On longest increasing subsequences in random permutations

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Dedicated to Leon Ehrenpreis

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ABSTRACT

The expected value of L_n, the length of the longest increasing subsequence of a random permutation of $\{1, \ldots, n\}$, has been studied extensively. This paper presents the results of both Monte Carlo and exact computations that explore the finer structure of the distribution of L_n. The results suggested that several of the conjectures that had been made about L_n were incorrect, and led to new conjectures, some of which have been proved recently by Jinho Baik, Percy Deift, and Kurt Johansson. In particular, the standard deviation of L_n is of order $n^{1/6}$, contrary to earlier conjectures.

This paper also explains some regular patterns in the distribution of L_n.

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1. Introduction

Let L_n denote the length of the longest increasing subsequence of a random permutation of $\{1, \ldots, n\}$. There is extensive literature about this random variable. Ulam [Ulam] was motivated to ask about the distribution of L_n by the famous result of Erdös and Szekeres that every permutation of $\{1, \ldots, n\}$ has either an increasing or a decreasing subsequence of length $\geq \sqrt{n}$. Monte Carlo computations led Ulam to the conjecture that L_n is usually on the order of $\sqrt{n}$. More extensive computations by Baer and Brock [BaerB] led them to the conjecture that the expected value of L_n is $\sim 2\sqrt{n}$ as $n \to \infty$. (Ulam had conjectured a different value for the constant of proportionality.) Hammersley [Ham] showed that L_n is asymptotic to $c\sqrt{n}$ in probability for some constant c, that $EL_n \sim c\sqrt{n}$ also, and that $\pi/2 \leq c \leq e$. Kingman [Kingman] (see also [Kingman2]) proved $(8/\pi)^{1/2} = 1.595\ldots \leq c < 2.49$. Logan and Shepp [LoganS] used calculus of variations methods to show that $c \geq 2$. Vershik and Kerov [VershikK1] (see also [KerovV]) used a method almost identical to that of Logan and Shepp to prove that $c \geq 2$, and a group theoretic and combinatorial argument to show that $c \leq 2$. A more directly combinatorial proof that $c \leq 2$ was obtained later by Pilpel [Pilpel]. Other proofs that $c = 2$ were recently obtained by Aldous and Diaconis [AldousD], Johansson [Joh], and Seppäläinen [Sep].

The Logan-Shepp and Vershik-Kerov results established that $c = 2$, and thus answered the main question in this area. However, they left open many other problems, especially about the distribution of L_n. Frieze [Frieze] was the first one to prove the conjecture that L_n is very concentrated near its mean. His result was improved by Bollobás and Brightwell [BB], who showed, among other things, that the variance of L_n is $O(n^{1/2}(\log n)^2(\log\log n)^{-2})$. (Bollobás and Brightwell proved a more general result, and we quote only the special case that is relevant for our discussion.) An interesting feature of the Frieze and Bollobás-Brightwell proofs is that they use martingale methods, and provide no information about EL_n itself. Talagrand [Tala] has recently sharpened the Bollobás-Brightwell result, so that the variance of L_n is known to be $O(n^{1/2})$. His methods are also indirect in that they prove only that the distribution of L_n is very concentrated, but do not show where the mean is located.

J.-H. Kim [Kim] has shown that for every $\epsilon > 0$,

$$Pr\left(L_n > \sum_{k=1}^{n} k^{-1/2} + \theta n^{1/6}\right) \leq \exp(-1.2\theta^{3/2}) \tag{1.1}$$

for $n^{-2/3+\epsilon} \leq \theta \leq 2n^{1/3}$ if $n \geq n_0(\epsilon)$, which provides a bound for one tail of the distribution, but without relating it to EL_n. Two-sided tail estimates have been provided more recently by Deuschel and Zeitouni [DeuZ].

Steele (unpublished) had originally conjectured that the variance of L_n is not only small, but is bounded. This was shown to be false by Bollobás and Janson, who proved that this variance is $\geq n^{1/8}(\log n)^{-3/4}$ for large n. Bollobás and Brightwell conjectured that the variance of L_n is $\geq n^{1/2}$. Since the Talagrand result [Tala] gives an upper bound for the variance of $O(n^{1/2})$, their conjecture says that this upper bound is best possible.

Vershik and Kerov [VershikK2] showed that $2\sqrt{n} - EL_n = O(n^{1/3})$. Pilpel's proof that $c \leq 2$ [Pilpel] shows that $EL_n \leq 2\sqrt{n}$ for all n. These results still left open the precise order of $2\sqrt{n} - EL_n$.

In 1992, Poonen, Widom, Wilf, and the first author [OdlyzkoPWW] developed an analytic method for studying the distribution of L_n. This motivated our computations, which were designed to extend those of Baer and Brock [BaerB]. The purpose was to obtain data to formulate more precise conjectures about the behavior of L_n, and hopefully to use it as a check on any asymptotic estimates that were to be made. Starting in 1993, we have intermittently done a series of computer calculations which are summarized in this note. More detailed data from our computations is available online at ⟨http://www.research.att.com/∼amo⟩, and will be supplemented by additional data that we are collecting to provide insight into other features of random permutations, Young tableaux, and related topics.

There have been no algorithmic advances since the time of Baer and Brock, and our methods are essentially the same as the ones they used. However, much faster computers have become available, and have allowed us to compute the distribution of L_n exactly for $n \leq 120$ (in contrast to $n \leq 36$ for [BaerB]) and to do Monte Carlo simulations for n up to 10^{10} (in constrast to 10^4

for [BaerB]). Our computational methods are described briefly in Section 4.

Table 1 summarizes the results of our Monte Carlo experiments. The scaled moments for each n are the moments of $(L_n - m_n)/s_n$, where m_n is the observed mean of the sample, and s_n the standard deviation (so that the 1-st and 2-nd moments are by definition 0 and 1).

The Monte Carlo data of Table 1 showed that the mean of L_n is about two standard deviations below $2n^{1/2}$. This was apparently first observed by H. L. Montgomery (personal communication to the first author). However, contrary to Montgomery's guess (based on smaller runs than ours) our data showed clearly that the distribution of L_n is not asymptotically normal, and is asymmetric. For a normal distribution, one would expect the odd-order scaled moments to be 0, and the $(2m)$-th order ones to be $(2m-1)(2m-3)\cdot\ldots\cdot 3\cdot 1$. While the even order moments are close to those of a normal distribution, the odd order ones are not. This difference is also visible in the data. For example, tables 2–4 as well as the tables in [BaerB] and the Monte Carlo runs show that the distribution function of L_n rises much faster to its peak than it falls afterwards. This impression is also confirmed by use of *qq*-plots.

The standard deviation of L_n appears to increase by a factor of about $5^{1/2}$ each time n increases by a factor of 100. This suggests that it grows like $n^{0.17}$ $((\log 5^{1/2})(\log 100)^{-1} = 0.1747\ldots)$, which is contrary to the Bollobás-Brightwell [BB] conjecture that it is $\geq n^{1/4}$. We conjectured back in 1993 that the standard deviation of L_n is asymptotic to a constant times $n^{1/6}$, and that $(2\sqrt{n} - L_n)/n^{1/6}$ converges to a nice distribution. This conjecture was presented in private conversations and public lectures, although was not published. (The same conjecture for the standard deviation of L_n was made independently by Kim [Kim].)

Although our and Kim's were the first explicit statements of the conjecture that the standard deviation of L_n is $O(n^{1/6})$, this conjecture arises naturally from some bounds of Vershik and Kerov [VershikK2]. Also, Kesten suggested around 1993 (unpublished) that $n^{1/6}$ might be the right order by analogy with some first-passage percolation results.

2. Asymptotic distribution of L_n

The approach of [OdlyzkoPWW] started with a generating function of Gessel [Gessel] and produced an explicit analytic formula for the distribution of L_n, a formula that was soon thereafter derived in a much more direct way by the second author [Rains]. However, this formula involved a complicated multidimensional integral. It led to very precise large deviations estimates for L_n, but not to any useful results about the behavior of L_n near its mode. It was also discovered (as a result of a conversation between the first author and Claude Itzykson) that the same multidimensional integral plays a crucial role in two-dimensional quantum gravity models [GrossW, MyersP, Neub, PerSa, PerSb]. The physics papers do have asymptotic estimates for this generating function, but those estimates are neither precise enough to obtain the asymptotic distribution of L_n, nor rigorous. Interestingly enough, one half of the main result of Gross and Witten [GrossW] can be deduced easily and rigorously from the estimates of Logan and Shepp [LoganS] and of Vershik and Kerov [VershikK1], but this was not recognized at the time, since the connection between the longest increasing subsequence problem and quantum gravity was not known.

Recently the problem of the distribution of L_n near its mode was solved rigorously and essentially completely by Jinho Baik, Percy Deift, and Kurt Johansson [BaikDJ]. Their work is a tour de force of mathematical analysis. It proceeds through the generating function of [OdlyzkoPWW, Rains], the theory of polynomials orthogonal on the unit circle (whose connection to the generating function was already known to the physicists [Neub, PerSa, PerSb]), very powerful and sophisticated Riemann-Hilbert Problem techniques, and the work of Tracy and Widom on eigenvalues of random matrices [TracyW]. Baik, Deift, and Johansson have completely determined the asymptotic distribution of $(2\sqrt{n} - L_n)/n^{1/6}$. Their results are not simple to state, as they are given in terms of the solution to a Painlevé II equation, and presumably are not expressible in elementary functions. A remarkable fact is that this asymptotic distribution is the same (aside from scale factors) as that which Tracy and Widom showed to hold for the gap between the largest eigenvalue of a random matrix from the Gaussian Unitary Ensemble and $(2n)^{1/2}$. No direct relation between the two problems is known, and the scaling factors make it unlikely there is one, so this is presumably an expression of the universality of the distribution. See [TracyW2] for more details.

Numerical computations by Craig Tracy show that the standard deviation of $L_n/n^{1/6}$ is asymptotic to 0.90177..., and the expected value of $(2\sqrt{n} - L_n)/n^{1/6}$ is asymptotic to -1.77108..., values that agree well with the numbers in Table 1. Fig. 1 compares the asymptotic distribution of $(2\sqrt{n} - L_n)/n^{1/6}$ to the Monte Carlo results for $n = 10^6$, and it can be seen that the agreement is excellent.

3. Numerology

Tables 2-4 give the exact values of $g_{n,k}$ for $n = 15$, 30, and 60. It is interesting to note the patterns in the final digits of these numbers; these patterns can all be explained by the following fact:

$$\chi^{\lambda}_{\kappa(p\mu)} \equiv \chi^{\lambda}_{\kappa\mu^p} \pmod p \ , \tag{3.1}$$

where $p\mu$ is the partition produced from μ by multiplying each element by p, and similarly for μ^p; $\kappa\mu$ is the (sorted) concatenation of the two partitions. This follows easily from the fact that S_n has integral representations, and so for all permutations π,

$$\chi^{\lambda}(\pi^p) \equiv \chi^{\lambda}(\pi)^p \equiv \chi^{\lambda}(\pi) \pmod p \ .$$

Consider, now, the special case κ empty and $\mu = k$ of 3.1. By squaring both sides and summing over λ, we get, in the notation of [Rains],

$$f_{(pn)k} \equiv f^{(p)}_{nk} \pmod p \ ;$$

as a special case,

$$f_{(p^r)k} - f_{(p^r)(k-1)} \equiv 1 \pmod p \ .$$

(By induction in r, we have $f_{(p^r)k} \equiv f^{(p^r)}_{1k}$; the latter is easily shown to be equal to k.) Similarly, one can fairly easily deduce other "numerological" results concerning the values (mod$p$) of f_{nk}, for $n = ap^r + b$, a, b small (In these cases, only a small, easily enumerated, set of shapes contributes (mod$p$).) When $p = 2$, everything actually works mod 4, since every term in the sum was squared. Thus, in particular, we have the following fact:

$$f_{nk} \equiv b_{nk} \pmod 4 \ .$$

4. Computations

The exact computations were performed using the Schensted correspondence and the hook formula, as in [BaerB]. Thus instead of computing all $n!$ permutations of n elements, it was only necessary to generate the $p(n)$ partitions of n. (Multiple precision arithmetic was required, which was performed using the GNU package.) The running time, on a Silicon Graphics computer with R10000 200 MHz chips, was about 10 seconds for $n = 60$, and 45,000 seconds for $n = 120$.

Roughly speaking, the Monte Carlo computations proceeded by generating random permutations, and computing the length of their longest increasing subsequence using the Schensted correspondence again.

One difficulty that arises is that the computation for 10^{10} requires the generation of around 10^{10} random 32-bit numbers per permutation (the exact number will be slightly greater, due to the times when two of the generated $\pi(i)$ agree in the first 32-bits, so more bits need be generated to distinguish them). For, say, 100 permutations, this means that 10^{12} random words need to be generated. This gives rise to two problems. The first, less serious, problem is that random number generation is frequently slow, causing the computation speed to be bound by the speed of random number generation. The more significant problem is that the readily available random number generators have periods of 2^{31} or 2^{48} (and the latter RNG is quite slow). Since we needed to generate over 2^{40} random numbers, there was a significant risk with such short periods that the results could be erroneous. This problem was fixed by combining a Marsaglia subtract-with-borrow generator (using code provided by Jim Reeds, who also provided helpful advice on random number generation in general) with the LCG routine lrand(), from v9 UNIX.

The running time for the Monte Carlo code was around 10 hours for each permutation for $n = 10^{10}$. (These times are for the same R10000 chips as were mentioned above, although many runs were performed on slower machines, using their idle cycles.)

Our programs have been adopted to produce further statistics, for example on the distribution of the length of the second row of a Young tableaux, which served as a check on the asymptotic estimates of [BaikDJ2]. Data is available at ⟨http://www.research.att.com/~amo/tables/index.html⟩ and will be supplemented as more runs are carried out and more statistics are collected.

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Table 1: Monte Carlo simulation data on the distribution of L_n and asymptotic values

	$n = 10^4$	$n = 10^5$	$n = 10^6$	$n = 10^7$	$n = 10^8$	$n = 10^9$	$n = 10^{10}$	*asymptotic*
no. permutations	10^7	$6 * 10^5$	10^5	10^5	10^4	2000	4000	
$2n^{1/2}-$ mean (L_n)	7.704	11.560	17.196	25.430	37.873	54.850	82.352	
$(2n^{1/2}-$ mean $(L_n))n^{-1/6}$	1.660	1.697	1.720	1.733	1.758	1.735	1.774	1.77109
st. dev. (L_n)	4.043	6.032	8.959	13.209	19.342	28.538	41.545	
(st. dev. $(L_n))n^{-1/6}$	0.871	0.885	0.896	0.900	0.898	0.902	0.895	0.90177
scaled moments								
3	0.249	0.237	0.238	0.222	0.204	0.251	0.269	0.224
4	3.108	3.092	3.135	3.068	3.139	3.072	3.007	3.094
5	2.531	2.394	2.497	2.174	2.115	2.455	2.277	2.280
6	17.217	16.952	17.694	16.224	17.310	16.557	14.922	16.908
7	27.826	26.323	28.655	22.155	23.301	23.437	19.417	25.051
8	145.110	141.505	153.789	123.732	139.010	125.014	100.303	139.552

Table 2: Exact distribution of L_n for $n = 15$

k	$g(15, k)$
1	1
2	9694844
3	8017098273
4	164161815768
5	485534447114
6	434119587475
7	172912977525
8	37558353900
9	4927007100
10	410474625
11	22128576
12	766221
13	16381
14	196
15	1

Table 3: Exact distribution of L_n for $n = 30$

k	$g(30, k)$
1	1
2	3814986502092303
3	1222388966728910010696665
4	1790036582998939530743648877
5	449044243619862872721423598179
6	10236819433951393776243660748875
7	50241067877038219983230124657600
8	86511371455863277882723853476200
9	70971582765623356071324810857700
10	33700117351593715495661064101700
11	10447178628714722178634866396630
12	2277900847905046253535807880680
13	366440157064983378222220318530
14	44912755712412555783652789980
15	4289203871330156652985437480
16	324301002215082697285357800
17	19633107355949074371195000
18	959064229546178387532600
19	37982369568044622191625
20	1222055891584247185425
21	31925927141978856309
22	675007128155925069
23	11475430101232224
24	155228816648544
25	1644397829384
26	13319151176
27	79490741
28	328861
29	841
30	1

Table 4: Exact distribution of L_n for $n = 60$

k	$g(60, k)$
1	1
2	1583850964596120042686772779038895
3	353580101123476924257628603730083960324608410748129
4	17080691328825216538079811628828842602913045806045692424793199
5	175243028250079660905018843213615929860825569549681884867765690541701
6	9336151984930708021143911217956813677819162164640452787627883005534760901
7	15180807338873516832021030140438444665815147021460591742801378406314408952231
8	2233494474948495690243110568745222983262159502283551689273891105099703764639203
9	60002895752771099779779088462943847999099581712023250349374731986619450937660387
10	468104440722126644812839632177556187281953330916322512459026291795529190084140003
11	1455327054374385756982545351864306579536481867901002010423776062240740978062678405
12	2259251055120372007733214696091079754018818083465717757461536975882962682765500625
13	2062265432178679983886852088922462401452557170316484374161761008379074310593517320
14	1243711511999821270591207565082889798761871176715300197918122808539228337822802740
15	537394830317050100339379519887032754646946119740464857911956705681737098244483360
16	175923103423553571947761906278676245128973950100129233119563346326464104855276860
17	45368617608497201905530039854748875664926717869975676357175086535901817870722812
18	9479603856030503157955685146063700672586357866208188042547738509639642888641224
19	1638759009110121823982506004487303838241549550728662507775364230503063574975316
20	238188858817559653907757766891040131636069359492222134403147802254653854771728
21	29480487670047223803921747890109023556319166001013451281038912984170310753120
22	3139198038828528286203710931455906285276708940460977128156446860885650927310
23	290030563932022002118220602447753575011631132138930828877529635970794362700
24	23413655153323993212944806641432538187723487636946542215904509952446312690
25	1661393542805513071742417067989822659686364866756920591886636429663178680
26	104145900363220144830466866571023152418199800997341553230438935249107258
27	5792237419925383613451898590561388009628831263062366370911868157401964
28	286869467381919681822222469760492255054776533189183154318863752312230
29	126918718554818286265934580251256068575961311143782738903573825981796
30	502968058628572662277191566575373626415624915252154954972665008452
31	17894607700299703295952617215614009799494649283846631618530060276
32	572672495594979262825338794738392020356281425452234267523956756
33	16511387220795567604685869448544444896127174737028045397681196
34	429447776828374945008891956588902876262471997021459648212556
35	10085939380850856133146998090323665735800777575608380654076
36	214047059046253016288888877319022168888880344026378396380
37	4106553547655147341710844172909909933553257664495111405
38	71234520883705192260893127544474396900805731744717605
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60	1

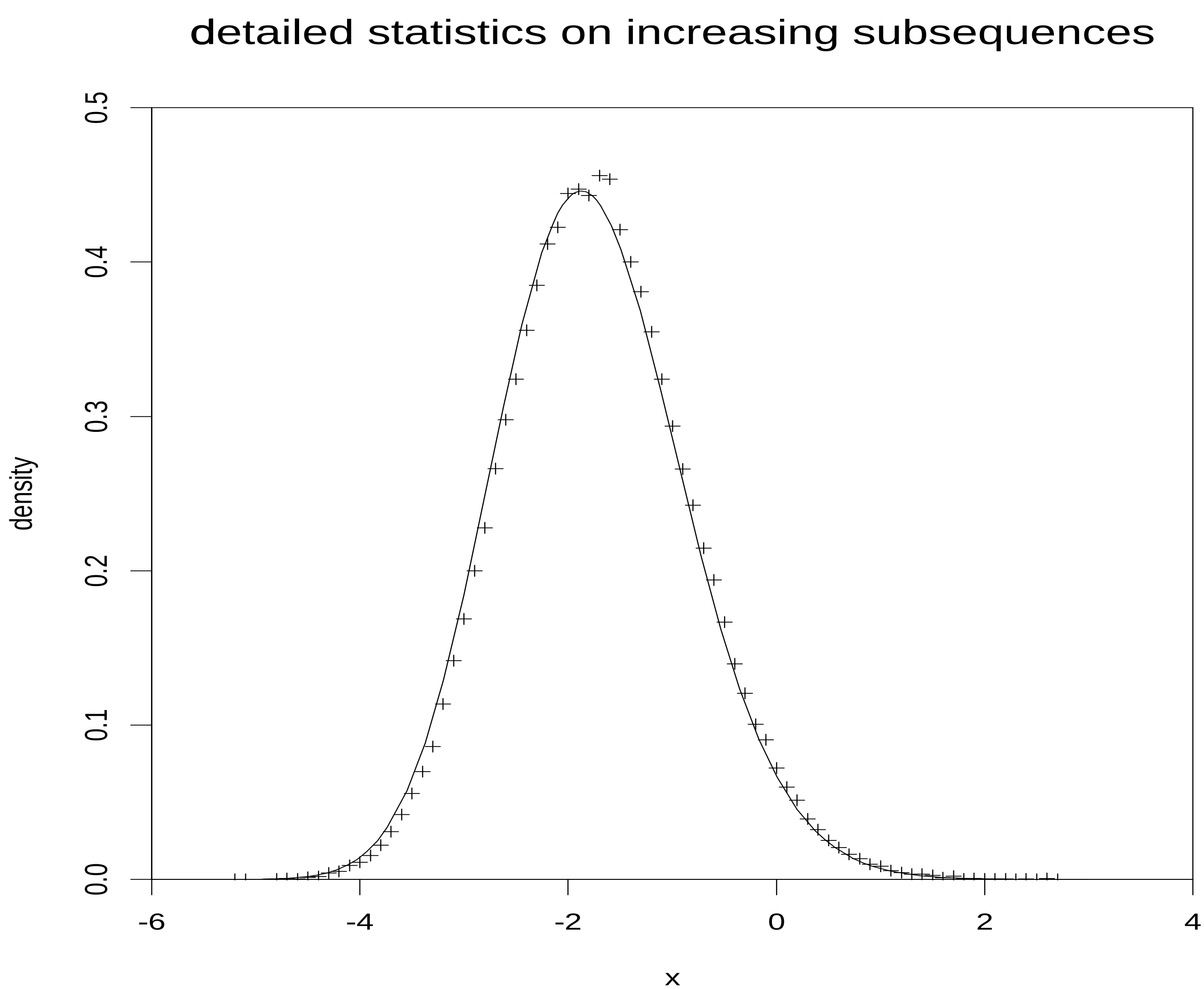

Figure 1: Asymptotic density function for for $n = 10^6$. The smooth curve is the asymptotic density function for $(2\sqrt{n} - L_n)/n^{1/6}$, based on theorem of Jinho Baik, Percy Deift, and Kurt Johansson. Data for the asymptotic distribution figure provided by Craig Tracy. Crosses represent the distribution of values of $(2\sqrt{n} - L_n)/n^{1/6}$ for $n = 10^5$ random permutations for $n = 10^6$.