}^{+}_{\epsilon_x,T}(v,u), \end{aligned}$$

where the *residual form* $\mathbb{RES}^{+}_{\epsilon_x,T}(v,u)$ is given by

$$\int_0^T \int_{\mathbb{Q}} sign(v-u)^{+} \, (\, r - (u_t + \nabla_{x'} \cdot \mathbf{f}(u) - \hat{\nu} \Delta_{x'} u) \,) \, \varphi_{\epsilon_x} \, dx \, dx' \, dt. \tag{3.19}$$

Noting that

$$\mathbb{RES}^{+}_{\epsilon_x,T}(v,u) \leq \int_0^T \int_{\mathbb{R}^d} (\, r - u_t - \nabla_{x'} \cdot \mathbf{f}(u) + \hat{\nu} \Delta_{x'} u \,)^{+} \, dx \, dt,$$

and minimizing with respect to ϵ_x, we get

$$\begin{aligned} \| e \|^{+}_{1,\epsilon_x,\nu,\hat{\nu},T} \leq \quad & \| e^{+}(T) \|_{L^1(\mathbb{R}^d)} + C(u) \, | \sqrt{\hat{\nu}} - \sqrt{\nu} \, | \\ & + \textstyle\int_0^T \int_{\mathbb{R}^d} (\, r - u_t - \nabla_{x'} \cdot \mathbf{f}(u) + \hat{\nu} \Delta_{x'} u \,)^{+} \, dx \, dt, \end{aligned}$$

where

$$C^2(u) = 8 \, | u |_{L^\infty(0,T;TV(\mathbb{R}^d))} \, | u |_{L^1(0,T;TV(\mathbb{R}^d))}.$$

We see that this *a posteriori* error estimate has three main terms in its right-hand side. The first measures the error made in the choice of the initial condition, the second measures the error made when a different diffusion coefficient is taken, and the last measures the residual that the approximate solution is trying to drive to zero.

The above *a posteriori* error estimate does hold when the diffusion coefficient ν tends to zero, as we wanted. However, one disadvantage of the above *a posteriori* error estimate, however, is that the approximate solution u has to be *smooth* so that the quantity

$$r - u_t - \nabla_{x'} \cdot \mathbf{f}(u) + \hat{\nu} \Delta_{x'} u,$$

be integrable. This is certainly inconvenient in practical applications.

It is possible, however, to relax the regularity conditions on u with respect to the space variables thanks to the fact that the space variables have been *doubled*. Indeed, since $v = v(x,t)$ is not a function of x', we can rewrite

$$sign(v-u)^+ \,(r - u_t - \nabla_{x'} \cdot \mathbf{f}(u) + \hat{\nu} \Delta_{x'} u,$$

as

$$sign(v-u)^+ \,(r - u_t) + \nabla_{x'} \cdot \mathbf{f}^+(v,u) - \hat{\nu} \Delta_{x'} (v - u)^+ + \hat{\nu} sign'(v-u)^+ \, |\nabla_{x'} u|^2 \,.$$

Hence

$$\begin{aligned} \mathbb{RES}^+_{\epsilon_x,T}(v,u) \quad = \quad & \int_0^T \int_{\mathbb{Q}} sign(v-u)^+ \,(r - u_t) \, \varphi_{\epsilon_x} \, dx \, dx' \, dt \\ & - \int_0^T \int_{\mathbb{Q}} F^+(v,u) \cdot \nabla_{x'} \varphi_{\epsilon_x} \, dx \, dx' \, dt \\ & - \hat{\nu} \int_0^T \int_{\mathbb{Q}} (v-u)^+ \Delta_{x'} \varphi_{\epsilon_x} \, dx \, dx' \, dt \\ & + \hat{\nu} \int_0^T \int_{\mathbb{Q}} sign'(v-u)^+ \, |\nabla_{x'} u|^2 \varphi_{\epsilon_x} \, dx \, dx' \, dt. \end{aligned}$$

We can see that it is not necessary anymore to impose smoothness conditions of the Laplacian of u, only on its gradient. However, we still require smoothness of the time derivative of u. In the next section, we show how to relax this requirement.

3.7 Relaxing the smoothness in time of the approximate solution u

In the previous computation, it is clear that we could not relax the smoothness in time of u because we could not integrate by parts in time. We can do that if we *double* the time variables.

So, we evaluate v_1 at (x,t) and v_2 at (x',t') and instead of dealing with

$$\| \, e^+(\tau) \, \|_{L^1(\mathbb{R}^d)},$$

or with

$$\rho^+_{\epsilon_x}(v_1(t), v_2(t)) = \int_{\mathbb{Q}} (\, v_1(x,t) - v_2(x',t)\,)^+ \,\varphi_{\epsilon_x}(x-x')\,dx\,dx',$$

we consider the quantity

$$\begin{aligned}\rho^+_{\epsilon_t,\epsilon_x}(v_1, v_2; \tau) \quad = \quad & \int_0^T\!\!\int_{\mathbb{Q}} (v_1(x,\tau) - v_2(x',t'))^+ \,\varphi_{\epsilon_x}(x-x')\,\varphi_{\epsilon_t}(\tau - t')\,dx\,dx'\,dt' \\ & + \int_0^T\!\!\int_{\mathbb{Q}} (v_1(x,t) - v_2(x',\tau))^+ \,\varphi_{\epsilon_x}(x-x')\,\varphi_{\epsilon_t}(t-\tau)\,dx\,dx'\,dt,\end{aligned}$$

where

$$\varphi_{\epsilon_t}(z) = \frac{1}{\epsilon_t}\,\eta(\frac{z}{\epsilon_t}),$$

where η is an even, nonnegative function with support in $[-1,1]$ and integral equal to one.

To obtain the new estimate, we proceed in several steps.

Step 1. We start as in the standard duality argument with the following trivial identity for $\rho^+_{\epsilon_t,\epsilon_x}(v_1, v_2; \tau)$:

$$\rho^+_{\epsilon_t,\epsilon_x}(v_1, v_2; T) = \rho^+_{\epsilon_t,\epsilon_x}(v_1, v_2; 0) + \int_0^T \frac{d}{d\tau}\rho^+_{\epsilon_t,\epsilon_x}(v_1, v_2; \tau)\,d\tau.$$

If we take into account that

$$(\frac{d}{dt} + \frac{d}{dt'})\,\varphi_{\epsilon_t}(t-t') = 0,$$

we obtain that

$$\begin{aligned}\rho^+_{\epsilon_t,\epsilon_x}(v_1, v_2; T) &= \rho^+_{\epsilon_t,\epsilon_x}(v_1, v_2; 0) \\ &\quad + \int_0^T\!\!\int_0^T\!\!\int_{\mathcal{Q}} sign(v_1 - v_2)^+\,((v_1)_t - (v_2)_{t'})\,\varphi_{\epsilon_x}\,\varphi_{\epsilon_t}\,dx\,dx'\,dt\,dt'.\end{aligned}$$

From this point, we proceed *exactly* as in the case in which only the space variables were doubled and easily obtain the following result:

$$\begin{aligned}&\rho^+_{\epsilon_t,\epsilon_x}(v_1, v_2; T) + \mathbb{FLT}^+_{\epsilon_t,\epsilon_x,\nu_1,\nu_2,T}(v_1, v_2)) \\ &= \rho^+_{\epsilon_t,\epsilon_x}(v_1, v_2; 0) + \Xi^+_{\epsilon_t,\epsilon_x,\nu_1,\nu_2,T}(v_1, v_2)\end{aligned}$$

where $\mathbb{FLT}^+_{\epsilon_t,\epsilon_x,\nu_1,\nu_2,T}(v_1, v_2)$ is given by

$$\int_0^T\!\!\int_0^T\!\!\int_{\mathbb{Q}} sign'(v_1 - v_2)^+\,|\,\sqrt{\nu_1}\,\nabla_x v_1 - \sqrt{\nu_2}\,\nabla_{x'} v_2\,|^2\,\varphi_{\epsilon_x}\varphi_{\epsilon_t}\,dx\,dx'\,dt\,dt' \quad (3.20)$$

and $\Xi^+_{\epsilon_t,\epsilon_x,\nu_1,\nu_2,T}(v_1, v_2)$ is given by

$$\int_0^T\int_0^T\int_{\mathbb{Q}} sign(v_1 - v_2)^+ \nabla_{x'}(\mathbf{f}_1(v_2) - \mathbf{f}_2(v_2) \cdot \nabla_{x'}\varphi_{\epsilon_x}\varphi_{\epsilon_t}\, dx\, dx'\, dt\, dt'$$
$$+(\sqrt{\nu_1} - \sqrt{\nu_2})^2 \int_0^T\int_0^T\int_{\mathbb{Q}} sign(v_1 - v_2)^+ \nabla_{x'}v_2 \cdot \nabla_{x'}\varphi_{\epsilon_x}\varphi_{\epsilon_t}\, dx\, dx'\, dt\, dt'$$
$$+ \int_0^T\int_0^T\int_{\mathbb{Q}} sign(v_1 - v_2)^+ \,(\, r_1 - r_2\,)\, \varphi_{\epsilon_x}\varphi_{\epsilon_t}\, dx\, dx'\, dt\, dt'.$$

It only remains to relate the quantity $\rho^+_{\epsilon_t,\epsilon_x}(v_1, v_2; \tau)$ to the error $\|\, e^+(\tau)\,\|_{L^1(\mathbb{R}^d)}$ for $\tau = 0$ and $\tau = T$. In what follows, we present two ways of doing that.

Step 2. If we assume that both v_1 and v_2 a smooth, it can be proven that the term

$$\left|\, \rho^+_{\epsilon_t,\epsilon_x}(v_1, v_2; T)/\mathbb{N}(T) - \|\, e^+(T)\,\|_{L^1(\mathbb{R}^d)}\,\right|$$

is bounded by

$$\begin{aligned} &\tfrac{1}{2}\left(\,|\, v_1(T)\,|_{TV(\mathbb{R}^d)} + |\, v_2(T)\,|_{TV(\mathbb{R}^d)}\,\right)\epsilon_x \\ +\quad &\tfrac{1}{2}\left(\,\|\,(v_1)_t\,\|_{L^\infty(T-\epsilon_t,T;L^1(\mathbb{R}^d)} + \|\,(v_2)_{t'}\,\|_{L^\infty(T-\epsilon_t,T;L^1(\mathbb{R}^d)}\,\right)\epsilon_t, \end{aligned}$$

and that the term

$$\left|\, \rho^+_{\epsilon_t,\epsilon_x}(v_1, v_2; 0)/\mathbb{N}(T) - \|\, e^+(0)\,\|_{L^1(\mathbb{R}^d)}\,\right|$$

is bounded by

$$\begin{aligned} &\tfrac{1}{2}\left(\,|\, v_1(0)\,|_{TV(\mathbb{R}^d)} + |\, v_2(0)\,|_{TV(\mathbb{R}^d)}\,\right)\epsilon_x \\ +\quad &\tfrac{1}{2}\left(\,\|\,(v_1)_t\,\|_{L^\infty(0,\epsilon_t;L^1(\mathbb{R}^d)} + \|\,(v_2)_{t'}\,\|_{L^\infty(0,\epsilon_t;L^1(\mathbb{R}^d)}\,\right)\epsilon_t, \end{aligned}$$

where

$$\mathbb{N}(T) = 2\int_0^{T/\epsilon_t} \eta(s)\, ds.$$

Note that for $T > \epsilon_t$, we have that $\mathbb{N}(T) = 1$.

Thus, we obtain the following result:

$$\begin{aligned} \|\, e\,\|^+_{1,\epsilon_t,\epsilon_x,\nu_1,\nu_2,T} \quad\leq\quad &\|\, e^+(0)\,\|_{L^1(\mathbb{R}^d)} \\ &+ \left(\,|\, v_1\,|_{L^\infty(0,T;TV(\mathbb{R}^d))} + |\, v_2\,|_{L^\infty(0,T;TV(\mathbb{R}^d))}\,\right)\epsilon_x \\ &+ \left(\|\,(v_1)_t\,\|_{L^\infty(0,T;L^1(\mathbb{R}^d))} + \|\,(v_2)_{t'}\,\|_{L^\infty(0,T;L^1(\mathbb{R}^d))}\right)\epsilon_t, \\ &+ \Xi^+_{\epsilon_t,\epsilon_x,\nu_1,\nu_2,T}(v_1, v_2)/\mathbb{N}(T). \end{aligned} \tag{3.21}$$

where

$$\|\, e\,\|^+_{1,\epsilon_t,\epsilon_x,\nu_1,\nu_2,T} = \|\, e^+(T)\,\|_{L^1(\mathbb{R}^d)} + \mathbb{FLT}^+_{\epsilon_t,\epsilon_x,\nu_1,\nu_2,T}(v_1, v_2)/\mathbb{N}(T) \tag{3.22}$$

Step 3. If we assume that only v_1 is smooth, we have to proceed in a very different way. The result is the following:

$$\lim_{\eta\mapsto\frac{1}{2}\chi_{[-1,1]}} \| e \|^{+}_{1,\epsilon_t,\epsilon_x,\nu_1,\nu_2,T} \leq 2 \, \| e^{+}(T) \|_{L^1(\mathbb{R}^d)} \tag{3.23}$$
$$+4 \, | v_1 |_{L^\infty(0,T;TV(\mathbb{R}^d))} \, \epsilon_x$$
$$+4 \, \| (v_1)_t \|_{L^\infty(0,T;L^1(\mathbb{R}^d))} \, \epsilon_t,$$
$$+4 \lim_{\eta\mapsto\frac{1}{2}\chi_{[-1,1]}} \Xi^{+}_{\epsilon_t,\epsilon_x,\nu_1,\nu_2,T}(v_1, v_2)/\mathbb{N}(T).$$

Let us compare this results with the one obtained in the previous step. Note how the moduli of continuity of v_2 are not present anymore and that the small price we are paying for this is the presence of the factor 2 in the right-hand side and that we must let the auxiliary function η tend to half the characteristic function of the interval $[-1, 1]$, $\frac{1}{2}\chi_{[-1,1]}$.

3.8 The *a posteriori* error estimate for non-smooth u

From the estimates obtained in the previous section, we can obtain the *a posteriori* error estimate we sought. To do that, we simply set $v_1 = v$ and $v_2 = u$ in the estimate (3.21) or in the estimate (3.23).

For example, if we choose the estimate (3.23), we have

$$\lim_{\eta\mapsto\frac{1}{2}\chi(-1,1)} \| e \|^{+}_{1,\epsilon_t,\epsilon_x,\nu,\hat{\nu},T} \leq 2 \, \| e^{+}(0) \|_{L^1(\mathbb{R}^d)}$$
$$+4 \, | v_1 |_{L^\infty(0,T;TV(\mathbb{R}^d))} \, \epsilon_x$$
$$+4 \, \| (v_1)_t \|_{L^\infty(0,T;L^1(\mathbb{R}^d))} \, \epsilon_t,$$
$$+4 \frac{(\sqrt{\nu} - \sqrt{\hat{\nu}})^2}{\epsilon_x} \, \| u \|_{L^1(0,T;TV(\mathbb{R}^d))}$$
$$+4 \lim_{\eta\mapsto\frac{1}{2}\chi(-1,1)} \mathbb{RES}_{\epsilon_t,\epsilon_x,T}(v, u)/\mathbb{N}(T).$$

where the *residual form* $\mathbb{RES}_{\epsilon_t,\epsilon_x,T}(v, u)$ can be expressed in two different ways. If the residual

$$R(u) = u_{t'} + \nabla_{x'} \cdot \mathbf{f}(u) - \hat{\nu}\Delta_{x'} u - r,$$

is an integrable function, we can write

$$\mathbb{RES}_{\epsilon_t,\epsilon_x,T}(v, u) = -\int_0^T\int_{\mathbb{R}} \Sigma(x', t') \, R(u(x', t')) \, dx' \, dt',$$

where

$$\Sigma(x', t') = \int_0^T\int_{\mathbb{R}} sign(v(x, t) - u(x', t')) \, \varphi(x, t, x', t') \, dx \, dt,$$

and

$$\varphi(x, t, x', t') = \varphi_{\epsilon_x}(x - x')\varphi_{\epsilon_t}(t - t').$$

If we want to relax the smoothness of the approximate solution u, we write the *residual form* as follows:

$$\mathbb{RES}_{\epsilon_t,\epsilon_x,T}(v,u) = \int_0^T \int_{\mathbb{R}} \Theta(v,u;x,t)\,dx\,dt$$

where

$$\begin{aligned}
\Theta(c,u;x,t\quad) = & \int_0^T \int_{\mathbb{R}} sign(c-u)^+\, r\, \varphi\, dx'\, dt' \\
& + \int_0^T \int_{\mathbb{R}} sign(c-u)^+\, \varphi_{t'}\, dx'\, dt' \\
& - \int_{\mathbb{R}} sign(c-u(t'=T))^+\, \varphi(t'=T)\, dx' \\
& + \int_{\mathbb{R}} sign(c-u(t'=0))^+\, \varphi(t'=0)\, dx' \\
& - \int_0^T \int_{\mathbb{R}} F^+(c,u)\cdot \nabla_{x'}\varphi\, dx'\, dt' \\
& - \hat{\nu} \int_0^T \int_{\mathbb{R}} (c-u)^+\, \Delta_{x'}\varphi\, dx'\, dt' \\
& + \hat{\nu} \int_0^T \int_{\mathbb{R}} sign'(c-u)^+\, |\nabla_{x'} u|^2 \varphi\, dx'\, dt'.
\end{aligned}$$

This is the form of the *a posteriori* error estimate we were looking for.

3.9 Concluding remarks

In this section, we have shown how to obtain continuous dependence results for the solution of the following initial value problem

$$\begin{aligned}
& v_t + \nabla\cdot\mathbf{f}(v) - \nu\Delta v = r, \qquad \text{in } \mathbb{R}^d \times (0,T), \\
& v(t=0) = v_0, \qquad \text{on } \mathbb{R}^d.
\end{aligned}$$

We have seen that the standard way to do that is through a *duality* argument. This requires the study of the solution of the so-called *adjoint* problem which is not a trivial matter. To avoid having to deal with the *adjoint* problem, a technique has been developed which, however, does not allow us to freely choose the norm in which we evaluate the error.

The classical example is the well-known L^2-energy technique that has been widely used to analyze parabolic problems. Unfortunately, this technique does teat the nonlinear convective terms in a convenient way. The use of L^1-like norms, however, naturally leads to a simple treatment of the nonlinear convective terms. To obtain continuous dependence results in these L^1-like norms that do not break down when the diffusion coefficient go to zero, we introduced the technique of the *doubling* of the space variables. Finally, to be able to compare the exact solution v with a non-smooth function u, we introduced the *doubling* of the time variables.

In this way, we obtained a theory of continuous dependence results for solutions of the initial value problem for nonlinear convection-diffusion equations. This theory allows us to obtain (i) continuous dependence results with respect to the initial data, with respect to the nonlinearities, and with respect to the right-hand side, (ii) regularity results for the exact solution, and (iii) *a posteriori* error estimates that are independent of the numerical scheme used to compute the approximate solution u. Since these results do not break down when the diffusion coefficients go to zero, they trivially hold for the *entropy* solution, as we show in the next chapter.

4. Continuous dependence for nonlinear convection

In this chapter, we study continuous dependence results for the *entropy* solution of the following initial-value problem

$$v_t + \nabla \cdot \mathbf{f}(v) = r, \qquad \text{in } \mathbb{R}^d \times (0,T), \tag{4.1}$$

$$v(t=0) = v_0, \qquad \text{on } \mathbb{R}^d. \tag{4.2}$$

First, we show how the *entropy* solution of the above problem *inherits* the continuous dependence results of the solutions of the parabolic problem (3.1) and (3.2). Then, we deduce from them regularity properties of the *entropy* solution and the *a posteriori* error estimate that we sought.

4.1 Existence and uniqueness of the entropy solution

It is well-known that for smooth enough data, the solution $_\nu$ of the parabolic problem

$$(v_\nu)t + \nabla \cdot \mathbf{f}(v_\nu) - \nu\, \Delta v_\nu = r, \qquad \text{in } \mathbb{R}^d \times (0,T),$$
$$v_\nu(t=0) = v_0, \qquad \text{on } \mathbb{R}^d,$$

belongs to $\mathcal{C}'(\imath,\mathcal{T};\mathcal{L}^\infty(\mathbb{R}^\ulcorner))$. In the last chapter, we saw that there is a constant C independent of ν such that

$$\| v_{\nu_1} - v_{\nu_2} \|_{L^\infty(0,T;L^1(\mathbb{R}^d))} \le C\, |\sqrt{\nu_1} - \sqrt{\nu_2}\,|.$$

As we said in the previous chapter, this means that the sequence $\{ v_\nu \}_{\nu>0}$ is a Cauchy sequence in $\mathcal{C}'(\imath,\mathcal{T};\mathcal{L}^\infty(\mathbb{R}^\ulcorner))$ and so it converges to a *unique* limit that we denote by v. This limit belongs to $\mathcal{C}'(\imath,\mathcal{T};\mathcal{L}^\infty(\mathbb{R}^\ulcorner))$ and is a *weak* solution of our initial value problem (4.1) and (4.2), as can easily be proven. This weak solution is the so-called *entropy* solution. Moreover, we immediately get that

$$\| v_\nu - v \|_{L^\infty(0,T;L^1(\mathbb{R}^d))} \le C\, \sqrt{\nu}\,.$$

It is now clear that to obtain continuous dependence results for the *entropy* solutions, we only have to let one the diffusion coefficients go to zero in the continuous dependence results obtained for the parabolic case.

4.2 The inherited continuous dependence results

We start by letting the diffusion coefficient ν to go to zero in the estimate (3.8), (3.9) we obtain

$$\|\, e \,\|^+_{1,T,0} = \| e^+(0) \|_{L^1(\mathbb{R}^d)} + \int_0^T \!\! \int_{\mathbb{R}^d} sign(v_1 - v_2)^+ \,(r_1 - r_2)\, dx\, dt, \tag{4.3}$$

where

$$\| e \|_{1,T,0}^{+} = \| e^{+}(T) \|_{L^1(\mathbb{R}^d)} + \lim_{\nu \mapsto 0} \nu \int_0^T \int_{\mathbb{R}^d} \nabla e \cdot \nabla sign(e)^{+} \, dx \, dt, \tag{4.4}$$

where, of course, v_ℓ is the *entropy* solution of the initial value problem (4.1) and (4.2) with initial data $v_{0,\ell}$ and with right-hand side r_ℓ for $\ell = 1, 2$. As in the case of the solutions of the parabolic problem, from the above estimate, we can deduce *a priori* estimates for the *entropy* solution as well as a continuous dependence with respect to the initial data, the nonlinearity $\mathbf{f}$, and the right-hand side r.

4.2.1 ***A priori*** **estimates of the entropy solution.** It is a trivial exercise to see that we have

$$v(x,t) \in I(v_0, r) = [a, b], \tag{4.5}$$

where

$$a = \inf_{x \in \mathbb{R}^d} v_0(x) - \| (-r)^{+} \|_{L^1(0,T;L^\infty(\mathbb{R}^d))},$$
$$b = \sup_{x \in \mathbb{R}^d} v_0(x) + \| r^{+} \|_{L^1(0,T;L^\infty(\mathbb{R}^d))},$$

and

$$\| v \|_{L^\infty(0,T;TV(\mathbb{R}^d))} \leq \| v_0 \|_{TV(\mathbb{R}^d)} + \| r \|_{L^1(0,T;TV(\mathbb{R}^d))}. \tag{4.6}$$

4.2.2 Continuous dependence. Proceeding as in the previous chapter, we get,

$$\begin{aligned} \| e \|_{1,T,0}^{+} \leq \; & \| e^{+}(0) \|_{L^1(\mathbb{R}^d)} + \int_0^T \int_{\mathbb{R}^d} (r_1 - r_2)^{+} \, dx \, dt \\ & + C \, \| \mathbf{f}_1'(w) - \mathbf{f}_2'(w) \|_{L^\infty(I(v_{2,0}, r_2))}, \end{aligned}$$

where

$$C = \| v_{2,0} \|_{L^\infty(0,T;TV(\mathbb{R}^d))} + \| r_2 \|_{L^1(0,T;TV(\mathbb{R}^d))}.$$

4.2.3 An ***a posteriori*** **error estimate for smooth** u**.** To obtain an *a posteriori* error estimate for smooth u, we take the limit as ν goes to zero to get

$$\begin{aligned} \lim_{\nu \mapsto 0} \| e \|_{1,\epsilon_x,\nu,\hat{\nu},T}^{+} \leq \; & \| e^{+}(T) \|_{L^1(\mathbb{R}^d)} + C(u) \sqrt{\hat{\nu}} \\ & + \int_0^T \int_{\mathbb{R}^d} (r - u_t - \nabla_{x'} \cdot \mathbf{f}(u) + \hat{\nu} \Delta_{x'} u)^{+} \, dx \, dt, \end{aligned} \tag{4.7}$$

where

$$C^2(u) = 8 \, | u |_{L^\infty(0,T;TV(\mathbb{R}^d))} \, | u |_{L^1(0,T;TV(\mathbb{R}^d))}.$$

4.2.4 An *a posteriori* error estimate for non-smooth u. When the approximate solution u is not smooth, we proceed in a similar way. For example, if we choose to pass to the limit in the estimate (3.21), we obtain

$$\begin{aligned} \lim_{\nu,\hat{\nu}\mapsto 0} \| e \|^{+}_{1,\epsilon_t,\epsilon_x,\nu,\hat{\nu},T} \quad \leq \quad & \| e^{+}(0) \|_{L^1(\mathbb{R}^d)} \\ & + \left(| v |_{L^\infty(0,T;TV(\mathbb{R}^d))} + | u |_{L^\infty(0,T;TV(\mathbb{R}^d))} \right) \epsilon_x \\ & + (\| v_t \|_{L^\infty(0,T;L^1(\mathbb{R}^d))} + \| u_{t'} \|_{L^\infty(0,T;L^1(\mathbb{R}^d))}) \, \epsilon_t, \\ & + \mathbb{RES}_{\epsilon_t,\epsilon_x,T}(v,u)/\mathbb{N}(T), \end{aligned} \tag{4.8}$$

and if we choose to pass to the limit in the estimate (3.23), we obtain

$$\begin{aligned} \lim_{\nu,\hat{\nu}\mapsto 0} \lim_{\eta\mapsto\frac{1}{2}\chi(-1,1)} \| e \|^{+}_{1,\epsilon_t,\epsilon_x,\nu,\hat{\nu},T} \quad \leq \quad & 2 \, \| e^{+}(0) \|_{L^1(\mathbb{R}^d)} \\ & +4 \, | v |_{L^\infty(0,T;TV(\mathbb{R}^d))} \, \epsilon_x \\ & +4 \, \| (v)_t \|_{L^\infty(0,T;L^1(\mathbb{R}^d))} \, \epsilon_t, \\ & +4 \lim_{\eta\mapsto\frac{1}{2}\chi(-1,1)} \mathbb{RES}_{\epsilon_t,\epsilon_x,T}(v,u)/\mathbb{N}(T). \end{aligned} \tag{4.9}$$

where the *residual form* $\mathbb{RES}_{\epsilon_t,\epsilon_x,T}(v,u)$ is given by

$$\mathbb{RES}_{\epsilon_t,\epsilon_x,T}(v,u) = \int_0^T \int_{\mathbb{R}^d} \Theta(v,u;x,t) \, dx \, dt$$

where

$$\begin{aligned} \Theta(c,u;x,t \quad) = \quad & \int_0^T \int_{\mathbb{R}^d} sign(c-u)^{+} \, r \, \varphi \, dx' \, dt' \\ & + \int_0^T \int_{\mathbb{R}^d} sign(c-u)^{+} \, \varphi_{t'} \, dx' \, dt' \\ & - \int_{\mathbb{R}^d} sign(c-u(t'=T))^{+} \, \varphi(t'=T) \, dx' \\ & + \int_{\mathbb{R}^d} sign(c-u(t'=0))^{+} \, \varphi(t'=0) \, dx' \\ & - \int_0^T \int_{\mathbb{R}^d} F^{+}(c,u) \cdot \nabla_{x'} \varphi \, dx' \, dt'. \end{aligned}$$

4.3 Concluding remarks

We have thus shown that the *entropy* solution *inherits* the continuous dependence results we obtained for the parabolic solutions in the previous chapter. We are thus ready to obtain *a posteriori* error estimates for the *entropy* solution.

5. *A posteriori* error estimates for continuous approximations

In this chapter, we apply the *a posteriori* error estimates obtained in the previous chapter to approximate solutions u that are taken to be continuous. Then, we perform several numerical experiments with the Engquist-Osher scheme and conclude that the estimate is particularly good when (i) the solution is smooth, even in the nonlinear case, and when (ii) the convective term is linear, even in the presence of discontinuities. We also conclude that when discontinuities are present and $\mathbf{f}$ is strictly nonlinear, the estimate is not useful. We end this chapter by providing an explanation of this phenomenon and by proposing a new error estimate that does not have this unwanted property.

5.1 The error estimate

From the error estimate (4.7), setting $\hat{\nu} = 0$, we easily get the following result.

Theorem 5.1. *Let v be the entropy solution and let u be a continuous approximation. Then,*

$$\| u(T) - v(T) \|_{L^1(\mathbb{R}^d)} \leq \Phi(v_0, u; T),$$

where

$$\Phi(v_0, u; T) = \quad \| u(0) - v_0 \|_{L^1(\mathbb{R}^d)} + \| Residual(u) \|_{L^1((0,T)\times\mathbb{R}^d)}$$

where

$$Residual(u) = u_t + \nabla \cdot f(u), \qquad on\ \Omega(T) = \mathbb{R}^d \times [0, T]. \tag{5.1}$$

If the residual on those smooth solutions is zero, that is, if the approximate solution u is also the entropy solution of the initial value problem (4.1) and (4.2), then Theorem (3.8) gives the familiar L^1-contraction property of the entropy solutions, namely,

$$\| u(T) - v(T) \|_{L^1(\mathbb{R}^d)} \leq \| u(0) - v_0 \|_{L^1(\mathbb{R}^d)},$$

as expected.

Now, let us apply Theorem (5.1). Suppose that we have a numerical scheme that produces an approximate solution u_h at times t^n, $n = 1, \ldots, N$. Let us denote by $\mathcal{U}$ the set of all the *interpolates* u such that $u(t^n) = u_h(t^n)$ for $n = 1, \ldots, N$. Then, a simple application of Theorem (5.1), gives the following *a posteriori* error estimate:

$$\| u_h(t^n) - v(t^n) \|_{L^1(\mathbb{R}^d)} \leq \inf_{u \in \mathcal{U}} \Phi(v_0, u; t^n), \quad n = 1, \ldots, N. \tag{5.2}$$

In this way, the problem of *a posteriori* error estimation becomes an *interpolation* problem. When a front-tracking method is used, the approximate solution is always taken to be smooth except at a finite number of surfaces which contain the discontinuities; this is why the above estimate is very well suited for these methods. Numerical experiments for front-tracking methods have not been performed, yet. However, applications of the above estimate to the Engquist-Osher scheme are available. We display those results next.

5.2 Application to the Engquist-Osher scheme

Next, we apply the *a posteriori* error estimate to the well-known Engquist-Osher scheme with the purpose of finding out how good is the estimate.

Thus, we pick $u^\star$ to be a member of $\mathcal{U}$ and study the behavior of the ratio

$$r(u^\star, t^N) = \frac{\Phi(v_0, u^\star; t^N)}{\| u_h(t^N) - v(t^N) \|_{L^1(\mathbb{R}^d)}},$$

as the discretization parameters go to zero. Of course, the optimal result is when this ratio is equal to one uniformly with respect to the discretization parameters. In what follows, we present some numerical results that show that this ideal situation can be achieved.

We consider the approximate solution given by the well-known Engquist-Osher scheme in uniform grids $\{t^n = n\,\Delta t\}_{n=0}^{N_t} \times \{x_i = i\,\Delta x\}_{i=0}^{N_x}$ defined on the space domain is the interval $[0,1)$ with periodic boundary conditions. We use two ways of interpolating the values u_i^n generated by the Engquist-Osher scheme. We denote by $u_1^\star$ the standard piecewise-bilinear function such that $u_1^\star(t^n, x_i) = u_i^n$. We also define a more sophisticated interpolation function $u_2^\star$ which takes into account the nature of the conservation law. We define $u_2^\star$ as follows. First, we take $u_2^\star(n\,\Delta t)$ to be the piecewise-linear function such that $u_2^\star(t^n, x_i) = u_i^n$. We extend $u_2^\star$ to the strips $(n\,\Delta t, (n+1)\,\Delta t) \times [0,1)$ as follows. We consider all the trapezoids formed by the lines $t = t^n$, $t = t^{n+1}$ and the approximate characteristics $x = x_i + f'(u(t^n, x_i))\,(t - t^n)$ and $x = x_i + f'(u(t^{n+1}, x_i))\,(t - t^{n+1})$. We then defined $u_2^\star$ inside of each of these trapezoids as the standard bilinear isoparametric interpolate of the values of u at the vertices. This construction is well-defined for the examples considered below.

All the integrals are computed by using the mid-point rule.

Example 1. We take $f(v) = v$, $v_0(x) = 1. + .5\sin(\pi(2x-1))$, and $T = .1$. The behavior of the ratio $r(u^\star, T)$ is displayed in Tables 1 and 2. We can see that with the simple interpolate $u_1^\star$, the ratio $r(u_1^\star, T)$ is not bigger that 1.3 and that with the more sophisticated interpolate $u_2^\star$, the ratio $r(u_2^\star, T)$ differs very little from 1. These results confirm that the error estimate is sharp and that even with a very simple interpolation procedure very good results can be obtained.

Example 2. We take $f(v) = v^2/2$, $v_0(x) = 1. + .5\sin(\pi(2x-1))$, and $T = .1$. The results obtained for the previous example also hold in this case, as we can see in Tables 3 and 4. (Note that ratios less than 1 might be obtained because we are evaluating the integrals in an approximate way.)

Example 3. We take $f(v) = v$, $T = 1/\sqrt{2}$, and

$$v_0(x) = \begin{cases} 1, & \text{for } x \in (.4, .6), \\ 0, & \text{elsewhere.} \end{cases}$$

The results obtained for the previous example also hold in this case, as we can see in Tables 3 and 4. (Note that ratios less than 1 might be obtained because we are evaluating the integrals in an approximate way.)

The results are displayed in Tables 5 and 6. We see, once more, that the ratio $r(u^\star, T)$ is close to 1 and that, unlike the previous cases, the results with $u_2^\star$ are only slightly better than the results obtained with $u_1^\star$. This might be explained by the fact that (i) the 'smooth region' of the exact solution is flat and so the contribution of the *Residual* in that region is negligible, and by the fact that (ii) the approximate solution is very smooth around the location of the exact discontinuity.

Example 4. We take $f(v) = v^2/2$, $v_0(x) = 1. + .5\sin(\pi(2x-1))$, and $T = .4$. In this case, the solution does have a single discontinuity and, unfortunately, the ratio blows up and the discretization parameters go to zero!

5.3 Explaining the numerical results

It is not difficult to explain the above numerical results with the help of the estimates obtained for the parabolic case. Indeed, the error estimate of Theorem 5.1 is nothing but the limit as the diffusion coefficient ν goes to zero of the first error estimate 3.8 we obtained for the parabolic regularization.

To see this, let us recall such estimate:

$$|\!|\!| e |\!|\!|_{1,T,\nu} = \| e(0) \|_{L^1(\mathbb{R})} + \int_0^T \int_{\mathbb{R}} sign(v_1 - v_2)\,(r_1 - r_2)\, dx\, dt,$$

where

$$|\!|\!| e |\!|\!|_{1,T,\nu} = \| e(T) \|_{L^1(\mathbb{R})} + \nu \int_0^T \int_{\mathbb{R}} \nabla e \cdot \nabla sign(e)\, dx\, dt.$$

If we set

$$r_1 = 0, \qquad r_2 = Residual(u),$$

and note that

$$|\!|\!| e |\!|\!|_{1,T,\nu} \geq \| e(T) \|_{L^1(\mathbb{R})},$$

we immediately obtain the estimate of Theorem 5.1.

If we want to study the sharpness of this estimate, it is obvious that we must consider what happens in the limit as ν goes to zero to the quantity

$$\Theta(\nu) = \nu \int_0^T \int_{\mathbb{R}} \nabla e \cdot \nabla sign(e)\, dx\, dt,$$

where, if we denote by v_ν the exact solution of the parabolic regularization,

$$e = v_\nu - u.$$

It can be immediately seen that when the exact solution v_ν remains smooth as the diffusion coefficient goes to zero, we get

$$\lim_{\nu \downarrow 0} \Theta(\nu) = 0.$$

This explains why we got sharp results in such cases.

In the case in which the entropy solution has discontinuities, we still get that

$$\lim_{\nu \downarrow 0} \Theta(\nu) = 0,$$

in the linear case $\mathbf{f}' \equiv constant$, since in this case,

$$\| \nabla v_\nu \| = O(1/\sqrt{\nu}).$$

To understand the nonlinear case is more delicate. To gain insight into this situation, let us place ourselves in the simple case of the traveling waves that we constructed when studying the traffic flow problem. In this case, we have that

$$v_\nu(x,t) = \phi(\frac{x - c\,t}{\nu}),$$

where

$$c = \frac{f(v^+) - f(v^-)}{v^+ - v^-}.$$

and where ϕ was the solution of the following ordinary differential equation:

$$\phi' = f(\phi) - \{\, f(v^+) - c\,(v^+ - \phi)\,\}.$$

Now, let us *assume* that for each value of the diffusion coefficient ν, the exact solution v_ν and the approximate solution u intersect only on a single curve $(x(t), t)$. Then, a simple *formal* computation gives us that

$$\Theta(\nu) = 2 \int_0^T \nu\, |\, v_{\nu,x}(x(t),t) - u_x(x(t),t)\,|\, dt.$$

Hence,

$$\lim_{\nu \downarrow 0} \Theta(\nu) = 2 \int_0^T |\lim_{\nu \downarrow 0} \nu\, v_\nu\, |(x(t),t)\, dt.$$

But, since v_ν is a traveling wave, we have that

$$\nu\, v_{\nu,x} = \phi',$$

and, noting that on the curve $(x(t), t)$ we have

$$v_\nu(x(t),t) = u(x(t),t),$$

we easily obtain that

$$\lim_{\nu \downarrow 0} \Theta(\nu) = 2 \int_0^T | f(u) - \{ f(v^+) - c\,(v^+ - u) \} |(x(t),t)\, dt.$$

Note that we have assumed that $u(x(t),t)$ lies between $v^+ = v(x(t)+0,t)$ and $v^- = v(x(t)-0,t)$ and hence, the above limit is different from zero (except, of course if u coincides with v^+ or with v^- on the discontinuity curve).

If we believe that what happened with the traveling wave solution also happens in the general case, this provides the explanation we were seeking since the value of the approximate solution u given by the Engquist-Osher scheme at the discontinuity of the exact solution always lies *between* the limits v^+ and v^-.

5.3.1 Conclusions. From the above discussion, we can draw several conclusions. The first is that to understand the hyperbolic conservation law case, it is crucial to understand the parabolic case: It was after the examination of the parabolic case that we identified the term

$$\Theta(\nu) = \nu \int_0^T \int_{\mathbb{R}} \nabla e \cdot \nabla sign(e)\, dx\, dt,$$

The second conclusion is that the above term is zero when (i) the exact solution is smooth, and when (ii) the convection is linear; in these cases, the above *a posteriori* error estimate seems to be sharp for the Engquist-Osher scheme. Otherwise, this term *has to be added* to the L^1-norm and must be part of the norm in which we measure the error.

Finally, our analytical computation for the case of the traveling waves, which gives that

$$\lim_{\nu \downarrow 0} \Theta(\nu) = 2 \int_0^T | f(u) - \{ f(v^+) - c\,(v^+ - u) \} |(x(t),t)\, dt,$$

suggests that the above term measures how close $u(x(t),t)$ is from being *outside* the discontinuity of the exact solution. Indeed, if the value $u(x(t),t)$ is never inside the *open* interval determined by $v^+ = v(x(t)+0,t)$ and $v^- = v(x(t)-0,t)$, the above term is identically zero. On the other hand, if $u(x(t),t)$ is always, say, in the middle of such interval, the term

$$\lim_{\nu \downarrow 0} \Theta(\nu),$$

will never go to zero!

5.4 Another error estimate

This unfortunate fact renders unpractical the use of the *a posteriori* error we have considered so far. To try to remedy this situation, we could try to apply the error estimate (4.7), this time with $\hat{\nu} \neq 0$; we easily get the following result.

Theorem 5.2. *Let v be the entropy solution and let u be a smooth approximation. Then,*

$$\| u(T) - v(T) \|_{L^1(\mathbb{R}^d)} \leq \Phi(v_0, u; T),$$

where

$$\Phi(v_0, u; T) = \| u(0) - v_0 \|_{L^1(\mathbb{R}^d)} + \| Residual(u) \|_{L^1((0,T)\times\mathbb{R}^d)} + C(u)\sqrt{\hat{\nu}},$$

where

$$Residual(u) = u_t + \nabla \cdot f(u) - \hat{\nu}\, \Delta u, \qquad \text{on } \Omega(T), \tag{5.3}$$

and

$$C^2(u) = 8\, | u |_{L^\infty(0,T;TV(\mathbb{R}))}\, | u |_{L^1(0,T;TV(\mathbb{R}))}.$$

Note that in the above result, the quantity $Residual(u)$ involves the term

$$\hat{\nu}\, \Delta u.$$

Accordingly, the numerical approximation should then be devised to render this *new* residual to zero. The diffusion parameter $\hat{\nu}$ is nothing but a simple case of the so-called *artificial viscosity* coefficient.

The exploration of this idea constitutes the subject of ongoing research.

Table 1
Example 1: Behavior of $r(u_1^\star, T)$ *with respect to* Δx *and* $\lambda = \Delta t/\Delta x$.

$1/\Delta x$	$\lambda = .25$	$\lambda = .50$	$\lambda = .75$
10	1.149	1.157	1.146
20	1.184	1.180	1.155
40	1.206	1.202	1.183
80	1.220	1.216	1.204
160	1.227	1.225	1.218
320	1.230	1.229	1.226
640	1.232	1.231	1.230

Table 2
Example 1: Behavior of $r(u_2^\star, T)$ *with respect to* Δx *and* $\lambda = \Delta t/\Delta x$.

$1/\Delta x$	$\lambda = .25$	$\lambda = .50$	$\lambda = .75$
10	1.02357	1.03430	1.04607
20	1.00642	1.00997	1.00633
40	1.00169	1.00275	1.00225
80	1.00044	1.00073	1.00043
160	1.00011	1.00019	1.00013
320	1.00003	1.00005	1.00004
640	1.00002	1.00002	1.00003

Table 3
Example 2: Behavior of $r(u_1^\star, T)$ *with respect to* Δx *and* $\lambda = 1.5\Delta t/\Delta x$.

$1/\Delta x$	$\lambda = .25$	$\lambda = .50$	$\lambda = .75$
10	1.177	1.161	1.165
20	1.197	1.187	1.175
40	1.232	1.218	1.197
80	1.249	1.238	1.215
160	1.257	1.249	1.225
320	1.262	1.253	1.231
640	1.264	1.257	1.234

Table 4
Example 2: Behavior of $r(u_2^\star, T)$ with respect to Δx and $\lambda = 1.5\Delta t/\Delta x$.

$1/\Delta x$	$\lambda = .25$	$\lambda = .50$	$\lambda = .75$
10	1.04012	1.03034	1.0504
20	1.00009	1.00025	1.01097
40	1.00071	0.99700	1.00065
80	1.00055	0.99875	0.99878
160	0.99947	0.99931	0.99895
320	1.00014	0.99968	0.99935
640	1.00019	0.99990	0.99964

Table 5
Example 3: Behavior of $r(u_1^\star, T)$ with respect to Δx and $\lambda = \Delta t/\Delta x$.

$1/\Delta x$	$\lambda = .25$	$\lambda = .50$	$\lambda = .75$
10	1.385	1.347	1.361
20	1.233	1.212	1.248
40	1.155	1.149	1.186
80	1.116	1.112	1.135
160	1.087	1.081	1.096
320	1.064	1.059	1.069
640	1.045	1.042	1.049

Table 6
Example 3: Behavior of $r(u_2^\star, T)$ with respect to Δx and $\lambda = \Delta t/\Delta x$.

$1/\Delta x$	$\lambda = .25$	$\lambda = .50$	$\lambda = .75$
10	1.301	1.265	1.287
20	1.159	1.141	1.184
40	1.088	1.085	1.132
80	1.057	1.059	1.094
160	1.038	1.040	1.066
320	1.028	1.029	1.048
640	1.020	1.021	1.034

6. *A posteriori* error estimates for discontinuous approximations

In this chapter, we apply the *a posteriori* error estimates obtained in chapter 4 to approximate solutions u that are be continuous. First, we consider the case in which the approximate solution has a finite number of discontinuities lying on smooth curves. Then, we consider the case in which the approximate solution is piecewise-constant. We end with some concluding remarks.

6.1 The case of a finite number of smooth discontinuity curves

In this section, we assume that

$$u \in C^0(0,T;L^1(\mathbb{R}^d)),$$

is a smooth function, except on a finite number of $(d-1)$-dimensional surfaces $C_i, i = 1,\ldots,I$ and develop an *a posteriori* error estimate for this type of approximate solution. To state the result, we need to introduce some notation.

We denote by n_{C_i} the unit normal to C_i such that its component on the t-axis is negative and define the jump of $G(u)$ at the point P on surface C_i, $[G(u)](P)$, to be the following quantity:

$$[G(u)](P) = \lim_{\rho\downarrow 0} \big(G(u(P+\rho\, n_{C_i})) - G(u(P-\rho\, n_{C_i}))\big).$$

Finally, we set

$$\begin{aligned} C_i(T) &= (0,T)\times\mathbb{R}^d \cap C_i,\\ C(T) &= \cup_{i=1}^{I} C_i(T),\\ \Omega(T) &= (0,T)\times\mathbb{R}^d \setminus C(T). \end{aligned}$$

With this notation, we have the following approximation result.

Theorem 6.1. *Let u be as above and let v be the entropy solution. Then,*

$$\| u(T) - v(T) \|_{L^1(\mathbb{R}^d)} \le \Phi(v_0,u;T),$$

where

$$\begin{aligned} \Phi(v_0,u;T) = \quad & \| u(0) - v_0 \|_{L^1(\mathbb{R}^d)} + \| Residual(u) \|_{L^1(\Omega(T))}\\ & + \textstyle\sum_{i=1}^{I} \| Residual(u) \|_{L^1(C_i(T))}, \end{aligned}$$

where

$$Residual(u) = u_t + \nabla\cdot f(u), \qquad \text{on } \Omega(T), \tag{6.1}$$

and

$$Residual(u) = \max\left\{0,\ \sup_{c\in[a,b]} \left\{([F(u,c)],[U(u-c)])\cdot n_{C_i}\right\}\right\} \qquad \text{on } C_i(T), \tag{6.2}$$

and $U(u-c) = |\,u-c\,|$, $F(u,c) = sign(u-c)\big(f(u)-f(c)\big)$. The interval $[a,b]$ is the range of the initial data v_0.

This result follows from setting ν and $\hat{\nu}$ equal to zero in the estimate (4.8), making a couple of integration by parts in the term

$$\mathbb{RES}_{\epsilon_t,\epsilon_x,T}(v,u),$$

and then letting the parameters ϵ_t and ϵ_x go to zero.

It is clear that the term $\|\, Residual(u)\, \|_{L^1(\Omega(T))}$ is nothing but the truncation error on the smooth regions of the approximation defined by u. The interpretation of the term $\|\, Residual(u)\, \|_{L^1(C_i(T))}$, however, requires a more detailed discussion: It measures how close to satisfying the entropy condition are the discontinuities of u on the curve $C_i(T)$.

To see this, let us consider the one-dimensional case, $d = 1$, for the sake of simplicity. If we write $C_i(T) = \{(x,t), x = x_i(t), 0 \le t_{i1} \le t \le t_{i2} \le T\}$, we have that if the quantity

$$([F(u,c)] - \frac{d\,x_i}{dt}\,[U(u-c)])(s,x_i(s)) \le 0 \qquad \text{for } s \in (0,T),$$

for all values of the real constant c, then the discontinuities of u on the curve $C_i(T)$ are entropy-satisfying discontinuities. In this case, we have that

$$\begin{aligned} \Theta_i &= \|\, Residual(u)\, \|_{L^1(C_i(T))} \\ &= \int_{t_{i1}}^{t_{i2}} \max\left\{0, \sup_{c\in[a,b]} \{([F(u,c)] - \frac{d\,x_i}{dt}\,[U(u-c)])(s,x_i(s))\}\right\} ds \\ &\le 0. \end{aligned}$$

This means that if all the discontinuity curves of the approximate solution u are entropy-satisfying discontinuities, the Theorem (3.8) becomes:

$$\|\, u(T) - v(T)\, \|_{L^1(\mathbb{R}^d)} \le \|\, u(0) - v_0\, \|_{L^1(\mathbb{R}^d)} + \|\, Residual(u)\, \|_{L^1(\Omega(T))}.$$

In other words, we only have to worry about the size of the residual on the smooth regions of the approximate solution.

If the residual on those smooth solutions is zero, that is, if the approximate solution u is also the entropy solution of the initial value problem (4.1) and (4.2), then the theorem above gives the familiar L^1-contraction property of the entropy solutions, namely,

$$\|\, u(T) - v(T)\, \|_{L^1(\mathbb{R}^d)} \le \|\, u(0) - v_0\, \|_{L^1(\mathbb{R}^d)},$$

as expected.

When a front-tracking method is used, the approximate solution is always taken to be smooth except at a finite number of surfaces which contain the discontinuities; this is why the above estimate is very well suited for these methods. Numerical experiments for front-tracking methods remain to be done. However, we do expect to have problems similar to the ones we encountered in the last chapter.

6.2 The case of a piecewise-constant approximation

To try to remedy this situation, we apply the estimate (4.9), but this time we work the term

$$\mathbb{RES}_{\epsilon_t,\epsilon_x,T}(v,u),$$

out in a very different way. Without entering into technical details, let us just say that this time, since the approximate solution is piecewise-constant, we cannot define the term *Residual*(u) as before; instead, we obtain a *weaker* version of this quantity which can be expressed solely in terms of the discontinuities of the fluxes. To be able to do that, the auxiliary function φ must *absorb a derivative*, so to speak, and, as a consequence, we obtain an estimate of the form

$$\mathbb{RES}_{\epsilon_t,\epsilon_x,T}(v,u) \leq C_1/\epsilon_t + C_2/\epsilon_x.$$

As we can see, the parameters ϵ_t and ϵ_x cannot longer be set to zero; instead, the optimal choice for them turns out to be the following:

$$\epsilon_t = O(\sqrt{C_1}), \qquad \epsilon_x = O(\sqrt{C_2}).$$

We carry out this idea for the Engquist-Osher method defined in general grids. Since the approximate solution is discontinuous, we allow ourselves to use different triangulations in different time intervals. Thus, for each partition of the interval $[0,T]$, $\{\, t_n \,\}_{0\leq n\leq N}$, we define

$$S_n = \mathbb{R} \times (t_n, t_{n+1}), \qquad n = 0,\ldots,N-1.$$

We set $\Delta t_n = t_{n+1} - t_n$. For each value of n, we define a triangulation $T_{h,n}$ of simplices. No compatibility at $t = t_n$ between the meshes of two consecutive slabs S_n and S_{n+1}, $n = 0,\ldots,N-2$, is required. We denote by T_e the simplex that shares the face e with the simplex T and write $n_{e,T}$ to denote the unit outer normal to T along its face e. Finally, we denote by h_T the diameter of the element T.

We assume that the approximate solution u has the following form:

$$u(x,t) = u^{T_n} \qquad \forall (x,t) \in T\times \in T_{h,n},$$

and determine these values by using the Engquist-Osher scheme. For $t = t_n$, we denote by $[u](x,t_n) = u(x,t_n+0) - u(x,t_n-0)$.

For this scheme, we have the following *a posteriori* error estimate.

Theorem 6.2. *Let u be as above and let v be the entropy solution. Then, we have*

$$\|v(T) - u(T)\|_{L^1(\mathbb{R}^d)} \leq \Phi(v_0, u; T),$$

where

$$\Phi(v_0,u;T) = 2\,\|v_0 - u_0\|_{L^1(\mathbb{R})} + C\big\{ |\, u_0\,|_{TV(\mathbb{R})}\, \Theta_0(u)\,\big\}^{1/2},$$

where C is a numerical constant, and

$$\begin{aligned}\Theta_0(u) \quad &= \sum_{n=0}^{N_\tau-1} \sum_{T\in T_{h,n}} h_T \sum_{e\in\partial T} \int_e |f_{e,T}^{EO}(u^T, u^{T_e}) - \tilde{f}(u^T)\cdot n_{e,T}|\, d\lambda \\ &\qquad + \sum_{n=0}^{N_\tau-1} \Delta t_n \int_{\mathbb{R}^d} \big|\, [u] \,\big|\, dx,\end{aligned}$$

When all the triangulations coincide, it has been proven that the nonlinear functional $\Phi(v_0, u; T)$ goes to zero as the discretization parameters go to zero. As a consequence, the problem we had with the previous error estimate is not present anymore. The question of the sharpness of the estimate, however, remains an open question. Let us just say that the norm in which we measure the error is *not* the L^1-norm but the norm

$$|\!|\!|\, e \,|\!|\!|^{+}_{1,\epsilon_t,\epsilon_x,0,0,T},$$

where the parameters ϵ_t and ϵ_x have been chosen as above! This question remains to be addressed.

7. Concluding remarks

7.1 Some bibliographical remarks

The modeling of the traffic flow in chapter 2 was extracted from Witham's book [27]. I could not find any reference for the discussion on traveling waves for the traffic flow equations, although this seems to be unbelievable.

The main part of these notes in contained in chapter 3. This chapter is based in the paper [9], devoted to continuous dependence for nonlinear, degenerate convection-diffusion equations, and on the paper [10], devoted to *a posteriori* error estimation of convection-diffusion equations.

The whole paper [7] is contained in chapter 5 and in the.first section of chapter 6. The explanation of the numerical results of chapter 5 is new and will most probably appear in [10]. The second part of chapter 6 is contained in [8] which is devoted to *a posteriori* error estimates for the discontinuous Galerkin and the streamline-diffusion methods.

7.2 Open problems

Let us end these notes by pointing out two important problems that could be of interest to researcher in this field.

The first is how to use *already existing* error estimates to define adaptive strategies. Although there have been many *a posteriori* error estimates for the scalar nonlinear conservation law, it is unbelievable the small amount of papers, [15], in which they have been used for adaptivity purposes. Examples of papers containing these estimates are [16], [17], [6], [26], [7], and [8] among others. The study of the *sharpness* of these estimates and their application of these estimates to define adaptive strategies is very interesting and has not been done.

The second problem is related to the so-called *adjoint* equation. In these notes, the *a posteriori* error estimates we have considered were obtained *without* solving an adjoint equation. As we have seen, the price we had to pay was to loose control on the norm we measured the error. On the other hand, there are methods for which the *adjoint* equation is solved. For example, in [22], a complete analysis of the solution of the *adjoint* equation is done for the one-dimensional case and strictly convex nonlinearities f; unfortunately, a more general case cannot be analyzed with that technique. In [14] the case of systems is treated, but the result holds only for a parabolic perturbation of the system of conservation laws and for the streamline-diffusion method. In [4], [5], [12], it is shown how to numerically solve the *adjoint* equation and use this information to to the adaptation. This method gives very impressive results and can be applied to quite general problems. However, it seems to be computationally very intensive and restricted to Galerkin methods.

The question is now if there is a *new* technique that would give us more control on the norm in which we measure the error, can be applied to general situations, and is not computationally intensive.

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