Devising **HDG** methods for Stokes flow: An overview

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Abstract

We provide a short overview of our recent work on the devising of hybridizable discontinuous Galerkin (**HDG**) methods for the Stokes equations of incompressible flow. First, we motivate and display the general form of the methods and show that they provide a well defined approximate solution for arbitrary polyhedral elements. We then discuss three different but equivalent formulations of the methods. Next, we describe a systematic way of constructing superconvergent **HDG** methods by using, as building blocks, the local spaces of superconvergent **HDG** methods for the Laplacian operator. This can be done, so far, for simplexes, parallelepipeds and prisms. Finally, we show how, by means of an elementwise computation, we can obtain divergence-free velocity approximations converging faster than the original velocity approximation when working with simplicial elements. We end by briefly discussing other versions of the methods, how to obtain **HDG** methods with $\mathbf{H}(\mathrm{div})$-conforming velocity spaces, and how to extend the methods to other related systems. Several open problems are described.

Keywords: hybridizable discontinuous Galerkin methods, Stokes equations, unstructured meshes, superconvergence, divergence-free approximations

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1. Introduction

In this paper, we give a short overview of our recent work on the devising of hybridizable discontinuous Galerkin (**HDG**) methods for the velocity gradient-velocity-pressure formulation of the Stokes equations, namely,

$$\mathrm{L} - \nabla \boldsymbol{u} = 0 \qquad \text{on } \Omega, \tag{1a}$$

$$-\nabla \cdot (\nu \mathrm{L}) + \nabla p = \boldsymbol{f} \qquad \text{on } \Omega, \tag{1b}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{on } \Omega, \tag{1c}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \partial\Omega, \tag{1d}$$

$$\int_\Omega p = 0, \tag{1e}$$

where $\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} = 0$. Here $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ is a bounded polygonal domain if $n = 2$, and a Lipschitz polyhedral domain if $n = 3$. We assume that ν is a constant and that f is smooth.

The paper is organized as follows. In Section 2, we begin by describing the characterization of the exact solution the **HDG** methods are obtained from. In Section 3, we use this characterization to define the methods and display very simple conditions that, for elements of arbitrary shape, ensure the existence and uniqueness of their solution. Next, in Section 4, we provide three different ways to presenting the methods according to which unknowns are considered independent and which ones dependent. Then, in Section 5, a fairly general construction of *superconvergent* methods in terms of superconvergent methods for the Laplace operator is presented. These are methods for which, roughly speaking, an elementwise post-processing of the velocity can be obtained which converges faster than the original approximation. Finally, in Section 6, restricting ourselves to simplicial elements, we show how the above-mentioned postprocessing can be defined which results in a globally divergence-free approximate velocity converging faster that the original approximation. We end in Section 7 by briefly considering other versions of the methods, by discussing a new way of obtaining method using $\mathbf{H}(div)$-conforming velocity spaces, and by commenting on how to extend the methods to other related systems.

2. The main idea for devising HDG methods

In this Section, we introduce a characterization of the exact solution whose discrete version gives rise to the **HDG** methods. Given any mesh

$\mathcal{T}_h$, which, for simplicity we take to be conforming, of the domain Ω, the characterization we seek states, roughly speaking, that the exact solution solves local Stokes problems which are suitably matched across inter-element boundaries. To find it, we begin with a simple observation.

2.1. A simple observation

Note that the exact solution satisfies the partial differential equations

$$\begin{aligned} \mathrm{L} - \nabla \boldsymbol{u} &= 0 \qquad \text{on } K, \\ -\nabla \cdot (\nu \mathrm{L}) + \nabla p &= \boldsymbol{f} \qquad \text{on } K, \\ \nabla \cdot \boldsymbol{u} &= 0 \qquad \text{on } K, \end{aligned}$$

on each of the elements K of the mesh $\mathcal{T}_h$. Moreover, it satisfies the transmission conditions

$$\begin{aligned} [\![-\nu \mathrm{L}\boldsymbol{n} + p\,\boldsymbol{n}]\!] &= 0 \quad \text{on } F, \\ [\![\boldsymbol{u} \otimes \boldsymbol{n}]\!] &= 0 \quad \text{on } F, \end{aligned}$$

for all the faces F of each of the elements $K \in \mathcal{T}_h$. Here, $[\![\cdot]\!]$ on F denotes the jump across the inter-element boundary F, that is

$$\begin{aligned} [\![-\nu \mathrm{L}\boldsymbol{n} + p\,\boldsymbol{n}]\!] &:= -\nu \mathrm{L}^- \boldsymbol{n}^- + p^-\,\boldsymbol{n}^- - \nu \mathrm{L}^+ \boldsymbol{n}^+ + p^+\,\boldsymbol{n}^+, \\ [\![\boldsymbol{u} \otimes \boldsymbol{n}]\!] &:= \boldsymbol{u}^- \otimes \boldsymbol{n}^- + \boldsymbol{u}^+ \otimes \boldsymbol{n}^+, \end{aligned}$$

where $\zeta^{\pm}$ is the trace on the face F of the generic function ζ from either of its sides. Finally, it satisfies Dirichlet boundary and global average conditions

$$\boldsymbol{u} = \boldsymbol{g} \text{ on } \partial\Omega, \qquad \int_{\Omega} p = 0.$$

Conversely, any function $(\mathrm{L}, \boldsymbol{u}, p)$ satisfying the above equations on each of the elements $K \in \mathcal{T}_h$, the transmission conditions on all the faces F of $K \in \mathcal{T}_h$ and the Dirichlet and global average conditions is nothing but the exact solution of the original problem.

2.2. Local and global problems

We are going now to use this simple result to obtain the characterization we seek. We proceed as follows. For an *arbitrary* function $\widehat{\boldsymbol{u}}$ defined on the set of all faces F of the elements K of $\mathcal{T}_h$, $\mathcal{E}_h$, and *any* function $\overline{p}$ defined on

Ω and constant on each element K of $\mathcal{T}_h$, we define the auxiliary function $(\widetilde{\mathrm{L}}, \widetilde{\boldsymbol{u}}, \widetilde{p})$ as the solution of the *local* problem

$$\widetilde{\mathrm{L}} - \nabla \widetilde{\boldsymbol{u}} = 0 \qquad \text{on } K, \tag{2a}$$

$$-\nabla \cdot (\nu \widetilde{\mathrm{L}}) + \nabla \widetilde{p} = \boldsymbol{f} \qquad \text{on } K, \tag{2b}$$

$$\nabla \cdot \widetilde{\boldsymbol{u}} = \frac{1}{|K|} \int_{\partial K} \widehat{\boldsymbol{u}} \cdot \boldsymbol{n} \quad \text{on } K, \tag{2c}$$

$$\widetilde{\boldsymbol{u}} = \widehat{\boldsymbol{u}} \qquad \text{on } \partial K, \tag{2d}$$

$$\frac{1}{K} \int_K \widetilde{p} = \overline{p}. \tag{2e}$$

Note that the divergence-free condition *has* to be modified for this problem to be solvable for arbitrary functions $\widehat{\boldsymbol{u}}$. If we want to keep the equation $\nabla \cdot \widetilde{\boldsymbol{u}} = 0$, the function $\widehat{\boldsymbol{u}}$ would have to satisfy the compatibility condition $\int_{\partial K} \widehat{\boldsymbol{u}} \cdot \widehat{\boldsymbol{n}} = 0$.

By the result in the previous subsection, the function $(\widehat{\boldsymbol{u}}, \overline{p})$ for which $(\widetilde{\mathrm{L}}, \widetilde{\boldsymbol{u}}, \widetilde{p})$ is nothing but the exact solution of the original problem, $(\mathrm{L}, \boldsymbol{u}, p)$, must be the solution of the *global* problem consisting in the transmission condition

$$[\![-\nu \widetilde{\mathrm{L}} \boldsymbol{n} + \widetilde{p}\, \boldsymbol{n}]\!] = 0 \quad \text{on } \mathcal{E}_h \setminus \partial\Omega, \tag{3a}$$

the divergence-free condition

$$\int_{\partial K} \widehat{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \quad \text{for } K \in \mathcal{T}_h, \tag{3b}$$

and the Dirichlet and global average conditions

$$\widehat{\boldsymbol{u}} = \boldsymbol{g} \quad \text{on } \partial\Omega, \tag{3c}$$

$$\int_\Omega \overline{p} = 0. \tag{3d}$$

Note that the second transmission condition, namely $[\![\widetilde{\boldsymbol{u}} \otimes \boldsymbol{n}]\!] = 0$ is automatically satisfied since on $\mathcal{E}_h \setminus \partial\Omega$ because $\widetilde{\boldsymbol{u}} = \widehat{\boldsymbol{u}}$ therein by the boundary condition of the local problems, (2d).

2.3. Characterization of the exact solution

Thus, we have that the exact solution can be characterized as the sum

$$(\mathrm{L}, \boldsymbol{u}, p) = (\mathrm{L}^{\widehat{\boldsymbol{u}}}, \boldsymbol{u}^{\widehat{\boldsymbol{u}}}, p^{\widehat{\boldsymbol{u}}}) + (\mathrm{L}^{\boldsymbol{f}}, \boldsymbol{u}^{\boldsymbol{f}}, p^{\boldsymbol{f}}) + (0, \mathbf{0}, \overline{p}),$$

where we denote by $(\mathrm{L}^{\widehat{\boldsymbol{u}}}, \boldsymbol{u}^{\widehat{\boldsymbol{u}}}, p^{\widehat{\boldsymbol{u}}})$ the solution $(\widetilde{\mathrm{L}}, \widetilde{\boldsymbol{u}}, \widetilde{p})$ of the local problem (2) with $\boldsymbol{f} := \mathbf{0}$ and $\overline{p} := 0$, and by $(\mathrm{L}^{\boldsymbol{f}}, \boldsymbol{u}^{\boldsymbol{f}}, p^{\boldsymbol{f}})$ the solution $(\widetilde{\mathrm{L}}, \widetilde{\boldsymbol{u}}, \widetilde{p})$ of the local problem (2) with $\widehat{\boldsymbol{u}} := \mathbf{0}$ and $\overline{p} := 0$. Note that the solution $(\widetilde{\mathrm{L}}, \widetilde{\boldsymbol{u}}, \widetilde{p})$ of the local problem (2) with $\widehat{\boldsymbol{u}} := \mathbf{0}$ and $\boldsymbol{f} := \mathbf{0}$ is $(0, \mathbf{0}, \overline{p})$.

Moreover, the function $(\widehat{\boldsymbol{u}}, \overline{p})$ is the solution of the global problem (3) which, given the last identity, we can rewrite as follows:

$$\begin{aligned}
-[\![-\nu \mathrm{L}^{\widehat{\boldsymbol{u}}}\boldsymbol{n} + p^{\widehat{\boldsymbol{u}}}\,\boldsymbol{n}]\!] - [\![\overline{p}\,\boldsymbol{n}]\!] &= [\![-\nu \mathrm{L}^{\boldsymbol{f}}\boldsymbol{n} + p^{\boldsymbol{f}}\,\boldsymbol{n}]\!] && \text{on } \mathcal{E}_h,\\
\int_{\partial K} \widehat{\boldsymbol{u}} \cdot \boldsymbol{n} &= 0 && \text{for } K \in \mathcal{T}_h,\\
\widehat{\boldsymbol{u}} &= \boldsymbol{g} && \text{on } \partial\Omega,\\
\int_{\Omega} \overline{p} &= 0.
\end{aligned}$$

This characterization of the exact solution is convenient for devising numerical methods because any discrete version of it will consist of local problems written in terms of approximations to $(\widehat{\boldsymbol{u}}, \overline{p})$, $(\widehat{\boldsymbol{u}}_h, \overline{p}_h)$, and a single global problem for $(\widehat{\boldsymbol{u}}_h, \overline{p}_h)$ *only*. This allows for a very efficient implementation of the method.

3. Definition of the HDG methods

In this Section, we introduce **HDG** methods by discretizing the local problems (2) by discontinuous Galerkin methods, and by enforcing the global problem (3) in a weak manner.

3.1. The approximating spaces

The **HDG** methods seek an approximation $(\mathrm{L}_h, \boldsymbol{u}_h, p_h)$ to the exact solution $(\mathrm{L}|_\Omega, \boldsymbol{u}|_\Omega, p|_\Omega)$ in the finite dimensional space $\mathrm{G}_h \times \boldsymbol{V}_h \times Q_h$ given by

$$\mathrm{G}_h = \{\mathrm{G} \in \mathrm{L}^2(\mathcal{T}_h) : \quad \mathrm{G}|_K \in \mathrm{G}(K) \quad \forall\, K \in \mathcal{T}_h\}, \tag{4a}$$

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{L}^2(\mathcal{T}_h) : \quad \boldsymbol{v}|_K \in \boldsymbol{V}(K) \quad \forall\, K \in \mathcal{T}_h\}, \tag{4b}$$

$$Q_h = \{q \in L^2(\mathcal{T}_h) : \quad q|_K \in Q(K) \quad \forall\, K \in \mathcal{T}_h\}, \tag{4c}$$

where the *local* spaces $\mathrm{G}(K)$, $\boldsymbol{V}(K)$, $Q(K)$ are general finite dimensional spaces.

The **HDG** methods also seek an approximation $(\widehat{\boldsymbol{u}}_h, \overline{p}_h)$ to the exact solution $(\boldsymbol{u}|_{\mathcal{E}_h}, \overline{p})$ in the space $\boldsymbol{M}_h \times Q_h^0$ where

$$Q_h^0 = \{q \in L^2(\mathcal{T}_h) : \quad q|_K \text{ is a constant} \quad \forall\, K \in \mathcal{T}_h\}, \tag{5a}$$
$$\boldsymbol{M}_h = \{\boldsymbol{\mu} \in \boldsymbol{L}^2(\mathcal{E}_h) : \quad \boldsymbol{\mu}|_F \in \boldsymbol{M}(F) \qquad \forall\, F \in \mathcal{E}_h\}, \tag{5b}$$

where the *local* space $\boldsymbol{M}(F)$ is a general finite dimensional space.

3.2. The local and the global problems

Writing $(\zeta, \eta)_K$ for the integral over the element K of $\zeta\eta$, and $\langle \zeta, \eta\rangle_{\partial K}$ for the corresponding integral over ∂K, it is not difficult to see that the exact solution of the local problem (2) satisfies the equations

$$\begin{aligned}
(\widetilde{\mathrm{L}}, \mathrm{G})_K + (\widetilde{\boldsymbol{u}}, \nabla\cdot\mathrm{G})_K - \langle \widehat{\boldsymbol{u}}, \mathrm{G}\boldsymbol{n}\rangle_{\partial K} &= 0,\\
\nu\,(\widetilde{\mathrm{L}}, \nabla\boldsymbol{v})_K - (\widetilde{p}, \nabla\cdot\boldsymbol{v})_K - \langle \nu\widetilde{\mathrm{L}}\boldsymbol{n} - \widetilde{p}\boldsymbol{n}, \boldsymbol{v}\rangle_{\partial K} &= (\boldsymbol{f}, \boldsymbol{v})_K,\\
-(\widetilde{\boldsymbol{u}}, \nabla q)_K + \langle \widehat{\boldsymbol{u}}\cdot\boldsymbol{n}, q\rangle_{\partial K} &= \frac{1}{|K|}\langle \widehat{\boldsymbol{u}}\cdot\boldsymbol{n}, 1\rangle_{\partial K}(1, q)_K\\
&= \langle \widehat{\boldsymbol{u}}\cdot\boldsymbol{n}, \overline{q}\rangle_{\partial K},\\
(\widetilde{p}, 1)_K &= (\overline{p}, 1)_K,
\end{aligned}$$

for all $(\mathrm{G}, \boldsymbol{v}, q) \in \mathrm{G}(K) \times \boldsymbol{V}(K) \times Q(K)$.

For this reason, for any given arbitrary function $(\widehat{\boldsymbol{u}}_h, \overline{p}_h)$ in the space $\boldsymbol{M}_h \times Q_h^0$, we define $(\mathrm{L}_h, \boldsymbol{u}_h, p_h)$ as the solution of the discrete local problem

$$(\mathrm{L}_h, \mathrm{G})_K + (\boldsymbol{u}_h, \nabla\cdot\mathrm{G})_K - \langle \widehat{\boldsymbol{u}}_h, \mathrm{G}\boldsymbol{n}\rangle_{\partial K} = 0, \tag{6a}$$
$$\nu\,(\mathrm{L}_h, \nabla\boldsymbol{v})_K - (p_h, \nabla\cdot\boldsymbol{v})_K - \langle \nu\widehat{\mathrm{L}}_h\boldsymbol{n} - \widehat{p}_h\boldsymbol{n}, \boldsymbol{v}\rangle_{\partial K} = (\boldsymbol{f}, \boldsymbol{v})_K, \tag{6b}$$
$$-(\boldsymbol{u}_h, \nabla q)_K + \langle \widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n}, q - \overline{q}\rangle_{\partial K} = 0, \tag{6c}$$
$$(p_h, 1)_K = (\overline{p}_h, 1)_K, \tag{6d}$$

for all $(\mathrm{G}, \boldsymbol{v}, q) \in \mathrm{G}(K) \times \boldsymbol{V}(K) \times Q(K)$, where

$$-\nu\widehat{\mathrm{L}}_h\boldsymbol{n} + \widehat{p}_h\boldsymbol{n} := -\nu\mathrm{L}_h\boldsymbol{n} + p_h\boldsymbol{n} + \mathrm{S}(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) \quad \text{on } \partial K. \tag{6e}$$

The following result gives sufficient conditions to the solution of this local problem to be well defined.

Theorem 1 ([1]). *The solution of the local problem* (6) *exists and is unique under the following sufficient conditions:*

(i) *The stabilization function* S *is uniformly positive definite.*

(ii) *The space* $\nabla \boldsymbol{V}(K)$ *is included in the space* $\mathrm{G}(K)$ *for all* $K \in \mathcal{T}_h$*.*

(iii) *The space* $\nabla Q(K)$ *is included in the space* $\boldsymbol{V}(K)$ *for all* $K \in \mathcal{T}_h$*.*

The matrix-valued function S has a crucial role ensuring the existence and uniqueness of the solution of the local problem. Indeed, thanks to the property of positive definiteness (i), the existence and uniqueness is ensured under the simple inclusion properties of the local spaces (ii) and (iii), *independent* of the shape of the elements $K \in \mathcal{T}_h$.

The definition of the global problem defining $(\widehat{\boldsymbol{u}}_h, \overline{p}_h)$ is more straightforward. Indeed, setting

$$\langle \cdot, \cdot \rangle_{\partial \mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle \cdot, \cdot \rangle_{\partial K}, \qquad \text{and} \qquad \langle \cdot, \cdot \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} := \sum_{K \in \mathcal{T}_h} \langle \cdot, \cdot \rangle_{\partial K \backslash \partial \Omega},$$

where $\partial \mathcal{T}_h := \{\partial K : \ K \in \mathcal{T}_h\}$, we have that the exact solution of the global problem satisfies

$$\begin{aligned}
\langle -\nu \widetilde{\mathrm{L}} \boldsymbol{n} + \widetilde{p}\, \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} &= 0, \\
\langle \widehat{\boldsymbol{u}} \cdot \boldsymbol{n}, \overline{q} \rangle_{\partial \mathcal{T}_h} &= 0, \\
\langle \widehat{\boldsymbol{u}}, \boldsymbol{\mu} \rangle_{\partial \Omega} &= \langle \boldsymbol{g}, \boldsymbol{\mu} \rangle_{\partial \Omega}, \\
(\overline{p}, 1)_{\Omega} &= 0,
\end{aligned}$$

for any $(\boldsymbol{\mu}, \overline{q})$ in $\boldsymbol{M}_h \times Q_h^0$.

As a consequence, it is very reasonable to take $(\widehat{\boldsymbol{u}}_h, \overline{p}_h)$ as the element in the space $\boldsymbol{M}_h \times Q_h^0$ satisfying the equations

$$\langle -\nu \widehat{\mathrm{L}}_h \boldsymbol{n} + \widehat{p}_h\, \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} = 0, \tag{7a}$$

$$\langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, \overline{q} \rangle_{\partial \mathcal{T}_h} = 0, \tag{7b}$$

$$\langle \widehat{\boldsymbol{u}}_h, \boldsymbol{\mu} \rangle_{\partial \Omega} = \langle \boldsymbol{g}, \boldsymbol{\mu} \rangle_{\partial \Omega}, \tag{7c}$$

$$(\overline{p}_h, 1)_{\Omega} = 0, \tag{7d}$$

for all $(\boldsymbol{\mu}, \overline{q})$ in $\boldsymbol{M}_h \times Q_h^0$.

The following result gives sufficient conditions for the global problem to be well posed.

Theorem 2 ([1]). *Suppose that the three conditions guaranteeing the existence and uniqueness of the solution of the local problems* (6) *of Theorem 6 hold. Then the solution of the global problem* (7) *exists and is unique.*

A remarkable feature of this result is that it holds *regardless* of the choice of the local spaces $\boldsymbol{M}(F)$. This is a direct consequence of the positive definiteness of the stabilization function S.

We have thus completed the definition of the **HDG** methods and have shown simple, sufficient conditions for their solution to exist and be unique. To end this Section, we give a characterization of the approximation they provide.

3.3. Characterization of the approximate solution

Mimicking what was done for the exact solution, we can write that

$$(\mathrm{L}_h, \boldsymbol{u}_h, p_h) = (\mathrm{L}_h^{\widehat{\boldsymbol{u}}}, \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}}, p_h^{\widehat{\boldsymbol{u}}}) + (\mathrm{L}_h^{\boldsymbol{f}}, \boldsymbol{u}_h^{\boldsymbol{f}}, p^{\boldsymbol{f}}) + (0, \boldsymbol{0}, \overline{p}_h),$$

with the obvious notation, where the function $(\widehat{\boldsymbol{u}}_h, \overline{p}_h)$ is the solution of the global problem (3) which, we rewrite as follows:

$$\begin{aligned}
-\langle -\nu \widehat{\mathrm{L}}_h^{\widehat{\boldsymbol{u}}} \boldsymbol{n} - \widehat{p}_h^{\widehat{\boldsymbol{u}}}\, \boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} + \langle \overline{p}_h, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h} &= \langle -\nu \widehat{\mathrm{L}}_h^{\boldsymbol{f}} \boldsymbol{n} + \widehat{p}_h^{\boldsymbol{f}}\, \boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega},\\
\langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, \overline{q}\rangle_{\partial\mathcal{T}_h} &= 0,\\
\langle \widehat{\boldsymbol{u}}_h, \boldsymbol{\mu}\rangle_{\partial\Omega} &= \langle \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega},\\
(\overline{p}_h, 1)_\Omega &= 0,
\end{aligned}$$

for all $(\boldsymbol{\mu}, \overline{q})$ in $\boldsymbol{M}_h \times Q_h^0$.

This implies that, as claimed, the **HDG** methods can be easily implemented. Indeed, we see that the only globally coupled degrees of freedom are those of the approximation of the velocity on $\mathcal{E}_h$, $\widehat{\boldsymbol{u}}_h$, and those of the local average of the pressure $\overline{p}_h$. This global problem can be solved by using, for example, the augmented Lagrangian method, see [1]. Once this global problem is solved, the approximate solution $(\mathrm{L}_h, \boldsymbol{u}_h, p_h)$ can be easily obtained in an element-by-element fashion by using the very first identity of this Subsection.

4. The three formulations of the HDG methods

In this Section, we present three quite different ways of formulating the **HDG** methods just devised since we believe it is important to recognize that they are all equivalent; they differ by the choice of the unknowns explicitly considered to be dependent on the others.

4.1. The formulation for $(\widehat{\boldsymbol{u}}_h, \overline{p}_h)$

This formulation is nothing but a slight rewriting of the formulation used to define the methods and is convenient for their implementation. As we have seen, it consists in *eliminating* $(\mathrm{L}_h, \boldsymbol{u}_h, p_h)$ from the equations by expressing it elementwise in terms of $(\widehat{\boldsymbol{u}}_h, \overline{p}_h)$ and then solving for the later unknown. It is contained in the following result. Therein, we use the following notation:

$$\boldsymbol{M}_h(\boldsymbol{g}) := \{\boldsymbol{\mu} \in \boldsymbol{M}_h : \langle \boldsymbol{\mu}, \boldsymbol{\lambda}\rangle_{\partial\Omega} = \langle \boldsymbol{g}, \boldsymbol{\lambda}\rangle_{\partial\Omega} \; \forall \boldsymbol{\lambda} \in \boldsymbol{M}_h\}.$$

Theorem 3 ([1]). *We have that*

$$(\mathrm{L}_h, \boldsymbol{u}_h, p_h) = (\mathrm{L}_h^{\widehat{\boldsymbol{u}}}, \boldsymbol{u}_h^{\widehat{\boldsymbol{u}}}, p_h^{\widehat{\boldsymbol{u}}}) + (\mathrm{L}_h^{\boldsymbol{f}}, \boldsymbol{u}_h^{\boldsymbol{f}}, p^{\boldsymbol{f}}) + (0, \boldsymbol{0}, \overline{p}_h),$$

where each of the terms of the right-hand side is the corresponding solution of the local problem (6)*, and where the function* $(\widehat{\boldsymbol{u}}_h, \overline{p}_h)$ *is the element in* $\boldsymbol{M}_h(\boldsymbol{g}) \times Q_h^0$ *satisfying*

$$\begin{aligned} a_h(\widehat{\boldsymbol{u}}_h, \boldsymbol{\mu}) + b_h(\boldsymbol{\mu}, \overline{p}_h) &= (\boldsymbol{f}, \boldsymbol{u}_h^{\boldsymbol{\mu}})_{\mathcal{T}_h} && \forall \boldsymbol{\mu} \in \boldsymbol{M}_h(\boldsymbol{0}), \\ -b_h(\widehat{\boldsymbol{u}}_h, \overline{q}) &= 0 && \forall \overline{q} \in Q_h^0, \\ (\overline{p}_h, 1)_\Omega &= 0, \end{aligned}$$

where

$$\begin{aligned} a_h(\boldsymbol{\lambda}, \boldsymbol{\mu}) &:= \nu\, (\mathrm{L}_h^{\boldsymbol{\lambda}}, \mathrm{L}_h^{\boldsymbol{\mu}})_{\mathcal{T}_h} + \langle \mathrm{S}(\boldsymbol{u}_h^{\boldsymbol{\lambda}} - \boldsymbol{\lambda}), (\boldsymbol{u}_h^{\boldsymbol{\mu}} - \boldsymbol{\mu})\rangle_{\partial\mathcal{T}_h} \\ b_h(\boldsymbol{\lambda}, \overline{q}) &:= -\langle \overline{q}, \boldsymbol{\lambda}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, \end{aligned}$$

for all $\boldsymbol{\lambda}$*,* $\boldsymbol{\mu}$ *in* $\boldsymbol{M}_h$ *and* $\overline{q}$ *in* Q_h^0*.*

Moreover, assume that the stabilization function S *is symmetric and that the three conditions guaranteeing the existence and uniqueness of the solution of the local problems* (6) *of Theorem 6 hold. Then the bilinear form* $a_h(\cdot,\cdot)$ *is symmetric and positive definite in* $\boldsymbol{M}_h(\boldsymbol{0}) \times \boldsymbol{M}_h(0)$*.*

This result allows us to realize that the saddle-point structure of the global problem for the **HDG** method reflects well the structure of the original problem. Indeed, and roughly speaking, the bilinear form $a_h(\widehat{\boldsymbol{u}}_h, \cdot)$ is a discrete version of $-\nu\Delta\boldsymbol{u}$, $b_h(\cdot, \overline{p}_h)$ of ∇p, and $b_h(\widehat{\boldsymbol{u}}_h, \cdot)$ of $-\nabla\cdot\boldsymbol{u}$. Thus, it is not a surprise that the form $a_h(\cdot,\cdot)$ is symmetric and positive definite on $\boldsymbol{M}_h(\boldsymbol{0}) \times \boldsymbol{M}_h(0)$ since the operator $-\nu\Delta$ is strongly elliptic and self-adjoint on $\boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega)$.

This result also allows us to realize that there is a minimization problem for the approximate velocity which is a discrete version of the corresponding minimization problem for the continuous case. Indeed, we can see that $\widehat{\boldsymbol{u}}_h$ is the only minimum on the space $\{\boldsymbol{\mu} \in \boldsymbol{M}_h(\boldsymbol{g}) : \ b_h(\boldsymbol{\lambda}, \overline{q}) = 0 \quad \forall\, \overline{q} \in Q_h\}$, of the quadratic functional $J_h(\boldsymbol{\lambda}) := \frac{1}{2}\, a_h(\boldsymbol{\lambda}, \boldsymbol{\lambda}) - (\boldsymbol{f}, \boldsymbol{u}_h^{\boldsymbol{\lambda}})_{\mathcal{T}_h}$. This is a discrete version of the fact that $\boldsymbol{u}$ is the only minimum on the space $\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v} = \boldsymbol{g} \text{ on } \partial\Omega,\ \nabla \cdot \boldsymbol{v} = 0\}$ of the functional $J(\boldsymbol{v}) := \frac{1}{2}\, (\nabla \boldsymbol{v}, \nabla \boldsymbol{v})_\Omega - (\boldsymbol{f}, \boldsymbol{v})_\Omega$.

4.2. The formulation for $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, p_h)$

This formulation consists in *eliminating* L_h from the equations by expressing it in terms of $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)$. To do that, we define matrix-valued function $\mathrm{L}_h^{\boldsymbol{v},\boldsymbol{\mu}} \in \mathrm{G}_h$ by using the first equation of the compact formulation, that is, by

$$(\mathrm{L}_h^{\boldsymbol{v},\boldsymbol{\mu}}, \mathrm{G})_{\mathcal{T}_h} = -(\boldsymbol{v}, \nabla \cdot \mathrm{G})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{v}}, \mathrm{G}\boldsymbol{n} \rangle_{\partial\mathcal{T}_h} \qquad \forall\, \mathrm{G} \in \mathrm{G}_h.$$

We have the following result.

Theorem 4. *We have that*

$$\mathrm{L}_h = \mathrm{L}_h^{\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h},$$

where the function $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, p_h)$ *is the element in* $\boldsymbol{M}_h \times \boldsymbol{M}_h(\boldsymbol{g}) \times Q_h$ *satisfying*

$$\begin{aligned} \mathsf{a}_h(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h; \boldsymbol{v}, \boldsymbol{\mu}) + \mathsf{b}_h(\boldsymbol{v}, \boldsymbol{\mu}; p_h) &= (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h} && \forall (\boldsymbol{v}, \boldsymbol{\mu}) \in \mathbf{V}_h \times \boldsymbol{M}_h(\mathbf{0}), \\ -\mathsf{b}_h(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h; q) &= 0 && \forall q \in Q_h, \\ (p_h, 1)_\Omega &= 0, \end{aligned}$$

where

$$\begin{aligned} \mathsf{a}_h(\boldsymbol{w}, \boldsymbol{\lambda}; \boldsymbol{v}, \boldsymbol{\mu}) &:= \nu\, (\mathrm{L}_h^{\boldsymbol{w},\boldsymbol{\lambda}}, \mathrm{L}_h^{\boldsymbol{v},\boldsymbol{\mu}})_{\mathcal{T}_h} + \langle \mathrm{S}(\boldsymbol{w} - \boldsymbol{\lambda}), (\boldsymbol{v} - \boldsymbol{\mu}) \rangle_{\partial\mathcal{T}_h} \\ \mathsf{b}_h(\boldsymbol{w}, \boldsymbol{\lambda}; q) &:= (\nabla q, \boldsymbol{w})_{\mathcal{T}_h} - \langle q, \boldsymbol{\lambda} \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_h}, \end{aligned}$$

for all $\boldsymbol{v}$, $\boldsymbol{w}$ *in* $\boldsymbol{V}_h$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ *in* $\boldsymbol{M}_h$ *and* q *in* Q_h.

Moreover, assume that the stabilization function S *is symmetric and that the three conditions guaranteeing the existence and uniqueness of the solution of the local problems* (6) *of Theorem 6 hold. Then the bilinear form* $\mathsf{a}_h(\cdot;\cdot)$ *is symmetric and positive definite in* $(\boldsymbol{V}_h \times \boldsymbol{M}_h(\mathbf{0})) \times (\boldsymbol{V}_h \times \boldsymbol{M}_h(0))$.

The proof of this result is similar to that of the first formulation but much easier. For this reason, we omit it.

Note the remarkable similarity of this result with that of the previous formulation. In particular, note that $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)$ is the only element of the space $\{(\boldsymbol{v}, \boldsymbol{\mu}) \in \boldsymbol{V}_h \times \boldsymbol{M}(\boldsymbol{g}) : \ \mathsf{b}_h(\boldsymbol{v}, \boldsymbol{\mu}; q) = 0\ \forall q \in Q_h\}$ that minimizes the functional $\mathsf{J}_h(\boldsymbol{v}, \boldsymbol{\mu}) := \frac{1}{2}\, \mathsf{a}_h(\boldsymbol{v}, \boldsymbol{\mu}; \boldsymbol{v}, \boldsymbol{\mu}) - (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}$.

4.3. The formulation for $(\mathrm{L}_h, p_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)$

It is not difficult to see that the **HDG** methods seeks an approximation $(\mathrm{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h)$ in the space $\mathrm{G}_h \times \boldsymbol{V}_h \times \boldsymbol{M}_h \times Q_h$ satisfying the equations

$$(\mathrm{L}_h, \mathrm{G})_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla\cdot\mathrm{G})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \mathrm{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{8a}$$

$$\nu\,(\mathrm{L}_h, \nabla\boldsymbol{v})_{\mathcal{T}_h} - (p_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle \nu\widehat{\mathrm{L}}_h\boldsymbol{n} - \widehat{p}_h\boldsymbol{n}, \boldsymbol{v}\rangle_{\partial\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}, \tag{8b}$$

$$-(\boldsymbol{u}_h, \nabla q)_K + \langle \widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n}, q\rangle_{\partial\mathcal{T}_h} = 0, \tag{8c}$$

$$\langle -\nu\widehat{\mathrm{L}}_h\boldsymbol{n} + \widehat{p}_h\,\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \tag{8d}$$

$$\langle \widehat{\boldsymbol{u}}_h, \boldsymbol{\mu}\rangle_{\partial\Omega} = \langle \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega}, \tag{8e}$$

$$(p_h, 1)_\Omega = 0, \tag{8f}$$

for all $(\mathrm{G}, \boldsymbol{v}, \boldsymbol{\mu}, q) \in \mathrm{G}_h \times \boldsymbol{V}_h \times \boldsymbol{M}_h \times Q_h$, where

$$-\nu\widehat{\mathrm{L}}_h\boldsymbol{n} + \widehat{p}_h\boldsymbol{n} := -\nu\mathrm{L}_h\boldsymbol{n} + p_h\boldsymbol{n} + \mathrm{S}(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) \quad \text{on } \partial\mathcal{T}_h. \tag{8g}$$

This way of expressing the methods is particularly well suited to the projection-error analyzes carried out in [2, 3, 4]. Here, in order to emphasize its strong relation to the two previous formulations, we rewrite it slightly in the following result.

Theorem 5. *We have that the function* $(\mathrm{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, p_h)$ *is the element in* $\mathrm{G}_h \times \boldsymbol{M}_h \times \boldsymbol{M}_h(\boldsymbol{g}) \times Q_h$ *satisfying*

$$\begin{aligned}
\mathrm{A}_h(\mathrm{L}_h, \mathrm{G}) - \mathrm{C}_h(\mathrm{G}; \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) &= 0 && \forall \mathrm{G}\in\mathrm{G}_h,\\
\nu\,\mathrm{C}_h(\mathrm{L}_h; \boldsymbol{v}_h, \boldsymbol{\mu}) + \mathrm{B}_h(\boldsymbol{v}, \boldsymbol{\mu}; p_h) + \mathrm{S}_h(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h; \boldsymbol{v}, \boldsymbol{\mu}) &= (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h} && \forall (\boldsymbol{v}, \boldsymbol{\mu}) \in \mathbf{V}_h \times \boldsymbol{M}_h(\mathbf{0}),\\
-\mathrm{B}_h(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h; q) &= 0 && \forall q \in Q_h,\\
(p_h, 1)_\Omega &= 0,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{A}_h(\mathrm{F}, \mathrm{G}) &:= (\mathrm{F}, \mathrm{G})_{\mathcal{T}_h},\\
\mathrm{B}_h(\boldsymbol{w}, \boldsymbol{\lambda}; q) &:= (\nabla q, \boldsymbol{w})_{\mathcal{T}_h} - \langle q, \boldsymbol{\lambda}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h},\\
\mathrm{C}_h(\mathrm{G}; \boldsymbol{v}_h, \boldsymbol{\mu})) &:= (\boldsymbol{v}, \nabla\cdot\mathrm{G})_{\mathcal{T}_h} - \langle \boldsymbol{\mu}, \mathrm{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h},\\
\mathrm{S}_h(\boldsymbol{w}, \boldsymbol{\lambda}; \boldsymbol{v}, \boldsymbol{\mu}) &:= \langle \mathrm{S}(\boldsymbol{w} - \boldsymbol{\lambda}), (\boldsymbol{v} - \boldsymbol{\mu})\rangle_{\partial\mathcal{T}_h},
\end{aligned}$$

for all F, G *in* G_h, $\boldsymbol{v}$, $\boldsymbol{w}$ *in* $\boldsymbol{V}_h$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ *in* $\boldsymbol{M}_h$ *and* q *in* Q_h.

5. Accuracy of the HDG methods

In this Section, we explore the issue of the accuracy of the **HDG** methods just introduced. As we just saw, very simple inclusion conditions on the local spaces $\mathrm{G}(K)$, $\boldsymbol{V}(K)$, $Q(K)$, and a positivity condition on the stabilization function S are enough to guarantee the existence and uniqueness of the approximate solution, regardless of the choice of the local spaces $M(F)$ and of the geometry of the elements K. However, the order of convergence of the approximations provided by the method have a more complicated relationship with the local spaces, stabilization function and geometry of the elements.

We discuss two main cases. In the first, we take the local spaces to be polynomials of a given degree defined on general polyhedral elements. Optimal orders of convergence are obtained for the velocity but suboptimal orders (by 1/2) are obtained for the velocity gradient and the pressure. In the second case, following [3], the local spaces, the stabilization function and the shape of the elements are taken in such a way that optimal convergence for all the variables is achieved. Moreover, a superconvergence property is also obtained for the velocity.

For simplicity, we assume the triangulation $\mathcal{T}_h$ to be conforming.

5.1. HDG methods on general meshes

We begin by considering the case in which the generic element $K \in \mathcal{T}_h$ is a regular-shaped polygon (in 2D) or a polyhedral (in 3D). We define the stabilization operator as

$$\mathrm{S} = \tau\, \mathrm{Id}.$$

Finally, we complete the definition of the method by taking the local spaces as:

$$\mathrm{G}(K) = \mathrm{P}_k(K), \quad \boldsymbol{V}(K) = \boldsymbol{\mathcal{P}}_k(K), \quad Q(K) = \mathcal{P}_k(K), \quad \boldsymbol{M}(F) = \boldsymbol{\mathcal{P}}_k(F).$$

Here $\mathcal{P}_m(D)$ denotes the space of polynomials of degree m defined on D, $\boldsymbol{\mathcal{P}}_m(D)$ denotes the space of d-component vector-valued functions whose components belong to $\mathcal{P}_m(D)$, and $\mathrm{P}_m(K)$ is the set of $d \times d$ matrix-valued functions whose entries belong to $\mathcal{P}_m(D)$. We denote this method by $\mathbf{HDG}_k$.

For this method, and the stabilization function and its inverse are bounded above uniformly, the orders of convergence for the velocity are optimal but those for the pressure and the velocity gradient are *suboptimal* by 1/2, see Table 1. Since a proof of this result has not appeared before, we provide it in the Appendix.

Table 1: Provable order of convergence for $\mathbf{HDG}_k$ on general polyhedral elements ($k \geq 0$)

$\|\mathrm{L} - \mathrm{L}_h\|_{L^2(\Omega)}$	$\|p - p_h\|_{L^2(\Omega)}$	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)}$
$k + 1/2$	$k + 1/2$	$k + 1$

5.2. Superconvergent **HDG** *methods*

We now discuss **HDG** methods for which the convergence properties can be proven to be *better*. These methods are called superconvergent.

Roughly speaking, what makes an **HDG** method ”superconvergent” are two properties. First, the elementwise average of the approximate velocity $\overline{\boldsymbol{u}}_h$ converges to the elementwise average of the exact velocity $\overline{\boldsymbol{u}}$ with a order, say $\ell+1$, which is strictly higher than that of the convergence of $\boldsymbol{u}_h$ converges to $\boldsymbol{u}$. Second, the approximate velocity gradient L_h converges to the exact velocity gradient L with an order equal or higher than ℓ. We can then find a new approximation to the velocity $\boldsymbol{u}_h^*$ converging with order $\ell + 1$, that is, faster than the original approximation.

To define this new approximation, we follow [5, 6, 7], and define $\boldsymbol{u}_h^*$ on the element $K \in \mathcal{T}_h$ as the element of finite dimensional space $\boldsymbol{V}^*(K)$ such that

$$(\nabla \boldsymbol{u}_h^*, \nabla \boldsymbol{v})_K = (\mathrm{L}_h, \nabla \boldsymbol{v})_K \quad \forall \boldsymbol{v} \in \boldsymbol{V}^*(K), \tag{9a}$$

$$(\boldsymbol{u}_h^*, \boldsymbol{v})_K = (\boldsymbol{u}_h, \boldsymbol{v})_K \qquad \forall \in \boldsymbol{\mathcal{P}}_0(K). \tag{9b}$$

By making sure that $\boldsymbol{V}_h^*(K)$ includes $\boldsymbol{\mathcal{P}}_\ell(K)$ for all elements $K \in \mathcal{T}_h$, we can make $\|\boldsymbol{u}_h^* - \boldsymbol{u}\|_{L^2(\Omega)}$ converge to zero with order $\ell + 1$; see [3].

Next, we describe the systematic way of constructing superconvergent **HDG** methods for the Stokes flow in terms superconvergent **HDG** methods for the model diffusion problem,

$$\boldsymbol{q} = \nabla u, \quad -\nabla \cdot \boldsymbol{q} = f \text{ on } \Omega, \quad u = g \text{ on } \partial\Omega,$$

obtained in [3]. These methods [8] seek approximations $(\boldsymbol{q}_h, u_h, \widehat{u}_h)$ of the exact solution $(\boldsymbol{q}|_\Omega, u|_\Omega, u|_{\mathcal{E}_h})$ whose local spaces are $\boldsymbol{V}^{\mathrm{D}}(K)$, $W^{\mathrm{D}}(K)$ and $M^{\mathrm{D}}(F)$, and whose stabilization function is τ^{D}. We associate to each of these methods a superconvergent **HDG** method constructed as follows.

Denote by $\mathrm{G}_i(K)$ the space of all the i-th rows of functions in $\mathrm{G}(K)$, and by $\boldsymbol{V}_i(K)$ and $\boldsymbol{M}_i(F)$ the space of the i-th component of functions in $\boldsymbol{V}(K)$

and $\boldsymbol{M}(F)$, respectively, for $i = 1, \ldots, d$. Then, we take the local spaces as

$$\mathrm{G}_i(K) := \boldsymbol{V}^{\mathrm{D}}(K), \quad \boldsymbol{V}_i(K) := W^{\mathrm{D}}(K), \quad \boldsymbol{M}_i(F) := M^{\mathrm{D}}(F), \tag{10a}$$

for $i = 1, \ldots, d$, and the stabilization function as

$$\mathrm{S} := \tau^{\mathrm{D}}\,\mathrm{Id}. \tag{10b}$$

The choice of the space for the pressure $Q(K)$ has to be done in such a way that

$$\sum_{j=1}^{d} \partial_j W^{\mathrm{D}}(K) \subset Q(K) \subset \cap_{j=1}^{d} \{v_j : \boldsymbol{v} \in \boldsymbol{V}^{\mathrm{D}}(K) : v_i = 0 \text{ for } i \neq j\}. \tag{10c}$$

5.3. Examples of superconvergent HDG methods

In Table 2, taken from [3], we display the orders of convergence for mixed methods and **HDG** methods using different elements K. We only show the space $Q(K)$. The the other spaces are given in [8] and a *better* version of the spaces $\mathbf{TNT}_{[k]}$ can be found in [9]. All the methods achieve optimal orders for L_h, p_h and superconvergence takes place, as claimed. Note that for the mixed methods, $\tau^{\mathrm{D}} = 0$, whereas for the **HDG** methods, $\tau^{\mathrm{D}} \geq 0$ is uniformly bounded from above.

6. Divergence-free velocities by postprocessing

In this Section, we consider the $\mathbf{HDG}_k$ method in Table 2, and consider an alternative to the postprocessing in the previous Section. It was introduced in [2] and, in three-space dimensions, is defined as follows. On the tetrahedron $K \in \mathcal{T}_h$, $\mathbf{u}_h^*$ is defined as the element of $\boldsymbol{\mathcal{P}}_{k+1}(K)$ such that

$$\langle(\mathbf{u}_h^* - \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n}, \mu\rangle_F = 0 \quad \forall\, \mu \in \mathcal{P}_k(F), \tag{11a}$$

$$\langle(\boldsymbol{n} \times \nabla)(\mathbf{u}_h^* \cdot \boldsymbol{n}) - \boldsymbol{n} \times (\{\!\{\mathrm{L}_h^t\}\!\}\boldsymbol{n}), (\boldsymbol{n} \times \nabla)\mu\rangle_F = 0 \quad \forall\, \mu \in \mathcal{P}_{k+1}(F)^{\perp}, \tag{11b}$$

for all faces F of K, and such that

$$(\mathbf{u}_h^* - \boldsymbol{u}_h, \nabla w)_K = 0 \quad \forall\, w \in \mathcal{P}_k(K), \tag{11c}$$

$$(\nabla \times \mathbf{u}_h^* - \mathbf{w}_h, (\nabla \times \boldsymbol{v})\, \mathrm{B}_K)_K = 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{S}}_k(K). \tag{11d}$$

Here

$$\mathcal{P}_{k+1}(F)^{\perp} := \{\mu \in \mathcal{P}_{k+1}(F) : \langle \mu, \widetilde{\mu}\rangle_F = 0, \quad \forall \widetilde{\mu} \in \mathcal{P}_k(F)\},$$

Table 2: Order of convergence for superconvergent **HDG** methods ($k \geq 1$)

method for diffusion	$Q(K)$	$\lVert \mathrm{L} - \mathrm{L}_h \rVert_{L^2(\Omega)}$	$\lVert p - p_h \rVert_{L^2(\Omega)}$	$\lVert \boldsymbol{u} - \boldsymbol{u}_h^\star \rVert_{L^2(\Omega)}$
		K simplex and $\boldsymbol{M}(F) = \boldsymbol{P}_k(F)$		
$\mathbf{BDFM}_{k+1}$	$\mathcal{P}_k(K)$	$k+1$	$k+1$	$k+2$
$\mathbf{RT}_k$	$\mathcal{P}_k(K)$	$k+1$	$k+1$	$k+2$
$\mathbf{HDG}_k$	$\mathcal{P}_k(K)$	$k+1$	$k+1$	$k+2$
$\mathbf{BDM}_k$ $k\geq 2$	$\mathcal{P}_k(K)$	$k+1$	$k+1$	$k+2$
		K square or cube and $\boldsymbol{M}(F) = \boldsymbol{P}_k(F)$		
$\mathbf{BDFM}_{[k+1]}$	$\mathcal{P}_k(K)$	$k+1$	$k+1$	$k+2$
$\mathbf{HDG}^P_{[k]}$	$\mathcal{P}_k(K)$	$k+1$	$k+1$	$k+2$
$\mathbf{BDM}_{[k]}$ $k\geq 2$	$\mathcal{P}_k(K)$	$k+1$	$k+1$	$k+2$
		K square or cube and $\boldsymbol{M}(F) = \boldsymbol{Q}_k(F)$		
$\mathbf{RT}_{[k]}$	$\mathcal{Q}_k(K)$	$k+1$	$k+1$	$k+2$
$\mathbf{TNT}_{[k]}$	$\mathcal{Q}_k(K)$	$k+1$	$k+1$	$k+2$
$\mathbf{HDG}^Q_{[k]}$	$\mathcal{Q}_k(K)$	$k+1$	$k+1$	$k+2$
		K prism and $\boldsymbol{M}(F)$ defined in [3]		
$\mathbf{BDFM}_{<k+1>}$	$\mathcal{P}_k(K)$	$k+1$	$k+1$	$k+2$
$\mathbf{RT}_{<k>}$	$\mathcal{P}_{k\|k}(K)$	$k+1$	$k+1$	$k+2$
$\mathbf{HDG}_{<k>}$	$\mathcal{P}_k(K)$	$k+1$	$k+1$	$k+2$

the operator $\mathbf{n} \times \nabla$ is the tangential gradient and the function $\{\!\{\mathrm{L}_h^t\}\!\}$ is the single-valued function on $\mathcal{E}_h$ equal to $((\mathrm{L}_h^t)^+ + (\mathrm{L}_h^t)^-)/2$ on the set $\mathcal{E}_h \setminus \partial\Omega$ and equal to L_h^t on $\partial\Omega$. Moreover,

$$\mathsf{w}_h := (\mathrm{L}_{32\,h} - \mathrm{L}_{23\,h}, \mathrm{L}_{13\,h} - \mathrm{L}_{31\,h}, \mathrm{L}_{21\,h} - \mathrm{L}_{12\,h})$$

is the approximation to the vorticity and B_K is the so-called *symmetric bubble matrix* introduced in [10], namely,

$$\mathrm{B}_K := \sum_{\ell=0}^{3} \lambda_{\ell-3}\lambda_{\ell-2}\lambda_{\ell-1}\nabla\lambda_\ell \otimes \nabla\lambda_\ell,$$

where λ_i are the barycentric coordinates associated with the tetrahedron K, the subindices being counted modulo 4. Finally, $\boldsymbol{\mathcal{S}}_k(K) := \sum_{\ell=1}^{k} \boldsymbol{S}_\ell(K)$

where $\boldsymbol{S}_\ell$ is the space of vector-valued homogeneous polynomials $\boldsymbol{v}$ of degree ℓ such that $\boldsymbol{v} \cdot \boldsymbol{x} = 0$, see [11, 12].

This elementwise post-processing has many properties we gather in the following result.

Theorem 6 ([2]). *We have that*

(i) $\mathbf{u}_h^*$ *is well defined.*

(ii) $\mathbf{u}_h^*$ *in* $\boldsymbol{H}(\mathrm{div})$.

(iii) $\nabla \cdot \mathbf{u}_h^* = 0$ *on* Ω.

(iv) $\|\boldsymbol{u} - \mathbf{u}_h^*\|_{L^2(\Omega)} \leq C\, h^{k+2}$, *for* $k \geq 1$ *and smooth solutions.*

The development of similar postprocessings for the other superconvergent methods remains an open problem.

7. Concluding remarks

7.1. Methods based on other formulations of the Stokes system

Although we have used a velocity gradient-velocity-pressure expression of the Stokes system, we could have used others. The main cases are to use the symmetric gradient of the velocity or the vorticity instead of the velocity gradient. Numerical experiments carried out in [13] show that the formulation using the velocity gradient is superior to that of the symmetric gradient which in turn is better than that of the vorticity; see also [14].

Indeed, since these three formulations have a global system of the same unknowns (they all solve for $\widehat{\boldsymbol{u}}_h$ and $\overline{p}_h$), size and structure, the only difference among them is how the local problems are defined. Assuming that the local problems do not represent the main computational cost of the method (note that they can be solved in parallel), and that the global problems of the three formulation can be solved with about the cost, the fact that the velocity gradient involves more unknowns than the other two formulations does not really play a significant role. The above-mentioned superiority of the velocity gradient formulation is thus a direct consequence of the fact that it is the only one that provides a superconvergence property of the velocity.

On the other hand, the use of the velocity gradient formulation does not allow in a natural way the imposition of the normal stress as a boundary condition. This problem has been partially addressed in [15], but although the optimal convergence of all the variables was retained, the superconvergence of the velocity was lost. How to devise superconvergent **HDG** methods for this boundary condition remains an open problem.

7.2. Methods based on other characterizations

Although we have used as data for defining the local problems on each given element the velocity on its boundary and the average of the pressure in the element, this is certainly not the only possibility. For example, we could have imposed the normal component of $-\nu\,\mathrm{L} + p\,\mathrm{Id}$ on the boundary of the element and the average of the velocity and the pressure on the element. The global problem would have then to be defined differently, by using suitably defined transmission conditions, but its unknowns would then be the data of the local problems. One wonders then if the resulting **HDG** method is necessarily different from the one we presented. The answer is no: The *same* **HDG** can be obtained with these two different characterizations of the exact solution. For example, an **HDG** method based on the vorticty formulation was studied in [16] which could be obtained with *four* different characterizations of the exact solution. Each of these give rise to a different way of characterizing the approximate solution (or,as we have seen, of *implementing* it), but the approximate solution, just as the exact one, remains the same. The issue of determining the most convenient way of implementing the **HDG** methods remains open.

7.3. Methods with $\mathbf{H}(\mathrm{div})$*-conforming spaces for the velocity*

In [4], a new approach for devising numerical methods using $\mathbf{H}(\mathrm{div})$-conforming spaces for the velocity was uncovered. The new method is obtained starting from the $\mathbf{HDG}_k$ in Table 2 with the following stabilization function

$$\mathrm{S} := \nu\,\tau_n\,\boldsymbol{n}\otimes\boldsymbol{n} + \nu\,\tau_t\,(\mathrm{Id} - \boldsymbol{n}\otimes\boldsymbol{n}),$$

where $\boldsymbol{n}$ is the normal to the faces on $\partial\mathcal{T}_h$, and by simply letting τ_n go to infinity. That this limiting process should lead to a reasonable numerical scheme was suggested by numerical experiments [1] and later by theoretical analysis [2] done on the HDG_k method pointing to the fact that the convergence properties of the method were *independent* of how big was τ_n. In [4], it was proven that, when τ_n goes to infinity, the quantity $\tau_n\,(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}$ does not blow up but *converges* to a new unknown δ_h. This forces the velocity $\boldsymbol{u}_h$ to lie in $\mathbf{H}(\mathrm{div})$ in the limit, and, more importantly, is strongly linked with the fact that the limiting method *retains* all the convergence properties of the original method $\mathbf{HDG}_k$, including the superconvergence property of the velocity.

Let us emphasize that **HDG** methods using $\mathbf{H}(\text{div})$-conforming spaces for the velocity can be obtained by using the simple approach discussed in the first two Section of this paper. However, the transformation of the term $\tau_n\,(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n}$ into a new unknown δ_h, which plays a crucial role ensuring the superconvergence property of the method, is almost impossible to fathom without using this simple approach. The extension of the approach proposed in [4] for the $\mathbf{HDG}_k$ method to the the other superconvergent **HDG** methods in Table 2 constitutes the subject of ongoing research.

7.4. Methods for related systems

The extension of the **HDG** methods to isotropic linear elasticity equations, namely,

$$\mathrm{L} - \nabla\boldsymbol{u} = 0, \quad \text{on } \Omega, \tag{12a}$$
$$-\nabla\cdot(\mu\mathrm{L}) + \nabla p = 0, \quad \text{on } \Omega, \tag{12b}$$
$$\epsilon p + \nabla\cdot\boldsymbol{u} = 0, \quad \text{on } \Omega, \tag{12c}$$
$$\boldsymbol{u} = \boldsymbol{g}, \quad \text{on } \partial\Omega, \tag{12d}$$

where $\epsilon = (1-2\nu)(1+\nu)/E$, E is the Young's modulus and $\nu \in (0, 1/2]$ the Poisson's ratio, is straightforward; see [17]. Note that the advantage of this formulation is that it holds for both compressible ($\nu \in (0, 1/2)$) and incompressible ($\nu = 1/2$) materials where this system of equations is nothing but the Stokes system. An extension to large deformation elasticity using a similar formulation has also been done in [17]. Note that the **HDG** methods for linear and large deformation elasticity proposed in [18, 19] use a different form of the equations than the one in [17].

Let us end by pointing out that the extension of the **HDG** methods to the Oseen and the incompressible Navier-Stokes equations does not present any major difficulty, see [20] and [21], respectively.

Acknowledgments. The first author would like to acknowledge the partial support of the National Science Foundation (Grant DMS-1115331) and of the University of Minnesota Supercomputing Institute.

Appendix A. An analysis for HDG methods on general meshes

In this section, we present an analysis for $\mathbf{HDG}_k$ methods on conforming meshes $\mathcal{T}_h$ whose elements K are *any* regular-shaped polygon (in 2D) or polyhedral (in 3D); see Subsection 5.1. As in [3], our analysis is based on estimating the projection of the errors. In contrast, we do not use projections tailored to the numerical traces, as this seems to be impossible to do when the generic element K is an arbitrary polygon or polyhedral. Instead, we use simple, standard L^2 projections, as was done for the original analysis of DG methods for diffusion problems on general meshes [22].

Thus, we use the L^2-projections into the spaces G_h, $\boldsymbol{V}_h$, Q_h and $\boldsymbol{M}_h$, which we denote by $\Pi_G, \boldsymbol{\Pi}_V, \Pi_Q, \boldsymbol{\Pi}_M$, respectively, and define the projection of the errors as:

$$\begin{aligned}
&\mathrm{E}_{\mathrm{L}} := \Pi_G \mathrm{L} - \mathrm{L}_h, \qquad e_p := \Pi_Q p - p_h, \qquad \boldsymbol{e}_u := \boldsymbol{\Pi}_V \boldsymbol{u} - \boldsymbol{u}_h,\\
&\nu \mathrm{E}_{\widehat{\mathrm{L}}}\,\boldsymbol{n} - e_{\widehat{p}}\,\boldsymbol{n} := \boldsymbol{\Pi}_M(\nu \mathrm{L}\boldsymbol{n} - p\boldsymbol{n}) - (\nu \widehat{\mathrm{L}}_h \boldsymbol{n} - \widehat{p}_h \boldsymbol{n}), \quad \boldsymbol{e}_{\widehat{u}} := \boldsymbol{\Pi}_M \boldsymbol{u} - \widehat{\boldsymbol{u}}_h.
\end{aligned}$$

Note that we have, by standard approximation theory, the following results:

$$\|u - \Pi_Y u\|_K \le C h^s \|u\|_{s,K} \tag{A.1a}$$

$$\|u - \Pi_Y u\|_{\partial K} \le C h^{s-\frac{1}{2}} \|u\|_{s,K}, \tag{A.1b}$$

for all $u \in H^s(K)$, $1 \le s \le k+1$. Here $Y = \mathrm{G}, \boldsymbol{V}, Q$. Finally,

We are now ready to carry out our a priori error analysis. We proceed in several steps.

Step 1: The error equations. We begin by obtaining the equations satisfied by the projection of the errors.

Lemma 1.

$$(\mathrm{E}_{\mathrm{L}}, \mathrm{G})_{\mathcal{T}_h} + (\boldsymbol{e}_u, \nabla\cdot \mathrm{G})_{\mathcal{T}_h} - \langle \boldsymbol{e}_{\widehat{u}}\,,\, \mathrm{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{A.2a}$$

$$(\nu \mathrm{E}_{\mathrm{L}}, \nabla \boldsymbol{v})_{\mathcal{T}_h} - (e_p, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} - \langle \nu \mathrm{E}_{\widehat{\mathrm{L}}}\,\boldsymbol{n} - e_{\widehat{p}}\,\boldsymbol{n}\,,\, \boldsymbol{v}\rangle_{\partial\mathcal{T}_h} = 0, \tag{A.2b}$$

$$-(\boldsymbol{e}_u, \nabla q)_{\mathcal{T}_h} + \langle \boldsymbol{e}_{\widehat{u}}\,,\, q\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{A.2c}$$

$$\langle \boldsymbol{e}_{\widehat{u}}, \boldsymbol{\mu}\rangle_{\partial\Omega} = 0, \tag{A.2d}$$

$$\langle \nu \mathrm{E}_{\widehat{\mathrm{L}}}\,\boldsymbol{n} - e_{\widehat{p}}\,\boldsymbol{n}\,,\, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \tag{A.2e}$$

$$(e_p, 1)_\Omega = 0, \tag{A.2f}$$

for all $(\mathrm{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h$.

PROOF. Note that the exact solution also solves the system (8). Then, using the definition of the L^2-projections, we immediately obtain that

$$\begin{aligned}
(\Pi_G \mathrm{L}, \mathrm{G})_{\mathcal{T}_h} + (\boldsymbol{\Pi}_V \boldsymbol{u}, \nabla \cdot \mathrm{G})_{\mathcal{T}_h} - \langle \boldsymbol{\Pi}_M \boldsymbol{u}\,,\, \mathrm{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= 0,\\
(\nu \Pi_G \mathrm{L}, \nabla \boldsymbol{v})_{\mathcal{T}_h} - (\Pi_Q p, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} - \langle \boldsymbol{\Pi}_M(\nu \mathrm{L}\boldsymbol{n} - p\boldsymbol{n})\,,\, \boldsymbol{v}\rangle_{\partial\mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h},\\
-(\boldsymbol{\Pi}_V \boldsymbol{u}, \nabla q)_{\mathcal{T}_h} + \langle \boldsymbol{\Pi}_M \boldsymbol{u}\,,\, q\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= 0,\\
\langle \boldsymbol{\Pi}_M \boldsymbol{u}, \boldsymbol{\mu}\rangle_{\partial\Omega} &= \langle \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega},\\
\langle \boldsymbol{\Pi}_M(\nu \mathrm{L}\boldsymbol{n} - p\boldsymbol{n})\,,\, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} &= 0,\\
(\Pi_Q p, 1)_\Omega &= 0,
\end{aligned}$$

for all $(\mathrm{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h$. If we now subtract the equations (8) from the above ones, we obtain the error equations (A.2). □

Step 2: Estimate of $\mathrm{L} - \mathrm{L}_h$. Next we apply an energy argument to obtain an error estimate for $\mathrm{L}-\mathrm{L}_h$. To this end, we first present the following identity.

Lemma 2. *We have:*

$$\begin{aligned}
\nu\|\mathrm{E_L}\|_0^2 + \langle \tau(\boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}}), \boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}}\rangle_{\partial\mathcal{T}_h} =& \langle \tau \boldsymbol{\Pi}_V \boldsymbol{u} - \boldsymbol{\Pi}_M \boldsymbol{u}, \boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}}\rangle_{\partial\mathcal{T}_h}\\
&- \langle \nu(\Pi_G \mathrm{L}\boldsymbol{n} - \boldsymbol{\Pi}_M \mathrm{L}\boldsymbol{n}, \boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}}\rangle_{\partial\mathcal{T}_h}\\
&+ \langle \Pi_Q p\boldsymbol{n} - \boldsymbol{\Pi}_M p\boldsymbol{n}, \boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}}\rangle_{\partial\mathcal{T}_h}.
\end{aligned}$$

PROOF. Taking $(\mathrm{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) := (\mathrm{E_L}, \boldsymbol{e}_u, e_p, \boldsymbol{e}_{\widehat{u}})$ in the error equations (A.2a) - (A.2e) and adding these equations we obtain, after some algebraic manipulation, that

$$\nu\|\mathrm{E_L}\|_0^2 + \langle \nu \mathrm{E_L}\boldsymbol{n} - e_p\boldsymbol{n} - (\nu \mathrm{E}_{\widehat{\mathrm{L}}}\,\boldsymbol{n} - e_{\widehat{p}}\,\boldsymbol{n}), \boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}}\rangle_{\partial\mathcal{T}_h} = 0.$$

By the definition of the projection of the errors, we have:

$$\begin{aligned}
\nu \mathrm{E_L}\boldsymbol{n} - e_p\boldsymbol{n} - (\nu \mathrm{E}_{\widehat{\mathrm{L}}}\,\boldsymbol{n} - e_{\widehat{p}}\,\boldsymbol{n}) =&\\
&\nu(\Pi_G \mathrm{L} - \mathrm{L}_h)\boldsymbol{n} - (\Pi_Q p - p_h)\boldsymbol{n} - \boldsymbol{\Pi}_M(\nu \mathrm{L}\boldsymbol{n} - p\boldsymbol{n}) + (\nu \widehat{\mathrm{L}}_h\boldsymbol{n} - \widehat{p}_h\boldsymbol{n})\\
&= -\tau(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) + \nu(\Pi_G \mathrm{L}\boldsymbol{n} - \boldsymbol{\Pi}_M \mathrm{L}\boldsymbol{n}) - (\Pi_Q p\boldsymbol{n} - \boldsymbol{\Pi}_M p\boldsymbol{n})\\
&= \tau(\boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}}) - \tau(\boldsymbol{\Pi}_V \boldsymbol{u} - \boldsymbol{\Pi}_M \boldsymbol{u}) + \nu(\Pi_G \mathrm{L}\boldsymbol{n} - \boldsymbol{\Pi}_M \mathrm{L}\boldsymbol{n})\\
&\quad - (\Pi_Q p\boldsymbol{n} - \boldsymbol{\Pi}_M p\boldsymbol{n}),
\end{aligned}$$

by the definition of the numerical trace (8g). The identity we want to prove now easily follows. □

We can now deduce the following estimate for $\mathrm{E_L}$.

Theorem 7. *We have:*

$$\|\mathrm{E_L}\|_0 + \|\tau^{\frac{1}{2}}(\boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}})\|_{\partial\mathcal{T}_h} \le Ch^{s-\frac{1}{2}}(\tau^{\frac{1}{2}}\|\boldsymbol{u}\|_s + \tau^{-\frac{1}{2}}\|\mathrm{L}\|_s + \tau^{-\frac{1}{2}}\|p\|_s),$$

for all $1 \le s \le k+1$.

Note that Theorem 7 states that, when the solution is very smooth, the error $\|\mathrm{L} - \mathrm{L}_h\|_0$ is of order $\mathcal{O}(h^{k+\frac{1}{2}}\max\{\tau^{\frac{1}{2}}, \tau^{-\frac{1}{2}}\})$. This shows that the best choice of τ is when it is of order one. Let us now prove this result.

PROOF. Applying the Cauchy-Schwartz inequality in the right-hand side of the energy identity of Lemma 2, we get

$$\nu\|\mathrm{E_L}\|_0^2 + \|\tau^{\frac{1}{2}}(\boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}})\|_{\partial\mathcal{T}_h}^2 \le \|\tau^{\frac{1}{2}}(\boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}})\|_{\partial\mathcal{T}_h}\,\Theta_h,$$

where

$$\begin{aligned}\Theta_h := {} & \tau^{\frac{1}{2}}\|\boldsymbol{\Pi}_V\boldsymbol{u} - \boldsymbol{\Pi}_M\boldsymbol{u}\|_{\partial\mathcal{T}_h} + \nu\tau^{-\frac{1}{2}}\|\Pi_G\mathrm{L}\boldsymbol{n} - \boldsymbol{\Pi}_M\mathrm{L}\boldsymbol{n}\|_{\partial\mathcal{T}_h} \\ & + \tau^{-\frac{1}{2}}\|\Pi_Q p\boldsymbol{n} - \boldsymbol{\Pi}_M p\boldsymbol{n}\|_{\partial\mathcal{T}_h}.\end{aligned}$$

Next, we apply standard inverse inequalities and the approximation property of the projections (A.1) to obtain the estimate

$$\|\boldsymbol{\Pi}_V\boldsymbol{u} - \boldsymbol{\Pi}_M\boldsymbol{u}\|_{\partial\mathcal{T}_h} \le 2\|\boldsymbol{u} - \boldsymbol{\Pi}_V\boldsymbol{u}\|_{\partial\mathcal{T}_h} \le Ch^{s-\frac{1}{2}}\|\boldsymbol{u}\|_s.$$

Since the other two terms defining Θ can be bounded in a similar manner, the estimate follows. This completes the proof. □

Step 3: Estimate of $p - p_h$. In order to obtain an estimate for the pressure, we use the standard *inf-sup* argument.

Theorem 8. *We have*

$$\|e_p\|_0 \le C(\|\mathrm{E_L}\|_0 + \tau^{\frac{1}{2}}h^{\frac{1}{2}}\|\tau^{\frac{1}{2}}(\boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}})\|_{\partial\mathcal{T}_h}) + Ch^s(\tau\|\boldsymbol{u}\|_s + \|\mathrm{L}\|_s + \|p\|_s),$$

for all $1 \le s \le k+1$.

An estimate of $p - p_h$ can be now by triangle inequality:

$$\|p - p_h\|_0 \le C(\|\mathrm{E_L}\|_0 + \tau^{\frac{1}{2}}h^{\frac{1}{2}}\|\tau^{\frac{1}{2}}(\boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}})\|_{\partial\mathcal{T}_h}) + Ch^s(\tau\|\boldsymbol{u}\|_s + \|\mathrm{L}\|_s + \|p\|_s),$$

for all $1 \le s \le k+1$. Let us prove this result.

PROOF. It is well known [23] that for any function $q \in L^2(\Omega)$ such that $(q,1)_\Omega = 0$ we have

$$\|q\|_0 \leq \kappa \sup_{\boldsymbol{w}\in \boldsymbol{H}^1(\Omega)\backslash\{\boldsymbol{0}\}} \frac{(q, \nabla\cdot \boldsymbol{w})}{\|\boldsymbol{w}\|_1},$$

for some constant κ independent of q. By the last error equation, (A.2f), we can take $q := e_p$ and get

$$\|e_p\|_0 \leq C \sup_{\boldsymbol{w}\in \boldsymbol{H}^1(\Omega)\backslash\{\boldsymbol{0}\}} \frac{(e_p, \nabla\cdot \boldsymbol{w})}{\|\boldsymbol{w}\|_1}.$$

Next, we work on the numerator of the right-hand side. We have

$$(e_p, \nabla\cdot\boldsymbol{w})_\Omega = (e_p, \nabla\cdot \boldsymbol{\Pi}_V\boldsymbol{w})_{\mathcal{T}_h} + \langle e_p, (\boldsymbol{w} - \boldsymbol{\Pi}_V\boldsymbol{w})\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}. \tag{A.3}$$

Taking $\boldsymbol{v} := \boldsymbol{\Pi}_V\boldsymbol{w}$ in the error equation (A.2b), and inserting this equation into above identity, we get

$$\begin{aligned}(e_p, \nabla\cdot\boldsymbol{w})_\Omega &= (\nu\mathrm{E_L}, \nabla\boldsymbol{\Pi}_V\boldsymbol{w})_{\mathcal{T}_h} - \langle \nu\mathrm{E}_{\widehat{\mathrm{L}}}\,\boldsymbol{n} - e_{\widehat{p}}\,\boldsymbol{n}, \boldsymbol{\Pi}_V\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}\\ &\quad + \langle e_p\boldsymbol{n}, \boldsymbol{w} - \boldsymbol{\Pi}_V\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}\\ &= (\nu\mathrm{E_L}, \nabla\boldsymbol{w})_{\mathcal{T}_h} - \langle \nu\mathrm{E}_{\widehat{\mathrm{L}}}\,\boldsymbol{n} - e_{\widehat{p}}\,\boldsymbol{n}, \boldsymbol{\Pi}_V\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}\\ &\quad - \langle \nu\mathrm{E_L}\boldsymbol{n} - e_p\boldsymbol{n}, \boldsymbol{w} - \boldsymbol{\Pi}_V\boldsymbol{w}\rangle_{\partial\mathcal{T}_h},\end{aligned}$$

after integrating by parts the first term of the right-hand side and using the definition of the L^2-projection $\boldsymbol{\Pi}_V$.

Now, by the error equation (A.2e), we know that $\nu\mathrm{E}_{\widehat{\mathrm{L}}}\,\boldsymbol{n} - e_{\widehat{p}}\,\boldsymbol{n}$ is continuous across the interior faces, so $\langle \nu\mathrm{E}_{\widehat{\mathrm{L}}}\,\boldsymbol{n} - e_{\widehat{p}}\,\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} = \langle \nu\mathrm{E}_{\widehat{\mathrm{L}}}\,\boldsymbol{n} - e_{\widehat{p}}\,\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\Omega} = 0$. Inserting this to the above expression, we obtain,

$$\begin{aligned}(e_p, \nabla\cdot\boldsymbol{w})_\Omega &= (\nu\mathrm{E_L}, \nabla\boldsymbol{w})_{\mathcal{T}_h} + \langle \nu\mathrm{E}_{\widehat{\mathrm{L}}}\,\boldsymbol{n} - e_{\widehat{p}}\,\boldsymbol{n} - (\nu\mathrm{E_L}\boldsymbol{n} - e_p\boldsymbol{n}), \boldsymbol{w} - \boldsymbol{\Pi}_V\boldsymbol{w}\rangle_{\partial\mathcal{T}_h},\\ &= (\nu\mathrm{E_L}, \nabla\boldsymbol{w})_{\mathcal{T}_h} - \langle \tau(\boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}}), \boldsymbol{w} - \boldsymbol{\Pi}_V\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}\\ &\qquad + \langle \tau(\boldsymbol{\Pi}_V\boldsymbol{u} - \boldsymbol{\Pi}_M\boldsymbol{u}), \boldsymbol{w} - \boldsymbol{\Pi}_V\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}\\ &\qquad - \langle \nu(\Pi_G\mathrm{L}\boldsymbol{n} - \boldsymbol{\Pi}_M\mathrm{L}\boldsymbol{n}), \boldsymbol{w} - \boldsymbol{\Pi}_V\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}\\ &\qquad + \langle \Pi_Q p\boldsymbol{n} - \boldsymbol{\Pi}_M p\boldsymbol{n}, \boldsymbol{w} - \boldsymbol{\Pi}_V\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

proceeding as in the proof of Lemma 2.

Applying the Cauchy-Schwartz inequality to the terms on the right-hand side, and proceeding as in the proof in Theorem 7, we obtain:

$$
\begin{aligned}
(e_p, \nabla\cdot \boldsymbol{w})_\Omega \le\ & \|\boldsymbol{w}\|_1(\|\mathrm{E_L}\|_0 + \tau^{\frac{1}{2}} h^{\frac{1}{2}}\|\tau^{\frac{1}{2}}(\boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}})\|_{\partial\mathcal{T}_h})\\
&+ C\|\boldsymbol{w}\|_1 h^s(\tau\|\boldsymbol{u}\|_s + \|\mathrm{L}\|_s + \|p\|_s),
\end{aligned}
$$

for all $1 \le s \le k+1$. This completes the proof. $\square$

Step 4: Estimate of $\boldsymbol{u}-\boldsymbol{u}_h$. To obtain the result, we obtain an intermediate result written in terms of the so-called dual problem we introduce next. Let $(\mathrm{Z}, \boldsymbol{\sigma}, \eta)$ be the solution of

$$
\begin{aligned}
\mathrm{Z} - \nabla\boldsymbol{\sigma} &= 0 \quad \text{on } \Omega, && \text{(A.4a)}\\
\nabla\cdot(\nu\mathrm{Z}) - \nabla\eta &= \boldsymbol{e}_u \ \ \text{on } \Omega, && \text{(A.4b)}\\
-\nabla\cdot\boldsymbol{\sigma} &= 0 \quad \text{on } \Omega, && \text{(A.4c)}\\
\boldsymbol{\sigma} &= 0 \quad \text{on } \partial\Omega. && \text{(A.4d)}
\end{aligned}
$$

We assume that, for some real number s, we have the regularity property

$$
\|\mathrm{Z}\|_1 + \|\boldsymbol{\sigma}\|_2 + \|\eta\|_1 \le C_{reg}\|\boldsymbol{e}_u\|_0. \tag{A.5}
$$

In the two-dimensional case, the above estimate follows from the results in [24] when the domain is convex. In the three-dimensional case, the above estimate follows from the results in [25] for any convex polyhedron.

We begin by introducing an identity obtained by a classic duality argument.

Lemma 3. *Let* $(\mathrm{Z}, \boldsymbol{\sigma}, \eta)$ *be the solution of* (A.4). *Then we have*

$$
\begin{aligned}
\|\boldsymbol{e}_u\|_0^2 =\ & \langle \boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}}, \nu(\mathrm{Z} - \Pi_G\mathrm{Z})\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}}, (\eta - \Pi_Q\eta)\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}\\
&- \langle \nu\mathrm{E_L}\boldsymbol{n} - e_p\boldsymbol{n} - (\nu\mathrm{E}_{\widehat{\mathrm{L}}}\,\boldsymbol{n} - e_{\widehat{p}}\,\boldsymbol{n}), \boldsymbol{\sigma} - \boldsymbol{\Pi}_V\boldsymbol{\sigma}\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

PROOF. By the dual problem (A.4), we have

$$
\|\boldsymbol{e}_u\|_0^2 = (\boldsymbol{e}_u, \nabla\cdot(\nu\mathrm{Z}))_\Omega - (\boldsymbol{e}_u, \nabla\eta)_\Omega + (\nu\mathrm{E_L}, \mathrm{Z} - \nabla\boldsymbol{\sigma})_\Omega + (e_p, \nabla\cdot\boldsymbol{\sigma})_\Omega.
$$

Integrating by parts, using the L^2 projections and integrating by parts again, we get, after rearranging terms, that

$$
\begin{aligned}
\|\boldsymbol{e}_u\|_0^2 =\ & (\boldsymbol{e}_u, \nabla\cdot(\nu\Pi_G\mathrm{Z}))_{\mathcal{T}_h} + (\mathrm{E_L}, \nu\Pi_G\mathrm{Z})_{\mathcal{T}_h} + \langle \boldsymbol{e}_u, \nu(\mathrm{Z} - \Pi_G\mathrm{Z})\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}\\
&-(\nu\mathrm{E_L}, \nabla\boldsymbol{\Pi}_V\boldsymbol{\sigma})_{\mathcal{T}_h} + (e_p, \nabla\cdot\boldsymbol{\Pi}_V\boldsymbol{\sigma})_{\mathcal{T}_h} - \langle \nu\mathrm{E_L}\boldsymbol{n} - e_p\boldsymbol{n}, \boldsymbol{\sigma} - \boldsymbol{\Pi}_V\boldsymbol{\sigma}\rangle_{\partial\mathcal{T}_h}\\
&-(\boldsymbol{e}_u, \nabla\Pi_Q\eta)_{\mathcal{T}_h} - \langle \boldsymbol{e}_u, (\eta - \Pi_Q\eta)\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}
\end{aligned}
$$

Taking $(\mathrm{G}, \boldsymbol{v}, q) := (\Pi_G \mathrm{Z}, \boldsymbol{\Pi}_V \boldsymbol{\sigma}, \Pi_Q \eta)$ in the error equations (A.2a)-(A.2c), inserting these equations into the above equation and simplifying, we obtain

$$
\begin{aligned}
\|\boldsymbol{e}_u\|_0^2 =& \langle \boldsymbol{e}_{\widehat{u}}, \nu \Pi_G \mathrm{Z} \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} + \langle \boldsymbol{e}_u, \nu(\mathrm{Z} - \Pi_G \mathrm{Z}) \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} \\
& - \langle \nu \mathrm{E}_{\widehat{\mathrm{L}}} \boldsymbol{n} - e_{\widehat{p}} \boldsymbol{n}, \boldsymbol{\Pi}_V \boldsymbol{\sigma} \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} - \langle \nu \mathrm{E}_{\mathrm{L}} \boldsymbol{n} - e_p \boldsymbol{n}, (\boldsymbol{\sigma} - \boldsymbol{\Pi}_V \boldsymbol{\sigma}) \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} \\
& - \langle \boldsymbol{e}_{\widehat{u}}, \Pi_Q \eta \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} - \langle \boldsymbol{e}_u, (\eta - \Pi_Q \eta) \boldsymbol{n} \rangle_{\partial \mathcal{T}_h}.
\end{aligned}
$$

Noting that the solution $(\mathrm{Z}, \boldsymbol{\sigma}, \eta)$ is continuous across the interior faces, and using the error equation about the Dirichlet boundary condition (A.2d) and that of the transmission condition (A.2e), we have

$$
\langle \boldsymbol{e}_{\widehat{u}}, \mathrm{Z} \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = \langle \boldsymbol{e}_{\widehat{u}}, \eta \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = \langle \nu \mathrm{E}_{\widehat{\mathrm{L}}} \boldsymbol{n} - e_{\widehat{p}} \boldsymbol{n}, \boldsymbol{\sigma} \rangle_{\partial \mathcal{T}_h} = 0.
$$

The identity we want to prove now follows after using this identity and rearranging terms. □

We are now ready to get the estimate for $\boldsymbol{e}_u$.

Theorem 9. *If the regularity inequality* (A.5) *holds, we have*

$$
\|\boldsymbol{e}_u\| \leq C(\tau^{-\frac{1}{2}} h^{\frac{1}{2}} + \tau^{\frac{1}{2}} h^{\frac{3}{2}}) \|\tau^{\frac{1}{2}} (\boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}})\|_{\partial \mathcal{T}_h} + C h^{s+1} (\tau \|\boldsymbol{u}\|_s + \|\mathrm{L}\|_s + \|p\|_s),
$$

for all $1 \leq s \leq k+1$.

PROOF. From the proof of Lemma 2, we have

$$
\begin{aligned}
\nu \mathrm{E}_{\mathrm{L}} \boldsymbol{n} - e_p \boldsymbol{n} - (\nu \mathrm{E}_{\widehat{\mathrm{L}}} \boldsymbol{n} - e_{\widehat{p}} \boldsymbol{n}) =& \tau(\boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}}) - \tau(\boldsymbol{\Pi}_V \boldsymbol{u} - \boldsymbol{\Pi}_M \boldsymbol{u}) \\
& + \nu(\Pi_G \mathrm{L} \boldsymbol{n} - \boldsymbol{\Pi}_M \mathrm{L} \boldsymbol{n}) - (\Pi_Q p \boldsymbol{n} - \boldsymbol{\Pi}_M p \boldsymbol{n}).
\end{aligned}
$$

Applying the above identity in Lemma 3, we obtain

$$
\begin{aligned}
\|\boldsymbol{e}_u\|_0^2 =& \langle \boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}}, \nu(\mathrm{Z} - \Pi_G \mathrm{Z}) \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} - \langle \boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}}, (\eta - \Pi_Q \eta) \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} \\
& - \langle \tau(\boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}}), \boldsymbol{\sigma} - \boldsymbol{\Pi}_V \boldsymbol{\sigma} \rangle_{\partial \mathcal{T}_h} + \langle \tau(\boldsymbol{\Pi}_V \boldsymbol{u} - \boldsymbol{\Pi}_M \boldsymbol{u}), \boldsymbol{\sigma} - \boldsymbol{\Pi}_V \boldsymbol{\sigma} \rangle_{\partial \mathcal{T}_h} \\
& - \langle \nu(\Pi_G \mathrm{L} \boldsymbol{n} - \boldsymbol{\Pi}_M \mathrm{L} \boldsymbol{n}), \boldsymbol{\sigma} - \boldsymbol{\Pi}_V \boldsymbol{\sigma} \rangle_{\partial \mathcal{T}_h} + \langle \Pi_Q p \boldsymbol{n} - \boldsymbol{\Pi}_M p \boldsymbol{n}, \boldsymbol{\sigma} - \boldsymbol{\Pi}_V \boldsymbol{\sigma} \rangle_{\partial \mathcal{T}_h}.
\end{aligned}
$$

If we now apply the Cauchy-Schwartz inequality on each term of the right-hand side, we get

$$
\begin{aligned}
\|\boldsymbol{e}_u\|_0^2 \leq & C \|\tau^{\frac{1}{2}} (\boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}})\|_{\partial \mathcal{T}_h} \Big\{ \tau^{-\frac{1}{2}} \|(\mathrm{Z} - \Pi_G \mathrm{Z}) \boldsymbol{n}\|_{\partial \mathcal{T}_h} + \tau^{-\frac{1}{2}} \|\eta - \Pi_Q \eta\|_{\partial \mathcal{T}_h} \\
& \qquad + \tau^{\frac{1}{2}} \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_V \boldsymbol{\sigma}\|_{\partial \mathcal{T}_h} \Big\} \\
& + C \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_V \boldsymbol{\sigma}\|_{\partial \mathcal{T}_h} \Big\{ \tau \|\boldsymbol{\Pi}_V \boldsymbol{u} - \boldsymbol{\Pi}_M \boldsymbol{u}\|_{\partial \mathcal{T}_h} + \|\Pi_G \mathrm{L} \boldsymbol{n} - \boldsymbol{\Pi}_M \mathrm{L} \boldsymbol{n}\|_{\partial \mathcal{T}_h} \\
& \qquad + \|\Pi_Q p \boldsymbol{n} - \boldsymbol{\Pi}_M p \boldsymbol{n}\|_{\partial \mathcal{T}_h} \Big\}.
\end{aligned}
$$

By standard inverse inequalities and the approximation properties of the projections (A.1), we obtain

$$\begin{aligned}\|\boldsymbol{e}_u\|_0^2 \leq C\|\tau^{\frac{1}{2}}(\boldsymbol{e}_u - \boldsymbol{e}_{\widehat{u}})\|_{\partial\mathcal{T}_h}(\tau^{-\frac{1}{2}}h^{\frac{1}{2}}\|\mathrm{Z}\|_1 + \tau^{-\frac{1}{2}}h^{\frac{1}{2}}\|\eta\|_1 + \tau^{\frac{1}{2}}h^{\frac{3}{2}}\|\boldsymbol{\sigma}\|_2)\\ + Ch^{\frac{3}{2}}\|\boldsymbol{\sigma}\|_2(\tau h^{s-\frac{1}{2}}\|\boldsymbol{u}\|_s + h^{s-\frac{1}{2}}\|\mathrm{L}\|_s + h^{s-\frac{1}{2}}\|p\|_s),\end{aligned}$$

and the estimate follows after using the regularity inequality (A.5). □

Step 4: Conclusion. From Theorems 7, 8, 9, we can see that if we take τ and $1/\tau$ of order one, the estimates of the errors are

$$\begin{aligned}\|\mathrm{L} - \mathrm{L}_h\|_0 &\leq Ch^{s-\frac{1}{2}}(\|\boldsymbol{u}\|_s + \|\mathrm{L}\|_s + \|p\|_s),\\ \|p - p_h\|_0 &\leq Ch^{s-\frac{1}{2}}(\|\boldsymbol{u}\|_s + \|\mathrm{L}\|_s + \|p\|_s),\\ \|\boldsymbol{u} - \boldsymbol{u}_h\|_0 &\leq Ch^{s}(\|\boldsymbol{u}\|_s + \|\mathrm{L}\|_s + \|p\|_s),\end{aligned}$$

for all $1 \leq s \leq k+1$. This shows that for general mesh, we can obtain optimal orders of convergence for the approximation to the velocity $\boldsymbol{u}$, and suboptimal orders, by 1/2, for the approximations to velocity gradient L and to the pressure p.

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