Zeros of the Riemann zeta function: Conjectures and computations

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Sieve of Eratosthenes:

$\boxed{1}$ 2 3 4 5 6 7 8 9 10 11 12 ...

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Mathematicians have tried in vain to discover some order in the sequence of prime numbers but we have every reason to believe that there are some mysteries which the human mind will never penetrate. To convince ourselves, we have only to cast a glance at tables of primes (which some have constructed to values beyond 100,000) and we should perceive that there reigns neither order nor rule.

Euler, 1751

As of 2002, we have learned that there is some order and rule among the primes, but there are still many mysteries that we have barely penetrated!

$\pi(x) =$ number of primes $\leq x$

Prime Number Theorem (PNT):

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty$$

Conjectured by Gauss and Legendre

Proved by Hadamard and de la Vallee Poussin (1896)

"Random" behavior of primes:

Suppose

$$Prob(n \text{ is prime}) \approx \frac{1}{\log n}.$$

Then expect

$$\pi(x) = \# \text{ primes} \leq x \sim \int_2^x \frac{dt}{\log t}$$

with an error term $\leq x^{\frac{1}{2}+o(1)}$ as $x \to \infty$, and a central limit theorem.

Riemann Hypothesis implies the expected error term estimate, but fine scale behavior of the remainder term is different than for coin tossing sequences

Euler ($\sim$1730 to $\sim$1780, motivated by a question of Mengoli, $\sim$1650)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

1) Computational methods
   (Euler-Maclaurin formula)

2) Exact formulas

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

3) Functional equation

$$\zeta(-3) \quad \text{``} = \text{''} \quad 1^3 + 2^3 + 3^3 + \cdots$$

$$\rightarrow \quad \zeta(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

4) Euler product

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad = \quad \prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^{2s}} + \cdots\right)$$

$$= \quad \prod_p (1 - p^{-s})^{-1}$$

Arithmetic consequence of Euler product:

$$\sum_p \frac{1}{p} = \infty\,.$$

Proof.

$$\begin{aligned}
\log \zeta(s) \quad &= \quad \sum_p -\log(1 - p^{-s}) \quad s > 1 \\
&= \quad \sum_p \{p^{-s} + O(p^{-2})\} \\
&= \quad \sum_p \frac{1}{p^s} + O(1)\,.
\end{aligned}$$

Riemann: ($\sim$1850)

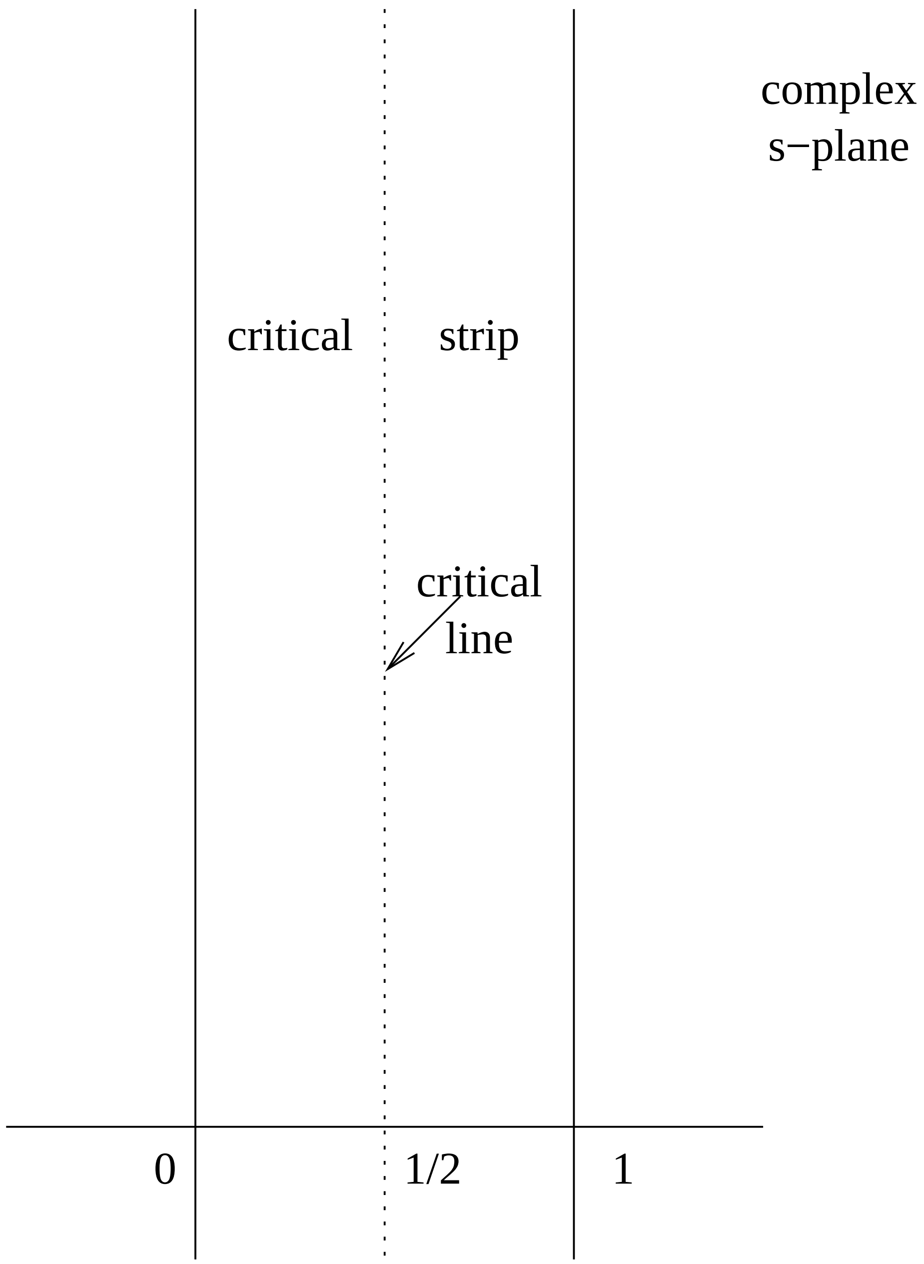
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C},\ \mathrm{Re}(s) > 1\,.$$

Showed $\zeta(s)$ can be continued analytically to $\mathbb{C} \setminus \{1\}$ and has a first order pole at $s = 1$ with residue 1. If

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)\ ,$$

then (functional equation)

$$\xi(s) = \xi(1-s)\ .$$



$$
\begin{aligned}
\xi(s) &= \pi^{-s/2}\Gamma(s/2)\zeta(s) \\
&= \pi^{-s/2}\Gamma(s/2)\prod_p (1-p^{-s})^{-1},\ \mathrm{Re}(s)>1, \\
\xi(s) &= \xi(1-s) \\
\xi(s) &= \overline{\xi(\bar{s})}
\end{aligned}
$$

$\pi(x) =$ number of primes $\leq x$.

$$\left(\sim \frac{x}{\log x} \text{ as } x \to \infty \text{ by PNT}\right)$$

$$\pi(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) + O(x^{1/2}\log x)$$

where ρ runs over the nontrivial zeros of $\zeta(s)$, and

$$\mathrm{Li}(v) = \int_2^v \frac{du}{\log u}\,.$$

$$\boxed{\begin{array}{c}\mathrm{Li}(x) \sim \dfrac{x}{\log x} \\ as\ x \to \infty\end{array}}$$

$$RH \Leftrightarrow |\pi(x) - \mathrm{Li}(x)| = O(x^{1/2}\log x)\,.$$

Main announced verifications of RH:

Gram 1903	15
Backlund 1914	79
Hutchinson 1925	138
Titchmarsh et al. 1935/6	1,041
Turing 1952	1,054
Lehmer 1956	25,000
Meller 1958	35,337
Lehman 1966	250,000
Rosser et al. 1968	3,500,000
Brent 1979	81,000,000
te Riele et al. 1986	1,500,000,000
van de Lune 2001	10,000,000,000
Wedeniwski 2002	75,000,000,000

first 2 zeros of the zeta function to about 200 decimal places:

14.134725141734693790457251983562470270784257115699243175685567460149963429809256764949010393171561012779202971548797436766142691469882254582505363239447137780413381237205970549621955865860200555566725836010 77...

21.022039638771554992628479593896902777334340524902781754629520403587598586068890799713658514180151419533725473642475891383865068603731321262118821624375741669256544711844071194031306725646227792614887337435 55...

10^{20}-th zero:

$$\frac{1}{2} + i\gamma \ ,$$

$$\gamma = 15,202,440,115,920,747,268.62902990$$

10^{22}-nd zero:

$$1,370,919,909,931,995,308,226.62751\ldots$$

additional zeros & some old preprints & reprints:

http://www.dtc.umn.edu/~odlyzko

New computations of actual values of zeros (modulo holes):

$$10^{20} - 30,769,710 \leq n$$

$$\leq 10^{20} + 1,634,308,174$$

$$10^{21} - 1,443,923 \leq n$$

$$\leq 10^{21} + 900,233,349$$

$$10^{22} - 61,110 \leq n$$

$$\leq 10^{22} + 10,461,101,664$$

$$10^{23} - 1,054,429,857 \leq n$$

$$\leq 10^{23} + 19,434,326,636$$

Verification of RH for

first n zeros:

Euler-Maclaurin: $n^{2+o(1)}$

Rieman-Siegel: $n^{\frac{3}{2}+o(1)}$

O.-Schöenhage: $n^{1+o(1)}$

Main challenges in current computations:

- data management

- data analysis

- social factors

A true (but misleading) statement:

2 zeros near zero number $10^{23} - 846,369,741$ cannot be located on the critical line

Die Legende, Riemann habe die Resultate seiner mathematischen Arbeit durch “groβe allgemeine” Ideen gefunden, ohne die formalen Hilfsmittel der Analysis zu benötigen, ist wohl jetzt nicht mehr so verbreitet wie zu Kleins Lebzeiten. Wie stark Riemanns analytische Technik war, geht besonders deutlich aus seiner Ableitung und Unformung der semi-konvergenten Reihe für $\zeta(s)$ hervor.

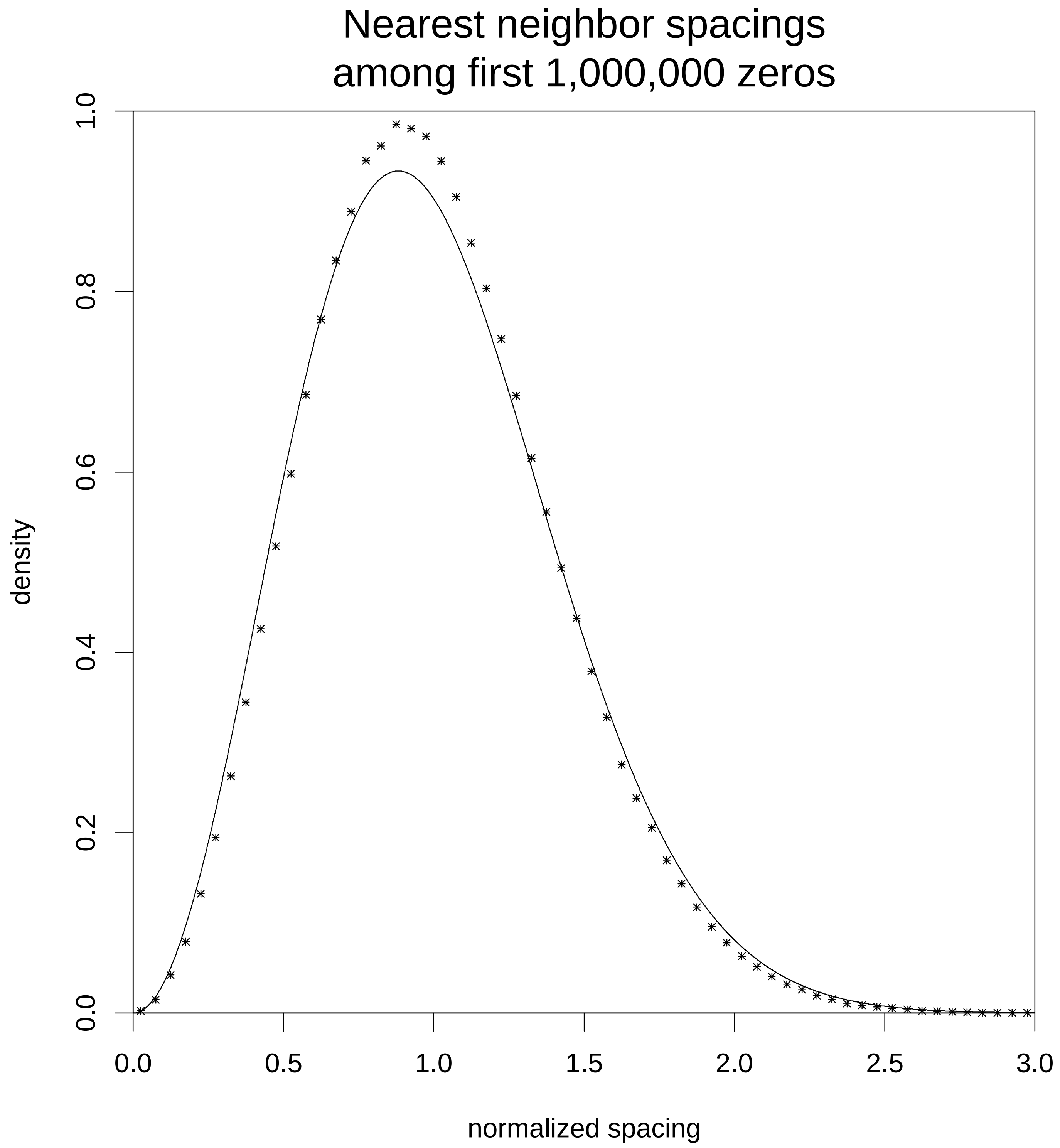
C. L. Siegel (1932)

Hilbert and Pólya conjecture:

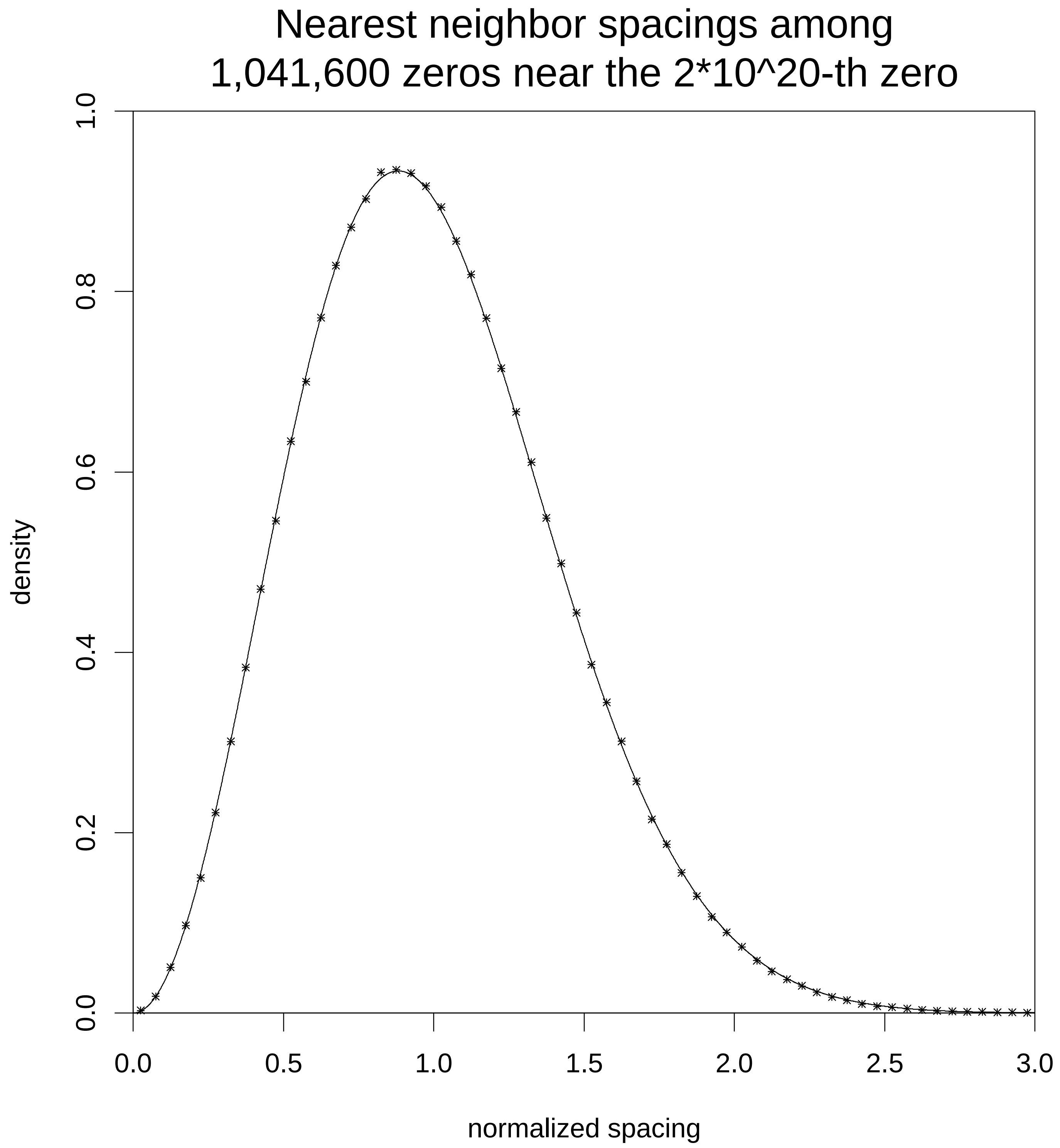
There exists a self-adjoint hermitian operator whose eigenvalues correspond to the zeros of the zeta function

$$\begin{array}{ccc} \text{zero} & & \text{eigenvalue} \\ \rho = \frac{1}{2} + i\gamma & \longleftrightarrow & \lambda = \frac{1}{4} + \gamma^2 \end{array}$$

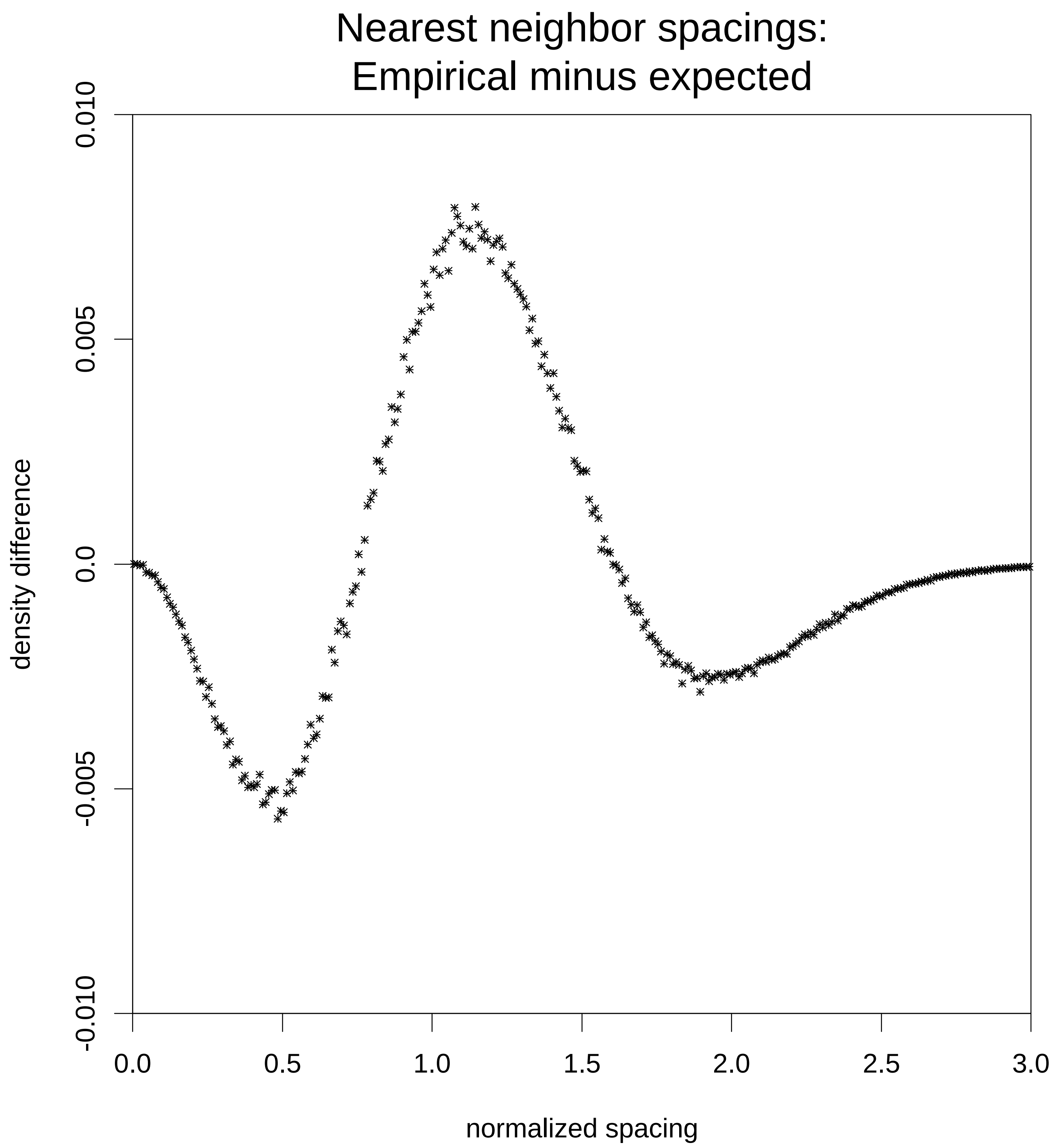
$$\gamma \in \mathbb{C}$$



Nearest neighbor spacings
among first 1,000,000 zeros
density
normalized spacing
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0



Nearest neighbor spacings among
1,041,600 zeros near the 2*10^20-th zero
density
normalized spacing
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0


Nearest neighbor spacings:
Empirical minus expected
density difference
normalized spacing
0.010
0.005
0.0
-0.005
-0.010
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Conjecture (mentioned by Riemann):

$$\pi(x) < li(x) = \int_0^x \frac{dt}{\log t}, \quad x \geq 1\,.$$

FALSE!

Bounds for first counterexample:

Littlewood		
Skewes	$\leq$	$e^{e^{e^{e^{\cdot^{\cdot}}}}}$
Lehman	$\leq$	10^{1180}
te Riele	$\leq$	10^{380}
Bays & Hudson	$\leq$	10^{317}

Expect no counterexample for $x \leq 10^{30}$.

Two conjectures (proved incompatible by Hensley and Richards):

1) $\pi(x+y) \leq \pi(x) + \pi(y)$
2) Hardy-Littlewood prime k-tuple conjecture:

   If $a_1, \ldots, a_k \in Z$, there are infinitely many $n \in Z^+$ such that $n + a_1, \ldots, n + a_k$ are all prime, unless there is a $\underbrace{\text{trivial obstruction}}_{\text{Ex.:}\{a_1,a_2,a_3\}=\{0,2,4\}}$

Hawkins random sieve

$\boxed{1}$ 2 3 4 5 6 7 8 9 10 11 ...

for each "prime" p, cross out every integer $> p$ with probability $\frac{1}{p}$

Hawkins random primes:

PNT, RH, twin prime conjecture, etc.:
all true with probability 1.

H. Maier result:

$$\limsup_{x\to\infty} \quad \frac{\pi(x+(\log x)^{\lambda})-\pi(x)}{(\log x)^{\lambda-1}} \quad > 1$$

$$\liminf_{x\to\infty} \quad \frac{\pi(x+(\log x)^{\lambda})-\pi(x)}{(\log x)^{\lambda-1}} \quad < 1\,.$$

Random model and Hawkins' sieve both predict:

1) Cramer's conjecture:

$$p_{n+1} - p_n \leq C(\log p_n)^2$$

2) For $\lambda \geq 4$,

$$\lim_{x\to\infty} \frac{\pi(x + \log x)^{\lambda}) - \pi(x)}{(\log x)^{\lambda-1}} = 1\,.$$

Contradicts Maier's result!

$$
\begin{aligned}
N(t) &= \#\text{ zeros } \rho \text{ with} \\
&\qquad 0 < \mathrm{Im}(\rho) \leq t \\
&= 1 + \frac{1}{\pi}\theta(t) + S(t) \\
\theta(t) &= \frac{t}{2}\log\frac{t}{2\pi e} - \frac{\pi}{8} + O\left(\frac{1}{t}\right) \\
&\qquad \text{as } t \to \infty
\end{aligned}
$$

$$|S(t)| = O(\log t)$$

$$S_1(t) = \int_{10}^{t} S(u)du = O(\log t)$$

$$\frac{1}{t}\int_{10}^{t} S(u)^2 du \sim c \log\log t$$

$$|S(t)| < 1 \quad \text{for} \quad t < 280$$

$$|S(t)| < 2 \quad \text{for} \quad t < 6.8 \times 10^6$$

$$\text{largest observed value of } |S(t)| \sim 3.2$$

Conclusion (inspired by a sign in a computing support office):

We are sorry that we have not been able to solve all of your problems, and we realize that you are about as confused now as when you came to us for help. However, we hope that you are now confused on a higher level of understanding than before.