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Asymptotics of integrals

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Standard methods in asymptotic expansions [1] of integrals are illustrated: *Watson's lemma* and *Laplace's method*. Watson's lemma dates from at latest [Watson 1918a], and Laplace's method at latest from [Laplace 1774]. The exposition here is revisionist, in that Laplace's method is reduced to Watson's lemma.

For example, a simple argument gives the main term [2] in the asymptotics for $\Gamma(s)$:

$$\Gamma(s) \;\sim\; \sqrt{2\pi}\, e^{-s}\, s^{s-\frac{1}{2}} \qquad\qquad (\text{as } |s|\to\infty, \text{ with } \mathrm{Re}(s)\ge\delta>0)$$

and obtain a useful result about ratios of gamma functions,

$$\frac{\Gamma(s+a)}{\Gamma(s)} \;\sim\; s^a \qquad\qquad (\text{as } |s|\to\infty, \text{ for fixed } a, \text{ for } \mathrm{Re}(s)\ge\delta>0)$$

The latter is awkward to obtain as a corollary from Stirling's formula. Laplace's method is further illustrated by an application to asymptotics of functions closely related to Bessel functions, namely, for any fixed spectral parameter $\nu\in\mathbb{R}$,

$$\sqrt{y}\int_0^\infty e^{-(u+\frac{1}{u})y}\, u^{i\nu}\,\frac{du}{u} \;\sim\; \sqrt{\pi}\cdot e^{-2y} \qquad\qquad (\text{as } y\to+\infty)$$

One point is avoidance of standard but immemorable arguments special to the gamma function. Of course, these special arguments do bear more forcefully upon gamma itself: see [Whittaker-Watson 1927] or [Lebedev 1963]. However, to the extent possible, we want to understand the asymptotics of gamma and other important special functions on general principles.

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[1] The simplest notion of *asymptotic* $F(s)$ for $f(s)$ as s goes to $+\infty$ on $\mathbb{R}$, or in a sector in $\mathbb{C}$, is a simpler function $F(s)$ such that $\lim_s f(s)/F(s)=1$, written $f\sim F$. One might require an error estimate, for example,

$$f\sim F \iff f(s)=F(s)\cdot\Big(1+O\Big(\frac{1}{|s|}\Big)\Big)$$

A more precise form is to say that

$$f(s) \;\sim\; f_0(s)\cdot\Big(\frac{c_0}{s^\alpha}+\frac{c_1}{s^{\alpha+1}}+\frac{c_2}{s^{\alpha+2}}+\ldots\Big)$$

with any auxiliary function f_0, is an *asymptotic expansion* for f when

$$f \;=\; f_0(s)\cdot\Big(\frac{c_0}{s^\alpha}+\frac{c_1}{s^{\alpha+1}}+\ldots+\frac{c_n}{s^{\alpha+n}}+O\Big(\frac{1}{|s|^{\alpha+n+1}}\Big)\Big)$$

[2] The *main* term in the asymptotics for $\Gamma(s)$ is due to Stirling. Higher terms are due to Binet, and perhaps Laplace.

1. *Heuristic for the main term in asymptotics for* $\Gamma(s)$

A memorable heuristic for Stirling's formula for the main term in the asymptotics of $\Gamma(s)$, namely

$$\Gamma(s) \;\sim\; e^{-s} \cdot s^{s-\frac{1}{2}} \cdot \sqrt{2\pi}$$

Using Euler's integral,

$$s \cdot \Gamma(s) \;=\; \Gamma(s+1) \;=\; \int_0^\infty e^{-u}\, u^{s+1}\, \frac{du}{u} \;=\; \int_0^\infty e^{-u}\, u^s\, du \;=\; \int_0^\infty e^{-u+s\log u}\, du$$

The trick is to replace the exponent $-u+s\log u$ by the quadratic polynomial in u best approximating it near its maximum, and evaluate the resulting integral. This replacement can be justified via Watson's lemma and Laplace's method, below, but the heuristic is simpler than the justification.

More precisely, the exponent takes its maximum where its derivative vanishes, at the unique solution $u_o = s$ of

$$-1 + \frac{s}{u} \;=\; 0$$

The second derivative in u of the exponent is $-s/u^2$, which takes value $-1/s$ at $u_o = s$. Thus, near $u_o = s$, the quadratic Taylor-Maclaurin polynomial in t approximating the exponent is

$$-s + s\log s - \frac{1}{2!\, s} \cdot (u-s)^2$$

Thus, we imagine that

$$s \cdot \Gamma(s) \;\sim\; \int_0^\infty e^{-s+s\log s - \frac{1}{2s}\cdot(u-s)^2}\, du \;=\; e^{-s} \cdot s^s \cdot \int_{-\infty}^\infty e^{-\frac{1}{2s}\cdot(u-s)^2}\, du$$

Note that the latter integral is taken over the whole real line. [3] To simplify the remaining integral, replace u by su and cancel a factor of s from both sides,

$$\Gamma(s) \;\sim\; e^{-s} \cdot s^s \cdot \int_{-\infty}^\infty e^{-s(u-1)^2/2}\, du$$

Then replace u by $u+1$, and then u by $u \cdot \sqrt{2\pi/s}$, obtaining

$$\int_{-\infty}^\infty e^{-s(u-1)^2/2}\, du \;=\; \int_{-\infty}^\infty e^{-su^2/2}\, du \;=\; \frac{\sqrt{2\pi}}{\sqrt{s}} \int_{-\infty}^\infty e^{-\pi u^2}\, du \;=\; \frac{\sqrt{2\pi}}{\sqrt{s}}$$

We obtain

$$\Gamma(s) \;\sim\; e^{-s} \cdot s^{s-\frac{1}{2}} \cdot \sqrt{2\pi}$$

Several aspects of this heuristic are dubious, so it is striking that it can be made rigorous, as below.

[3] Evaluation of the integral over the whole line, and simple estimates on the integral over $(-\infty, 0]$, show that the integral over $(-\infty, 0]$ is of a lower order of magnitude than the whole. Thus, the leading term of the asymptotics of the integral over the whole line is the same than the integral from 0 to $+\infty$.

2. *Watson's lemma*

The often-rediscovered *Watson's lemma*[4] gives asymptotic expansions valid in half-planes in $\mathbb{C}$ for *Laplace transform* integrals. For example, for smooth h on $(0, +\infty)$ with all derivatives of *polynomial growth*, and expressible for small $x > 0$ as

$$h(x) \;=\; x^\alpha \cdot g(x) \qquad\qquad \text{(for } x > 0\text{, some } \alpha \in \mathbb{C})$$

where $g(x)$ is differentiable[5] on $\mathbb{R}$ near 0. Thus, $h(x)$ has an asymptotic expansion at 0^+

$$h(x) \;\sim\; x^\alpha \cdot \sum_{n=0}^{\infty} c_n\, x^n \qquad\qquad \text{(Taylor-Maclaurin } \textit{asymptotic expansion} \text{ for } x \to 0^+)$$

Watson's Lemma gives an *asymptotic expansion* of the Laplace transform of h:

$$\int_0^\infty e^{-sx}\, h(x)\, \frac{dx}{x} \quad\sim\quad \frac{\Gamma(\alpha)\, c_0}{s^\alpha} + \frac{\Gamma(\alpha+1)\, c_1}{s^{\alpha+1}} + \frac{\Gamma(\alpha+2)\, c_2}{s^{\alpha+2}} + \ldots \qquad\qquad \text{(for } \mathrm{Re}(s) > 0)$$

The error estimates below give

$$\int_0^\infty e^{-sx}\, h(x)\, \frac{dx}{x} \;=\; \int_0^\infty e^{-sx}\, x^\alpha\, g(x)\, \frac{dx}{x} \;=\; \frac{\Gamma(\alpha)\, g(0)}{s^\alpha} + O\Big(\frac{1}{|s|^{\mathrm{Re}(\alpha)+1}}\Big)$$

Similar conclusions hold for errors after finite sum of terms.

The idea is straightforward: the expansion is obtained from

$$\int_0^\infty e^{-sx}\, h(x)\, \frac{dx}{x} \;=\; \int_0^\infty e^{-sx}\, x^\alpha \big(c_0 + \ldots + c_n x^n\big)\, \frac{dx}{x} \;+\; \int_0^\infty e^{-sx}\, x^\alpha\, \Big(g(x) \;-\; \big(c_0 + \ldots + c_n x^n\big)\Big)\, \frac{dx}{x}$$

The first integral gives the asymptotic expansion, and for $\mathrm{Re}(s) > 0$ the second integral can be integrated by parts and trivially bounded to give the error term.

To understand the error, let $\varepsilon \geq 0$ be the smallest such that

$$N \;=\; \mathrm{Re}(\alpha) + n - \varepsilon \;\in\; \mathbb{Z}$$

The subtraction of the initial polynomial and re-allocation of the $1/x$ from the measure makes $x^{\alpha-1}(g(x) - (c_0 + \ldots + c_n x^n)$ vanish to order N at 0. This, with the exponential e^{-sx} and the presumed polynomial growth of h and its derivatives, allows integration by parts N times without boundary terms, giving

$$\int_0^\infty e^{-sx}\, h(x)\, dx \quad=\quad \frac{\Gamma(\alpha)\, c_0}{s^\alpha} + \frac{\Gamma(\alpha+1)\, c_1}{s^{\alpha+1}} + \ldots + \frac{\Gamma(\alpha+n)\, c_n}{s^{\alpha+n}}$$
$$+\;\; \frac{1}{s^N} \int_0^\infty e^{-sx}\, \Big(\frac{\partial}{\partial x}\Big)^N \Big(x^\alpha \cdot \big(g(x) - (c_0 + \ldots + c_n x^n)\big)\Big)\, dx$$

[4] This lemma appeared in the treatise [Watson 1922] on page 236, citing [Watson 1918a], page 133. Curiously, the aggregate bibliography of [Watson 1922] omitted [Watson 1918a], and the footnote mentioning it gave no title. Happily, [Watson 1918a] is mentioned by title in [Blaustein-Handelsman 1975]. In the bibliography at the end, we note [Watson 1917], [Watson 1918a], [Watson 1918b].

[5] g need not be *real-analytic* near 0, only *smooth* to the right of 0, so it and its derivatives have finite Taylor-Maclaurin expansions approximating it as $x \to 0^+$.

Although the indicated leftover term is typically larger than the last term in the asymptotic expansion, it is smaller than the *next*-to-last term, so the desired conclusion holds: for $h(x)$ with asymptotic expansion at 0^+

$$h(x) \;\sim\; x^\alpha \cdot \sum_{n=0}^\infty c_n\, x^n \qquad\qquad \text{(Taylor-Maclaurin } \textit{asymptotic expansion} \text{ for } x \to 0^+)$$

and it and its derivatives of polynomial growth as $h \to +\infty$, the Laplace transform has asymptotic expansion

$$\int_0^\infty e^{-sx}\, h(x)\, dx \;=\; \frac{\Gamma(\alpha)\, c_0}{s^\alpha} + \frac{\Gamma(\alpha+1)\, c_1}{s^{\alpha+1}} + \ldots + \frac{\Gamma(\alpha+n)\, c_n}{s^{\alpha+n}} + O\Big(\frac{1}{|s|^{\mathrm{Re}\,(\alpha)+n+1}}\Big) \qquad \text{(for } n = 1, 2, 3, \ldots)$$

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3. *Watson's lemma and* $\Gamma(s)/\Gamma(s+a)$

A useful asymptotic awkward to derive from Stirling's formula for $\Gamma(s)$, but easy to obtain from Watson's lemma, is an asymptotic for Euler's beta integral [6]

$$B(s,a) \;=\; \int_0^1 x^{s-1}\, (1-x)^{a-1}\, dx \;=\; \frac{\Gamma(s)\,\Gamma(a)}{\Gamma(s+a)}$$

Fix a with $\mathrm{Re}\,(a) > 0$, and consider this integral as a function of s. Setting $x = e^{-u}$ gives an integrand fitting Watson's lemma,

$$B(s,a) \;=\; \int_0^\infty e^{-su}\, (1-e^{-u})^{a-1}\, du \;=\; \int_0^\infty e^{-su}\, (u - \frac{u^2}{2!} + \ldots)^{a-1}\, du$$

$$= \int_0^\infty e^{-su}\, u^a \cdot (1 - \frac{u}{2!} + \ldots)^{a-1}\, \frac{du}{u} \;\sim\; \frac{\Gamma(a)}{s^a} \qquad\qquad \text{(for fixed } a)$$

taking just the first term in an asymptotic expansion, using Watson's lemma. Thus,

$$\frac{\Gamma(s)\,\Gamma(a)}{\Gamma(s+a)} \;\sim\; \frac{\Gamma(a)}{s^a} \qquad\qquad \text{(for fixed } a)$$

giving

$$\frac{\Gamma(s)}{\Gamma(s+a)} \;\sim\; \frac{1}{s^a} \qquad\qquad \text{(for fixed } a)$$

That is,

$$\frac{\Gamma(s)}{\Gamma(s+a)} \;=\; \frac{1}{s^a} + O\Big(\frac{1}{|s|^{\mathrm{Re}\,(a)+1}}\Big) \qquad\qquad \text{(for fixed } a)$$

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[6] We recall how to obtain the expression for beta in terms of gamma. With $x = u/(u+1)$ in the beta integral,

$$B(s,a) \;=\; \int_0^\infty u^{s-1}\, (u+1)^{-(s-1)-(a-1)-2}\, du \;=\; \int_0^\infty u^{s-1}\, (u+1)^{-s-a}\, du$$

$$= \frac{1}{\Gamma(s+a)} \int_0^\infty \int_0^\infty u^s\, e^{-v(u+1)}\, v^{s+a}\, \frac{dv}{v}\, \frac{du}{u}$$

using $\int_0^\infty e^{-vy}\, v^b\, dv/v = \Gamma(b)/y^b$. Replacing u by u/v gives $B(s,a) = \Gamma(s)\Gamma(a)/\Gamma(s+a)$.

4. *Main term in asymptotics by Laplace's method*

Laplace's method [7] obtains asymptotics in s for integrals

$$\int_0^\infty e^{-s\cdot f(u)}\,du \qquad\qquad (\text{for } f \text{ real-valued, } \mathrm{Re}(s) > 0)$$

Information attached to u *minimizing* $f(u)$ dominate. For a *unique* minimum, at u_o, with $f''(u_o) > 0$, the main term of the asymptotic expansion is

$$\int_0^\infty e^{-s\cdot f(u)}\,du \quad\sim\quad e^{-sf(u_o)}\cdot\frac{\sqrt{2\pi}}{\sqrt{f''(u_o)}}\cdot\frac{1}{\sqrt{s}} \qquad\qquad (\text{for } |s|\to\infty, \text{ with } \mathrm{Re}(s)\ge\delta>0)$$

This reduces to a variant [8] of Watson's lemma, breaking the integral at points where the derivative f' changes sign, and changing variables to convert each fragment to a Watson-lemma integral. The function f must be smooth, with all derivatives of at most polynomial growth *and* at most polynomial *decay*, as $u \to +\infty$.

[4.0.1] Example: An integral $\int_0^\infty e^{-sy^2}\,h(y)\,\frac{dy}{y}$ is not quite in the form to apply the simplest version of Watson's lemma. Replacing y by $\sqrt{x}$ corrects the exponential

$$\int_0^\infty e^{-sy^2}\,h(y)\,\frac{dy}{y} \;=\; \int_0^\infty e^{-sx}\,\tfrac{1}{2}h(\sqrt{x})\,\frac{dx}{x}$$

but the asymptotic expansion of $h(\sqrt{x})$ at 0^+ will be in powers of $\sqrt{x}$. This is harmless, by a variety of possible adaptations.

[4.0.2] Remark: In fact, in the discussion below, the *odd* powers of $x^{\frac{1}{2}}$ will *cancel.*

For simplicity assume exactly *one* point u_o at which $f'(u_o) = 0$, and that $f''(u_o) > 0$, and that $f(u)$ goes to $+\infty$ at 0^+ and at $+\infty$. [9] Since $f'(u) > 0$ for $u > u_o$ and $f'(u) < 0$ for $0 < u < u_o$, there are functions F, G *smooth near* 0 such that

$$\begin{cases} F\big(\sqrt{f(u)-f(u_o)}\big) &=& u & \qquad (\text{for } u_o < u < +\infty) \\ G\big(\sqrt{f(u)-f(u_o)}\big) &=& u & \qquad (\text{for } 0 < u < u_o) \end{cases}$$

Let $y = \sqrt{f(u)-f(u_o)}$ in both integrals, noting that $F(y) = u$ gives $\frac{dy}{du}\cdot F'(y) = 1$, obtaining integrals *almost* as in Watson's lemma:

$$\int_0^\infty e^{-s\cdot f(u)}\,du \;=\; e^{-sf(u_o)}\Big(\int_0^{u_o} e^{-sy^2}\,du + \int_{u_o}^\infty e^{-sy^2}\,du\Big) \;=\; e^{-sf(u_o)}\int_0^\infty e^{-sy^2}\,\Big(F'(y)+G'(y)\Big)\,dy$$

$$=\; e^{-sf(u_o)}\Big(\int_0^{u_o} e^{-sy^2}\,du + \int_{u_o}^\infty e^{-sy^2}\,du\Big) \;=\; e^{-sf(u_o)}\int_0^\infty e^{-sy^2}\,y\Big(F'(y)+G'(y)\Big)\,\frac{dy}{y}$$

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[7] Perhaps the first appearance of this is in [Laplace 1774].

[8] See [Miller 2006] for a thorough discussion of variants of Watson's lemma.

[9] The hypothesis of exactly *one* point u_o at which $f'(u_o) = 0$, that $f''(u_o) > 0$, and that $f(u)$ goes to $+\infty$ at 0^+ and at $+\infty$, holds in two important examples, namely, $f(u) = u - \log u$ for Euler's integral for $\Gamma(s)$, and $f(u) = u + \frac{1}{u}$ for the Bessel function.

Since F, G are smooth near $y = 0$, they do have Taylor-Maclaurin asymptotics in y near 0. To convert the integrals to integrals of the form in Watson's lemma, replace y by $\sqrt{x}$. This would seem to require extending Watson's lemma to tolerate asymptotic expansion of $F'(\sqrt{x}) + G'(\sqrt{x})$ in powers of $x^{\frac{1}{2}}$, but, in fact, the *odd* powers of $x^{\frac{1}{2}}$ *cancel*. Derivatives of f must increase *or decrease* only polynomially as $u \to +\infty$. An asymptotic near $x = 0$ of the form

$$\tfrac{1}{2}\big(F'(\sqrt{x}) + G'(\sqrt{x})\big) \quad\sim\quad c_0 + c_1x^1 + c_2x^2 + c_3x^3 + \ldots \qquad\qquad (\text{as } x \to 0^+)$$

follows from a Taylor-Maclaurin expansion of $F'(y) + G'(y)$. Watson's lemma gives asymptotic expansion

$$\int_0^\infty e^{-s\cdot f(u)}\, du \;=\; e^{-sf(u_o)} \int_0^\infty e^{-sy^2}\, y\Big(F'(y)+G'(y)\Big)\,\frac{dy}{y} \;=\; e^{-sf(u_o)} \int_0^\infty e^{-sx}\, \tfrac{1}{2}x^{\frac{1}{2}}\Big(F'(\sqrt{x})+G'(\sqrt{x})\Big)\,\frac{dx}{x}$$

$$\sim\quad \frac{\Gamma(\frac{1}{2})\,c_0}{s^{\frac{1}{2}}} + \frac{\Gamma(\frac{3}{2})\,c_1}{s^{\frac{3}{2}}} + \frac{\Gamma(\frac{5}{2})\,c_2}{s^{\frac{5}{2}}} + \ldots \qquad\qquad (\text{for } \mathrm{Re}(s) > 0)$$

To determine only the leading coefficient $F'(0)$, $F(y) = u$ gives $F'(y)\cdot\frac{dy}{du} = 1$, so $F'(y) = 1/\frac{dy}{du}$. From

$$y \;=\; \sqrt{f(u) - f(u_o)} \;=\; \Big(\frac{f''(u_o)}{2!}\cdot(u-u_o)^2 + O\big((u-u_o)^3\big)\Big)^{1/2}$$

$$=\; (u-u_o)\cdot\sqrt{\frac{f''(u_o)}{2}}\cdot\Big(1 + O(u-u_o)\Big)^{1/2} \;=\; (u-u_o)\cdot\sqrt{\frac{f''(u_o)}{2}}\cdot\Big(1 + O(u-u_o)\Big)$$

the derivative is

$$\frac{dy}{du} \;=\; \sqrt{\frac{f''(u_o)}{2}} + O(u-u_o)$$

and

$$F'(y) \;=\; \frac{1}{\frac{dy}{du}} \;=\; \sqrt{\frac{2}{f''(u_o)}} \;+\; O(u-u_o)$$

At $y = 0$, also $u - u_o = 0$, so

$$F'(0) \;=\; \sqrt{\frac{2}{f''(u_o)}}$$

The same argument applied to G gives $G'(0) = F'(0)$, and Watson's lemma gives

$$\int_0^\infty e^{-s\,f(u)}\,du \;\sim\; e^{-sf(u_o)}\cdot\frac{\Gamma(\frac{1}{2})\cdot\sqrt{\frac{2}{f''(u_o)}}}{\sqrt{s}} \;=\; e^{-sf(u_o)}\cdot\frac{\sqrt{2\pi}}{f''(u_o)^{\frac{1}{2}}}\cdot\frac{1}{\sqrt{s}}$$

Last, this outcome would be obtained by replacing $f(u)$ by its quadratic approximation

$$f(u) \quad\sim\quad f(u_o) + \frac{f''(u_o)}{2!}\cdot(u-u_o)^2$$

Integrating over the whole line,

$$\int_{-\infty}^\infty e^{-s\cdot\left(f(u_o)+\frac{1}{2}f''(u_o)(u-u_o)^2\right)}\,du \;=\; e^{-sf(u_o)}\int_{-\infty}^\infty e^{-s\cdot\frac{1}{2}f''(u_o)(u-u_o)^2}\,du \;=$$

$$=\; e^{-sf(u_o)}\int_{-\infty}^\infty e^{-s\cdot\frac{1}{2}f''(u_o)u^2}\,du \;=\; e^{-sf(u_o)}\cdot\frac{\sqrt{\pi}}{\sqrt{\frac{1}{2}f''(u_o)}}\cdot\frac{1}{\sqrt{s}} \;=\; e^{-sf(u_o)}\cdot\frac{\sqrt{2\pi}}{\sqrt{f''(u_o)}}\cdot\frac{1}{\sqrt{s}}$$

This does indeed agree. Last, verify that the integral of the exponentiated quadratic approximation over $(-\infty, 0]$ is of a lower order of magnitude. Indeed, for $u \le 0$ and $u_o > 0$ we have $(u - u_o)^2 \ge u^2 + u_o^2$, and $f''(u_o) > 0$ by assumption, so

$$e^{-sf(u_o)} \int_{-\infty}^{0} e^{-s\cdot\left(\frac{1}{2}f''(u_o)(u-u_o)^2\right)}\,du \;\le\; e^{-sf(u_o)}\cdot e^{-s\cdot\frac{1}{2}f''(u_o)\cdot u_o^2} \int_{-\infty}^{0} e^{-s\cdot\frac{1}{2}f''(u_o)u^2}\,du$$

$$\le\; e^{-sf(u_o)}\cdot e^{-s\cdot\frac{1}{2}f''(u_o)\cdot u_o^2} \int_{-\infty}^{\infty} e^{-s\cdot\frac{1}{2}f''(u_o)u^2}\,du \;=\; e^{-sf(u_o)}\cdot e^{-s\cdot\frac{1}{2}f''(u_o)\cdot u_o^2}\cdot \frac{\sqrt{2\pi}}{\sqrt{f''(u_o)}}\cdot\frac{1}{\sqrt{s}}$$

Thus, the integral over $(-\infty, 0]$ has an additional exponential decay by comparison to the integral over the whole line.

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5. *Stirling's formula for main term in asymptotics for* $\Gamma(s)$

Stirling's formula for main term in asymptotics for $\Gamma(s)$ can be obtained in this context. For *real* $s > 0$, replacing u by su expresses Euler's integral for $\Gamma(s)$ as a product of an exponential and a Watson's-lemma integral:

$$s\cdot\Gamma(s) \;=\; \Gamma(s+1) \;=\; \int_0^\infty e^{-u}\,u^s\,du \;=\; \int_0^\infty e^{-u+s\log u}\,du$$

$$=\; \int_0^\infty e^{-su+s\log u+s\log s}\,s\,du \;=\; s\cdot e^{s\log s}\int_0^\infty e^{-s(u-\log u)}\,du$$

so

$$\Gamma(s) \;=\; e^{s\log s}\int_0^\infty e^{-s(u-\log u)}\,du$$

For complex s with $\mathrm{Re}(s) > 0$, both $s\cdot\Gamma(s)$ and the integral $s\cdot e^{s\log s}\int_0^\infty e^{-s(u+\log u)}\,du$ are holomorphic in s, and they *agree* for *real* s. The *identity principle* gives equality for $\mathrm{Re}(s) > 0$. With $f(u) = u - \log u$, the derivative $f'(u) = 1 - \frac{1}{u}$ has unique zero at $u_o = 1$, and $f''(1) = 0 + \frac{1}{1} = 1$. Thus,

$$\Gamma(s) \;\sim\; e^{s\log s}\cdot\left(e^{-sf(u_o)}\cdot\frac{\sqrt{2\pi}}{\sqrt{f''(u_o)}}\cdot\frac{1}{\sqrt{s}}\right) \;=\; e^{s\log s}\cdot e^{-s}\cdot\frac{\sqrt{2\pi}}{1}\cdot\frac{1}{\sqrt{s}} \;=\; \sqrt{2\pi}\,e^{(s-\frac{1}{2})\log s}\,e^{-s}$$

[5.0.1] Remark: As noted earlier, the odd powers of $x^{\frac{1}{2}}$ cancel, so $\frac{1}{2}(F'(\sqrt{x}) + G(\sqrt{x}))$ has an expansion $c_0 + c_1x + c_2x^2 + \ldots$, and the error estimate in the asymptotic expansion is

$$\Gamma(s) \;\sim\; \sqrt{2\pi}\,e^{(s-\frac{1}{2})\log s}\,e^{-s}\cdot\left(1 + O\left(\frac{1}{|s|}\right)\right)$$

6. *Laplace's method illustrated on Bessel functions*

The above discussion extends to treat the standard integral [10]

$$y \longrightarrow \sqrt{y}\int_0^\infty e^{-(u+\frac{1}{u})y}\, u^{i\nu}\, \frac{du}{u}$$

The exponent $-(u+\frac{1}{u})y$ is of the desired form, with the earlier s replaced by y, but the $u^{i\nu}$ in the integrand does not fit into the simpler Laplace' method. Thus, consider integrals

$$\int_0^\infty e^{-sf(u)}\, g(u)\, du$$

where f is real-valued, but g may be complex-valued. The minimum values of $f(u)$ should still dominate, and the leading term of the asymptotics should be (assuming a *unique* minimum at u_o)

$$\int_0^\infty e^{-s\cdot f(u)}\, g(u)\, du \;\sim\; e^{-sf(u_o)} \cdot \frac{\sqrt{2\pi}\cdot g(u_o)}{\sqrt{f''(u_o)}} \cdot \frac{1}{\sqrt{s}}$$

As in the simpler case, reduce to Watson's lemma by breaking the integral where f' changes sign, and change variables to convert each fragment to a Watson-lemma integral. The course of the argument uncovers conditions on f and g.

For simplicity assume that there is exactly *one* point u_o at which $f'(u_o) = 0$, that $f''(u_o) > 0$, and that $f(u)$ goes to $+\infty$ at 0^+ and at $+\infty$. Thus, on these intervals there are smooth square roots $\sqrt{f(u) - f(u_o)}$ and smooth functions F, G such that

$$\begin{cases} F(\sqrt{f(u) - f(u_o)}) \;=\; u & (\text{for } u_o < u < +\infty) \\ G(\sqrt{f(u) - f(u_o)}) \;=\; u & (\text{for } 0 < u < u_o) \end{cases}$$

With $y = \sqrt{f(u) - f(u_o)}$ in both integrals,

$$\int_0^\infty e^{-sf(u)}\, g(u)\, du \;=\; e^{-sf(u_o)}\int_0^{u_o} e^{-s\,(f(u)-f(u_o))}\, g(u)\, du + e^{-sf(u_o)}\int_{u_o}^\infty e^{-s\,(f(u)-f(u_o))}\, g(u)\, du$$

$$=\; e^{-sf(u_o)}\int_0^\infty e^{-s\,y^2}\Big(g(F(y))\, F'(y) + g(G(y))\, G'(y)\Big)\, dy$$

The assumptions $f'(u_o) = 0$ and $f''(u_o) > 0$ assure that $F(y)$ has a Taylor series expansion near 0, which gives an asymptotic expansion at 0^+

$$g(F(y))\, F'(y) \;=\; \frac{g(F(0))\, F'(0)}{2} + \ldots \qquad (\text{small } y > 0)$$

[10] This integral is almost the *Bessel function*

$$K_\nu(y) \;=\; \frac{1}{2}\int_0^\infty e^{(u+\frac{1}{u})y/2}\, u^\nu\, \frac{du}{u}$$

The function K_ν is variously called a *Bessel function of imaginary argument* or *MacDonald's function* or *modified Bessel function of third kind.* Being interested mainly in the case that the parameter ν here is purely imaginary, in the text we replace ν by $i\nu$. The leading factor of $\sqrt{y}$ arises in applications, where such an integral appears as a *Whittaker function.*

The exponential e^{-sy^2} is not quite right to apply Watson's lemma, so replace y by $\sqrt{x}$, obtaining an asymptotic expansion in powers of $\sqrt{x}$. The main term of the Watson's lemma asymptotics for the integral involving F would be

$$\int_0^\infty e^{-sx}\,\tfrac{1}{2}\sqrt{x}\,g(F(\sqrt{x}))\,F'(\sqrt{x})\,\frac{dx}{x} \;\sim\; \frac{\Gamma(\frac{1}{2})\,g(F(0))\,F'(0)}{2}\cdot\frac{1}{\sqrt{s}}$$

As before, $F'(0)$ from $F(y)=u$ and $F'(y)\cdot\frac{dy}{du}=1$ give

$$F'(y) \;=\; \frac{1}{\dfrac{dy}{du}} \;=\; \sqrt{\frac{2}{f''(u_o)}}\cdot\Big(1+O(u-u_o)\Big)$$

and $F'(0)=\sqrt{\frac{2}{f''(u_o)}}$ and $G'(0)=F'(0)$. Thus,

$$\int_0^\infty e^{-s\,f(u)}\,g(u)\,du \;\sim\; e^{-sf(u_o)}\cdot\frac{\Gamma(\frac{1}{2})\cdot 2\cdot g(u_o)\cdot\sqrt{\frac{2}{f''(u_o)}}}{2\sqrt{s}} \;=\; e^{-sf(u_o)}\cdot\frac{\sqrt{2\pi}\cdot g(u_o)}{\sqrt{f''(u_o)}}\cdot\frac{1}{\sqrt{s}}$$

Returning to

$$\sqrt{y}\int_0^\infty e^{-(u+\frac{1}{u})y}\,u^{i\nu}\,\frac{du}{u}$$

we have critical point $u_o=1$, and $f(u_o)=2$ and $f''(u_o)=2$. Applying the just-derived asymptotic,

$$\sqrt{y}\int_0^\infty e^{-(u+\frac{1}{u})y}\,u^{i\nu}\,\frac{du}{u} \;\sim\; \sqrt{y}\cdot\left(e^{-2y}\cdot\frac{\sqrt{2\pi}\cdot 1^{i\nu}}{\sqrt{2}}\cdot\frac{1}{\sqrt{y}}\right) \;=\; \sqrt{\pi}\cdot e^{-2y} \qquad\qquad (\text{as } y\to+\infty)$$

Even though the exponent is plausible, it would be easy to lose track of the power of y, which might matter. Note that the leading constant does not depend upon the index ν.

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