A short introduction to continuous dependence results for Hamilton-Jacobi equations

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1 Introduction

In this paper, we provide a *short introduction* to the techniques that allow us to obtain continuous dependence results for the Hamilton-Jacobi equations. These are equations of the form

$$u_t + H(t, x, u, \nabla u) = 0 \qquad \text{in } \mathbf{R}^d,$$

where $u \mapsto H(\cdot, \cdot, u, \cdot)$ is non-decreasing. They arise in many areas of applied mathematics like optimal control [24], differential games [11], seismic wave propagation [25, 23, 20], terrain navigation (computation of minimum time transit paths) of robots [4], and financial mathematics [12] among many others. They also appear when modeling evolving interfaces in geometry, fluid mechanics, computer vision [21, 15], and materials science [22]; and are essential when dealing with level set methods [19, 17], which are numerical methods that have reached a widespread popularity nowadays for solving interface evolution equations. This is why they have been the focus of attention of many researchers. A basic introduction to the study of these equations can be found in the book [10]. More advanced introductory

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material can be found in the first two chapters of the 1997 book [3]; the bibliography notes therein can be used as an excellent guide for further reading.

It is thus important to study these equations and hence, it is essential to make sure that the corresponding Cauchy problem is well-posed and to be able to known how its solution depends on the Hamiltonian. As we shall see, these results, called continuous dependence results, not only allow us to know properties of the exact solution but also give us a way to know the quality of any numerical approximation to it. They are thus relevant from both the theoretical and computational points of view. Moreover, the fact that weak solutions of the Cauchy problem for the Hamilton-Jacobi equation under consideration are not necessarily unique makes this study crucial. Indeed, among all the weak solutions, we are only interested in the so-called *viscosity* solution which is obtained as a limit of solutions of well-posed parabolic problems. As a consequence, the continuous dependence results we are interested in have reflect this fact.

In this paper, since it is impossible to carry out a reasonable study of these continuous dependence results for the above Hamilton-Jacobi equation, we are going to restrict ourselves to the following steady-state model Hamilton-Jacobi equation

$$u + H(\nabla u) = f, \tag{1}$$

where u and f are periodic in each space coordinate with period 1. This will allow us to provide a shorter, simpler and more clear presentation of the main ideas associated with continuous dependence results for these equations. Moreover it is not difficult to extend the results for this model Hamilton-Jacobi equation to the general case.

The continuous dependence results we consider have the form

$$\| u - v \| \leq \Phi(v), \tag{2}$$

where $\| \cdot \|$ is a norm, u is the exact solution, v is an (almost) arbitrary function and Φ is a non-linear functional. Note that if v is an approximation to the exact solution, estimate (2) gives a rigorous measure of the quality of the approximation; these estimates, called *a posteriori* error estimates turn out to be very useful in practice.

Such estimates can also be used to obtain theoretical properties of the exact solution. Indeed, since we expect to have that $\Phi(u) = 0$, it is reasonable to expect the functional $\Phi(v)$ to depend solely on the *residual* of v, namely,

$$R(v) = v + H(\nabla v) - f.$$

Now if v satisfies the model equation with g instead of f, estimate (2) gives a continuous dependence result with respect to the right-hand side, and if v satisfies the model equation with $\bar{H}$ instead of H, estimate (2) gives a continuous dependence result with respect to the Hamiltonian. We thus see that important properties of the exact solution can be easily deduced from the continuous dependence result (2), as claimed.

We are also interested in studying continuous dependence results for approximations u_h to the exact solution. We assume that the approximate solution u_h is defined by

$$u_h + \widehat{H}(\nabla_h u_h) = f,$$

where ∇_h is an approximation to ∇. We get continuous dependence results of the form

$$\| u_h - v \|_h \leq \Phi_h(v), \tag{3}$$

where $\| \cdot \|_h$ is a discrete version of the norm $\|\cdot\|$ and $\Phi_h(\cdot)$ a discrete version of $\Phi(\cdot)$. Note that if v is taken to be the exact solution of the Hamilton-Jacobi equation u, then (3) is an estimate of the quality of the approximation u_h to u which only depends on the exact solution u. These estimates, called *a priori* error estimates, can tell us what kind of accuracy it is reasonable to expect from the approximation. Moreover by using the above continuous dependence result, properties similar to those obtained for the exact solution can also be obtained for its approximation.

Since we should have $\Phi_h(u_h) = 0$, we expect the functional $\Phi_h(v)$ to depend solely on the *truncation error* of v,

$$T(v) = v + \widehat{H}(\nabla_h v) - f.$$

Just as in the continuous case, continuous dependence results for the approximate solution u_h with respect to the initial data, the right-hand side and the approximate Hamiltonian $\bar{H}$ can be obtained from this estimate.

In this paper, we obtain the continuous dependence result (2) for the exact solution. Then we show that for the so-called monotone numerical schemes, continuous dependence results (3) can be obtained in a remarkably similar way.

To obtain the continuous dependence result for the Hamilton-Jacobi equations, we could use the elegant theory of viscosity solutions; see [8], [7]. However, we have chosen to follow a path that appears to be more natural since it explicitly takes into account that fact that the viscosity solution is the limit as $\nu \downarrow 0$ of the solution of the elliptic problem,

$$u + H(\nabla u) - \nu \Delta u = f.$$

So, first we obtain a continuous dependence result of the form

$$\| u - v \| \leq \Phi_\nu(v), \tag{4}$$

where v is a general function and Φ_ν a non-linear functional, and then we show that when we let ν go to zero in (4), we obtain the desired continuous dependence result (2).

The main reason to take this approach is that it emphasizes the fact that the viscosity solution is a limit of solutions of elliptic problems, and, accordingly, that the estimate (2) is *also* a limit of the estimate (4). We shall see that one advantage of this approach is that it is enough to obtain the continuous dependence results for very smooth solutions; this certainly avoids many technicalities induced by a possible lack of smoothness. An additional interesting advantage of this approach is that both the uniqueness of the viscosity solution as well as one of its classical characterizations can be *deduced* from the estimate (2). In this way, we emphasize that *to obtain continuous dependence results for the singular limit when $\nu \downarrow 0$, that* *is, for the viscosity solution, it is enough to obtain continuous dependence results for smooth solutions of an elliptic problem.* A similar idea was developed in [5] for hyperbolic scalar conservation laws.

The paper is organized as follows. In §2, we consider *smooth* solutions of the elliptic model problem and show how to find a very simple continuous dependence result of the type (4). We then show that although this result gives rise to many interesting properties it breaks down when $\nu \downarrow 0$. This motivates a modification of the continuous dependence result which does not break down when $\nu \downarrow 0$; it is developed in §3 along the lines presented in [6]. In §4, we pass to the limit in ν to obtain the desired estimate (2) for the viscosity solution of the steady-state Hamilton-Jacobi equation. The main result of this section is obtained, by different means, and can be viewed as an extension of some of the results in [7]. In §5, we introduce monotone numerical schemes and obtain the estimate of the form (3) by mimicking the techniques used in the continuous case. This section contains a new unified approach to a priori error estimates which contains some of the error estimates for monotone numerical schemes obtained in [9], [1], and in [14]. In §6, we conclude by giving some references for the reader interested in extensions of these results to more complicated Hamiltonians and to the time-dependent case.

2 A first continuous dependence result for elliptic equations

In this section, we begin our program by considering the problem of how to compare the solution of the elliptic problem

$$u + H(\nabla u) - \nu \Delta u = f \qquad \text{in } \mathbf{R}^d, \tag{5}$$

where u and f are periodic in each coordinate with period 1, to an arbitrary smooth function v, which we assume to be periodic in each coordinate with period 1. We assume that the Hamiltonian H and the right-hand side f are $\mathcal{C}^\infty(\mathbf{R}^d)$ functions since a unique $\mathcal{C}^\infty(\mathbf{R}^d)$ solution u exists for this data; see Friedman [13].

We obtain a remarkably simple continuous dependence result which allows us to obtain (i) a maximum principle for the exact solution, (ii) the so-called L^∞-contraction property for exact solutions, (iii) an a priori estimate on the $W^{1,\infty}(\Omega)$-semi-norm of the exact solution, (iv) a continuous dependence result of the exact solution with respect to the Hamiltonian H, and (v) an a posteriori error estimate.

Unfortunately, this continuous dependence result does *not* allow us to obtain a continuous dependence result with respect to the coefficient ν. Moreover, we show that the continuous dependence result *breaks down* as we let the coefficient ν go to zero; this continuous dependence result is *not* preserved under this limit process.

2.1 A simple continuous dependence result

The continuous dependence result we present gives an upper bound for the following semi-norms:

$$|\, u - v \,|_- = \sup_{x\in\Omega} (\, u(x) - v(x)\,)^+, \qquad |\, u - v \,|_+ = \sup_{x\in\Omega} (\, v(x) - u(x)\,)^+,$$

where $w^+ \equiv \max\{0, w\}$ and $\Omega = [0,1)^d$. An upper bound of the uniform norm can be easily obtained by noting that

$$\| u - v \|_{L^\infty(\Omega)} = \max_{\sigma \in \{-,+\}} |u - v|_\sigma . \tag{6}$$

The result is stated in terms of the *residual* of the function v, namely,

$$R(v; x) = v(x) + H(\nabla v(x)) - \nu \Delta v(x) - f(x). \tag{7}$$

We are now ready to state the result.

Theorem 1 (First Continuous Dependence Result for elliptic equations). *Let u be the solution of the equation (5) and let v be any $\mathcal{C}^2(\mathbf{R}^d)$ function that is periodic in each coordinate with period 1. Then, for $\sigma \in \{-,+\}$, we have*

$$| u - v |_\sigma \leq \Phi_\sigma(v), \tag{8}$$

where

$$\Phi_\sigma(v) = \sup_{x \in \mathbf{R}^d} \big(\sigma\, R(v; x)\big)^+ . \tag{9}$$

It is important to point out the following consequences of this result:

a. The a posteriori error estimate. Note that if v is considered to be an approximation to the exact solution u, Theorem 1 allows us to estimate the L^∞-error between u and v solely in terms of v. Since this can be done only *after* the computation of v, estimates like this are called *a posteriori error estimates*. They are very useful in practical situations as they give a rigorous upper bound of the error in terms of known quantities.

b. Dependence on the right-hand side: The maximum principle. Let us denote by u_f the solution of the equation (5) for a given Hamiltonian H. If we take $v = u_g$, since $R(v, x) = g(x) - f(x)$, Theorem 1 becomes

$$| u_f - u_g |_- \leq \sup_{x \in \mathbf{R}^d} \big(f(x) - g(x)\big)^+, \qquad | u_f - u_g |_+ \leq \sup_{x \in \mathbf{R}^d} \big(g(x) - f(x)\big)^+. \tag{10}$$

The above inequalities imply the maximum principle

$$-\sup_{x \in \Omega} (H(0) - f(x))^+ \leq u_f(y) \leq \sup_{x \in \Omega} (f(x) - H(0))^+ \qquad \forall\, y \in \Omega. \tag{11}$$

c. Dependence on the right-hand side: The L^∞-contraction property. The estimates (10) also imply the so-called L^∞-contraction property of the solutions of the equation (5), namely,

$$\| u_f - u_g \|_{L^\infty(\Omega)} \leq \| f - g \|_{L^\infty(\Omega)} . \tag{12}$$

Note that by using this property, we can consider solutions of the equation (5) with $f \in \mathcal{C}^0(\mathbf{R}^d)$ only. Indeed, if $\{ f_n \}$ is a sequence of $\mathcal{C}^\infty(\mathbf{R}^d)$ periodic functions with period 1 in each coordinate that converges in $L^\infty(\mathbf{R}^d)$ to $f \in \mathcal{C}^0(\mathbf{R}^d)$, then by the above inequality, $\{ u_{f_n} \}$ is also a sequence of $\mathcal{C}^\infty(\mathbf{R}^d)$ periodic functions with period 1 in each coordinate that converges in $L^\infty(\mathbf{R}^d)$ to a $\mathcal{C}^0(\mathbf{R}^d)$ periodic functions with period 1 in each coordinate, u_f. We can then *define* u_f to be a solution of the equation (5).

Also if we take $g(x) = f(x+\delta p)$ for any given non-zero $p \in \mathbf{R}^d$, since $u_g(x) = u_f(x+\delta p)$, we get that

$$\frac{\| u_f(\cdot) - u_f(\cdot + \delta p) \|_{L^\infty(\Omega)}}{|\,\delta\,|} \leq \frac{\| f(\cdot) - f(\cdot + \delta p) \|_{L^\infty(\Omega)}}{|\,\delta\,|},$$

and so, by taking the limit when $|\,\delta\,| \downarrow 0$ and since p is arbitrary, we obtain

$$|\, u_f \,|_{W^{1,\infty}(\Omega)} \leq |\, f \,|_{W^{1,\infty}(\Omega)}. \tag{13}$$

Note that $|\cdot|_{W^{1,\infty}(\Omega)}$ is a semi-norm; indeed, if $|\, f \,|_{W^{1,\infty}(\Omega)} = 0$, then $f$is a constant. From the above inequality, this implies that u is also a constant since in this case, we have $u = f - H(0)$.

b. Dependence on the Hamiltonian. Let us denote by u_H the solution of the equation (5) for a given f. If we take $v = u_{\bar{H}}$ in Theorem 1, since $R(v,x) = H(\nabla v(x)) - \bar{H}(\nabla v(x))$, we get

$$\| u_H - u_{\bar{H}} \|_{L^\infty(\Omega)} \leq \sup_{p \in \mathbf{R}^d : \| p \| \leq |\, f \,|_{W^{1,\infty}(\Omega)}} \left| H(p) - \bar{H}(p) \right|. \tag{14}$$

Note that we have used the bound (13). By using this property, we can consider solutions of the equation (5) with $H \in \mathcal{C}^0(\mathbf{R}^d)$; we only have to proceed in a way similar to that displayed in the previous paragraph.

2.2 Proof of the First Continuous Dependence Theorem for Elliptic Equations

We prove the result for $\sigma = -$; the proof for the case $\sigma = +$ is similar. Define the auxiliary function

$$\psi(x) = u(x) - v(x),$$

and choose $\widehat{x} \in \Omega$ so that

$$\psi(\widehat{x}) \geq \psi(y), \qquad \forall y \in \Omega;$$

such a point exists since ψ is continuous and periodic on Ω. Moreover, since $\nabla\psi(\widehat{x}) = 0$, we can set $\widehat{p} = \nabla u(\widehat{x}) = \nabla v(\widehat{x})$.

We assume that $|\, u - v \,|_- > 0$, otherwise there is nothing to prove. In this case, we have

$$\begin{aligned} |\, u - v \,|_- &= \sup_{x \in \Omega} \{u(x) - v(x)\} \\ &= \sup_{x \in \Omega} \psi(x) \\ &= u(\widehat{x}) - v(\widehat{x}) \end{aligned}$$

$$
\begin{aligned}
&= \big[u(\widehat{x}) + H(\widehat{p}) - \nu\Delta u(\widehat{x}) - f(\widehat{x})\big] \\
&\quad -\big[v(\widehat{x}) + H(\widehat{p}) - \nu\Delta v(\widehat{x}) - f(\widehat{x})\big] \\
&\quad +\big[\nu\Delta u(\widehat{x}) - \nu\Delta v(\widehat{x})\big] \\
&= R(u;\widehat{x}) - R(v;\widehat{x}) + \big[\nu\Delta u(\widehat{x}) - \nu\Delta v(\widehat{x})\big],
\end{aligned}
$$

by the definition of $\widehat{p}$ and R in (7).

Since u is the exact solution of (5), $R(u;\widehat{x}) = 0$, and since $u - v$ attains a maximum at $\widehat{x}$,

$$\big[\nu\Delta u(\widehat{x}) - \nu\Delta v(\widehat{x})\big] \le 0;$$

so we get

$$|\, u - v\,|_- \le (-R(v;\widehat{x}))^+,$$

and the result follows. This completes the proof of Theorem 1.

2.3 A key observation and the breakdown of the estimate

Although the stability Theorem 1 is remarkable in its simplicity, it is useless when we let the coefficient ν tend to zero. A key observation to understand why is this the case is to realize that the proof of Theorem 1 is actually the proof of a *stronger* result which can be stated in terms of the functionals

$$
\begin{aligned}
\|\, u - v\,\|_- &= \sup_{x:|\, u-v\,|_- = u(x)-v(x)} (\, u(x) - v(x)\,)^+ + \nu\big[-\Delta u(x) + \Delta v(x)\big]^+, \\
\|\, u - v\,\|_+ &= \sup_{x:|\, u-v\,|_+ = v(x)-u(x)} (\, v(x) - u(x)\,)^+ + \nu\big[-\Delta v(x) + \Delta u(x)\big]^+.
\end{aligned}
$$

These are stronger than the functionals $|\, u - v\,|_-$ and $|\, u - v\,|_+$, respectively.

Theorem 2 (Strengthened First Continuous Dependence Result). *Let u be the solution of* (5) *and let v be any $\mathcal{C}^2(\mathbf{R}^d)$ function that is periodic in each coordinate with period* 1. *Then for $\sigma \in \{-,+\}$, we have*

$$\|\, u - v\,\|_\sigma \le \Phi_\sigma(v), \tag{15}$$

where

$$\Phi_\sigma(v) = \sup_{x\in\mathbf{R}^d} \big(\sigma\, R(v;x)\big)^+. \tag{16}$$

Note that this result shows that the functional $\Phi_\sigma(v)$ is measuring *more* than just the semi-norm $|\, u - v\,|_\sigma$ since it adds ν times the Laplacian of the error. This additional term is not necessarily of the same order as that of the semi-norm $|\, u-v\,|_\sigma$; in fact, it might be considerably bigger. This is precisely why the estimate of Theorem 1 (and that of Theorem 2) fails when ν goes to zero.

To illustrate this phenomenon, let us consider the following simple but illuminating example. Consider

$$\zeta_\nu(x) = -\nu \ln\big(\exp(x/\nu) + 2 + \exp(-x/\nu)\big),$$

which is the smooth solution of the parabolic equation

$$u + \frac{1}{2}(u')^2 - \nu\, u'' = f_\nu,$$

where $f_\nu(x) = \zeta_\nu(x) + 1/2$. Since $\lim_{\nu\downarrow 0} \zeta_\nu(x) = \zeta_0(x) = -|\,x\,|$, the second order derivative of ζ_ν at $x = 0$ has to become infinite as $\nu \downarrow 0$. This "kink" of ζ_0 is responsible for the breakdown of the continuous dependence result under consideration as we show next.

Although we cannot apply Theorem 2 since these functions are not periodic, it is not difficult to verify that Theorem 2 holds for $u = \zeta_\nu$ and $v = \zeta_{\bar\nu}$ for any positive parameters ν and $\bar\nu$. Moreover in the case $\nu < \bar\nu$, a simple computation gives that

$$|\!|\!|\, \zeta_\nu - \zeta_{\bar\nu} \,|\!|\!|_+ = \Phi_+(\zeta_{\bar\nu}) = 0,$$

and

$$|\, \zeta_\nu - \zeta_{\bar\nu} \,|_- = (\bar\nu - \nu)\ln(2), \qquad |\!|\!|\, \zeta_\nu - \zeta_{\bar\nu} \,|\!|\!|_- = \Phi_-(\zeta_{\bar\nu}) = (\bar\nu - \nu)\ln(2) + \frac{\bar\nu - \nu}{2\,\bar\nu},$$

where we have used the identity $\zeta''_\nu(0) = \frac{1}{2\nu}$ to compute the last term.

Note that the last equality shows that Theorem 2 is *sharp*. Note also that as a consequence of the above computations,

$$\lim_{\nu\downarrow 0} |\, \zeta_\nu - \zeta_{\bar\nu} \,|_- = |\, \zeta_0 - \zeta_{\bar\nu} \,|_- = \bar\nu \ln(2), \quad \text{and} \quad \lim_{\nu\downarrow 0} \Phi_-(\zeta_{\bar\nu}) = \bar\nu \ln(2) + \frac{1}{2}.$$

In other words, the result of Theorem 1 cannot give any useful information about how close $\zeta_{\bar\nu}$ is to ζ_0. This is what we mean by the breakdown of this estimate when ν goes to zero.

3 A continuous dependence result that holds as $\nu \downarrow 0$

The results of the last section motivate the search for a new continuous dependence result that is actually useful when $\nu \downarrow 0$. We derive a *modification* of Theorem 1 that allows us to compare solutions of the elliptic problem with different values of the parameter ν.

3.1 The new continuous dependence result

To state the new continuous dependence result, we need to introduce two quantities. The first is the *generalized residual* defined by

$$R^{\bar\nu}_\epsilon(u; x, p) = u(x) + H(p) - \bar\nu \Delta u(x) - f(x - \epsilon\, p) - \frac{\epsilon}{2}\,|\,p\,|^2. \tag{17}$$

Note that $R^{\bar\nu}_\epsilon$ is just the residual R when $\epsilon = 0$ and $\bar\nu = \nu$. We introduce this quantity because we are interested in dealing with functions v that are solutions of the elliptic problem (5) with $\bar\nu \neq \nu$!

The second quantity is the *paraboloid* P_v defined by

$$P_v(x,p,\kappa;y) = v(x) + (y-x)\cdot p + \frac{\kappa}{2}|\,y-x\,|^2, \qquad y \in \mathbf{R}^d, \tag{18}$$

where x is a point in $\mathbf{R}^d$, p is a vector of $\mathbf{R}^d$, and κ is a real number.

We are now ready to state the new continuous dependence result.

Theorem 3 (Second Continuous Dependence Result for Elliptic Equations). *Let u be the solution of the equation (5) and let v be any $\mathcal{C}^2(\mathbf{R}^d)$ function that is periodic in each coordinate with period 1. Then for $\sigma \in \{-,+\}$, we have*

$$|\,u - v\,|_\sigma \le \inf_{\bar\nu\ge 0,\epsilon>0} \Phi_\sigma^{\bar\nu}(v;\epsilon), \tag{19}$$

where

$$\Phi_\sigma^{\bar\nu}(v;\epsilon) = \sup_{(x,p)\in\mathcal{A}_\sigma(v;\epsilon)} \left(\sigma\, R_{\sigma\epsilon}^{\bar\nu}(v;x,p) + \frac{(\sqrt{\nu}-\sqrt{\bar\nu})^2}{\epsilon}\, d\right)^+, \tag{20}$$

and the set $\mathcal{A}_\sigma(v;\epsilon)$ is the set of points $(x,p) \in \mathbf{R}^d \times \mathbf{R}^d$ satisfying

$$\sigma\,\{v(y) - P_v(x,p,\sigma/\epsilon\,;\,y)\} \le 0, \qquad \forall\, y \in \mathbf{R}^d. \tag{21}$$

Moreover if $\bar\nu = 0$, this estimate holds for all $v \in \mathcal{C}^0(\mathbf{R}^d)$.

Some important remarks are in order:

a. The set $\mathcal{A}_\sigma(v;\epsilon)$ and the semi-differentials of v. By definition, the point (x,p) belongs to the set $\mathcal{A}_-(v;\epsilon)$ if the paraboloid $P_v(x,p,-1/\epsilon\,;\,\cdot)$ does not lie above the function v. This implies that for all $y \in \mathbf{R}^d$,

$$0 \le v(y) - P_v(x,p,-1/\epsilon\,;\,y) = v(y) - \left\{v(x) + (y-x)\cdot p - \frac{1}{2\epsilon}|\,y-x\,|^2\right\},$$

or, equivalently,

$$v(y) - \{v(x) + (y-x)\cdot p\} \ge -\frac{1}{2\epsilon}|\,y-x\,|^2.$$

This implies that p belongs to the *sub-differential* $D^-v(x)$ of v at x which is defined to be the set of vectors $p \in \mathbf{R}^d$ such that

$$\lim_{y\to x}\,(v(y) - \{v(x) + (y-x)\cdot p\}) \ge 0. \tag{22}$$

In the case $\sigma = +$, the point (x,p) belongs to the set $\mathcal{A}_+(v;\epsilon)$ if the paraboloid $P_v(x,p,1/\epsilon\,;\,\cdot)$ does not lie below the function v. In this case, p belongs to the *super-differential* $D^+v(x)$ of v at x which is defined to be the set of vectors $p \in \mathbf{R}^d$ such that

$$\lim_{y\to x}\,(v(y) - \{v(x) + (y-x)\cdot p\}) \le 0. \tag{23}$$

Note that if v is smooth at x, the paraboloid $P_v(x,p,\sigma/\epsilon\,;\,\cdot)$ is *tangent* to v at x since in this case $D^+v(x) = D^-v(x) = \{\nabla v(x)\}$.

b. The dependence of the set $\Phi_\sigma^{\bar\nu}(v;\epsilon)$ on ϵ. Note that if $\epsilon \le \bar\epsilon$ then

$$\mathcal{A}_\sigma(v;\epsilon) \supset \mathcal{A}_\sigma(v;\bar\epsilon).$$

This means that the function $\Phi_\sigma^{\bar\nu}(v;\epsilon)$ can decrease as ϵ increases. On the other hand, since

$$R_{\sigma\epsilon}^{\bar\nu}(v;x,p) = R_0^{\bar\nu}(v;x,p) + f(x) - f(x-\sigma\,\epsilon\,p) - \frac{\sigma\epsilon}{2}\,|\,p\,|^2,$$

the function $\Phi_\sigma^{\bar\nu}(v;\epsilon)$ can also increase as ϵ increases. Theorem 3 improves the result of Theorem 1 precisely because in many cases $\inf_{\epsilon>0}\Phi_\sigma^{\bar\nu}(v;\epsilon)$ is actually significantly smaller than $\lim_{\epsilon\downarrow 0}\Phi_\sigma^{\bar\nu}(v;\epsilon)$. An analytical example is presented in [2].

c. Recovering Theorem 1 from Theorem 3. For $\bar\nu = \nu$,

$$|\,u-v\,|_\sigma \le \lim_{\epsilon\downarrow 0}\Phi_\sigma^{\nu}(v;\epsilon) = \Phi_\sigma(v).$$

This means that we recover Theorem 1. To see this, we proceed as follows. First since $\nu > 0$ and $v \in \mathcal{C}^2(\mathbf{R}^d)$, it is not difficult to prove that

$$\lim_{\epsilon\downarrow 0}\mathcal{A}_\sigma(v;\epsilon) = \left\{(x,\nabla v(x)), x\in\mathbf{R}^d\right\},$$

and since $p = \nabla v(x)$,

$$\lim_{\epsilon\downarrow 0}\left(\sigma\,R_{\sigma\epsilon}^{\nu}(v;x,p)\right)^+ = \left(\sigma\,R(v;x)\right)^+.$$

Now, we show that Theorem 3 does *not* break down when ν goes to zero. To do that, it suffices to show that we can use it to compare solutions of (5) with different values of the parameter ν. We prove the following result.

Corollary 4. *Let u_ν denote the solution of* (5), *then*

$$\|\,u_\nu - u_{\bar\nu}\,\|_{L^\infty(\Omega)} \le \sqrt{2\,d}\,|\,f\,|_{W^{1,\infty}(\Omega)}\left|\sqrt{\nu}-\sqrt{\bar\nu}\right|.$$

In essence, this says that $\{\,u_\nu\,\}_{\nu>0}$ is a Cauchy sequence.

Proof. Since for any $p\in\mathbf{R}^d$,

$$\begin{aligned} R_{\sigma\epsilon}^{\bar\nu}(u_{\bar\nu};x,p) &= f(x) - f(x-\sigma\epsilon\,p) - \frac{\epsilon}{2}\,|\,p\,|^2 \\ &\le \epsilon\,|\,p\,|\,|\,f\,|_{W^{1,\infty}(\Omega)} - \frac{\epsilon}{2}\,|\,p\,|^2 \\ &\le \frac{\epsilon}{2}\,|\,f\,|^2_{W^{1,\infty}(\Omega)}, \end{aligned}$$

Theorem 3 implies

$$\begin{aligned} |\,u_\nu - u_{\bar\nu}\,|_\sigma &\le \inf_{\epsilon\ge 0}\;\Phi_\sigma^{\bar\nu}(u_{\bar\nu};\epsilon) \\ &\le \inf_{\epsilon\ge 0}\;\left(\frac{\epsilon}{2}\,|\,f\,|^2_{W^{1,\infty}(\Omega)} + \frac{(\sqrt{\nu}-\sqrt{\bar\nu})^2}{\epsilon}\,d\right) \\ &= \sqrt{2\,d}\,|\,f\,|_{W^{1,\infty}(\Omega)}\left|\sqrt{\nu}-\sqrt{\bar\nu}\right|, \end{aligned}$$

and the result follows from (6). This completes the proof. □

3.2 Proof of the Second Continuous Dependence Theorem for Elliptic Equations

We prove the result for $\sigma = -$; the proof for the case $\sigma = +$ is similar. We use the so-called *doubling-of-the-variables* technique, which consists in considering the auxiliary function

$$\psi(x,y) = u(x) - v(y) - \frac{|x-y|^2}{2\epsilon},$$

instead of the auxiliary function $\psi(x,x) = u(x) - v(x)$ used before.

We choose $(\widehat{x},\widehat{y}) \in \Omega \times \Omega$ so that

$$\psi(\widehat{x},\widehat{y}) \geq \psi(x,y), \qquad \forall\, x,y \in \Omega;$$

such a point exists since ψ is continuous and periodic on $\Omega \times \Omega$. Set $\widehat{p} = (\widehat{x}-\widehat{y})/\epsilon$.

We assume that $|\, u - v\,|_- > 0$, otherwise there is nothing to prove. Also, we assume that $v \in \mathcal{C}^2(\mathbf{R}^d)$. In this case, we have

$$\begin{aligned}
|\, u - v\,|_- &= \sup_{x\in\Omega} \big\{u(x) - v(x)\big\} \\
&= \sup_{x\in\Omega} \psi(x,x) \\
&\leq \sup_{x,y\in\Omega} \psi(x,y) \\
&= \psi(\widehat{x},\widehat{y}) \\
&= u(\widehat{x}) - v(\widehat{y}) - \frac{|\widehat{x}-\widehat{y}|^2}{2\epsilon} \\
&= \big[u(\widehat{x}) + H(\widehat{p}) - \nu\Delta u(\widehat{x}) - f(\widehat{x})\big] \\
&\qquad -\big[v(\widehat{y}) + H(\widehat{p}) - \bar{\nu}\Delta v(\widehat{y}) - f(\widehat{x}) + \frac{|\widehat{x}-\widehat{y}|^2}{2\epsilon}\,\big] \\
&\qquad +\big[\nu\Delta u(\widehat{x}) - \bar{\nu}\Delta v(\widehat{y})\big] \\
&= \big[u(\widehat{x}) + H(\widehat{p}) - \nu\Delta u(\widehat{x}) - f(\widehat{x})\big] \\
&\qquad -\big[v(\widehat{y}) + H(\widehat{p}) - \bar{\nu}\Delta v(\widehat{y}) - f(\widehat{y}+\epsilon\, p) + \frac{|\, p\,|^2}{2\epsilon}\,\big] \\
&\qquad +\big[\nu\Delta u(\widehat{x}) - \bar{\nu}\Delta v(\widehat{y})\big]
\end{aligned}$$

by definition of $\widehat{p}$. Hence by the definition (17) of $R^{\bar{\nu}}_{\epsilon}(\cdot;\cdot,\cdot)$, we get

$$|\, u - v\,|_- \leq R^{\nu}_0(u;\widehat{x},\widehat{p}) - R^{\bar{\nu}}_{-\epsilon}(v;\widehat{y},\widehat{p}) + \big[\nu\Delta u(\widehat{x}) - \bar{\nu}\Delta v(\widehat{y})\big].$$

Since the mapping $x \mapsto \psi(x,\widehat{y})$ has a maximum at $x = \widehat{x}$, we have $\nabla_x\psi(\widehat{x},\widehat{y}) = 0$ and so

$$\widehat{p} = \nabla u(\widehat{x}).$$

Now since u is the exact solution of (5), $R_0^\nu(u;\widehat{x},\widehat{p})=0$, and hence,

$$|\,u-v\,|_- \le -R^{\bar\nu}_{-\epsilon}(v;\widehat{y},\widehat{p}) + \big[\nu\Delta u(\widehat{x}) - \bar\nu\Delta v(\widehat{y})\big].$$

To deal with the last term of the right-hand side, we use the fact that the Hessian of ψ at $(\widehat{x},\widehat{y})$, $\mathcal{H}\psi(\widehat{x},\widehat{y})$, is negative semi-definite. Note that

$$\mathcal{H}\psi(\widehat{x},\widehat{y}) = \begin{pmatrix} \mathcal{H}u(\widehat{x}) - \frac{1}{\epsilon}I & \frac{1}{\epsilon}I \\ \frac{1}{\epsilon}I & -\mathcal{H}v(\widehat{y}) - \frac{1}{\epsilon}I \end{pmatrix},$$

where I is the identity matrix and $\mathcal{H}u$ and $\mathcal{H}v$ are the Hessians of u and v respectively.

Now, we let e_i denote the d-dimensional vector whose j-th entry is δ_{ij} and set $\eta_i^t = (\sqrt{\nu}e_i^t, \sqrt{\bar\nu}e_i^t)$. Since $\eta_i^t\,\mathcal{H}\psi(\widehat{x},\widehat{y})\,\eta_i \le 0$, we have

$$\sum_{i=1}^d \eta_i^t\,\mathcal{H}\psi(\widehat{x},\widehat{y})\,\eta_i \le 0,$$

or

$$\big[\nu\Delta u(\widehat{x}) - \bar\nu\Delta v(\widehat{y})\big] \le \frac{(\sqrt{\nu}-\sqrt{\bar\nu})^2}{\epsilon}\,d.$$

It is important to stress the fact that obtaining this elegant inequality is a *crucial* step for proving the theorem and that it is the doubling-of-the-variables technique that allows this to happen!

We can now write

$$|\,u-v\,|_- \le -R^{\bar\nu}_{-\epsilon}(v;\widehat{y},\widehat{p}) + \frac{(\sqrt{\nu}-\sqrt{\bar\nu})^2}{\epsilon}\,d.$$

To prove the estimate, it only remains to show that $(\widehat{y},\widehat{p})$ belongs to $\mathcal{A}_\sigma(v;\epsilon)$. To do that, we note that since $\psi(\widehat{x},\widehat{y}) \ge \psi(\widehat{x},y)$ for all $y\in\Omega$, we have

$$v(y) \ge v(\widehat{y}) + \frac{|\widehat{x}-\widehat{y}|^2}{2\epsilon} - \frac{|\widehat{x}-y|^2}{2\epsilon} = v(\widehat{y}) + \widehat{p}\cdot(y-\widehat{y}) - \frac{|y-\widehat{y}|^2}{2\epsilon},$$

which is nothing but the so-called *paraboloid test*:

$$v(y) - Pv(\widehat{y},\widehat{p},-1/\epsilon;y) \le 0, \qquad \forall\, y\in\mathbf{R}^d.$$

It is also important to stress the fact that it is the doubling-of-the-variables technique that allows us to obtain the paraboloid test, which is a *very* important feature of the error estimate!

This completes the proof of Theorem 3 for $v\in\mathcal{C}^2(\mathbf{R}^d)$.

Now in the case $\bar\nu=0$, it is clear that v does not need to be in $\mathcal{C}^2(\mathbf{R}^d)$; we only need to consider $v\in\mathcal{C}^0(\mathbf{R}^d)$. Indeed it is not difficult to verify that the proof above is still valid provided we modify the argument about the Hessian of ψ as follows. Since the mapping $x\mapsto\psi(x,\widehat{y})$ has a maximum at $x=\widehat{x}$, we have

$$\Delta_x\psi(\widehat{x},\widehat{y}) \le 0,$$

that is,

$$\Delta u(\widehat{x}) \le d/\epsilon.$$

This completes the proof of Theorem 3 for $\bar\nu=0$ and $v\in\mathcal{C}^0(\mathbf{R}^d)$.

4 Continuous Dependence Results for Hamilton-Jacobi Equations

Corollary 4 shows that there exists a unique limit of the sequence $\{u_\nu\}_{\nu>0}$ in $\mathcal{C}^0(\mathbf{R}^d)$ that is periodic in each coordinate with period 1. This limit is precisely the viscosity solution of the Hamilton-Jacobi equation

$$u + H(\nabla u) = f \qquad \text{in } \mathbf{R}^d.$$

In this section, we show how the viscosity solution *inherits* the continuous dependence results for the solutions u_ν. We also show that the continuous dependence result for the viscosity solution can be used to characterize it.

4.1 Preservation of continuous dependence under the limit $\nu \downarrow 0$

The following result is a direct consequence of Theorem 3 and Corollary 4. This shows how the continuous dependence result contained in Theorem 3 is *preserved* when we take the limit $\nu \downarrow 0$. In what follows, we write $R_{\sigma\epsilon}$ instead of $R^0_{\sigma\epsilon}$

Theorem 5 (Continuous Dependence Result for Hamilton-Jacobi equations). *Let u be the viscosity solution of* (1) *and let v be any $\mathcal{C}^0(\mathbf{R}^d)$ function that is periodic in each coordinate with period* 1*. Then for $\sigma \in \{-,+\}$, we have*

$$|\, u - v\,|_\sigma \le \inf_{\epsilon>0} \Phi_\sigma(v;\epsilon), \tag{24}$$

where

$$\Phi_\sigma(v;\epsilon) = \sup_{(x,p)\in\mathcal{A}_\sigma(v;\epsilon)} \left(\sigma\, R_{\sigma\epsilon}(v;x,p)\right)^+, \tag{25}$$

and the set $\mathcal{A}_\sigma(v;\epsilon)$ is the set of points $(x,p) \in \mathbf{R}^d \times \mathbf{R}^d$ satisfying

$$\sigma\,\{v(y) - P_v(x,p,\sigma/\epsilon\,;\,y)\} \le 0, \qquad \forall y \in \mathbf{R}^d. \tag{26}$$

We note that this result implies that *all* the properties described in subsection 2.1 for the exact solution of the elliptic equation (5) hold for the viscosity solution of the Hamilton-Jacobi equation (1).

4.2 Characterization of the viscosity solution

We can obtain the following result from Theorem 5.

Corollary 6 (Characterization of the viscosity solution). *The viscosity solution of* (1) *is the only function u in $\mathcal{C}^0(\mathbf{R}^d)$ that is periodic in each coordinate with period* 1*, such that for all x in $\mathbf{R}^d$,*

$$u(x) + H(p) - f(x) \le 0, \quad \forall\, p \in D^+u(x),$$

and

$$u(x) + H(p) - f(x) \geq 0, \quad \forall\, p \in D^- u(x).$$

Proof. Let u be the viscosity solution of (1) and let v be a function such that for all x in $\mathbf{R}^d$,

$$v(x) + H(p) - f(x) \geq 0, \quad \forall\, p \in D^- v(x).$$

Then Theorem 5 implies

$$\begin{aligned} |\, u - v\,|_- &\leq \inf_{\epsilon>0} \Phi_-(v;\epsilon) \\ &= \inf_{\epsilon>0} \sup_{(x,p)\in\mathcal{A}_\sigma(v;\epsilon)} (-R_{-\epsilon}(v;x,p))^+ \\ &= \inf_{\epsilon>0} \sup_{(x,p)\in\mathcal{A}_\sigma(v;\epsilon)} (f(x+\epsilon\, p) - f(x))^+ \\ &\leq \inf_{\epsilon>0} |\, f\,|_{W^{1,\infty}(\Omega)}\, |\, v\,|_{W^{1,\infty}(\Omega)}\, \epsilon \\ &= 0. \end{aligned}$$

Similarly if v satisfies

$$v(x) + H(p) - f(x) \geq 0, \quad \forall\, p \in D^- v(x),$$

we conclude in an analogous manner that

$$|\, u - v\,|_+ \leq 0.$$

This implies that $u \equiv v$ and the Corollary is proved. □

5 Continuous Dependence Results for Monotone Numerical Schemes

We now consider numerical approximations of the Hamilton-Jacobi equation.

5.1 Monotone Numerical Schemes

The numerical schemes we consider determine the values of a function u_h on a grid G_h that is periodic with period 1 in each of the canonical directions of $\mathbf{R}^d$. These schemes take the form

$$u_h(y) + \widehat{H}_y(\partial_{\delta_y} u_h(y)) = f(y), \qquad \forall\, y \in G_h, \tag{27}$$

where $\widehat{H}_y(\partial_{\delta_y} u_h(y))$ is a discrete version of $H(\nabla u(y))$ and

$$\partial_{\delta_y} u_h(y) = (\partial_{\delta_{1,y}} u_h(y), \ldots, \partial_{\delta_{N_y,y}} u_h(y)),$$

where

$$\partial_{\delta_{i,y}} u_h(y) = \frac{u_h(y) - u_h(y - \delta_{i,y})}{|\, \delta_{i,y}\,|} \text{ where } y - \delta_{i,y} \in G_h \qquad i = 1, \ldots, N_y.$$

To prove the desired continuous dependence results, we assume that the numerical Hamiltonian $\widehat{H}$ has the following properties:

(i) *Consistency*: $\widehat{H}_y(\partial_{\delta_y} u_h(y)) = H(p)$ if $\nabla u_h = p \in \mathbf{R}^d$.

(ii) *Monotonicity*: $\widehat{H}_y$ is non-decreasing in each of its arguments.

(iii) *Global smoothness*: $|\,\widehat{H}_y(z_1) - \widehat{H}_y(z_2)\,| \leq L\,\|\,z_1 - z_2\,\|_{\ell^\infty}$.

The first property ensures that we are approximating the viscosity solution corresponding to the correct Hamilton-Jacobi equation. The second property is the *key* property that allows us to obtain the continuous dependence result we seek. The third property is not really necessary but simplifies the proofs.

The existence of at least one solution u_h of (27) can be proved by using the classic Leray-Schauder fixed point theorem; see the Appendix. The uniqueness will follow from the continuous dependence result stated in the next subsection.

The classical example of a monotone scheme for Hamilton-Jacobi equation is the so-called Lax-Friedrichs scheme. The Lax-Friedrichs scheme, see [9, 18, 2], on the uniform Cartesian grid

$$G_h = \{(x_0 + (i-1)\Delta x, y_0 + (j-1)\Delta y)\,\}$$

reads as follows:

$$\begin{aligned} v_{i,j} &+ H\left(\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x}, \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right) \\ &- \omega_x \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{\Delta x^2} - \omega_y \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\Delta y^2} = f_{i,j}, \end{aligned}$$

where

$$\begin{aligned} \omega_x &= \sup_{(x,y)\in\mathbf{R}^d} \frac{1}{2}\left| H_1\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)\right| \Delta x, \\ \omega_y &= \sup_{(x,y)\in\mathbf{R}^d} \frac{1}{2}\left| H_2\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)\right| \Delta y, \end{aligned}$$

and $H_i(p_1, p_2) = \frac{\partial H}{\partial p_i}(p_1, p_2)$ for $i = 1, 2$.

Since the grid G_h is Cartesian, for each $y = (i,j) \in G_h$ we have $N_y = 4$. The quantities $\partial_{\delta_i, y} v(y)$ are thus

$$\begin{aligned} \partial_{-\Delta x} v_{i,j} &= \frac{v_{i,j} - v_{i+1,j}}{\Delta x}, \qquad & \partial_{\Delta x} v_{i,j} &= \frac{v_{i,j} - v_{i-1,j}}{\Delta x}, \\ \partial_{-\Delta y} v_{i,j} &= \frac{v_{i,j} - v_{i,j+1}}{\Delta y}, \qquad & \partial_{\Delta y} v_{i,j} &= \frac{v_{i,j} - v_{i,j-1}}{\Delta y}; \end{aligned}$$

and therefore,

$$\begin{aligned} \widehat{H} = H &\left(\frac{1}{2}(\partial_{\Delta x} v_{i,j} - \partial_{-\Delta x} v_{i,j}), \frac{1}{2}(\partial_{\Delta y} v_{i,j} - \partial_{-\Delta y} v_{i,j})\right) \\ &+ \frac{\omega_x}{\Delta x}(\partial_{-\Delta x} v_{i,j} + \partial_{\Delta x} v_{i,j}) + \frac{\omega_y}{\Delta y}(\partial_{-\Delta y} v_{i,j} + \partial_{\Delta y} v_{i,j}). \end{aligned}$$

It is easy to verify that this $\widehat{H}$ satisfies properties (i), (ii) and (iii).

Other examples of schemes satisfying these three properties are the monotone schemes of Crandall and Lions [9], the intrinsic monotone scheme of Abgrall [1] and the monotone schemes of Kossioris, Makridakis and Souganidis [14].

5.2 A Continuous Dependence Result for Monotone schemes

The main result of this section gives upper bounds for discrete versions of the semi-norms used for the continuous case:

$$|\, u_h - v\,|_{-,G_h} = \sup_{x\in G_h} (\, u_h(x) - v(x)\,)^+, \qquad |\, u_h - v\,|_{+,G_h} = \sup_{x\in G_h} (\, v(x) - u_h(x)\,)^+,$$

where $\eta^+ \equiv \max\{0,\eta\}$. Note that an upper bound on the L^∞-norm of $u_h - v$ can be obtained via

$$\|\, u_h - v\,\|_{L^\infty(G_h)} = \max\{|\, u_h - v\,|_{+,G_h}, |\, u_h - v\,|_{-,G_h}\}.$$

These upper bounds are given in terms of something like a generalization of the classical truncation error. Next, we define this quantity.

We begin by introducing the expression

$$\mathcal{D}_{\delta,y;\epsilon,p} v(\widehat{y}) = \frac{v(\widehat{y}) - v(y)}{|\,\delta\,|} + \frac{\Theta_\epsilon(y - \widehat{y} + \delta, p)}{|\,\delta\,|},$$

where

$$\Theta_\epsilon(z,p) = p\cdot z + \frac{1}{2\epsilon}|\, z\,|^2.$$

By choosing $y = \widehat{y} - \delta$ and $y = \widehat{y}$, the above expression can be used to approximate directional derivatives, super- or sub-differentials, or finite differences of the function v.

Indeed for the first choice, we have

$$\mathcal{D}_{\delta,\hat{y}-\delta;\epsilon,p} v(\widehat{y}) = \frac{v(\widehat{y}) - v(\widehat{y}-\delta)}{|\,\delta\,|} + \frac{\Theta_\epsilon(0,p)}{|\,\delta\,|} = \partial_\delta v(\widehat{y}). \tag{28}$$

In other words when $y = \widehat{y} - \delta$, the quantity under consideration is nothing but a finite difference of u in the δ-direction. Of course if u is a smooth function, we have

$$\mathcal{D}_{\delta,\hat{y}-\delta;\epsilon,p} v(\widehat{y}) = \nabla v(\widehat{y}) \cdot \frac{\delta}{|\,\delta\,|} + \mathcal{O}(|\,\delta\,|),$$

for small $|\,\delta\,|$. That is, $\mathcal{D}_{\delta,\hat{y}-\delta;\epsilon,p} v(\widehat{y})$ approximates the partial derivative of v at $\widehat{y}$ in the δ-direction.

For the second choice, we have

$$\mathcal{D}_{\delta,\hat{y};\epsilon,p} v(\widehat{y}) = \frac{v(\widehat{y}) - v(\widehat{y})}{|\,\delta\,|} + \frac{\Theta_\epsilon(\delta,p)}{|\,\delta\,|} = p\cdot\frac{\delta}{|\,\delta\,|} + \frac{|\,\delta\,|}{2\epsilon}, \tag{29}$$

and so if $|\,\delta\,|$ is small,

$$\mathcal{D}_{\delta,\hat{y};\epsilon,p}v(\widehat{y}) = p\cdot\frac{\delta}{|\,\delta\,|} + \mathcal{O}(|\,\delta\,|).$$

Since in our applications, p will belong to super- or sub-differentials of viscosity solutions, the quantity $\mathcal{D}_{\delta,\hat{y};\epsilon,p}v(\widehat{y})$ provides an approximation to the component of p in the δ-direction.

Now we introduce the *generalized truncation error* of v at the point $y\in\mathbf{R}^d$,

$$T_{\mathbf{y};\epsilon}(v;y,p) = v(y) + \widehat{H}_{y-\epsilon\,p}(\mathcal{D}_{\delta_{y-\epsilon\,p},\mathbf{y};\epsilon,p}v(y)) - f(y-\epsilon\,p) - \frac{|\,p\,|^2}{2}\epsilon, \qquad (30)$$

where $\mathbf{y} = (y_1,\ldots,y_{N_y})$ and

$$\mathcal{D}_{\delta_y,\mathbf{y};\epsilon,p}v(y) = (\mathcal{D}_{\delta_1,y_1;\epsilon,p}v(y),\ldots,\mathcal{D}_{\delta_{N_y},y_{N_y};\epsilon,p}v(y)).$$

Note that for the above expression to have meaning, the point $x = y-\epsilon\,p$ must belong to the grid G_h. Note also that if we set $y_i = y-\delta_i$ for $i=1,\ldots,N_y$ and $\epsilon=0$, we get

$$T_{\mathbf{y};0}(u;y,p) = u(y) + \widehat{H}_y(\partial_{\delta_y}u(y)) - f(y),$$

by using (28). Since this expression is nothing but the classical truncation error of v at $y\in G_h$, our terminology is justified.

We are now ready to state the main result of this section.

Theorem 7 (Continuous Dependence Result for Monotone Schemes). *Let u_h be the approximate solution given by (27) and let v be an arbitrary continuous function that is periodic with period 1 in each of the canonical directions of $\mathbf{R}^d$. Then for $\sigma\in\{-,+\}$, we have*

$$|\,u_h - v\,|_{\sigma,G_h} \le \inf_{\epsilon>0}\Psi_\sigma(v;\epsilon), \qquad (31)$$

where

$$\Psi_\sigma(v;\epsilon) = \sup_{(y,p)\in\mathcal{A}_{h,\sigma}(v;\epsilon)}\ \inf_{\mathbf{y}\in\mathbf{R}^{N_y\times d}}\left(\sigma\,T_{\mathbf{y};\sigma\epsilon}(v;y,p)\right)^+, \qquad (32)$$

and the set $\mathcal{A}_{h,\sigma}(v;\epsilon)$ is the set of points (y,p) satisfying

$$(y-\sigma\,\epsilon\,p,p)\in G_h\times\mathbf{R}^d, \qquad (33)$$

$$\sigma\,\{v(\zeta) - P_v(y,p,\sigma/\epsilon\,;\,\zeta)\}\le 0, \qquad \forall\,\zeta\in\mathbf{R}^d. \qquad (34)$$

Clearly, this result is a discrete version of Theorem 5 for the Hamilton-Jacobi equation. In this sense, we can say that the continuous dependence result of Theorem 5 has been preserved under discretization of the Hamilton-Jacobi equation by a monotone scheme.

a. The a priori error estimate. Note that if v is taken to be the viscosity solution u of the Hamilton-Jacobi equations (1), Theorem 7 allows us to estimate the L^∞-error between u_h and u solely in terms of u. Since this can be done only *before* the computation of u_h, estimates like this are called *a priori error estimates*. Their usefulness resides in the fact that they indicate the accuracy that can be expected from the approximation.

In our case, we have the following result.

Corollary 8 (A Priori Error Estimate). *Let u be the viscosity solution of the equation* (1) *and let v be the approximate solution given by the monotone scheme* (27). *Then if $f \in W^{1,\infty}(\Omega)$,*

$$\| u - v \|_{L^\infty(G_h)} \le L^{1/2} \, | f |_{W^{1,\infty}(\Omega)} \, h^{1/2},$$

where $h = \max_{y \in G_h} \max_{i=1,\dots,N_y} | \delta_i |$ and L is the Lipschitz constant of the numerical Hamiltonian. Moreover if $u \in W^{2,\infty}(\Omega)$,

$$\| u - v \|_{L^\infty(G_h)} \le \frac{d}{2} L \, | u |_{W^{2,\infty}(\Omega)} \, h.$$

The proof of this result is given in subsection 5.3.

b. Uniqueness of the approximation u_h. We can prove the uniqueness of the approximate solution u_h by a simple application of Theorem 7. Assume that there are two different solutions v_1 and v_2. Setting $u_h = v_1$ and $v = v_2$ in Theorem 7, we get

$$| v_1 - v_2 |_{\sigma, G_h} \le \Psi(v_1; 0) = \sup_{y \in G_h} (-\sigma (v_1(y) + \widehat{H}_y(\partial_{\delta_y} v_1(y)) - f(y)))^+ = 0.$$

This implies $v_1 = v_2$ and the uniqueness of u_h follows.

c. A maximum principle. If we set $v = 0$ and $\epsilon = 0$ in Theorem 7, we get

$$-\sup_{\zeta \in G_h} (H(0) - f(\zeta))^+ \le u_h(y) \le \sup_{\zeta \in G_h} (f(\zeta) - H(0))^+ \qquad \forall \, y \in G_h.$$

d. The L^∞-contraction property We denote the solution of (27) with right-hand side f by $u_{h,f}$. Setting $u_h = u_{h,f}$, $v = u_{h,g}$ and $\epsilon = 0$ in Theorem 7, we get

$$| u_{h,f} - u_{h,g} |_{\sigma, G_h} \le | f - g |_{\sigma, G_h}.$$

This immediately implies

$$\| u_{h,f} - u_{h,g} \|_{L^\infty(G_h)} \le \| f - g \|_{L^\infty(G_h)},$$

which is nothing but the discrete version of the so-called L^∞-contraction property for viscosity solutions.

5.3 Proof of the a Priori Error Estimate

The converge rate of the numerical scheme depends on the smoothness of the exact solution u; typically, the rate of convergence decreases when the smoothness of the exact solution is degraded.

To illustrate this phenomenon in our setting, we consider two important cases: when $u \in W^{2,\infty}(\Omega)$ and when we only assume that $f \in W^{1,\infty}(\Omega)$. Note that in the latter case, the viscosity solution u belongs to $W^{1,\infty}(\Omega)$ but not necessarily to $u \in W^{2,\infty}(\Omega)$. Indeed the first property is a direct consequence of the L^∞-contraction property, and the second of the fact that there are viscosity solutions with $f \in W^{1,\infty}(\Omega)$ that display discontinuities in its derivatives.

a. The case $u \in W^{2,\infty}(\Omega)$.

Taking $y_i = y - \delta_i$ for $i = 1, \ldots, N_y$ and $\epsilon = 0$, we obtain by the definition of the generalized truncation error (30),

$$\Theta = T_{\mathbf{y};0}(u; y, p) = u(y) + \widehat{H}_y(\partial_{\delta_y} u(y)) - f(y) = \widehat{H}_y(\partial_{\delta_y} u(y)) - H(\nabla u(y)),$$

since u is the smooth viscosity solution. Next, we set $\mathcal{L}u(\zeta) = u(y) + \nabla u(y) \cdot (\zeta - y)$, and invoke the consistency property of the numerical Hamiltonian to get

$$\Theta = \widehat{H}_y(\partial_{\delta_y} u(y)) - \widehat{H}_y(\partial_{\delta_y} \mathcal{L}u(y)).$$

Finally by using the smoothness of $\widehat{H}$ and a simple Taylor expansion, we obtain

$$\begin{aligned} |\,\Theta\,| &\leq L \, \left\| \partial_{\delta_y} u(y) - \partial_{\delta_y} \mathcal{L}u(y) \right\|_{\ell^\infty} \\ &= L \max_{i=1,\ldots,N_y} \left| \frac{u(y) - u(y - \delta_i)}{|\,\delta_i\,|} - \nabla u(y) \cdot \frac{\delta_i}{|\,\delta_i\,|} \right| \\ &\leq \frac{d}{2} L \, |\,u\,|_{W^{2,\infty}(\Omega)} \, h. \end{aligned}$$

This completes the proof of Corollary 8 in the case of $u \in W^{2,\infty}(\Omega)$.

b. The case $f \in W^{1,\infty}(\Omega)$. For each $(y, p) \in \mathcal{A}_{h,\sigma}(u, \epsilon)$ and $\sigma \in \{+, -\}$, consider the quantity $\Theta = \sigma \, T_{\mathbf{y};\sigma\epsilon}(u; y, p)$ which by definition (30) is

$$\sigma \left(u(y) + \widehat{H}_y(\mathcal{D}_{\delta_{y-\sigma\epsilon\,p}, \mathbf{y}; \sigma\epsilon, p} u(y)) - f(y - \sigma\epsilon\, p) \right) - \frac{1}{2}\epsilon |\,p\,|^2.$$

Since $p \in D^\sigma u(y)$, we have

$$\sigma(u(y) + H(p) - f(y)) \leq 0,$$

by definition of the viscosity solution. This implies that

$$\begin{aligned} \Theta &\leq \sigma \left(\widehat{H}_y(\mathcal{D}_{\delta_{y-\sigma\epsilon\,p}, \mathbf{y}; \sigma\epsilon, p} u(y)) - H(p) - f(y - \sigma\epsilon\, p) + f(y) \right) - \frac{1}{2}\epsilon |\,p\,|^2 \\ &\leq \sigma \left(\widehat{H}_y(\mathcal{D}_{\delta_{y-\sigma\epsilon\,p}, \mathbf{y}; \sigma\epsilon, p} u(y)) - H(p) \right) + \epsilon |\,f\,|_{W^{1,\infty}(\Omega)} \, |\,p\,| - \frac{1}{2}\epsilon |\,p\,|^2 \\ &\leq \sigma \left(\widehat{H}_y(\mathcal{D}_{\delta_{y-\sigma\epsilon\,p}, \mathbf{y}; \sigma\epsilon, p} u(y)) - H(p) \right) + \frac{\epsilon}{2} |\,f\,|^2_{W^{2,\infty}(\Omega)}. \end{aligned}$$

Now if we set $\mathcal{L}u(\zeta) = u(y) + p \cdot (\zeta - y)$, we have

$$H(p) = \widehat{H}_{y-\sigma\epsilon\, p}(\partial_{\delta_{y-\sigma\epsilon\, p}} \mathcal{L}u(y)),$$

by the consistency of the numerical Hamiltonian, and so

$$\begin{aligned}\Theta &\le \sigma\left(\widehat{H}_y(\mathcal{D}_{\delta_{y-\sigma\epsilon\,p},\mathbf{y};\sigma\epsilon,p}u(y)) - \widehat{H}_{y-\sigma\epsilon\,p}(\partial_{\delta_{y-\sigma\epsilon\,p}}\mathcal{L}u(y))\right) + \frac{\epsilon}{2}|\,f\,|^2_{W^{1,\infty}(\Omega)}\\ &\le L\,\|\,\mathcal{D}_{\delta_{y-\sigma\epsilon\,p},\mathbf{y};\sigma\epsilon,p}u(y) - \partial_{\delta_{y-\sigma\epsilon\,p}}\mathcal{L}u(y)\,\|_{\ell^\infty} + \frac{\epsilon}{2}|\,f\,|^2_{W^{1,\infty}(\Omega)},\end{aligned}$$

by the smoothness property of $\widehat{H}$. Finally, since

$$\partial_{\delta_i}\mathcal{L}u(y) = p\cdot\frac{\delta_i}{|\,\delta_i\,|} \qquad \text{and} \qquad \mathcal{D}_{\delta_i,\widehat{x};\sigma\epsilon,p}u(y) = p\cdot\frac{\delta_i}{|\,\delta_i\,|} + \frac{|\,\delta_i\,|}{2\sigma\epsilon},$$

by (29), taking $y_i = y$ yields

$$\Theta \le L\,\frac{h}{2\epsilon} + \frac{\epsilon}{2}\,|\,f\,|^2_{W^{1,\infty}(\Omega)}.$$

The result follows by minimizing the right-hand side with respect to ϵ. This completes the proof of Corollary 8 in the case of $f \in W^{1,\infty}(\Omega)$.

5.4 Proof of the Continuous Dependence Result for Monotone Schemes

The proof of this result is a discrete version of the proof of the second continuous dependence theorem for elliptic equations.

We prove the result for $\sigma = -$; the proof for the case $\sigma = +$ is similar. We assume that $|\,u_h - v\,|_{-,G_h} > 0$, otherwise there is nothing to prove. Given $\epsilon > 0$, we define the auxiliary function

$$\psi(x,y) = u(x) - v(y) - \frac{|x-y|^2}{2\epsilon},$$

and let $(\widehat{x},\widehat{y}) \in \mathbf{R}^d \times G_h$ be such that

$$\psi(\widehat{x},\widehat{y}) \ge \psi(x,y), \qquad \forall\,(x,y) \in G_h \times \mathbf{R}^d.$$

The existence of such a point follows from the fact that both u_h and v are continuous and periodic with the same period. Set $\widehat{p} = (\widehat{x}-\widehat{y})/\epsilon$.

Since $|\,u_h - v\,|_{-,G_h} > 0$, we have

$$\begin{aligned}|\,u_h - v\,|_{-,G_h} &= \sup_{x\in G_h}\left\{u_h(x) - v(x)\right\}\\ &= \sup_{x\in G_h}\psi(x,x)\\ &\le \sup_{(x,y)\in G_h\times\mathbf{R}^d}\psi(x,y)\\ &= \psi(\widehat{x},\widehat{y})\\ &= u_h(\widehat{x}) - v(\widehat{y}) - \frac{|\widehat{x}-\widehat{y}|^2}{2\epsilon}\end{aligned}$$

$$
\begin{aligned}
&= \big[u_h(\widehat{x}) + \widehat{H}_{\hat{x}}(\partial_{\delta_{\hat{x}}} u_h(\widehat{x})) - f(\widehat{x})\big] \\
&\quad -\big[v(\widehat{y}) + \widehat{H}_{\hat{x}}(\mathcal{D}_{\delta_{\hat{x}},\mathbf{y};-\epsilon,\hat{p}} v(\widehat{y})) + f(\widehat{x}) - \frac{|\widehat{x}-\widehat{y}|^2}{2\epsilon}\big] \\
&\quad +\big[\widehat{H}_{\hat{x}}(\mathcal{D}_{\delta_{\hat{x}},\mathbf{y};-\epsilon,\hat{p}} v(\widehat{y})) - \widehat{H}_{\hat{x}}(\partial_{\delta_{\hat{x}}} u_h(\widehat{x}))\big] \\
&= -\big[v(\widehat{y}) + \widehat{H}_{\hat{y}+\epsilon\,\hat{p}}(\mathcal{D}_{\delta_{\hat{y}+\epsilon\,\hat{p}},\mathbf{y};-\epsilon,\hat{p}}\, v(\widehat{y})) - f(\widehat{y}+\epsilon\,\widehat{p}) + \frac{|\,\widehat{p}\,|^2}{2}\epsilon\big] \\
&\quad +\big[\widehat{H}_{\hat{x}}(\mathcal{D}_{\delta_{\hat{x}},\mathbf{y};-\epsilon,\hat{p}} v(\widehat{y})) - \widehat{H}_{\hat{x}}(\partial_{\delta_{\hat{x}}} u_h(\widehat{x}))\big]
\end{aligned}
$$

by the definition of the approximate solution u_h, (27), and the definition of $\widehat{p}$. Hence by the definition of the generalized truncation error (30), we get

$$|\,u_h - v\,|_{-,G_h} \le -T_{\mathbf{y};-\epsilon}(v;\widehat{y},\widehat{p}) + \big[\widehat{H}_{\hat{x}}(\mathcal{D}_{\delta_{\hat{x}},\mathbf{y};-\epsilon,\hat{p}}\, v(\widehat{y})) - \widehat{H}_{\hat{x}}(\partial_{\delta_{\hat{x}}} u_h(\widehat{x}))\big].$$

Next, we show that the last term in the above inequality is non-positive. First, we show that

$$\partial_{\delta_{\hat{x}}} u_h(\widehat{x}) \ge \mathcal{D}_{\delta_{\hat{x}},\mathbf{y};-\epsilon,\hat{p}}\, v(\widehat{y}).$$

We begin by noting that the inequality

$$\psi(\widehat{x},\widehat{y}) \ge \psi(\widehat{x}-\delta,y), \qquad \forall\,(\widehat{x}-\delta,y) \in G_h \times \mathbf{R}^d,$$

can be rewritten as follows:

$$u_h(\widehat{x}) - u_h(\widehat{x}-\delta) \ge v(\widehat{y}) - v(y) + \Theta, \tag{35}$$

where

$$\Theta = \frac{|\widehat{x}-\widehat{y}|^2}{2\epsilon} - \frac{|y-\widehat{x}+\delta|^2}{2\epsilon} = \widehat{p}\cdot(y-\widehat{y}+\delta) - \frac{1}{2\epsilon}|y-\widehat{y}+\delta\,|^2 = \Theta_{-\epsilon}(y-\widehat{y}+\delta,\widehat{p}).$$

Since this implies that

$$\partial_\delta u_h(\widehat{x}) = \frac{u_h(\widehat{x}) - u_h(\widehat{x}-\delta)}{|\,\delta\,|} \le \frac{v(\widehat{y}) - v(y)}{|\,\delta\,|} + \frac{\Theta_{-\epsilon}(y-\widehat{y}+\delta,\widehat{p})}{|\,\delta\,|} = \mathcal{D}_{\delta,y;-\epsilon,\hat{p}}\, v(\widehat{y}),$$

using in a *crucial* way the monotonicity of the numerical Hamiltonian $\widehat{H}$, we get

$$\big[\widehat{H}_{\hat{x}}(\mathcal{D}_{\delta_{\hat{x}},\mathbf{y};-\epsilon,\hat{p}} v(\widehat{y})) - \widehat{H}_{\hat{x}}(\partial_{\delta_{\hat{x}}} u_h(\widehat{x}))\big] \le 0.$$

This implies

$$|\,u_h - v\,|_{-,G_h} \le -T_{\mathbf{y};-\epsilon}(v;\widehat{y},\widehat{p}).$$

The estimate follows if we show that $(\widehat{y},\widehat{p})$ belongs to $\mathcal{A}_{h,-}(v;\epsilon)$. By definition of $\widehat{p}$, we have that $\widehat{y}+\epsilon\,\widehat{p} = \widehat{x} \in G_h$. Now we set $\delta = 0$ in (35) to get

$$0 \ge v(\widehat{y}) - v(y) + \widehat{p}\cdot(y-\widehat{y}) - \frac{1}{2\epsilon}|\,y-\widehat{y}\,|^2,$$

or equivalently,

$$v(y) - P_v(\widehat{y},\widehat{p},-1/\epsilon;y) \ge 0, \qquad \forall\, y \in \mathbf{R}^d.$$

This completes the proof of Theorem 7. $\square$

6 Some extensions

Extensions of these continuous dependence results to more complicated Hamiltonians can be done by following [7]. In that paper, an extension to the time-dependent case is also carried out. See also the paper [6] which contains dependence results of viscosity solutions of more general parabolic equations.

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Appendix: Existence of the approximate solution of the monotone scheme

Proposition 9 (Existence of u_h). *There exists a solution u_h of the monotone scheme* (27)*.*

Proof. We establish the existence by using the classic Leray-Schauder fixed-point theorem (see, for example, page 162 in [16]).

First, we enumerate the points y_i of the grid G_h and identify the function v defined on G_h by $(v_1, \dots, v_N)$, where $v_i = v(y_i)$. Then we consider the mapping $\mathcal{F} : \mathbf{R}^N \to \mathbf{R}^N$ defined by $(\mathcal{F}(v))(y) = -\widehat{H}_y(\partial_{\delta_y} v(y)) + f(y)$, $\forall\, y \in G_h$. A fixed point of this map is a solution of the numerical scheme (27) and vice versa. Choosing $r = \| H(0) - f \|_{l^\infty(\mathbf{R}^N)}$, the Leray-Schauder fixed-point theorem guarantees the existence of a fixed point of $\mathcal{F}$ in the ball $\overline{B}(r) = \{v \in \mathbf{R}^N : \| v \|_{\ell^\infty} \leq r\}$ if $\mathcal{F}$ is a continuous mapping from $B(r) = \{v \in \mathbf{R}^N : \| v \|_{\ell^\infty} < r\} \subset \mathbf{R}^N$ to $\mathbf{R}^N$ and $\mathcal{F}(v) \neq \lambda v$ whenever $\lambda > 1$ and $v \in \partial B(r) = \{v \in \mathbf{R}^N : \| v \|_{\ell^\infty} = r\}$.

Since we assume that the numerical Hamiltonian is globally Lipschitz, it is clear that $\mathcal{F}$ is a continuous mapping. Assuming $v \in \partial B(r)$, i.e., $\| v \|_{\ell^\infty} = r$, there must be at least one index i_0 $(1 \leq i_0 \leq N)$ such that $v_{i_0} = v(y_{i_0}) = r$ or $v_{i_0} = v(y_{i_0}) = -r$. If $v_{i_0} = r$, then $v_{i_0} \geq v_i$ for $1 \leq i \leq N$. Therefore, $\frac{v_{i_0} - v_i}{|\delta_{i_0,i}|} \geq 0$ for $1 \leq i \leq N$. Furthermore, *by the monotonicity* of $\widehat{H}_y$, we have $\widehat{H}_{y_{i_0}}(\partial_{\delta_{y_{i_0}}} v(y_{i_0})) \geq \widehat{H}_{y_{i_0}}(0) = H(0)$. Hence,

$$-\widehat{H}_{y_{i_0}}(\partial_{\delta_{y_{i_0}}} v(y_{i_0})) + f(y_{i_0}) \leq -H(0) + f(y_{i_0}) \leq r < \lambda r = \lambda v_{i_0},$$

that is, $(\mathcal{F}(v))(y_{i_0}) \neq \lambda v(y_{i_0})$. Similarly if $v_{i_0} = -r$, we still have $(\mathcal{F}(v))(y_{i_0}) \neq \lambda v(y_{i_0})$. Thus $(\mathcal{F}(v)) \neq \lambda v$ for $v \in \partial B(r)$. All in all, the assumptions of the Leray-Schauder theorem hold, hence we conclude that there exists a solution $v = u_h$ given by the monotone scheme (27) in the ball $\overline{B}(r)$. $\square$

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