Explicit Tauberian estimates for functions with positive coefficients

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ABSTRACT

If $f(x) = \sum a_n x^n$ has $a_n \geq 0$ for all n, then for each $x > 0$ for which the series converges we have $a_n \leq x^{-n} f(x)$ for each n. By choosing that x which minimizes the upper bound one obtains a "saddle point estimate" for each a_n that has been known to be close to best possible in several cases. This paper presents a lower bound for summatory functions of the coefficients that is derived by elementary methods. It is not as sharp as the estimates that one obtains from most modern Tauberian theorems. However, this method can be used when Tauberian theorems are not applicable, for example, when one is dealing not with a single generating function but a sequence of them. Applications to partitions, integers without large prime factors, and other problems are presented.

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We now show how to derive Theorem 3 in the smaller range $\exp((\log x)^{\epsilon}) \le y \le \exp((\log x)^{1/3})$ from Proposition 1 and Theorem 1. We again let $g(s) = \log F(s)$. The prime number theorem shows (cf. [4, 10, 12]) that for $3/4 \le s < 1$, $y^{1-s} \ge (\log x)^{\epsilon/10}$,

$$\begin{aligned} g(s) &= li(y^{1-s})(1 + O((\log y)^{-1})) + O(\log|1-s|) , && (3.24) \\ g'(s) &\sim -(1-s)^{-1} y^{1-s} , && (3.25) \\ g''(s) &\sim (1-s)^{-1} y^{1-s} \log y , && (3.26) \\ g'''(s) &\sim -(1-s)^{-1} y^{1-s} (\log y)^2 && (3.27) \end{aligned}$$

as $x \to \infty$, uniformly for s and y in the specified ranges. We need to find the minimum value of $g(s) + s \log x$. This occurs at $g'(s) \sim -\log x$, so that the minimizing s satisfies

$$(1-s)^{-1} y^{1-s} \sim \log x . \qquad (3.28)$$

For this value of s, we see that $|g'''(s)| = o((g''(s))^{3/2})$, so the hypotheses of Theorem 1 are satisfied. The error term $sA^{1/2}$ in (1.10) will be $o(u \log u)$ for $y \le \exp((\log x)^{1/3})$. For larger values of y, we obtain poorer error terms.

The main point of this subsection was to show how Theorem 1 can give a lower bound almost for free once the upper bound of Proposition 1 has been established. Most of the slight extra cost is due not to the conditions on derivatives that are required by Theorem 1, but to the need to obtain the optimal bound from Proposition 1.

breaks down. For this choice of parameters, it is easy to see that the conditions of Theorem 1 apply, and so the number of lattice points inside $tS_\sigma + (w_1, \ldots w_n)$ behaves like c^n, where c varies in a bounded interval (for σ fixed) as $(w_1, \ldots, w_n)$ varies. This is dealt with in detail in [11] for $\sigma = 2$, and is only mentioned for other σ in [5] (since for the main results of that paper only the upper bound of Proposition 1 was needed). Lattice points in other kinds of bodies, such as those studied in [5], can also be estimated by using Theorem 1.

3.3 Integers without large prime factors

Let $\Psi(x, y)$ denote the number of positive integers $\leq x$ and free of prime factors $> y$. This function has been investigated extensively because of its applicability to sieve methods, integer factoring algorithms, and other problems. For extensive references and the best currently known bounds, see [10]. If we let

$$F(s) = \prod_{p \leq y} (1 - p^{-s})^{-1} \ , \tag{3.20}$$

then $F(s)$ can be represented as in (1.1) with $\Psi(x, y) = \mu(\log x)$. The best currently known estimates for $\Psi(x, y)$ have been obtained by Hildebrand and Tenenbaum [10] using complex integration, but at the cost of complicated proofs. Most of the papers in the literature, such as [4, 14], use Proposition 1 to obtain an upper bound for $\Psi(x, y)$, and then prove lower bounds by combinatorial methods, counting integers of certain special kinds. These methods yield weaker results, but ones that are usually sufficient for applications. The advantage of these methods is that they are simple. The argument of Pomerance [12] is particularly short. He proves the following result.

Theorem 3 *If $\epsilon > 0$ is fixed, and y satisfies*

$$\exp((\log x)^\epsilon) \leq y \leq \exp((\log x)^{1-\epsilon}) \ , \tag{3.21}$$

then

$$\Psi(x, y) = x \exp(-(1 + o(1)) u \log u) \tag{3.22}$$

uniformly as $x \to \infty$, where

$$u = \frac{\log x}{\log y} \ . \tag{3.23}$$

Theorem 2 *Suppose that the $a(n)$ are defined by (3.13),*

$$\sum_{k \le x} b(k) \sim C x^u (\log x)^v \quad \text{as} \quad x \to \infty \ , \tag{3.14}$$

where $C > 0$, $u > 0$, and $b(k) \ge 0$ for all k. Then

$$\log \left(\sum_{n \le m} a(n) \right) \sim u^{-1} \{ C u \Gamma(u+2) \zeta(u+1) \}^{1/(u+1)} \cdot (u+1)^{(u-v)/(u+1)} m^{u/(u+1)} (\log m)^{v/(u+1)} \tag{3.15}$$

as $m \to \infty$.

Brigham proved Theorem 2 by showing that

$$F(s) \sim C s^{-u} (-\log s)^v \Gamma(u+1) \zeta(u+1) \quad \text{as} \quad s \to 0^+ \tag{3.16}$$

and then invoking the Hardy-Ramanujan Tauberian theorem [9]. We can obtain another derivation by using Theorem 1. The crucial part, the estimate (3.16), is the same in both proofs, and the very crude bounds on the derivatives of $\log F(s)$ that are needed to apply Theorem 1 are much easier to establish than (3.16). The advantage of using Theorem 1 is that it can be applied in many more situations than just those covered by the Hardy-Ramanujan theorem.

3.2 Lattice points in superballs

For any $\sigma > 0$, a σ-superball S_σ in $\mathbb{R}^n$ is defined by

$$S_\sigma = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^{n} |x_j|^\sigma \le 1 \right\} \ . \tag{3.17}$$

The number of integer lattice points in $tS_\sigma + (w_1, \ldots, w_n)$ is easily seen to be $\mu(t^\sigma)$, where

$$\int_0^\infty e^{-sx} d\mu(x) = \prod_{j=1}^{n} f_j(s) \ , \tag{3.18}$$

$$f_j(s) = \sum_{k=-\infty}^{\infty} \exp(-s|k - w_j|^\sigma) \ . \tag{3.19}$$

This is a case where the generating function $F(s) = \Pi f_j(s)$ varies with n and so none of the standard Tauberian theorems apply. As is explained in [5] and [11], the most interesting case to study is $t = (\alpha n)^{1/\sigma}$ for a fixed σ, $\alpha > 0$ and $n \to \infty$, since in that case Gauss' principle that the number of lattice points in a "nice" body in $\mathbb{R}^n$ is approximated well by its volume

(The simplest way to obtain these estimates is to use the Euler-Maclaurin sum formula. For other, more precise methods, see [1, 2].) Hence, for $y \to \infty$, $s(y) \sim \pi/(6y)^{1/2}$, the conditions of Theorem 1 hold, and so (1.1) gives us a lower bound for

$$p(1) + p(2) + \cdots + p(y)$$

that is within a multiplicative factor of $\exp(cy^{1/4})$ of the upper bound given by Proposition 1.

So far we have not said what the upper bound of Proposition 1 is. The reason is that the quality of this bound depends on how carefully one estimates $g(s)$ at the optimal s. Since $F(s) = \exp(g(s))$, even small errors in $g(s)$ yield huge errors in the bound for $\mu(y)$. Using the bound (3.6) gives a useful but poor estimate. Thus here we have a common situation that arises in applying Theorem 1; the estimate of Proposition 1 requires more care to derive than the additional estimates required by Theorem 1.

Careful estimates of $F(s)$ show [1, 2] that

$$F(s) \sim (2\pi)^{-1/2} s^{1/2} \exp(\pi^2/(6s)) \quad \text{as} \quad s \to 0^+ \ . \tag{3.10}$$

Therefore Proposition 1 gives a bound of

$$p(1) + \cdots + p(y) \le 2^{-3/4} e^{-1/4} y^{-1/4} (1 + o(1)) \exp(2\pi 6^{-1/2} y^{1/2}) \ , \tag{3.11}$$

which is off only by a factor of $y^{1/4}$ from the correct value, since the asymptotic formula for $p(n)$ [1, 2] shows that

$$p(1) + \cdots + p(n) \sim 2^{-3/2} \pi^{-1} y^{-1/2} \exp(2\pi 6^{-1/2} y^{1/2}) \ \text{ as } \ y \to \infty \ . \tag{3.12}$$

On the other hand, the lower bound of Theorem 1 is off by a factor of $\exp(cy^{1/4})$.

We next consider more general partition problems. Let

$$F(s) = \prod_{k=1}^{\infty} (1 - e^{-ks})^{-b(k)} = \sum_{n=0}^{\infty} a(n) e^{-ns} \ , \tag{3.13}$$

where the $b(k)$ are nonnegative integers. When $b(k) = 1$ for all k, $a(n)$ is the ordinary partition function $p(n)$. When $b(k) = k$, $a(n)$ is the number of plane partitions of n. Many other partition problems can be put in this form. Brigham [3] proved a widely used theorem that applies in this setting.

so that (1.1) holds with $\mu(x) = \lfloor x \rfloor + 1$. The upper bound $\exp(sy)/(1-\exp(-s))$ for $\mu(y)$ of Proposition 1 is minimized for

$$s = \log(1 + y^{-1}) = y^{-1} + O(y^{-2}) \ , \tag{3.2}$$

and shows that

$$\mu(y) \leq (e + o(1))y \quad \text{as} \quad y \to \infty \ , \tag{3.3}$$

which is too large by a factor of e. The lower bound (1.1) of Theorem 1 would tell us that $\mu(y) \geq \epsilon y$ for some $\epsilon > 0$, since $A \sim ey^2$ as $y \to \infty$. However, we cannot apply Theorem 1 here, since the hypothesis (1.9) of Theorem 1 is not satisfied, as the third derivative of $\log F(s)$ is comparable to $A^{3/2}$. It is possible to modify the proof of Theorem 1 to obtain a nontrivial lower bound, but it requires more careful estimates.

3.1 Partitions

This section applies Theorem 1 to enumeration of partitions, and shows how this method compares with other ones. Unlike in the next two sections, in this one we will always deal with asymptotics of coefficients of a single generating function, so one of the main advantages of the new method will not be apparent.

First, let us consider $p(n)$, the number of ordinary (unordered) partitions of an integer n into positive summands. Then we have the well known formula

$$F(s) = \sum_{n=1}^{\infty} p(n) e^{-ns} = \prod_{k=1}^{\infty} (1 - e^{-ks})^{-1} \ . \tag{3.4}$$

Let, as in the proof of Theorem 1, $g(s) = \log F(s)$. Then

$$g(s) = \sum_{k=1}^{\infty} - \log(1 - e^{-ks}) \ , \tag{3.5}$$

and we find that for $s \to 0^+$,

$$g(s) \sim \frac{\pi^2}{6s} \ , \tag{3.6}$$

$$g'(s) \sim \frac{-\pi^2}{6s^2} \ , \tag{3.7}$$

$$g''(s) \sim \frac{\pi^2}{3s^3} \ , \tag{3.8}$$

$$g'''(s) \sim \frac{-\pi^2}{s^4} \ . \tag{3.9}$$

where

$$B = \max_{s_1 < s < s_3} |g'''(s)| \ . \tag{2.22}$$

The same arguments, mutatis mutandis, show that inequality (2.16) follows from

$$(s_3 - s_2)^2 g''(s_2) \geq 2 \log 4 + 2B|s_3 - s_2|^3 \ . \tag{2.23}$$

We select s_1 and s_2 by

$$s_2 - s_1 = s_3 - s_2 = 10(g''(s_1))^{-1/2} \ . \tag{2.24}$$

Since

$$|g''(s_2) - g''(s_1)| \leq (s_2 - s_1)B \ , \tag{2.25}$$

the conditions of Theorem 1 guarantee that (2.21) and (2.23) do hold, which proves (2.15) and (2.16), and therefore (2.17).

By (2.14) and (2.17), we now have

$$\mu(y_1) - \mu(y_3) \geq F(s_2) \exp(y_3 s_2)/2 \ . \tag{2.26}$$

However, $F(s_2)$ is close to $F(s_1)$;

$$g(s_2) = g(s_1) + (s_2 - s_1)g'(s_1) + \frac{1}{2}(s_2 - s_1)^2 g''(s_6) \tag{2.27}$$

for some s_6, $s_1 \leq s_6 \leq s_2$, so

$$g(s_2) \geq g(s_1) - (s_2 - s_1)y_1 - 60 \ , \tag{2.28}$$

and so

$$F(s_2) \exp(y_3 s_2)/2 \geq F(s_1) \exp(y_1 s_1 - (y_1 - y_3)s_2 - 61) \ . \tag{2.29}$$

Now for some s_7, $s_1 \leq s_7 \leq s_3$, we have

$$0 \leq y_1 - y_3 = g'(s_3) - g'(s_1) = (s_3 - s_1)g''(s_7) \ ,$$

and so we obtain the claim of Theorem 1.

3 Applications

3.0 Integers

To show the power and limitations of the method of this paper, we consider the trivial case of

$$F(s) = \sum_{n=0}^{\infty} e^{-ns} = \frac{1}{1 - e^{-s}} \ , \tag{3.1}$$

which obviously holds for $x \geq y_1$, and so $h(x) \leq 0$ for $x \geq y_1$. A similar argument shows that $h(x) \leq 0$ for $x \leq y_3$.

Since $h(x) \leq 0$ for $x \leq y_3$ and $x \geq y_1$, and

$$h(x) \leq \exp(y_2 s_2 - x s_2) \tag{2.13}$$

for all x, we see that

$$h(x) \leq \exp(y_2 s_2 - y_3 s_2)$$

for all x. This shows that

$$\mu(y_1) - \mu(y_3) \geq H \exp(-(y_2 - y_3) s_2) \ . \tag{2.14}$$

It remains to obtain a lower bound for H.

If we can choose s_2 and s_3 so that

$$\exp(y_2 s_2) F(s_2) \quad \geq \quad 4 \exp(y_1 s_1 + y_2 s_2 - y_1 s_2) F(s_1) \ , \tag{2.15}$$

$$\exp(y_2 s_2) F(s_2) \quad \geq \quad 4 \exp(y_3 s_3 + y_2 s_2 - y_3 s_2) F(s_3) \ , \tag{2.16}$$

then we will have

$$H \geq \exp(y_2 s_2) F(s_2)/2 \ , \tag{2.17}$$

which will give the desired bound of the theorem. We consider the inequality (2.15) first. We need to prove that

$$g(s_2) \geq g(s_1) + y_1(s_1 - s_2) + \log 4 \ . \tag{2.18}$$

Now

$$g(s_1) = g(s_2) + (s_1 - s_2) g'(s_2) + \frac{1}{2}(s_1 - s_2)^2 g''(s_2) + \frac{1}{6}(s_1 - s_2)^3 g'''(s_4) \tag{2.19}$$

for some s_4, $s_1 \leq s_4 \leq s_2$. Furthermore,

$$y_1 = -g'(s_1) = -g'(s_2) - (s_1 - s_2) g''(s_2) - \frac{1}{2}(s_1 - s_2)^2 g'''(s_5) \tag{2.20}$$

for some s_5, $s_1 \leq s_5 \leq s_2$. Thus inequality (2.18) will follow if

$$(s_1 - s_2)^2 g''(s_2) \geq 2 \log 4 + 2B |s_1 - s_2|^3 \ , \tag{2.21}$$

for all $s > 0$, since by the Cauchy-Schwarz inequality

$$(F'(s))^2 = \left(\int_0^\infty xe^{-sx}d\mu(x)\right)^2 \leq \int_0^\infty e^{-sx}d\mu(x) \int_0^\infty x^2 e^{-su}d\mu(u) \ ,$$

and equality cannot hold because of the assumption on $\mu(x)$.

Since $\mu(x)$ is nondecreasing and $\mu(x) \to \infty$ as $x \to \infty$, we have

$$g'(s) \to -\infty \qquad \text{as} \qquad s \to 0^+ \ , \tag{2.6}$$

$$g'(s) \to -x_0 \qquad \text{as} \qquad s \to \infty \ , \tag{2.7}$$

where

$$x_0 = \inf\{x : \mu(x) > 0\} \ . \tag{2.8}$$

Therefore for every $y > x_0$, there is a unique solution $s = s(y) > 0$ to

$$-\, g'(s) = y \ . \tag{2.9}$$

Moreover, $s(y) < s(y')$ if $y > y' > x_0$.

To prove Theorem 1, let $y_1 = y$, $s_1 = s(y_1)$, and choose $0 < s_1 < s_2 < s_3$, with $y_j = -g'(s_j)$, $j = 2, 3$, so that $y_1 > y_2 > y_3 > x_0$. (The precise choice of the s_j and thus of the y_j for $j = 2$ and 3 will be made later. We note here that the roles of s_1 and s_3 were switched by mistake in [11].) Define

$$\begin{aligned} H = \exp(y_2 s_2)F(s_2) \quad &- \quad \exp(y_1 s_1 + y_2 s_2 - y_1 s_2)F(s_1) \\ &- \quad \exp(y_3 s_3 + y_2 s_2 - y_3 s_2)F(s_3) \ , \end{aligned} \tag{2.10}$$

so that

$$H = \int_0^\infty h(x)d\mu(x) \tag{2.11}$$

with

$$\begin{aligned} h(x) = \exp(y_2 s_2 - x s_2) \quad &- \quad \exp(y_1 s_1 + y_2 s_2 - y_1 s_2 - x s_1) \\ &- \quad \exp(y_3 s_3 + y_2 s_2 - y_3 s_2 - x s_3) \ . \end{aligned} \tag{2.12}$$

We first show that $h(x) \leq 0$ for $x \geq y_1$. It suffices to prove that in this range,

$$y_2 s_2 - x s_2 \leq y_1 s_1 + y_2 s_2 - y_1 s_2 - x s_1 \ .$$

This inequality is equivalent to

$$y_1(s_2 - s_1) \leq x(s_2 - s_1) \ ,$$

combination of Proposition 1 and Theorem 1. Theorem 1 makes assumptions on the derivatives of $\log F(s)$, unlike the theorem of [9]. (The existence of these derivatives is trivial, as is noted in Section 2.) However, in many applications, such as those of Section 3, verifying that the conditions on derivatives hold is easy, usually much easier than estimating the minimum of $\exp(sy)F(s)$.

2 Proof of theorem

The proof of the proposition came from writing

$$e^{sy}F(y) = \int_o^\infty e^{s(y-x)}d\mu(x)$$

and noting that all the contributions to the integral are nonnegative, and that for $0 \le x \le y$ the "weight" $\exp(s(y-x))$ is ≥ 1. To obtain a lower bound, we will consider

$$H = \int_0^\infty h(x)d\mu(x) \ , \tag{2.1}$$

where $h(x)$ is ≥ 0 only for $w \le x \le y$. Then we obtain

$$\left(\max_x h(x)\right)(\mu(y) - \mu(w)) \ge H \ . \tag{2.2}$$

The problem is to choose $h(x)$ so that the bound obtained from (2.2) is useful. (In particular we need $H > 0$.)

Since the integral in (1.1) converges for all $s > 0$, the integrals

$$\int_0^\infty s^k e^{-sx}d\mu(x)$$

converge for all $s > 0$ if $k \in \mathbb{Z}^+$, and so $F(s)$ is in $C^\infty(0,\infty)$. Furthermore, $F(s) > 0$ for all $s > 0$. Let

$$g(s) = \log F(s) \ . \tag{2.3}$$

Then $g(s)$ is also in $C^\infty(0,\infty)$, and

$$g'(s) = \frac{F'(s)}{F(s)} < 0 \tag{2.4}$$

for $s > 0$. On the other hand,

$$g''(s) = \frac{F''(s)F(s) - (F'(s))^2}{F(s)^2} > 0 \tag{2.5}$$

Theorem 1 *Suppose that $\mu(x)$ is nondecreasing, $\mu(x) \to \infty$ as $x \to \infty$, and the integral in (1.1) converges for all $s > 0$. Let $x_0 = inf\{x : \mu(x) > 0\}$. Then for any $y > x_0$, there is a unique $s = s(y) > 0$ that minimizes $\exp(sy)F(s)$. Let*

$$A = \frac{\partial^2}{\partial s^2} \log F(s)\Big|_{s=s(y)} . \tag{1.7}$$

If $A \geq 10^6$ and for all t with

$$s(y) \leq t \leq s(y) + 20A^{-1/2} \tag{1.8}$$

we have

$$\left| \frac{\partial^3}{\partial s^3} \log F(s)\Big|_{s=t} \right| \leq 10^{-3} A^{3/2} , \tag{1.9}$$

then

$$\mu(y) - \mu(y - 30A^{-1/2}) \geq F(s(y)) \exp(ys(y) - 30s(y)A^{1/2} - 100) . \tag{1.10}$$

The constants in Theorem 1 are far from best possible, and no effort was made to optimize them, since the only goal was to obtain explicit estimates that can be applied to obtain asymptotic estimates.

The above theorem is a generalization of the lower bound method used in [11] to estimate the number of lattice points in high-dimensional spheres. That this method could be extended to some related problems was mentioned in [5]. No details were provided, though, since only the upper bound of Proposition 1 was needed to obtain the main results of [5]. Since it has recently become clear that there are many more applications, it seems desirable to present in an explicit form a general version of the bound.

The bound of Theorem 1 can be improved, as was already mentioned in [11], by choosing better weights. However, this requires more work, and when it is necessary to obtain improved bounds, one can usually use other methods, such as complex integration.

The bound of the proposition is usually much closer to the true size of $\mu(y)$ than the bound of the theorem. The method of proof of the theorem is crude and leaves a lot of slack. It is similar to the proofs of some of the early Tauberian theorems, especially that of Hardy and Ramanujan [9]. It yields results that are comparable to those of [9] in strength in many applications, but much more general and simpler to prove. The Hardy-Ramanujan result is one of the few Tauberian theorems in the literature that apply to functions $F(s)$ that grow rapidly as $s \to 0^+$. It also gives asymptotics only for $\log \mu(y)$, and not for $\mu(y)$ itself, just like the they do provide information in many situations where Tauberian theorems do not apply. Furthermore, these new results have a very simple proof. A strong assumption on $\mu(x)$ is made, namely that $\mu(x)$ is nondecreasing, but that is common to many Tauberian theorems as well.

From now on we will assume that $\mu(x) \geq \mu(y)$ for all $x \geq y \geq 0$. In the cases of power series (1.3) or Dirichlet series (1.5), this implies that $a_n \geq 0$, and in general that the "measure" $d\mu(x)$ is nonnegative. This assumption is crucial for what follows. We will also assume that the integral in (1.1) converges for all $s > 0$ and that $\mu(x) \to \infty$ as $x \to \infty$, so that the integral diverges for all $s \leq 0$. This will let us study the asymptotics of $\mu(x)$. In situations where the integral (1.1) converges for $s > s_0$ for some s_0, we can renormalize by redefining $\mu(x)$, but then the estimates we obtain will not be for the original $\mu(x)$.

The genesis of the new method is in the following well-known and trivial upper bound.

Proposition 1 *Suppose that $\mu(x)$ is nondecreasing, and the integral in (1.1) converges for all $s > 0$. Then for any $y > 0$ and any $s > s_0$,*

$$\mu(y) \leq e^{sy} F(s) \ . \tag{1.6}$$

Proof Note that

$$e^{sy} F(s) = \int_0^\infty e^{s(y-x)} d\mu(x) \geq \int_0^y d\mu(x) = \mu(y) \ .$$

This result is old. In number theory, for $F(s)$ in the form (1.5), it was used by Rankin in the study of integers without large prime factors [14], and is often referred to as "Rankin's method." However, Hardy and Ramanujan [9] had used this argument earlier, and similar arguments have been used frequently in probability theory and other fields.

The upper bound of Proposition 1 is optimized for each x by choosing that s for which $\exp(sx)F(s)$ is minimized. For $s \to 0$ and $s \to \infty$, $\exp(sx)F(s) \to \infty$, so we can expect that there will be some intermediate point that minimizes this expression. (It is easy to show that there is a unique minimizing s; see Section 2.) Since $F(s)$ is analytic for complex s with $\text{Re}(s) > 0$, and $\mu(x)$ is nondecreasing, there is a "saddle point" at this minimizing s, and in some cases the saddle point method can be used to obtain precise estimates. It has been remarked by many authors that the upper bound obtained from using the saddle point in (1.6) is often surprisingly good. The main result of this paper is to show that this is a general phenomenon.

Explicit Tauberian estimates for functions with positive coefficients

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1 Introduction

Consider a function $F(s)$ defined by the Laplace-Stieltjes transform

$$F(s) = \int_0^\infty e^{-sx} d\mu(x) \ , \tag{1.1}$$

where $\mu(x)$ is of bounded variation on every finite interval. This is the representation used most frequently in Tauberian theory. It covers most of the important cases. If $z = e^{-s}$ and

$$\mu(x) = \sum_{0 \le n \le x} a_n \ , \tag{1.2}$$

then we reduce to a power series

$$F(s) = \sum_{n=0}^{\infty} a_n z^n \ . \tag{1.3}$$

If

$$\mu(x) = \sum_{\substack{n \ge 1 \\ \log n \le x}} a_n \ , \tag{1.4}$$

then we reduce to a Dirichlet series

$$F(s) = \sum_n a_n n^{-s} \ . \tag{1.5}$$

Tauberian theorems obtain information about the function $\mu(x)$ from information about $F(s)$. (See [6, 13] for results and references.) Their advantage is that usually only weak assumptions about $F(s)$ have to be made. A major disadvantage is that the error bounds in Tauberian theorems are usually poor, and so they yield only the leading term of the asymptotic expansion. It is usually impossible to apply them in situations where one is dealing not with a single function $F(s)$ but a sequence of such functions. As the examples in Section 3 show, there are many cases where one does need to deal with varying generating functions.

The aim of this paper is to present an elementary method that yields satisfactory estimates in a variety of situations. These estimates are usually not very accurate. On the other hand,