HDG methods for hyperbolic problems

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Abstract

We give a short overview of the development of the so-called hybridizable discontinuous Galerkin methods for hyperbolic problems. We describe the methods, discuss their main features and display numerical results which illustrate their performance. We do this in the framework of wave propagation problems. In particular, we show that these methods are amenable to static condensation, and hence to efficient implementation, both for time-dependent (with implicit time-marching schemes) as well as for time-harmonic problems; we also show that they can be used with explicit time-marching schemes. We discuss an unexpected, recently uncovered superconvergence property and introduce a new postprocessing for time-harmonic Maxwell's equations. We end by providing bibliographical notes.

Keywords: discontinuous Galerkin methods, hybridization, hyperbolic problems.

1 Introduction

We give a short overview of the development of the so-called hybridizable discontinuous (HDG) methods for hyperbolic problems. The HDG methods are discontinuous Galerkin methods which were originally devised for numerically approximating steady-state problems and implicit time-marching schemes for time-dependent problems. Their distinctive feature is that they are amenable to *static condensation* and hence to efficient implementation. They turned out to be more accurate that other DG methods, as will be shown below.

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The HDG methods were introduced in [14] in the framework of steady-state diffusion as part of the effort that started at the end of last century to devise efficient DG methods for second-order elliptic problems. The development of the HDG methods was then spearheaded by the authors who extended them to a variety of problems in computational fluid dynamics including convection-diffusion [33, 34], the incompressible Navier-Stokes equations [36, 40], and the compressible Euler and Navier-Stokes equations [43, 41]; to partial differential equations in continuum mechanics, see [32] and the references therein; and to wave propagation problems in the time-domain [37, 46] as well as to the frequency domain [39, 42].

In this paper, we describe the HDG methods, highlight some of their main features and provide numerical experiments displaying their performance. In particular, we show that they can be efficiently implemented, that they can be used with either implicit or explicit time-marching schemes, and that they do possess recently uncovered superconvergence properties. We do this for the acoustic wave equation in Section 2, for the elastic wave equation in Section 3, and for the time-harmonic Maxwell's equation in Section 4. In Section 5, we end with a few bibliographic notes.

2 The Acoustics Wave Equation

In this section we describe HDG methods for the numerical solution of the acoustic wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot (\kappa \nabla u) = f, \quad \text{in } \Omega \times (0, T]. \tag{1}$$

By introducing the velocity $v = u_t$ and the flux $\boldsymbol{q} = -\kappa \nabla u$, we can write (1) as the following system of first-order equations:

$$\begin{aligned} \kappa^{-1} \frac{\partial \boldsymbol{q}}{\partial t} + \nabla v &= 0, && \text{in } \Omega \times (0, T], \\ \rho \frac{\partial v}{\partial t} + \nabla \cdot \boldsymbol{q} &= f, && \text{in } \Omega \times (0, T]. \end{aligned} \tag{2a}$$

The exact solution $(v, \boldsymbol{q})$ satisfies the following initial conditions

$$\begin{aligned} v(\boldsymbol{x}, t = 0) &= v_0(\boldsymbol{x}), \\ \boldsymbol{q}(\boldsymbol{x}, t = 0) &= \boldsymbol{q}_0(\boldsymbol{x}), \end{aligned} \tag{2b}$$

and a Robin boundary condition

$$-\boldsymbol{q} \cdot \boldsymbol{n} + \alpha v = g, \quad \text{on } \partial\Omega \times (0, T]. \tag{2c}$$

The coefficient α varies on the boundary $\partial\Omega$ and represents different types of boundary conditions. Specifically, the Neumann boundary condition corresponds to $\alpha = 0$, the Dirichlet boundary condition to $1/\alpha = 0$, and the first-order absorbing boundary condition to $\alpha = \sqrt{\kappa\rho}$. We assume that $\kappa(\boldsymbol{x}), \rho(\boldsymbol{x})$, and $\alpha(\boldsymbol{x})$ are scalar positive functions.

We begin with the spatial discretization of the wave equation (2) and the temporal integration of the semi-discrete form using both explicit and implicit time-stepping methods. We end by presenting numerical experiments to demonstrate their performance.

2.1 Spatial discretization

Let $\mathcal{T}_h$ be a collection of disjoint elements that partition Ω. We denote by $\partial\mathcal{T}_h$ the set $\{\partial K : K \in \mathcal{T}_h\}$. For an element K of the collection $\mathcal{T}_h$, $F = \partial K \cap \partial\Omega$ is the boundary face if the $d-1$ Lebesgue measure of F is nonzero. For two elements K^+ and K^- of the collection $\mathcal{T}_h$, $F = \partial K^+ \cap \partial K^-$ is the interior face between K^+ and K^- if the $d-1$ Lebesgue measure of F is nonzero. Let $\mathcal{E}_h^o$ and $\mathcal{E}_h^\partial$ denote the set of interior and boundary faces, respectively. We denote by $\mathcal{E}_h$ the union of $\mathcal{E}_h^o$ and $\mathcal{E}_h^\partial$.

Let $\mathcal{P}_k(D)$ denote the set of polynomials of degree at most k on a domain D. We are going to use the following discontinuous finite element spaces:

$$\begin{aligned} W_h &= \{w \in L^2(\Omega) \quad : \ w|_K \in W(K), \ \forall K \in \mathcal{T}_h\}, \\ \boldsymbol{V}_h &= \{\boldsymbol{p} \in (L^2(\Omega))^d : \ \boldsymbol{p}|_K \in \boldsymbol{V}(K), \ \ \forall K \in \mathcal{T}_h\}. \end{aligned}$$

Some appropriate choices for the local space $W(K) \times \boldsymbol{V}(K)$ on K include

$$W(K) \times \boldsymbol{V}(K) \equiv \begin{cases} \mathcal{P}_k(K) \times (\mathcal{P}_k(K))^d, \\ \mathcal{P}_{k-1}(K) \times (\mathcal{P}_k(K))^d, \\ \mathcal{P}_k(K) \times \left((\mathcal{P}_k(K))^d + \boldsymbol{x}\mathcal{P}_k(K)\right). \end{cases}$$

These spaces correspond to the equal-order elements, the BDM elements [1], and the RT elements [44], respectively. In addition, we introduce a traced finite element space

$$M_h = \{\mu \in L^2(\mathcal{E}_h) \ : \ \mu|_F \in \mathcal{P}_k(F), \ \forall F \in \mathcal{E}_h\}.$$

For functions $\boldsymbol{w}$ and $\boldsymbol{v}$ in $(L^2(D))^d$, we denote $(\boldsymbol{w}, \boldsymbol{v})_D = \int_D \boldsymbol{w} \cdot \boldsymbol{v}$. For functions w and v in $L^2(D)$, we denote $(w, v)_D = \int_D wv$ if D is a domain in $\mathbb{R}^d$ and $\langle w, v\rangle_D = \int_D wv$ if D is a domain in $\mathbb{R}^{d-1}$. We then introduce

$$(w, v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (w, v)_K, \qquad \langle \mu, \eta\rangle_{\partial\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \langle \mu, \eta\rangle_{\partial K},$$

for w, v defined on $\mathcal{T}_h$ and μ, η defined on $\partial\mathcal{T}_h$.

The HDG methods for the wave equation (2) seek to define $(\boldsymbol{q}_h, v_h, \widehat{v}_h)(t) \in \boldsymbol{V}_h \times W_h \times M_h$, for $t \in [0, T]$, as a solution of the following system

$$\left(\kappa^{-1} \frac{\partial \boldsymbol{q}_h}{\partial t}, \boldsymbol{r}\right)_{\mathcal{T}_h} - (v_h, \nabla \cdot \boldsymbol{r})_{\mathcal{T}_h} + \langle \widehat{v}_h, \boldsymbol{r} \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{3a}$$

$$\left(\rho \frac{\partial v_h}{\partial t}, w\right)_{\mathcal{T}_h} - (\boldsymbol{q}_h, \nabla w)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, w\rangle_{\partial\mathcal{T}_h} = (f, w)_{\mathcal{T}_h}, \tag{3b}$$

$$-\langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \mu\rangle_{\partial\mathcal{T}_h \backslash \partial\Omega} + \langle -\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} + \alpha\widehat{v}_h, \mu\rangle_{\partial\Omega} = \langle g, \mu\rangle_{\partial\Omega}, \tag{3c}$$

for all $(\boldsymbol{r}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h$ and all $t \in (0,T]$, where the numerical flux is defined as

$$\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} = \boldsymbol{q}_h \cdot \boldsymbol{n} + \tau(v_h - \widehat{v}_h), \quad \text{on } \partial\mathcal{T}_h. \tag{3d}$$

If the stabilization function is taken as $\tau = \sqrt{\kappa\rho}$, we obtain the well known upwinding flux.

Note that the equations (2a) require v and the normal component of $\boldsymbol{q}$ to be continuous across the set of interior faces $\mathcal{E}_h^o \times (0,T)$. The HDG method takes into account these *transmission conditions* by requiring that the corresponding numerical traces $\widehat{v}_h$ and the normal component of $\widehat{\boldsymbol{q}}_h$ be single valued on that set. The first condition is satisfied by taking $\widehat{v}_h(t)$ in M_h and the second by impossing equation (3c) for any $t \in [0,T]$. For other ways of defining HDG methods, see [8] and the references therein.

This semidiscretization gives rise to a system of ODEs to be solved by using some time-marching methods. As we are going to see in the next subsection, the form presented here is useful when using implicit time-marching methods because it takes advantage of the fact that the HDG methods are amenable to static condensation. When using explicit time-marching methods, a better way of presenting the method is the following: Find $(\boldsymbol{q}_h, v_h) \in \boldsymbol{V}_h \times W_h$ such that for all $K \in \mathcal{T}_h$,

$$\left(\kappa^{-1}\frac{\partial \boldsymbol{q}_h}{\partial t}, \boldsymbol{r}\right)_K - (v_h, \nabla\cdot\boldsymbol{r})_K + \langle \widehat{v}_h, \boldsymbol{r}\cdot\boldsymbol{n}\rangle_{\partial K} = 0, \quad \forall\, \boldsymbol{r} \in \boldsymbol{V}(K), \tag{4a}$$

$$\left(\rho\frac{\partial v_h}{\partial t}, w\right)_K - (\boldsymbol{q}_h, \nabla w)_K + \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w\rangle_{\partial K} = (f, w)_K, \quad \forall\, w \in W(K), \tag{4b}$$

where, for any given face $F \in \partial K$,

$$\widehat{v}_h = \begin{cases} \dfrac{\tau^+ v_h^+ + \tau^- v_h^-}{\tau^+ + \tau^-} + \dfrac{1}{\tau^+ + \tau^-}(\boldsymbol{q}_h^+ \cdot \boldsymbol{n}^+ + \boldsymbol{q}_h^- \cdot \boldsymbol{n}^-), & \text{if } F \in \mathcal{E}_h^o, \\ \dfrac{\tau}{\tau+\alpha} v_h + \dfrac{1}{\tau+\alpha}(\mathsf{P}g + \boldsymbol{q}_h \cdot \boldsymbol{n}), & \text{if } F \in \partial\Omega, \end{cases} \tag{4c}$$

and

$$\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} = \boldsymbol{q}_h \cdot \boldsymbol{n} + \tau(v_h - \widehat{v}_h) \quad \text{on } \partial K. \tag{4d}$$

Here $\mathsf{P}g$ denotes the L^2 projection of g onto the space M_h, and

$$v_h^\pm|_F = v_h|_{F\in\partial K^\pm}, \quad \text{and} \quad \boldsymbol{q}_h^\pm|_F = \boldsymbol{q}_h|_{F\in\partial K^\pm},$$

where K^+ and K^- are two elements sharing the face F. Hence, v_h^- and $\boldsymbol{q}_h^-$ (respectively, v_h^+ and $\boldsymbol{q}_h^+$) are nothing but the value of v_h and $\boldsymbol{q}_h$ on the face F from the element K^- (respectively, K^+). We can easily show that when the stabilization function is taken to be a constant on each face lying on $\partial\mathcal{T}_h$, the system (4) is equivalent to the original formulation (3) [37, 46].

2.2 Temporal discretization

We now show how to obtain a fully discrete scheme by discretizing the above system of ODEs by several different time-marching methods, two being implicit and the other two explicit.

BDF methods

We will only discuss the backward-Euler method since higher-order BDF methods follow a similar way. Using the backward-Euler scheme for the discretization of the time derivative in (3), we find that the approximate solution $(\boldsymbol{q}_h^n, v_h^n, \widehat{v}_h^n) \in \boldsymbol{V}_h \times W_h \times M_h$ at time step n satisfies the following equations

$$\left(\frac{\boldsymbol{q}_h^n}{\kappa \Delta t}, \boldsymbol{r}\right)_{\mathcal{T}_h} - (v_h^n, \nabla \cdot \boldsymbol{r})_{\mathcal{T}_h} + \langle \widehat{v}_h^n, \boldsymbol{r} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = \left(\frac{\boldsymbol{q}_h^{n-1}}{\kappa \Delta t}, \boldsymbol{r}\right)_{\mathcal{T}_h}, \tag{5a}$$

$$\left(\rho \frac{v_h^n}{\Delta t}, w\right)_{\mathcal{T}_h} - (\boldsymbol{q}_h^n, \nabla w)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{q}}_h^n \cdot \boldsymbol{n}, w \rangle_{\partial \mathcal{T}_h} = \left(f^n + \rho \frac{v_h^{n-1}}{\Delta t}, w\right)_{\mathcal{T}_h}, \tag{5b}$$

$$-\langle \widehat{\boldsymbol{q}}_h^n \cdot \boldsymbol{n}, \mu \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} + \langle -\widehat{\boldsymbol{q}}_h^n \cdot \boldsymbol{n} + \alpha \widehat{v}_h^n, \mu \rangle_{\partial \Omega} = \langle g^n, \mu \rangle_{\partial \Omega}, \tag{5c}$$

for all $(\boldsymbol{r}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h$, where

$$\widehat{\boldsymbol{q}}_h^n \cdot \boldsymbol{n} = \boldsymbol{q}_h^n \cdot \boldsymbol{n} + \tau(v_h^n - \widehat{v}_h^n), \quad \text{on } \partial \mathcal{T}_h. \tag{5d}$$

Here $(\boldsymbol{q}_h^n, v_h^n, \widehat{v}_h^n)$ represents the numerical approximation to the exact solution $(\boldsymbol{q}(t^n), u(t^n), \widehat{v}(t^n))$ at time t^n. We then find $u_h^n \in W_h$ such that

$$\frac{1}{\Delta t}(u_h^n, w)_{\mathcal{T}_h} = (v_h^n, w)_{\mathcal{T}_h} + \frac{1}{\Delta t}(u_h^{n-1}, w)_{\mathcal{T}_h}, \quad \forall w \in W_h. \tag{6}$$

The fully discrete system (5) can be efficiently solved by locally eliminating $(\boldsymbol{q}_h, \boldsymbol{u}_h)$ to obtain a linear system in terms of the globally coupled degrees of freedom of $\widehat{v}_h$. We refer to [37] for a detailed discussion.

DIRK methods

Next, we apply the DIRK formula represented by the coefficients (a_{ij}, b_i, c_i), $1 \le i \le q, 1 \le j \le i$, to the semidiscrete system (3). We denote by $(\boldsymbol{q}_h^{n.i}, v_h^{n,i}, \widehat{v}_h^{n,i})$ the numerical approximation to the exact solution $(\boldsymbol{q}(t^{n,i})|_{\mathcal{T}_h}, v(t^{n,i})|_{\mathcal{T}_h}, v(t^{n,i})|_{\mathcal{E}_h})$, where $t^{n,i} = t^{n-1} + c_i \Delta t, 1 \le i \le q$. Given the approximate solution at time t^{n-1}, $(\boldsymbol{q}_h^{n-1}, v_h^{n-1}, \widehat{v}_h^{n-1})$, we find the intermediate solutions $(\boldsymbol{q}_h^{n.i}, v_h^{n,i}, \widehat{v}_h^{n,i}) \in \boldsymbol{V}_h \times W_h \times M_h$ satisfying

$$\left(\frac{\boldsymbol{q}_h^{n,i}}{\kappa \Delta t}, \boldsymbol{v}\right)_{\mathcal{T}_h} - (v_h^{n,i}, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \left\langle \widehat{v}_h^{n,i}, \boldsymbol{v} \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h} = \left(\frac{\boldsymbol{p}_h^{n,i}}{\kappa}, \boldsymbol{v}\right)_{\mathcal{T}_h}, \tag{7a}$$

$$\left(\frac{\rho v_h^{n,i}}{a_{ii} \Delta t}, w\right)_{\mathcal{T}_h} - (\boldsymbol{q}_h^{n,i}, \nabla w)_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{q}}_h^{n,i} \cdot \boldsymbol{n}, w \right\rangle_{\partial \mathcal{T}_h} = (f^{n,i} + \rho s_h^{n,i}, w)_{\mathcal{T}_h}, \tag{7b}$$

$$-\left\langle \widehat{\boldsymbol{q}}_h^{n,i} \cdot \boldsymbol{n}, \mu \right\rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} + \left\langle -\widehat{\boldsymbol{q}}_h^{n,i} \cdot \boldsymbol{n} + \alpha \widehat{v}_h^{n,i}, \mu \right\rangle_{\partial \Omega} = \left\langle g^{n,i}, \mu \right\rangle_{\partial \Omega}, \tag{7c}$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h$, where

$$\widehat{\boldsymbol{q}}_h^{n,i} \cdot \boldsymbol{n} = \boldsymbol{q}_h^{n,i} \cdot \boldsymbol{n} + \tau(v_h^{n,i} - \widehat{v}_h^{n,i}), \quad \text{on } \partial\mathcal{T}_h. \tag{7d}$$

The terms $s_h^{n,i}$ and $\boldsymbol{p}_h^{n,i}$ on the right-hand side of (7) are given by

$$\begin{aligned} s_h^{n,i} &= \frac{v_h^{n-1}}{a_{ii}\Delta t} + \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} \Big(\frac{v_h^{n,j}}{a_{jj}\Delta t} - s_h^{n,j} \Big), \quad i = 1, \ldots, q, \\ \boldsymbol{p}_h^{n,i} &= \frac{\boldsymbol{q}_h^{n-1}}{a_{ii}\Delta t} + \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} \Big(\frac{\boldsymbol{q}_h^{n,j}}{a_{jj}\Delta t} - \boldsymbol{p}_h^{n,j} \Big), \quad i = 1, \ldots, q. \end{aligned}$$

The discrete systems (7) must be solved sequentially from $i = 1, 2, \ldots, q$. Hence, a q-stage DIRK scheme requires us to solve q discrete systems which are very similar to the system (5) resulting from the backward-Euler method.

Once the intermediate solutions have been computed, the numerical solution $(\boldsymbol{q}_h^n, v_h^n)$ at time t^n is given by

$$(\boldsymbol{q}_h^n, v_h^n) = (\boldsymbol{q}_h^{n-1}, v_h^{n-1}) + \Delta t \sum_{i=1}^{q} b_i (\boldsymbol{y}_h^{n,i}, z_h^{n,i}), \tag{8}$$

where

$$\begin{aligned} \boldsymbol{y}_h^{n,i} &= \frac{\boldsymbol{q}_h^{n,i} - \boldsymbol{q}_h^{n-1}}{a_{ii}\Delta t} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} \boldsymbol{y}_h^{n,j}, \quad i = 1, \ldots, q, \\ z_h^{n,i} &= \frac{v_h^{n,i} - v_h^{n-1}}{a_{ii}\Delta t} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} z_h^{n,j}, \quad i = 1, \ldots, q. \end{aligned}$$

Finally, we compute u_h^n by using the same DIRK scheme to discretize the ODE $\partial u_h / \partial t = v_h$.

Adams–Bashforth methods

The Adams–Bashforth (AB) methods are linear multistep explicit methods. The forward-Euler method is a first-order AB method. Here we discuss the forward-Euler method since higher-order AB methods can be constructed in a similar way. Given the solution at the previous time step $(\boldsymbol{q}_h^n, v_h^n, u_h^n)$, we first compute the approximate traces as

$$\widehat{v}_h^n = \begin{cases} \dfrac{\tau^+ v_h^{+n} + \tau^- v_h^{-n}}{\tau^+ + \tau^-} - \dfrac{1}{\tau^+ + \tau^-} (\boldsymbol{q}_h^{+n} \cdot \boldsymbol{n}^+ + \boldsymbol{q}_h^{-n} \cdot \boldsymbol{n}^-), & \text{if } F \in \mathcal{E}_h \backslash \partial\Omega, \\ \dfrac{\tau}{\tau + \alpha} v_h^n + \dfrac{1}{\tau + \alpha} (\mathsf{P} g^n + \alpha \boldsymbol{q}_h^n \cdot \boldsymbol{n}), & \text{if } F \in \partial\Omega, \end{cases} \tag{9}$$

and $\widehat{\boldsymbol{q}}_h^n \cdot \boldsymbol{n} = \boldsymbol{q}_h^n \cdot \boldsymbol{n} + \tau(v_h^n - \widehat{v}_h^n)$ for all faces F of $\mathcal{E}_h$. We then determine the numerical solution $(\boldsymbol{q}_h^{n+1}, v_h^{n+1}, u_h^{n+1}) \in \boldsymbol{V}(K) \times W(K) \times W(K)$ at the next time step by solving

$$
\begin{aligned}
\Big(\frac{1}{\kappa}\frac{\boldsymbol{q}_h^{n+1}-\boldsymbol{q}_h^n}{\Delta t^n}, \boldsymbol{r}\Big)_K - (v_h^n, \nabla\cdot\boldsymbol{r})_K + \langle \widehat{v}_h^n, \boldsymbol{r}\cdot\boldsymbol{n}\rangle_{\partial K} &= 0,\\
\Big(\rho\frac{v_h^{n+1}-v_h^n}{\Delta t^n}, w\Big)_K - (\boldsymbol{q}_h^n, \nabla w)_K + \langle \widehat{\boldsymbol{q}}_h^n\cdot\boldsymbol{n}, w\rangle_{\partial K} &= (f^n, w)_K,\\
\Big(\frac{u_h^{n+1}-u_h^n}{\Delta t^n}, z\Big)_K - (v_h^n, z)_K &= 0,
\end{aligned}
\tag{10}
$$

for all $(\boldsymbol{r}, w, z) \in \boldsymbol{V}(K) \times W(K) \times W(K)$ and for all elements $K \in \mathcal{T}_h$.

It is clear that we compute the numerical solution at any time step in an element-by-element fashion. Therefore, explicit HDG methods have the same computational complexity as other explicit DG methods. Higher-order AB methods can be used as well, provided that the numerical solutions at the earlier time steps are available.

2.3 SSP-RK methods

Lastly, we describe the SSP-RK(q, q) scheme [4, 27] to integrate the semidiscrete system (4) in time. For $i = 0, \ldots, q-1$, we compute

$$
\widehat{v}_h^{n,i-1} = \begin{cases}
\dfrac{\tau^+ v_h^{+n,i-1} + \tau^- v_h^{-n,i-1}}{\tau^+ + \tau^-} & \\
+\frac{1}{\tau^+ + \tau^-}(\boldsymbol{q}_h^{+n,i-1}\cdot\boldsymbol{n}^+ + \boldsymbol{q}_h^{-n,i-1}\cdot\boldsymbol{n}^-), & \text{if } F \in \mathcal{E}_h \backslash \partial\Omega,\\
\dfrac{\tau}{\tau+\alpha} v_h^{n,i-1} + \dfrac{1}{\tau+\alpha}(\mathsf{P}g^{n,i-1} + \alpha \boldsymbol{q}_h^{n,i-1}\cdot\boldsymbol{n}), & \text{if } F \in \partial\Omega,
\end{cases}
\tag{11}
$$

and $\widehat{\boldsymbol{q}}_h^{n,i-1} \cdot \boldsymbol{n} = \boldsymbol{q}_h^{n,i-1} \cdot \boldsymbol{n} + \tau(v_h^{n,i-1} - \widehat{v}_h^{n,i-1})$ for all faces F of $\mathcal{E}_h$; we then determine $(\boldsymbol{q}_h^{n,i}, v_h^{n,i}, v_h^{n,i}) \in \boldsymbol{V}(K) \times W(K) \times W(K)$ as the solution of

$$
\begin{aligned}
\Big(\frac{1}{\kappa}\frac{\boldsymbol{q}_h^{n,i}-\boldsymbol{q}_h^{n,i-1}}{\Delta t}, \boldsymbol{r}\Big)_K - \Big(v_h^{n,i-1}, \nabla\cdot\boldsymbol{r}\Big)_K + \Big\langle \widehat{v}_h^{n,i-1}, \boldsymbol{r}\cdot\boldsymbol{n}\Big\rangle_{\partial K} &= 0,\\
\Big(\rho\frac{v_h^{n,i}-v_h^{n,i-1}}{\Delta t}, w\Big)_K - \Big(\boldsymbol{q}_h^{n,i-1}, \nabla w\Big)_K + \Big\langle \widehat{\boldsymbol{q}}_h^{n,i-1}\cdot\boldsymbol{n}, w\Big\rangle_{\partial K} &= (f^{n,i-1}, w)_K,\\
\Big(\frac{v_h^{n,i}-v_h^{n,i-1}}{\Delta t}, z\Big)_K - (v_h^{n,i-1}, z)_K &= 0,
\end{aligned}
\tag{12}
$$

for all $(\boldsymbol{r}, w, z) \in \boldsymbol{V}(K) \times W(K) \times W(K)$ and for all elements $K \in \mathcal{T}_h$. We finally set

$$
(\boldsymbol{q}_h^n, v_h^n, v_h^n) = \sum_{i=0}^{s} \alpha_{q,i}(\boldsymbol{q}_h^{n,i}, v_h^{n,i}, v_h^{n,i}),
\tag{13}
$$

where the coefficients $\alpha_{q,i}$ are precisely those corresponding to the SSPRK scheme (q, q) [4, 27], namely

$$\begin{aligned} &\alpha_{1,0} = 1, \qquad \alpha_{q,i} = \frac{1}{i}\alpha_{q-1,i-1}, \quad i = 1, \dots, q-2, \\ &\alpha_{q,q} = \frac{1}{q!}, \qquad \alpha_{q,q-1} = 0, \qquad \alpha_{q,0} = 1 - \sum_{i=1}^{q-1} \alpha_{q,i}. \end{aligned} \tag{14}$$

The SSP-RK(q, q) scheme has q stages and q orders of accuracy. Each stage of the SSP-RK(q, q) scheme is exactly the forward-Euler method described earlier.

2.4 Postprocessing

The numerical results we present in the next subsection are going to involve two elementwise postprocessings defined as follows. The first is a new approximation to u: On every simplex $K \in \mathcal{T}_h$, we take $u_h^{n\,*} \in \mathcal{P}_{k+1}(K)$, such that

$$\begin{aligned} (\nabla u_h^{n\,*}, \nabla w)_K &= (\boldsymbol{q}_h^n, \nabla w)_K\,, \quad \forall\, w \in \mathcal{P}_{k+1}(K), \\ (u_h^{n\,*}, 1)_K &= (u_h^n, 1)_K\,. \end{aligned} \tag{15}$$

The second is a new approximation to u_t: On every simplex $K \in \mathcal{T}_h$, we take $v_h^{n\,*} \in \mathcal{P}_{k+1}(K)$, such that

$$\begin{aligned} (\nabla v_h^{n\,*}, \nabla w)_K &= -\,(v_h^n, \Delta w)_K + \langle \widehat{v}_h^n, \nabla w \cdot \boldsymbol{n} \rangle_{\partial K}, \quad \forall\, w \in \mathcal{P}_{k+1}(K), \\ (v_h^{n\,*}, 1)_K &= (v_h^n, 1)_K\,. \end{aligned} \tag{16}$$

As we are going to see, it turns out that both postprocessings u_h^* and v_h^* have better orders of convergence that the original approximations u_h and v_h, respectively. Note that this local postprocessing can be performed at suitable time steps, where these more accurate approximations are needed.

2.5 Numerical results

We consider the wave equation on a unit square $\Omega = (0, 1)\times(0, 1)$ with boundary condition $v = 0$ on $\partial\Omega$ and initial condition $u_0 = 0$ and $v_0 = \sin(\pi x) \sin(\pi y)$. For $\rho = \kappa = 1$ and $f = 0$, the problem has the following exact solution

$$u = \frac{1}{\sqrt{2}\pi} \sin(\pi x) \sin(\pi y) \sin(\sqrt{2}\pi t), \qquad v = \sin(\pi x) \sin(\pi y) \cos(\sqrt{2}\pi t).$$

We use triangular meshes obtained by splitting a regular $n \times n$ Cartesian grid into a total of $2n^2$ triangles, giving uniform element sizes of $h = 1/n$.

We present the L^2-errors and orders of convergence for the numerical approximations in Table 1 for the DIRK schemes and Table 2 for the SSP-RK schemes. We observe that the approximate field variables converge with the optimal order $k + 1$, while the postprocessed displacement and velocity converge with order $k+2$. The HDG methods yield optimal convergence for the approximate gradient, while many other DG methods provide suboptimal convergence of order k. Furthermore, the postprocessed displacement and velocity converge one order higher than the original approximations.

These convergence properties were first reported in [37] and later proven (for the semidiscrete case) in [18]. A priori error estimates for $v - v_h^*$ remain an open problem though.

k	$1/h$	$\lVert u - u_h \rVert_{\mathcal{T}_h}$ error	order	$\lVert v - v_h \rVert_{\mathcal{T}_h}$ error	order	$\lVert \boldsymbol{q} - \boldsymbol{q}_h \rVert_{\mathcal{T}_h}$ error	order	$\lVert u - u_h^* \rVert_{\mathcal{T}_h}$ error	order	$\lVert v - v_h^* \rVert_{\mathcal{T}_h}$ error	order
2	2	7.29e-3	−−	1.72e-2	−−	3.01e-2	−−	6.16e-3	−−	1.71e-2	−−
	4	4.80e-4	3.92	2.16e-3	2.99	2.00e-3	3.91	2.77e-4	4.48	1.99e-3	3.11
	8	4.47e-5	3.42	1.86e-4	3.54	1.84e-4	3.44	7.02e-6	5.30	1.40e-4	3.83
	16	5.24e-6	3.09	1.81e-5	3.36	2.15e-5	3.10	2.54e-7	4.79	8.73e-6	4.00
	32	6.36e-7	3.04	2.08e-6	3.12	2.61e-6	3.04	1.44e-8	4.14	5.36e-7	4.03
3	2	5.80e-4	−−	1.60e-3	−−	2.67e-3	−−	1.97e-4	−−	1.59e-3	−−
	4	3.12e-5	4.22	8.22e-5	4.29	1.38e-4	4.27	4.92e-6	5.33	8.05e-5	4.30
	8	1.78e-6	4.13	5.20e-6	3.98	7.74e-6	4.16	1.37e-7	5.17	3.78e-6	4.41
	16	1.06e-7	4.07	3.32e-7	3.97	4.56e-7	4.08	4.05e-9	5.08	1.14e-7	5.05
	32	6.46e-9	4.04	2.09e-8	3.99	2.77e-8	4.04	1.24e-10	5.03	1.50e-9	6.24

Table 1: History of convergence results using DIRK$(k+1, k+2)$ schemes.

k	$1/h$	$\lVert u - u_h \rVert_{\mathcal{T}_h}$ error	order	$\lVert v - v_h \rVert_{\mathcal{T}_h}$ error	order	$\lVert \boldsymbol{q} - \boldsymbol{q}_h \rVert_{\mathcal{T}_h}$ error	order	$\lVert u - u_h^* \rVert_{\mathcal{T}_h}$ error	order	$\lVert v - v_h^* \rVert_{\mathcal{T}_h}$ error	order
2	2	4.13e-3	−−	9.84e-3	−−	1.65e-2	−−	2.13e-3	−−	8.64e-3	−−
	4	4.01e-4	3.37	1.06e-3	3.22	1.65e-3	3.32	1.02e-4	4.38	5.19e-4	4.06
	8	4.44e-5	3.17	1.27e-4	3.06	1.83e-4	3.18	4.82e-6	4.40	2.80e-5	4.21
	16	5.24e-6	3.08	1.60e-5	2.99	2.15e-5	3.09	2.59e-7	4.22	1.61e-6	4.12
	32	6.36e-7	3.04	2.02e-6	2.99	2.61e-6	3.04	1.53e-8	4.08	9.81e-8	4.04
3	2	5.75e-4	−−	1.62e-3	−−	2.66e-3	−−	1.82e-4	−−	1.33e-3	−−
	4	3.12e-5	4.21	8.22e-5	4.30	1.38e-4	4.27	4.63e-6	5.29	3.59e-5	5.21
	8	1.78e-6	4.13	5.21e-6	3.98	7.74e-6	4.15	1.31e-7	5.15	1.03e-6	5.13
	16	1.06e-7	4.07	3.32e-7	3.97	4.56e-7	4.08	3.88e-9	5.07	3.05e-8	5.07
	32	6.46e-9	4.04	2.09e-8	3.99	2.77e-8	4.04	1.19e-10	5.03	8.97e-10	5.09

Table 2: History of convergence results using SSP-RK$(k+2, k+2)$ schemes.

3 The Elastic Wave Equations

The elastic wave equations are different from the scalar acoustic wave equation in that they are vectorial and have two different wave speeds, namely, pressure (primary) wave speed and shear (secondary) wave speed. Although there are several different formulations of the elastic wave equations, we will focus on HDG methods for the displacement gradient-velocity-pressure formulation.

Let $\boldsymbol{u}$ represent the displacement field, λ and μ the Lamé moduli, ρ the density of the elastic isotropic material, and $\boldsymbol{b}$ a time-dependent body force. Let Ω be an open bounded domain in $\mathbb{R}^d$ and T a fixed final time. The motion of the elastic isotropic body Ω is governed by:

$$\rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \nabla\cdot[\mu\nabla\boldsymbol{u} + (\mu+\lambda)(\nabla\cdot\boldsymbol{u})\mathbf{I}] \quad = \quad \boldsymbol{b}, \qquad \text{in } \Omega\times(0,T]. \tag{17}$$

We introduce the velocity field $\boldsymbol{v} = \partial\boldsymbol{u}/\partial t$, the displacement gradient tensor $\mathbf{H} = \nabla\boldsymbol{u}$, and the hydrostatic pressure $p = (\mu+\lambda)(\nabla\cdot\boldsymbol{u})$. We then rewrite (17) as the first-order system

$$\begin{aligned}
\frac{\partial \mathbf{H}}{\partial t} - \nabla\boldsymbol{v} &= 0, && \text{in } \Omega\times(0,T],\\
\rho\frac{\partial \boldsymbol{v}}{\partial t} - \nabla\cdot(\mu\mathbf{H} + p\mathbf{I}) &= \boldsymbol{b}, && \text{in } \Omega\times(0,T],\\
\epsilon\frac{\partial p}{\partial t} - \nabla\cdot\boldsymbol{v} &= 0, && \text{in } \Omega\times(0,T],
\end{aligned} \tag{18}$$

where $\epsilon = 1/(\mu+\lambda)$, and $\mathbf{I}$ is the second-order identity tensor. Associated with this system is the boundary condition

$$(\mu\boldsymbol{H} + p\boldsymbol{I})\cdot\boldsymbol{n} + \alpha\boldsymbol{v} = \boldsymbol{g}, \quad \text{on } \partial\Omega\times(0,T],$$

and initial condition

$$\boldsymbol{v} = \boldsymbol{v}_0, \quad \mathbf{H} = \mathbf{H}_0, \quad p = p_0, \quad \text{on } \Omega\times\{t=0\}.$$

For simplicity of exposition, we assume that $\epsilon > 0$, which in essence means that the elastic solid is either compressible or nearly incompressible. The incompressible limit $\epsilon = 0$ requires the average pressure condition and can be treated by the augmented Lagrangian method [38, 35].

3.1 Spatial discretization

In addition to the finite element spaces defined in Section 2.2, we introduce the following new finite element spaces:

$$\begin{aligned}
\mathbf{G}_h &= \{\mathbf{N}\in(L^2(\mathcal{T}_h))^{d\times d} &&: \mathbf{N}|_K \in (W(K))^{d\times d},\ \forall K\in\mathcal{T}_h\},\\
\boldsymbol{M}_h &= \{\boldsymbol{\mu}\in(L^2(\mathcal{E}_h))^{d} &&: \boldsymbol{\mu}|_F \in (\mathcal{P}_k(F))^{d},\ \forall F\in\mathcal{E}_h\}.
\end{aligned}$$

We then define volume and boundary inner products associated with $\mathbf{G}_h$ as

$$(\mathbf{N},\mathbf{L})_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h}(\mathbf{N},\mathbf{L})_K, \qquad \langle\mathbf{N},\mathbf{L}\rangle_{\partial\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h}\langle\mathbf{N},\mathbf{L}\rangle_{\partial K},$$

for $\mathbf{N},\mathbf{L}\in(L^2(\mathcal{T}_h))^{d\times d}$. Note that $(\mathbf{N},\mathbf{L})_D$ denotes the integral of $\mathrm{tr}(\mathbf{N}^T\mathbf{L})$ over D, where tr is the trace operator.

The HDG methods then find an approximation $(\mathbf{H}_h, \boldsymbol{v}_h, p_h, \widehat{\boldsymbol{v}}_h) \in \mathbf{G}_h \times \boldsymbol{V}_h \times W_h \times \boldsymbol{M}_h$ at time t such that

$$\left(\frac{\partial \mathbf{H}_h}{\partial t}, \mathbf{N}\right)_{\mathcal{T}_h} + (\boldsymbol{v}_h, \nabla \cdot \mathbf{N})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{v}}_h, \mathbf{N} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = 0, \tag{19a}$$

$$\begin{aligned}\left(\rho \frac{\partial \boldsymbol{v}_h}{\partial t}, \boldsymbol{w}\right)_{\mathcal{T}_h} &+ (\mu \mathbf{H}_h + p_h \mathbf{I}, \nabla \boldsymbol{w})_{\mathcal{T}_h} \\ &- \left\langle (\mu \widehat{\mathbf{H}}_h + \widehat{p}_h \mathbf{I}) \cdot \boldsymbol{n}, \boldsymbol{w} \right\rangle_{\partial \mathcal{T}_h} = (\boldsymbol{b}, \boldsymbol{w})_{\mathcal{T}_h},\end{aligned} \tag{19b}$$

$$\left(\epsilon \frac{\partial p_h}{\partial t}, q\right)_{\mathcal{T}_h} + (\boldsymbol{v}_h, \nabla q)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{v}}_h \cdot \boldsymbol{n}, q \rangle_{\partial \mathcal{T}_h} = 0, \tag{19c}$$

$$\left\langle (\mu \widehat{\mathbf{H}}_h + \widehat{p}_h \mathbf{I}) \cdot \boldsymbol{n}, \boldsymbol{\mu} \right\rangle_{\partial \mathcal{T}_h} + \langle \alpha \widehat{\boldsymbol{v}}_h, \boldsymbol{\mu} \rangle_{\partial \Omega} = \langle \boldsymbol{g}, \boldsymbol{\mu} \rangle_{\partial \Omega}, \tag{19d}$$

for all $(\mathbf{N}, \boldsymbol{w}, q, \boldsymbol{\mu}) \in \mathbf{G}_h \times \boldsymbol{V}_h \times W_h \times \boldsymbol{M}_h$, where

$$(\mu \widehat{\mathbf{H}}_h + \widehat{p}_h \mathbf{I}) \cdot \boldsymbol{n} = (\mu \mathbf{H}_h + p_h \mathbf{I}) \cdot \boldsymbol{n} - \mathbf{S}(\boldsymbol{v}_h - \widehat{\boldsymbol{v}}_h). \tag{19e}$$

Here $\mathbf{S}$ is a second-order tensor consisting of stabilization parameters which can be set to $\sqrt{(\mu+\lambda)\rho}\boldsymbol{I}$.

The semidiscrete form (19) can be equivalently reformulated into finding $(\mathbf{H}_h, \boldsymbol{v}_h, p_h)$ such that for all $K \in \mathcal{T}_h$,

$$\left(\frac{\partial \mathbf{H}_h}{\partial t}, \mathbf{N}\right)_K + (\boldsymbol{v}_h, \nabla \cdot \mathbf{N})_K - \langle \widehat{\boldsymbol{v}}_h, \mathbf{N} \cdot \boldsymbol{n} \rangle_{\partial K} = 0, \tag{20a}$$

$$\begin{aligned}\left(\rho \frac{\partial \boldsymbol{v}_h}{\partial t}, \boldsymbol{w}\right)_K &+ (\mu \mathbf{H}_h + p_h \mathbf{I}, \nabla \boldsymbol{w})_K \\ &- \left\langle (\mu \widehat{\mathbf{H}}_h + \widehat{p}_h \mathbf{I}) \cdot \boldsymbol{n}, \boldsymbol{w} \right\rangle_{\partial K} = (\boldsymbol{b}, \boldsymbol{w})_K,\end{aligned} \tag{20b}$$

$$\left(\epsilon \frac{\partial p_h}{\partial t}, q\right)_K + (\boldsymbol{v}_h, \nabla q)_K - \langle \widehat{\boldsymbol{v}}_h \cdot \boldsymbol{n}, q \rangle_{\partial K} = 0, \tag{20c}$$

where, for any given face $F \in \partial K$,

$$\widehat{\boldsymbol{v}}_h = \begin{cases} \dfrac{\tau^+ \boldsymbol{v}_h^+ + \tau^- \boldsymbol{v}_h^-}{\tau^+ + \tau^-} \\ -\frac{1}{\tau^+ + \tau^-}((\mu \mathbf{H}_h^+ + p_h^+ \mathbf{I}) \cdot \boldsymbol{n}^+ + (\mu \mathbf{H}_h^- + p_h^- \mathbf{I}) \cdot \boldsymbol{n}^-), & \text{if } F \in \mathcal{E}_h^o, \\ \dfrac{\tau}{\tau + \alpha} \boldsymbol{v}_h + \dfrac{1}{\tau + \alpha}(\mathsf{P}\boldsymbol{g} - (\mu \mathbf{H}_h + p_h \mathbf{I}) \cdot \boldsymbol{n}), & \text{if } F \in \partial \Omega, \end{cases} \tag{20d}$$

and

$$(\mu \widehat{\mathbf{H}}_h + \widehat{p}_h \mathbf{I}) \cdot \boldsymbol{n} = (\mu \mathbf{H}_h + p_h \mathbf{I}) \cdot \boldsymbol{n} - \mathbf{S}(\boldsymbol{v}_h - \widehat{\boldsymbol{v}}_h) \quad \text{on } \partial K. \tag{20e}$$

In this formulation, both $\widehat{\boldsymbol{v}}_h$ and $(\mu \widehat{\mathbf{H}}_h + \widehat{p}_h \mathbf{I}) \cdot \boldsymbol{n}$ are explicitly determined from the numerical solution $(\mathbf{H}_h, \boldsymbol{v}_h, p_h)$.

While the first formulation (19) is useful for implicit time-stepping methods, the second formulation (20) is convenient for explicit time-stepping methods. Since the temporal discretization in this case is very similar to that in the scalar wave equation, it will not be discussed here.

3.2 Local Postprocessing

As with the acoustic wave equation, we can define two new approximations which will converge faster than the corresponding original approximations. The postprocessed displacement field $\boldsymbol{u}_h^{n\,*} \in (\mathcal{P}_{k+1}(K))^d$ satisfies, on every simplex $K \in \mathcal{T}_h$,

$$\begin{aligned} (\nabla \boldsymbol{u}_h^{n\,*}, \nabla \boldsymbol{w})_K &= (\mathbf{H}_h^n, \nabla \boldsymbol{w})_K\,, \quad \forall\, \boldsymbol{w} \in (\mathcal{P}_{k+1}(K))^d, \\ (\boldsymbol{u}_h^{n\,*}, 1)_K &= (\boldsymbol{u}_h^n, 1)_K\,. \end{aligned} \tag{21}$$

The postprocessed velocity field $\boldsymbol{v}_h^{n\,*} \in (\mathcal{P}_{k+1}(K))^d$ is obtained by locally solving

$$\begin{aligned} (\nabla \boldsymbol{v}_h^{n\,*}, \nabla \boldsymbol{w})_K &= -(\boldsymbol{v}_h^n, \Delta \boldsymbol{w})_K + \langle \widehat{\boldsymbol{v}}_h^n, \nabla \boldsymbol{w} \cdot \boldsymbol{n} \rangle_{\partial K} \quad \forall\, \boldsymbol{w} \in (\mathcal{P}_{k+1}(K))^d, \\ (\boldsymbol{v}_h^{n\,*}, 1)_K &= (\boldsymbol{v}_h^n, 1)_K\,. \end{aligned} \tag{22}$$

Since the local postprocessing can be carried out at any particular timestep and performed at the element level, the postprocesssed displacement and velocity are very inexpensive to compute. Note that the postprocessing is effective only if the temporal accuracy is of order $k+2$.

3.3 Numerical Results

We consider the elastic wave equations on a unit square $\Omega = (0,1) \times (0,1)$ with $\mu = 1$ and $\rho = 1$. The exact solution is given by

$$\begin{aligned} u_1 &= -x^2 y(2y-1)(x-1)^2(y-1)\sin(\pi t), \\ u_2 &= xy^2(2x-1)(x-1)(y-1)^2\sin(\pi t). \end{aligned}$$

The source term $\boldsymbol{b}$ is determined from the above solution. The Dirichlet boundary data are determined as the restriction of the exact solution on the boundary. Likewise the initial data is taken as an instantiation of the exact solution at time $t = 0$. Our triangular meshes are constructed upon regular $n \times n$ Cartesian grids $(h = 1/n)$. The stabilization parameter is set to $\tau = 1$.

		$\lVert \boldsymbol{u} - \boldsymbol{u}_h \rVert_{\mathcal{T}_h}$		$\lVert \boldsymbol{v} - \boldsymbol{v}_h \rVert_{\mathcal{T}_h}$		$\lVert \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \rVert_{\mathcal{T}_h}$		$\lVert \boldsymbol{u} - \boldsymbol{u}_h^* \rVert_{\mathcal{T}_h}$		$\lVert \boldsymbol{v} - \boldsymbol{v}_h^* \rVert_{\mathcal{T}_h}$	
k	$1/h$	error	order	error	order	error	order	error	order	error	order
1	4	3.79e-4	−−	1.94e-3	−−	2.08e-3	−−	1.74e-4	−−	1.28e-3	−−
	8	1.12e-4	1.76	4.51e-4	2.11	5.07e-4	2.04	2.53e-5	2.78	1.74e-4	2.88
	16	3.04e-5	1.88	1.06e-4	2.09	1.26e-4	2.01	3.27e-6	2.95	2.18e-5	2.99
	32	7.90e-6	1.94	2.60e-5	2.03	3.16e-5	2.00	4.12e-7	2.99	2.96e-6	2.89
	64	2.01e-6	1.97	6.45e-6	2.01	7.93e-6	2.00	5.16e-8	3.00	3.99e-7	2.89
2	4	5.14e-5	−−	2.26e-4	−−	3.27e-4	−−	1.78e-5	−−	2.41e-4	−−
	8	8.01e-6	2.68	2.90e-5	2.96	4.21e-5	2.96	1.20e-6	3.89	7.10e-6	5.08
	16	1.10e-6	2.87	3.67e-6	2.98	5.25e-6	3.00	7.39e-8	4.02	4.53e-7	3.97
	32	1.43e-7	2.94	4.60e-7	3.00	6.54e-7	3.01	4.52e-9	4.03	2.70e-8	4.07
	64	1.82e-8	2.97	5.75e-8	3.00	8.14e-8	3.00	2.78e-10	4.02	1.68e-9	4.01

Table 3: History of convergence results for a compressible material $(\lambda = 1)$.

k	$1/h$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathcal{T}_h}$ error	order	$\|\boldsymbol{v}-\boldsymbol{v}_h\|_{\mathcal{T}_h}$ error	order	$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathcal{T}_h}$ error	order	$\|\boldsymbol{u}-\boldsymbol{u}_h^*\|_{\mathcal{T}_h}$ error	order	$\|\boldsymbol{v}-\boldsymbol{v}_h^*\|_{\mathcal{T}_h}$ error	order
	4	3.75e-4	−−	1.94e-3	−−	2.2e-3	−−	1.72e-4	−−	1.26e-3	−−
	8	1.12e-4	1.75	4.49e-4	2.11	5.41e-4	2.02	2.57e-5	2.74	1.71e-4	2.89
1	16	3.04e-5	1.88	1.06e-4	2.08	1.33e-4	2.02	3.37e-6	2.93	2.13e-5	3.00
	32	7.90e-6	1.94	2.60e-5	2.03	3.33e-5	2.00	4.26e-7	2.98	2.87e-6	2.89
	64	2.01e-6	1.97	6.45e-6	2.01	8.33e-6	2.00	5.34e-8	2.99	3.85e-7	2.90
	4	5.11e-5	−−	2.24e-4	−−	3.67e-4	−−	1.80e-5	−−	2.40e-4	−−
	8	7.98e-6	2.68	2.88e-5	2.96	4.82e-5	2.93	1.22e-6	3.89	6.91e-6	5.12
2	16	1.09e-6	2.87	3.66e-6	2.98	6.12e-6	2.98	7.44e-8	4.03	4.20e-7	4.04
	32	1.43e-7	2.94	4.59e-7	2.99	7.89e-7	2.96	4.52e-9	4.04	2.48e-8	4.08
	64	1.82e-8	2.97	5.75e-8	3.00	9.95e-8	2.99	2.78e-10	4.02	1.48e-9	4.07

Table 4: History of convergence results for a nearly incompressible material ($\lambda = 1000$).

We present the L^2-errors and orders of convergence of the numerical approximations at the final time $T = 0.5$ in Table 3 for $\lambda = 1$ (compressible case) and in Table 4 for $\lambda = 1000$ (nearly incompressible case). These results are obtained using the DIRK(2,3) scheme for $k = 1$ and the DIRK(3,4) scheme for $k = 2$, and a fixed ratio $h/\Delta t = 4$. We observe that the approximate field variables converge with the optimal order $k + 1$ even for the nearly incompressible case. Furthermore, we observe that both the postprocessed displacement and velocity converge with order $k + 2$, which are one order higher than the original approximations. Since the local postprocessing is inexpensive, the HDG methods provide better convergence and accuracy than other DG methods.

These convergence properties were first reported in [37]. For the semidiscrete case, they can be obtained by an analysis similar to that for the acoustic wave equation in [18]. Again, a priori error estimates for $\boldsymbol{v} - \boldsymbol{v}_h^*$ remain an open problem.

4 The Electromagnetic Wave Equations

In this section, we restrict our attention to devising HDG methods for the Maxwell's equations in frequency domain. Numerical treatment of the Maxwell's equations in time domain follows from the discussion in this section and the second section.

Let us consider the time-harmonic Maxwell's equations in a connected and bounded domain $\Omega \in \mathbb{R}^3$ with zero charge density and zero conductivity:

$$\nabla \times \boldsymbol{E} + i\mu\omega \boldsymbol{H} = 0, \qquad \nabla \times \boldsymbol{H} - i\epsilon\omega \boldsymbol{E} = \boldsymbol{J}, \qquad \text{in } \Omega \subset \mathbb{R}^3, \tag{23}$$

where $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{J}$ are the electric field, magnetic field, and current source, respectively. In addition, ω is the frequency, ϵ the permittivity, and μ the permeability of the medium. We assume that the electromagnetic field satisfies the following impedance condition

$$-\boldsymbol{n} \times \boldsymbol{H} + \alpha \boldsymbol{n} \times \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{g}, \quad \text{on } \partial\Omega, \tag{24}$$

for some given scalar function α and vectorial function $\boldsymbol{g}$.

4.1 Numerical discretization

To define the numerical approximation of the Maxwell's equations (23), we introduce the following approximation spaces

$$\begin{aligned} \boldsymbol{V}_h &:= \{\boldsymbol{v} \in [L^2(\mathcal{T}_h)]^3 \;:\; \boldsymbol{v}|_K \in [\mathcal{C}_k(K)]^3, \;\forall\, K \in \mathcal{T}_h\}, \\ \boldsymbol{M}_h^t &:= \{\boldsymbol{\eta} \in [L^2(\mathcal{E}_h)]^3 \;:\; \boldsymbol{\eta}|_F \in [\mathcal{C}_k(F)]^3, \;(\boldsymbol{\eta}\cdot\boldsymbol{n})|_F = 0, \;\forall\, F \in \mathcal{E}_h\}. \end{aligned} \tag{25}$$

Heret $\mathcal{C}_k(D)$ denote the space of *complex-valued* polynomials of degree at most k on D. We then define the inner products for our approximation spaces as

$$\begin{aligned} (w,v)_{\mathcal{T}_h} &:= \sum_{K\in\mathcal{T}_h}\int_K w\bar{v}, & (\boldsymbol{w},\boldsymbol{v})_{\mathcal{T}_h} &:= \sum_{j=1}^{3}(w_j,v_j)_{\mathcal{T}_h}, \\ \langle w,v\rangle_{\partial\mathcal{T}_h} &:= \sum_{K\in\mathcal{T}_h}\int_{\partial K} w\bar{v}, & \langle \boldsymbol{w},\boldsymbol{v}\rangle_{\partial\mathcal{T}_h} &:= \sum_{j=1}^{3}\langle w_j,v_j\rangle_{\partial\mathcal{T}_h}. \end{aligned} \tag{26}$$

Here the bar denotes a complex conjugate which is applied only to the second argument of the inner products.

Note that $\boldsymbol{M}_h^t$ consists of vector-valued functions whose normal component is zero on any face $F \in \mathcal{E}_h$. Hence, a member of $\boldsymbol{M}_h^t$ can be characterized by two tangential vectors on the faces: if $\boldsymbol{t}_1^F$ and $\boldsymbol{t}_2^F$ denote independent tangent vectors on F, we can write the restriction of $\boldsymbol{\eta} \in \boldsymbol{M}_h^t$ on F as

$$\boldsymbol{\eta}|_F = \eta_1^F \boldsymbol{t}_1^F + \eta_2^F \boldsymbol{t}_2^F, \tag{27}$$

where $\eta_1^F \in \mathcal{C}_k(F)$ and $\eta_2^F \in \mathcal{C}_k(F)$ are complex-valued polynomials of degree at most k on F. Hence, the vector-valued function $\boldsymbol{\eta} \in \boldsymbol{M}_h^t$ is characterized by two scalar functions η_1 and η_2.

The HDG method seeks $(\boldsymbol{E}_h, \boldsymbol{H}_h, \widehat{\boldsymbol{E}}_h^t) \in \boldsymbol{V}_h \times \boldsymbol{V}_h \times \boldsymbol{M}_h^t$ such that

$$(i\omega\mu\boldsymbol{H}_h,\boldsymbol{R})_{\mathcal{T}_h} + (\boldsymbol{E}_h,\nabla\times\,\boldsymbol{R})_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{E}}_h^t, \boldsymbol{R}\times\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} = 0, \tag{28a}$$

$$(\boldsymbol{H}_h,\nabla\times\boldsymbol{W})_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{H}}_h, \boldsymbol{W}\times\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} - (i\epsilon\omega\boldsymbol{E}_h,\boldsymbol{W})_{\mathcal{T}_h} = (\boldsymbol{J},\boldsymbol{W})_{\mathcal{T}_h}, \tag{28b}$$

$$-\left\langle \boldsymbol{n}\times\widehat{\boldsymbol{H}}_h, \boldsymbol{\eta}\right\rangle_{\partial\mathcal{T}_h} + \left\langle \alpha\widehat{\boldsymbol{E}}_h^t, \boldsymbol{\eta}\right\rangle_{\partial\Omega} = \langle \boldsymbol{g},\boldsymbol{\eta}\rangle_{\partial\Omega}, \tag{28c}$$

for all $(\boldsymbol{R}, \boldsymbol{W}, \boldsymbol{\eta}) \in \boldsymbol{V}_h \times \boldsymbol{V}_h \times \boldsymbol{M}_h^t$, where

$$\widehat{\boldsymbol{H}}_h = \boldsymbol{H}_h + \tau(\boldsymbol{E}_h - \widehat{\boldsymbol{E}}_h^t)\times\boldsymbol{n}. \tag{28d}$$

Here the stabilization parameter is chosen as $\tau = \sqrt{\frac{\epsilon\omega^2}{\mu}}$. This completes the HDG method for solving the time-harmonic Maxwell's equations.

The structure of the HDG method makes itself amenable to an efficient implementation. Note that the first two equations in (28) can be thought as characterizing $(\boldsymbol{E}_h, \boldsymbol{H}_h)$ in terms of $\widehat{\boldsymbol{E}}_h$. The equation (28c) is then the equation that determines the actual values of the unknown $\widehat{\boldsymbol{E}}_h$. In this manner, the only globally-coupled degrees of freedom are those of $\widehat{\boldsymbol{E}}_h$. As a result, the HDG method can provide more accurate solutions at much lower computational cost than standard frequency-domain DG method.

4.2 Local postprocessing

We propose a new local postprocessing to obtain new approximations of the electric and magnetic fields, which converges with an additional order in the $\mathcal{H}^{\text{curl}}(\mathcal{T}_h)$-norm. A remarkable feature of this new local postprocessing is that it is effective even in three dimensions, whereas the local postprocessing introduced in our previous work [39] is only applicable in two dimensions.

We find the new approximate electric field $\boldsymbol{E}_h^*$ as the element of $[\mathcal{C}_{k+1}(K)]^3$ such that for all $K \in \mathcal{T}_h$,

$$(\nabla \times \boldsymbol{E}_h^*, \boldsymbol{W})_K = -(i\mu\omega \boldsymbol{H}_h, \boldsymbol{W})_K, \quad \forall \boldsymbol{W} \in \nabla \times [\mathcal{C}_{k+1}(K)]^3, \tag{29a}$$

$$(\boldsymbol{E}_h^*, \boldsymbol{Y})_K = (\boldsymbol{E}_h, \boldsymbol{Y})_K, \quad \forall \boldsymbol{Y} \in \nabla \mathcal{C}_{k+2}(K). \tag{29b}$$

Similarly, we find the new approximate magnetic field $\boldsymbol{H}_h^*$ as the element of $[\mathcal{C}_{k+1}(K)]^3$ such that for all $K \in \mathcal{T}_h$,

$$(\nabla \times \boldsymbol{H}_h^*, \boldsymbol{W})_K = (i\epsilon\omega \boldsymbol{E}_h + \boldsymbol{J}, \boldsymbol{W})_K, \quad \forall \boldsymbol{W} \in \nabla \times [\mathcal{C}_{k+1}(K)]^3, \tag{30a}$$

$$(\boldsymbol{H}_h^*, \boldsymbol{Y})_K = (\boldsymbol{H}_h, \boldsymbol{Y})_K, \quad \forall \boldsymbol{Y} \in \nabla \mathcal{C}_{k+2}(K). \tag{30b}$$

It is obvious that $\nabla \times \boldsymbol{E}_h^*$ and $\nabla \times \boldsymbol{H}_h^*$ are nothing but the projection of $i\mu\omega \boldsymbol{H}_h$ and $i\epsilon\omega \boldsymbol{E}_h + \boldsymbol{J}$, respectively, onto the space of divergence-free functions in $[\mathcal{P}_{k+1}(K)]^3$. Therefore, we expect that both $\boldsymbol{E}_h^*$ and $\boldsymbol{H}_h^*$ converge with order $k+1$ in the $\mathcal{H}^{\text{curl}}(\mathcal{T}_h)$-norm, whereas $\boldsymbol{E}_h$ and $\boldsymbol{H}_h$ converge with order k in the $\mathcal{H}^{\text{curl}}(\mathcal{T}_h)$-norm.

4.3 Numerical results

We consider the time-harmonic Maxwell's equations on a unit cube $\Omega = (0,1)^3$ with $\mu = 1, \epsilon = 2, \alpha = 0$, and $\omega = 1$. For $\boldsymbol{J} = 0$ the problem has the exact solution

$$\begin{aligned}
E_x &= \sin(\omega y)\sin(\omega z), & H_x &= i\sin(\omega x)(\cos(\omega y) - \cos(\omega z)),\\
E_y &= \sin(\omega x)\sin(\omega z), & H_y &= i\sin(\omega y)(\cos(\omega z) - \cos(\omega x)),\\
E_z &= \sin(\omega y)\sin(\omega x), & H_z &= i\sin(\omega z)(\cos(\omega x) - \cos(\omega y)),
\end{aligned}$$

The boundary data $\boldsymbol{g}$ is determined from the exact solution. The tetrahedral meshes are constructed upon regular $n \times n \times n$ Cartesian grids ($h = 1/n$) by splitting each cube into 6 tetrahedral.

We present the L^2-errors and orders of convergence of the numerical approximations in Table 5 and the postprocessed quantities in Table 6. We observe that the approximate electric and magnetic fields converge with order $k+1$ in the

L^2-norm, but only order k in the $\mathcal{H}^{\mathrm{curl}}(\mathcal{T}_h)$-norm. Furthermore, we observe that the postprocessed electric and magnetic fields converge with order $k+1$ in the $\mathcal{H}^{\mathrm{curl}}(\mathcal{T}_h)$-norm, which are one order higher than the original approximations. The theoretical justification of these results is still an open problem.

		$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\mathcal{T}_h}$		$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\mathcal{H}^{\mathrm{curl}}(\mathcal{T}_h)}$		$\|\boldsymbol{H}-\boldsymbol{H}_h\|_{\mathcal{T}_h}$		$\|\boldsymbol{H}-\boldsymbol{H}_h\|_{\mathcal{H}^{\mathrm{curl}}(\mathcal{T}_h)}$	
k	$1/h$	error	order	error	order	error	order	error	order
1	2	2.94e-2	$--$	9.90e-2	$--$	8.41e-3	$--$	2.20e-1	$--$
	4	7.77e-3	1.92	4.46e-2	1.15	2.18e-3	1.95	1.10e-1	1.00
	6	1.94e-3	2.00	2.14e-2	1.06	5.85e-4	1.90	5.52e-2	1.00
	8	4.81e-4	2.01	1.05e-2	1.02	1.54e-4	1.93	2.76e-2	1.00
2	2	9.49e-4	$--$	1.32e-2	$--$	6.56e-4	$--$	3.28e-2	$--$
	4	1.33e-4	2.84	3.37e-3	1.97	8.74e-5	2.91	8.15e-3	2.01
	6	1.90e-5	2.81	8.47e-4	1.99	1.12e-5	2.96	2.03e-3	2.00
	8	2.87e-6	2.73	2.12e-4	2.00	1.42e-6	2.98	5.09e-4	2.00
3	2	8.72e-5	$--$	1.40e-3	$--$	5.51e-5	$--$	1.74e-3	$--$
	4	5.59e-6	3.96	1.73e-4	3.02	3.51e-6	3.97	2.28e-4	2.93
	6	3.53e-7	3.99	2.15e-5	3.01	2.23e-7	3.98	2.92e-5	2.97
	8	2.22e-8	3.99	2.67e-6	3.00	1.41e-8	3.99	3.69e-6	2.98

Table 5: History of convergence results for the approximate solution.

		$\|\boldsymbol{E}-\boldsymbol{E}_h^*\|_{\mathcal{T}_h}$		$\|\boldsymbol{E}-\boldsymbol{E}_h^*\|_{\mathcal{H}^{\mathrm{curl}}(\mathcal{T}_h)}$		$\|\boldsymbol{H}-\boldsymbol{H}_h^*\|_{\mathcal{T}_h}$		$\|\boldsymbol{H}-\boldsymbol{H}_h^*\|_{\mathcal{H}^{\mathrm{curl}}(\mathcal{T}_h)}$	
k	$1/h$	error	order	error	order	error	order	error	order
1	2	3.19e-2	$--$	3.44e-2	$--$	1.05e-2	$--$	6.26e-2	$--$
	4	8.42e-3	1.92	9.05e-3	1.93	2.69e-3	1.97	1.67e-2	1.90
	6	2.10e-3	2.00	2.27e-3	1.99	7.05e-4	1.93	4.21e-3	1.99
	8	5.23e-4	2.01	5.68e-4	2.00	1.83e-4	1.95	1.05e-3	2.00
2	2	9.56e-4	$--$	1.58e-3	$--$	8.34e-4	$--$	2.06e-3	$--$
	4	1.34e-4	2.84	2.07e-4	2.93	1.08e-4	2.95	2.82e-4	2.87
	6	1.91e-5	2.81	2.76e-5	2.91	1.38e-5	2.97	3.85e-5	2.87
	8	2.88e-6	2.73	3.81e-6	2.86	1.74e-6	2.99	5.46e-6	2.82
3	2	8.36e-5	$--$	1.03e-4	$--$	4.88e-5	$--$	1.75e-4	$--$
	4	5.43e-6	3.95	6.71e-6	3.95	3.20e-6	3.93	1.13e-5	3.94
	6	3.44e-7	3.98	4.26e-7	3.98	2.05e-7	3.97	7.20e-7	3.98
	8	2.17e-8	3.99	2.69e-8	3.99	1.29e-8	3.98	4.54e-8	3.99

Table 6: History of convergence results for the postprocessed solution.

5 Bibliographic notes

Time-dependent wave propagation

The devising of HDG methods for the acoustic wave equation was carried out as an almost immediate consequence of the introduction of HDG methods for steady-state diffusion. After all, both equations share the same second-order strongly elliptic operator. However, not all convergence properties which hold for HDG methods for steady-state diffusion problems [9, 16, 15, 5, 6, 17, 19] can be immediately obtained for time-dependent wave equations. In particular, the wave equation does not have a smoothing effect which could generate superconvergence results, as happens for the heat equation, see [3]. However, in [18], it was shown how to obtain the superconvergence results we have illustrated in Section 2; a comparison with other mixed and DG methods can also be found there. Although therein we only used simplexes and spaces of polynomials of degree k, similar convergence and superconvergence results do hold for meshes made of elements of arbitrary shape. This can be obtained by using the so-called theory of M-decompositions developed in [12, 10, 11]. In a similar way, HDG methods for the elastic wave equation can be easily obtained once HDG methods for linear elasticity [45, 32, 20, 26] are obtained.

The first HDG methods for wave propagation were proposed in [37], where implicit time-marching methods were used, and in [46], where explicit time-marching methods were used. In both papers, the superconvergence properties of the semidiscrete method uncovered in [18] were shown to hold for the corresponding implicit and explicit time-marching schemes, respectively.

The HDG methods we have presented here can be also used with PML absorbing boundary conditions, as shown in [37]. HDG methods which are not dissipative, and have similar superconvergence properties, have been developed in [13].

Time-harmonic wave propagation

HDG methods for time-harmonic hyperbolic equations are also strongly related to the HDG methods for steady-state diffusion problems. The first HDG method for the Helmholtz equation was introduced in [28]. The same optimal convergence and superconvergence properties of the associated steady-state diffusion were proven. In [25], a wide family of discontinuous Galerkin methods, which included the HDG methods, were proven to be stable regardless of the wave number. The methods used piecewise linear approximations. In [21], an analysis of the HDG methods for the Helmholtz equations was carried which shows that the method is stable for any wave number, mesh and polynomial degree and which recovers the orders of convergence and superconvergence obtained previously in [28]. A method for arbitrarily large wave numbers is proposed in [42].

The first HDG for the time-harmonic Maxwell's equations was proposed in [39] in two-space dimensions. The extension of the method to three-dimensions was done in [30]. A variation was introduced in [31]. HDG method for the time-harmonic equations of elastodynamics can be found in [29].

Further reading material

A systematic way of defining HDG methods for Friedrichs' systems has been developed in [2]. A general construction of DG methods for these problems can be found in [22, 23, 24]. An overview of the development of DG (and in particular, HDG) methods for fluid dynamics can be found in [7]. An overview of the HDG methods for steady-state diffusion can be found in [8]. Therein, the relation between static condensation, hybridization and the devising of HDG methods is carefully explored.

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