PREFACE

It is often said (following Gauss) that number theory is the queen of mathematics. Until comparatively recently, most people tended to view number theory as the very paradigm of pure mathematics. With the advent of computers, however, number theory has been finding an increasing number of applications in practical settings, such as in cryptography, random number generation, coding theory, and even concert hall acoustics. Yet other applications are still emerging — providing number theorists with some major new areas of opportunity.

The 1996 IMA summer program on *Emerging Applications of Number Theory* was held July 15–26, 1996 at the University of Minnesota in Minneapolis with the aim of stimulating further work with some of these newest (and most attractive) applications.

For reasons of synergy, it was decided to concentrate on number theory's recent links with:

(a) wave phenomena in quantum mechanics (more specifically, quantum chaos); and

(b) graph theory (especially expander graphs and related spectral theory).

Partly because of these links, there has been a tendency for these two areas to exhibit a growing number of structural and technical similarities in recent years; it was thus natural that the two areas be treated together.

There were about 70 participants. Many came from a background in either computing or physical science, and this helped produce an unusually invigorating (and eye-opening) atmosphere.

The first week's activities focussed mainly on area (a), i.e. **quantum mechanics**. Whereas classical mechanics is based primarily on ordinary differential equations in its simplest applications, quantum mechanics has — as its name indicates — drawn from its very beginning on mathematics of a more discrete type, in particular spectral theory of differential operators, group theory, and algebra. The relevance of number theory in such a setting is thus not entirely unexpected. As the phenomenon of chaos in classical mechanics became better understood, the spirit of number theory was found to enter physics even at the classical level, *e.g.*, in the study of critical resonances and the use of symbolic dynamics. Manifestations of classical chaos are visible not only in quantum mechanics (hence the expression "quantum chaos"), but also in a variety of other wave theories such as optics, electromagnetism, and acoustics.

The distribution of energy levels and eigenfrequencies, as well as their relation with classical periodic orbits (through the trace formula), has led to new methods in spectroscopy which are at least partially based on number theory. Prime examples of this are particles — say electrons — moving either in a cavity, or on a Riemann surface of negative curvature, and the resultant description using zeta functions. Applications of these concepts have recently been made to small (mesoscopic) switching elements, quantum computers, quantum dot devices, polymers, and so on.

The lectures during week 1 touched on a good portion of this. The plenary lectures, in particular, were invaluable for the state-of-the-art surveys they provided to listeners. (Number theorists, for instance, found it very stimulating to be able to hear first-hand accounts of progress taking place in the lab.)

During the second week, the focus gradually shifted over to one of **graph theory**. Application-wise, graphs are frequently used to model communication networks, both among people and processors. In many types of analyses, the optimal network often turns out to be one having properties similar to random networks. One good example of such a property is "expansion," which guarantees an absence of "hot spots" in the network. It turns out that random graphs have a great deal of expansion, but it is a famous open problem to give *explicit* constructions of graphs with as much expansion. A lot of excitement has been generated in recent years with the discovery of explicit constructions having quite good expansion; these constructions require number theory and the spectral theory of graphs (viz., eigenvalue considerations) to prove that they do, in fact, have the necessary properties. These developments also have ties to the spectral theory of differential operators, algebraic geometry, representation theory, and the trace formula.

The graph-theoretic part of the program concentrated largely on the use of number theory to construct graphs having desirable features such as the good expansion property mentioned above. There are many ways number theory can be used to produce large classes of such graphs. To show that these graphs have the desired properties, one typically combines the spectral theory of graphs with number-theoretical estimates to prove that these graphs' eigenvalues are small, and then checks that any graph *with* small eigenvalues necessarily has good behavior. There are a number of relationships between eigenvalues and graph properties that are not yet well understood; several of the 2nd week lectures touched on this (e.g., from the point-of-view of combinatorial trace formulae and zeta functions). In addition, there are a number of new applications of expanders, including to coding theory, which were reported on.

The week 2 lecturers did an admirable job of not only surveying recent developments, but also pinpointing some of the most important technical/structural similarities between areas (a) and (b).

This volume contains the refereed versions of papers from the meeting, including contributions from 16 of the 29 one-hour, plenary speakers.

Taken together, the papers offer a rather good sense of what transpired at the meeting — both at the lectures and in smaller discussions.

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