The Absolute Value of Functions

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Abstract. A real-valued function f defined on a topological space is called *absolutely polynomial* if its absolute value can be written as a polynomial in f with continuous coefficients. One motivation for studying such functions comes from the theory of rings of continuous functions. While many real functions are absolutely polynomial, we provide a number of interesting explicit examples which are not. The absolutely polynomial criterion turns out to be quite delicate, and we develop the theory in some detail. Our study of absolutely polynomial functions is then widened to more general topological spaces. Our results provide pertinent counterexamples in the theory of rings of quotients of Φ–algebras.

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1. Motivation.

The original purpose of this paper was to provide a negative answer to the following question.

Problem: Can the function $\left|\sin\frac{1}{x}\right|$ be written as a polynomial in $\sin\frac{1}{x}$ with continuous functions as coefficients?

On the surface, this may seem to be merely a technical issue in real analysis. However, the motivation is much deeper, since a negative answer has important ring-theoretic consequences. A function that satisfies the condition stated in the Problem will be called *absolutely polynomial*, and we begin by showing that $\sin\frac{1}{x}$ is not absolutely polynomial. However, many standard functions, including all rational functions and all piecewise continuous functions, are absolutely polynomial. This distinction served to motivate a detailed investigation into the properties of real functions that distinguish those that are absolutely polynomial.

The familiar functions which are absolutely polynomial are of degree one or two. We begin by producing an example of a ("generalized") piecewise linear function of bounded variation that is not absolutely polynomial, even though small nearby perturbations are absolutely polynomial. The key to understanding the concept lies in the notion of the first (and subsequent) jump ratio functions, as defined in section 3. One is led naturally to study, in turn, monotone functions, functions of bounded variation, and regulated functions, which are natural generalizations of piecewise continuous functions. A general criterion, formulated in terms of the jump ratio functions, is provided for a regulated function (that is continuous everywhere it is defined) to be absolutely polynomial of degree n. Monotone functions are seen to be absolutely polynomial of degree at most 4. This section concludes with an explicit family of regulated functions that are absolutely polynomial of arbitrarily high degree, as well as a regulated function that is locally (but not globally) absolutely polynomial. Usually, one tries to avoid discontinuities and pathology. We need them (to be bad enough) in order to get valid interesting examples.

Following our discussion of real-valued functions, we expand our investigations to a wide class of topological spaces. A topological space will be called *absolutely closed* if every function on it is absolutely polynomial. The only examples known, though, are discrete spaces. More interesting is the question of whether a point in a topological space is *absolutely isolated*, meaning that every continuous function which is defined everywhere except at the point is absolutely polynomial. In particular, we discuss generalizations of our results concerning real rational functions to other topological spaces.

Finally, we present some nontrivial algebraic consequences of our results. An important motivation for the original problem was the question of how the category of Φ–algebras behaves under the formation of rings of quotients. Given a topological space X, we let $\mathcal{C}(X)$ denote the algebra of continuous functions defined on all of X. If a given function $f\colon X \to \mathbb{R}$ with dense open domain is not absolutely polynomial, then the polynomial ring $\mathcal{C}(X)[\,f\,]$ is not a Φ–algebra, even though it is a quotient ring of the Φ–algebra $\mathcal{C}(X)$. Consequently, any topological space with a non-absolutely isolated point has such a property — that one can construct a ring of quotients of the Φ–algebra of continuous functions

$\mathcal{C}(X)$ which is not a Φ–algebra. We conclude with some open problems and directions for further study.

2. The Key Result.

We begin by presenting the basic result in real analysis that answers the original question. The proof of this result forms the foundation of our generalizations.

Proposition 2.1. *There do not exist any real-valued functions $g_0(x), \ldots, g_n(x)$ which are continuous and defined for all $x \in \mathbb{R}$ such that*

$$\left| \sin \frac{1}{x} \right| = \sum_{k=0}^{n} g_k(x) \left(\sin \frac{1}{x} \right)^k \qquad \text{for all} \qquad x \neq 0. \tag{2.1}$$

Proof: Suppose we can find $g_k(x)$ satisfying (2.1). Set

$$c_k = g_k(0)$$

and define

$$r(z) = \left| \, | \, z \, | - \sum_{k=0}^{n} c_k z^k \right| . \tag{2.2}$$

Since $|\, z \,|$ does not coincide with any polynomial on any interval containing 0, there exist $\varepsilon > 0$ and $-1 \leq z_0 \leq 1$ such that

$$r(z_0) > \varepsilon. \tag{2.3}$$

Choose $\delta > 0$ such that

$$|\, g_k(x) - c_k \,| < \frac{\varepsilon}{n+1}, \qquad \text{for} \qquad |\, x \,| < \delta. \tag{2.4}$$

Then

$$\left| \sum_{k=0}^{n} \left[g_k(x) - c_k \right] \left(\sin \frac{1}{x} \right)^k \right| < \varepsilon, \qquad |\, x \,| < \delta.$$

Therefore, if (2.1) holds,

$$\left| \left| \sin \frac{1}{x} \right| - \sum_{k=0}^{n} c_k \left(\sin \frac{1}{x} \right)^k \right| < \varepsilon, \qquad |\, x \,| < \delta. \tag{2.5}$$

However, if we choose N sufficiently large so that

$$x = \frac{1}{2N\pi + \sin^{-1} z_0} < \delta,$$

then $\sin \frac{1}{x} = z_0$, and hence (2.5) becomes

$$r(z_0) = \left| \, |\, z_0 \,| - \sum_{k=0}^{n} c_k z_0^k \right| < \varepsilon.$$

But this contradicts (2.3). *Q.E.D.*

Clearly, the basic method of proof is not particular to the function $\sin \frac{1}{x}$, or to the choice of absolute value. In fact, the same proof can be used to establish a general result, of which Proposition 2.1 is merely one special case.

Theorem 2.2. *Let $f(x)$ be defined on the set $\operatorname{dom} f \subset \mathbb{R}$. Let $I_\delta = (x_0 - \delta, x_0 + \delta)$ denote the open subinterval of width $2\delta > 0$ centered at a point x_0, and define $f(I_\delta) = \{\, z = f(x) \mid x \in I_\delta \cap \operatorname{dom} f \,\}$. Assume that the set $J = \bigcap_{\delta > 0} f(I_\delta)$ contains an open subinterval J_0, which implies that f is discontinuous at x_0. Let $\psi(z)$ be a continuous function which is not identically equal to a polynomial on J_0. Then there do not exist continuous functions $g_0(x), \ldots, g_n(x) \in \mathcal{C}(\mathbb{R})$ such that*

$$\psi(f(x)) = \sum_{k=0}^{n} g_k(x) f(x)^k \qquad \text{for all} \qquad x \in \operatorname{dom} f. \tag{2.6}$$

In particular, if J contains an open subinterval $(-\varepsilon, \varepsilon)$, then there do not exist continuous functions $g_0(x), \ldots, g_n(x)$ such that

$$|\, f(x) \,| = \sum_{k=0}^{n} g_k(x) f(x)^k \qquad \text{for all} \qquad x \in \operatorname{dom} f.$$

3. Absolutely Polynomial Functions.

Let us now concentrate on the general question of whether the absolute value† of a given function $f: \mathbb{R} \to \mathbb{R}$ can be written as a polynomial in f with continuous coefficients. Note that the domain of the function $\operatorname{dom} f \subset \mathbb{R}$ need not be the entire real line. We propose the following definition to study this phenomenon. Let $\mathcal{C}(\mathbb{R})$ denote the ring of all continuous real-valued functions defined on *all* of $\mathbb{R}$, so $g \in \mathcal{C}(\mathbb{R})$ requires $\operatorname{dom} g = \mathbb{R}$.

Definition 3.1. A function $f: \mathbb{R} \to \mathbb{R}$ will be called *absolutely polynomial* if there exist continuous functions $g_0, \ldots, g_n \in \mathcal{C}(\mathbb{R})$ such that

$$|\, f(x) \,| = \sum_{k=0}^{n} g_k(x)\, f(x)^k \qquad \text{for all} \qquad x \in \operatorname{dom} f. \tag{3.1}$$

The *degree* of an absolutely polynomial function is the minimal n for which an identity of the form (3.1) holds.

For example, if $|\, f \,| \in \mathcal{C}(\mathbb{R})$ is continuous on all of $\mathbb{R} = \operatorname{dom} f$, then f is automatically absolutely polynomial of degree 0, since we can take $g_0 = |\, f \,|$. More generally, the absolutely polynomial functions of degree 0 are those whose absolute value $|\, f \,|$ can be continuously extended to all of $\mathbb{R}$. (The characterization of functions which can be continuously extended to $\mathbb{R}$ is discussed in more detail below.)

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† This is the most interesting case, and so we shall concentrate on it in this section. Most results can be extended to other functions ψ, as described in Theorem 2.2, without difficulty. The appropriate modifications of our results and constructions are left to the reader.

Proposition 3.2. *A function f is absolutely polynomial of degree* 0 *if and only if there exists $g_0 \in \mathcal{C}(\mathbb{R})$ such that* $|\,f\,| = g_0\,|\,\mathrm{dom}\, f$.

If $f(x) \geq 0$ for all $x \in \mathrm{dom}\, f$ and f is not continuously extendable to $\mathbb{R}$, then f is linearly (degree one) absolutely polynomial, since $|\,f\,| = f$; a similar result holds any non-positive function $f \leq 0$. Thus, only functions of variable sign are candidates for not being absolutely polynomial. Moreover, every function is the sum of two linearly absolutely polynomial functions — being the difference of its positive and negative parts.

Proposition 2.1 shows that $\sin \frac{1}{x}$ is not absolutely polynomial. More generally, Theorem 2.2 implies that functions with certain "essential" discontinuities are not absolutely polynomial. However, milder discontinuities are permitted. For example, the function $\frac{1}{x}$ is quadratically (but not linearly) absolutely polynomial. One has

$$\left|\,\frac{1}{x}\,\right| = |\,x\,| \cdot \left(\frac{1}{x}\right)^2,$$

and hence (3.1) holds with $n = 2$, $g_0(x) = g_1(x) = 0$, and $g_2(x) = |\,x\,|$. Clearly, a function f is absolutely polynomial of degree ≤ 2 if and only if $1/f$ is absolutely polynomial of degree ≤ 2.

Note that the sum of two absolutely polynomial functions (when defined) need not be absolutely polynomial; for example, $\sin \frac{1}{x}$ is the sum of the non-negative function $2 + \sin \frac{1}{x}$ and the nonpositive function -2. Similarly, the product of two absolutely polynomial functions, or even a continuous function with an absolutely polynomial function, need not be absolutely polynomial. An example is given by multiplying the function $\frac{1}{x}$ by the continuous function

$$p(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases} \tag{3.2}$$

Indeed, $\frac{1}{x} \cdot p(x) = \sin \frac{1}{x}$ is our original non-absolutely polynomial example. On the other hand, the product of an absolutely polynomial function and a *nonvanishing* continuous function remains absolutely polynomial.

A more interesting question is whether the (odd) powers of an absolutely polynomial function are absolutely polynomial. Of course, even powers of a function are non-negative, and so trivially linearly absolutely polynomial. The only result we have been able to establish is in the linear case:

Proposition 3.3. *If f is linearly absolutely polynomial, then every power f^n is also linearly absolutely polynomial.*

Proof: We only need consider odd powers f^{2m+1}. By assumption, we have $|\,f\,| = a + bf$ for $a, b \in \mathcal{C}(\mathbb{R})$. We rewrite this as $|\,f\,| - bf = a$ and take powers of both sides:

$$\sum_{k=0}^{2m+1} \binom{2m+1}{k} b^k\,|\,f\,|^{2m+1-k} f^k = (|\,f\,| - bf)^{2m+1} = a^{2m+1}. \tag{3.3}$$

Note that $|\,f\,|^{2m+1-k} f^k = f^{2m+1}$ when k is even, while $|\,f\,|^{2m+1-k} f^k = |\,f\,|^{2m+1}$ when k is odd. Therefore, we can rewrite (3.3) as

$$\gamma\,|\,f\,|^{2m+1} = \alpha + \beta f^{2m+1}, \tag{3.4}$$

where

$$\alpha = a^{2m+1}, \qquad \beta = \sum_{i=0}^{m} \binom{2m+1}{2i+1} b^{2i+1}, \qquad \gamma = \sum_{i=0}^{m} \binom{2m+1}{2i} b^{2i}.$$

Since $\gamma > 0$ everywhere, (3.4) implies that $|\,f\,|^{2m+1} = (\alpha/\gamma) + (\beta/\gamma) f^{2m+1}$, so f^{2m+1} is linearly absolutely polynomial. Note finally that an odd power of a function is continuously extendable if and only if the function itself is continuously extendable, and so Proposition 3.2 shows that the absolutely polynomial degree of the odd powers of f is the same as that of f. *Q.E.D.*

We do not know whether the odd powers of a quadratically absolutely polynomial function are necessarily absolutely polynomial.

Definition 3.4. Define the *support* of a function $q\colon \mathbb{R} \to \mathbb{R}$ to be the set $\operatorname{supp} q = \{x \mid q(x) \neq 0\} \subset \operatorname{dom} q$. The *zero set* of q is the set $\mathcal{Z}(q) = \{x \mid q(x) = 0\} = \operatorname{dom} q \setminus \operatorname{supp} q$.

Quotients of absolutely polynomial functions, even when both numerator and denominator are continuous but have overlapping zero sets, need not be absolutely polynomial. For example, if we divide (3.2) by $q(x) = x$ we again recover $p(x)/q(x) = \sin \frac{1}{x}$, which is not absolutely polynomial. However, certain types of quotients of continuous functions, including rational functions, are always absolutely polynomial.

Theorem 3.5. *Suppose $p, q \in \mathcal{C}(\mathbb{R})$ are continuous functions and suppose there exist continuous functions $u, v \in \mathcal{C}(\mathbb{R})$ such that*

$$up + vq = 1 \qquad \text{for all} \qquad x \in \mathbb{R}. \tag{3.5}$$

Then the quotient p/q, which is defined on $\operatorname{supp} q$, is at most quadratically absolutely polynomial.

Proof: We have

$$\left|\,\frac{p}{q}\,\right| = \frac{|\,p\,|\,|\,q\,|}{q^2} = |\,p\,|\,|\,q\,| \left(\frac{up + vq}{q}\right)^2 = |\,p\,|\,|\,q\,| \left(u\,\frac{p}{q} + v\right)^2. \tag{3.6}$$

The latter expression is clearly a polynomial of degree 2 in p/q with continuous coefficients. *Q.E.D.*

For example, the function $\tan x = \sin x / \cos x$ is quadratically absolutely polynomial, because $\sin^2 x + \cos^2 x = 1$. Incidentally, it is easy to see that $\tan x$ is not linearly absolutely polynomial.

Theorem 3.6. *Suppose $p, q \in \mathcal{C}(\mathbb{R})$ are continuous functions, with disjoint zero sets $\mathcal{Z}(p) \cap \mathcal{Z}(q) = \varnothing$. Then the quotient p/q is absolutely polynomial.*

Proof: The function $h = p^2 + q^2$ is always positive, and so $1/h \in \mathcal{C}(\mathbb{R})$ is continuous. Moreover, one has $(p/h)p + (q/h)q = 1$, and so one can use Theorem 3.5 with $u = p/h$, $v = q/h$. *Q.E.D.*

Remark: When the above holds, not only is p/q absolutely polynomial, but so are its powers. Indeed, we raise (3.5) to the $2n^{\text{th}}$ power, which makes 1 a linear combination p^n and q^n with continuous coefficients.

Corollary 3.7. *Any rational function over the real field is absolutely polynomial.*

Proof: Any rational function can be written as a quotient p/q of relatively prime polynomials, which implies that p and q have disjoint zero sets. Alternatively, one can use the Euclidean algorithm to construct real polynomials u and v such that $up + vq = 1$ and appeal to Theorem 3.5. *Q.E.D.*

In order to investigate more general singularities, we first require some basic definitions. Given a set $S \subset \mathbb{R}$, we let S^+ denote the set of right-hand accumulation points, so that $x \in S^+$ if and only if there exists a decreasing sequence of points $x_j \in S$ with $x_j \to x^+$ converging to x from the right. Similarly, S^- will denote the set of all left-hand accumulation points. Note that $S^+ \cup S^- \cup S = \overline{S}$ equals the closure of S, while $S \setminus (S^+ \cup S^-)$ are the isolated points of S. Let $S^\pm = S^+ \cap S^-$ denote the set of two-sided accumulation points, and $S^0 = S^\pm \cap S$ those that belong to S.

The following terminology has been adapted from that in Bourbaki, [**1**; §II.1.3], and Dieudonné, [**2**; §VII.6], who considered the particular case of functions whose domain is an interval.

Definition 3.8. A function $f\colon \mathbb{R} \to \mathbb{R}$ will be called *regulated* if, for every $x \in (\operatorname{dom} f)^+$, the right hand limit $f(x^+)$ exists, and for every $x \in (\operatorname{dom} f)^-$, the left hand limit $f(x^-)$ exists.

Every monotone function is regulated, as is every function of bounded variation, being the difference of two monotone functions, [**11**; p. 86]. According to a remark in Bourbaki, [**1**; p. II.6], the regulated functions defined on an interval form an algebra — indeed, they form what we will later call a Φ–algebra, cf. Definition 5.1, that is uniformly closed, [**1**; p. II.5]. The following basic characterization of regulated functions can be found in [**1**, **2**].

Theorem 3.9. *A function f whose domain is a compact interval is regulated if and only if it is the uniform limit of step functions.*

Classically, the condition that a real function f be "continuous at a point x" requires that $x \in (\operatorname{dom} f)^0$ is a two-sided accumulation point in its domain, and $f(x^+) = f(x^-) = f(x)$. We wish to extend this usual notion of continuity to other types of points, and the following definition provides a natural generalization.

Definition 3.10. Let $f\colon \mathbb{R} \to \mathbb{R}$ be regulated. A point $x_0 \in \mathbb{R}$ will be called a point of *generalized continuity* of f if either f is continuous at x_0, or f is not defined at x_0, but has a limit as $x \to x_0$.

Note that points of generalized continuity all belong to the closure $\overline{\operatorname{dom} f}$ of the domain of f. In particular, all isolated points in the domain of f are included as points of generalized continuity.

Definition 3.11. The *continuity set* $\mathcal{C}(f) \subset \mathbb{R}$ of a regulated function $f\colon \mathbb{R} \to \mathbb{R}$ is the set of all points of generalized continuity $x \in \overline{\operatorname{dom} f}$. The *discontinuity set* is defined as $\mathcal{D}(f) = \overline{\operatorname{dom} f} \setminus \mathcal{C}(f)$. The set of *two-sided discontinuities* of f is

$$\mathcal{D}(f)^* = \left\{ x \in (\operatorname{dom} f)^\pm \;\middle|\; f(x^+) \neq f(x^-) \right\}. \tag{3.7}$$

Note that f may or may not be defined at points on its discontinuity set. For us, the most important (and easiest to analyze) case is when $\mathcal{D}(f) \cap \operatorname{dom} f = \varnothing$, so that f is not actually defined at any discontinuity point. This implies that every discontinuity is a two-sided discontinuity, so $\mathcal{D}(f) = \mathcal{D}(f)^*$, cf. (3.7).

Remark: If f is not regulated, then one enlarges its discontinuity set $\mathcal{D}(f)$ to include all right hand accumulation points $x \in (\operatorname{dom} f)^+$ where the right hand limit $f(x^+)$ does not exist, and, similarly, all left hand accumulation points $x \in (\operatorname{dom} f)^-$ where $f(x^-)$ does not exist.

Definition 3.12. A function f is called *generalized continuous* if $\mathcal{D}(f) = \varnothing$, so that every point in $\overline{\operatorname{dom} f}$ is a point of generalized continuity.

Remark: In particular, every generalized continuous function is regulated. In Bourbaki, [**1**; Théorème II.3], it is proved that the discontinuity set of a regulated function whose domain is an interval is a countable set.

The generalized continuous functions are those that can be continuously *interpolated* — meaning continuously extended to all of $\mathbb{R}$.

Definition 3.13. A *continuous extension* of a function $f\colon \mathbb{R} \to \mathbb{R}$ is a function $h \in \mathcal{C}(\mathbb{R})$ such that $h \mid \operatorname{dom} f = f$.

Proposition 3.14. *A function f admits a continuous extension if and only if f is generalized continuous.*

Proof: The direct statement is easy. To prove the converse, let $a = \inf\{\operatorname{dom} f\}$, $b = \sup\{\operatorname{dom} f\}$, so that $\operatorname{dom} f \subset [\,a, b\,]$. Given $a < x < b$, define

$$m_x = \sup\,\{ y \in \operatorname{dom} f \mid x \geq y \}, \qquad M_x = \inf\,\{ y \in \operatorname{dom} f \mid x \leq y \}.$$

Note that $m_x = M_x$ if and only if either $x \in \operatorname{dom} f \cup (\operatorname{dom} f)^\pm$ is either in the domain of f or is a two-sided accumulation point thereof. The function

$$h(x) = \begin{cases} f(a^+), & x \leq a, \\ f(b^-), & x \geq b, \\ f(x), & x \in \operatorname{dom} f, \\ f(x^+) = f(x^-), & x \in (\operatorname{dom} f)^\pm, \\ \dfrac{(x - m_x) f(M_x) + (M_x - x) f(m_x)}{M_x - m_x}, & a < x < b, \quad m_x \neq M_x, \end{cases}$$

defines a continuous extension to f. *Q.E.D.*

Proposition 3.14 generalizes the classical interpolation result that any function whose domain is a discrete subset of $\mathbb{R}$ has a continuous extension. Combining Propositions 3.2 and 3.14, we deduce the following characterization of absolutely polynomial functions of degree 0.

Corollary 3.15. *A function f is absolutely polynomial of degree 0 if and only if $|\,f\,|$ is generalized continuous.*

We next recall the standard definition of piecewise continuity, which is usually stated just in the case that the domain of f is an interval, or, more generally, a set of the form $\operatorname{dom} f = I \setminus D$, where $I \subset \mathbb{R}$ is an interval, and $D \subset I$ a discrete subset.

Definition 3.16. A function $f: \mathbb{R} \to \mathbb{R}$ is called *piecewise continuous* if it is regulated and its discontinuity set $\mathcal{D}(f)$ is a discrete set which has no accumulation point in $\mathbb{R}$.

Theorem 3.17. *A piecewise continuous function on $\mathbb{R}$ is absolutely polynomial of degree at most 2. The degree is equal to*

(a) 0 *if $|\,f\,|$ is generalized continuous,*

(b) 1 *if a) fails, and at each $x \in \mathcal{D}(f) \cap (\operatorname{dom} f)^0$ either*

i) $f(x)$, $f(x^+)$, and $f(x^-)$ all have the same sign (including 0), or

ii) exactly two of the three values $f(x)$, $f(x^+)$, $f(x^-)$ are equal,

(c) 2 *if there exist one or more points in $x \in \mathcal{D}(f) \cap (\operatorname{dom} f)^0$ where $f(x)$, $f(x^+)$, $f(x^-)$, are different, not all having the same sign.*

Proof: Assume that case (a) does not hold. We then begin by assuming that all the discontinuities in f are two-sided, $\mathcal{D}(f) = \mathcal{D}(f)^*$, and, moreover, f is not defined at any discontinuity point, $\mathcal{D}(f) \cap \operatorname{dom} f = \varnothing$. This means that the only discontinuities $x_j \in \mathcal{D}(f)$ are where $f(x_j^+) \neq f(x_j^-)$ and $f(x_j)$ is not defined. We shall prove that in this case f is linearly absolutely polynomial. Let

$$\alpha_j = f(x_j^+) - f(x_j^-) \neq 0, \qquad \beta_j = |\,f(x_j^+)\,| - |\,f(x_j^-)\,|, \tag{3.8}$$

denote the jumps in f and $|\,f\,|$ at the point $x_j \in \mathcal{D}(f)$, respectively. Define the *jump ratio* at x_j to be

$$\rho_j = \frac{\beta_j}{\alpha_j} = \frac{|\,f(x_j^+)\,| - |\,f(x_j^-)\,|}{f(x_j^+) - f(x_j^-)}. \tag{3.9}$$

Note that if both $f(x_j^-)$ and $f(x_j^+)$ are positive, then $\alpha_j = \beta_j$, so $\rho_j = 1$, while if $f(x_j^-)$ and $f(x_j^+)$ are both negative, $\alpha_j = -\beta_j$, so $\rho_j = -1$. Choose $g_1 \in \mathcal{C}(\mathbb{R})$ to be any continuous function that interpolates the jump ratios:

$$g_1(x_j) = \rho_j. \tag{3.10}$$

Note that g_1 exists because $\mathcal{D}(f)$ is discrete, so the x_j have no accumulation points. The claim is that the function

$$\widetilde{g}_0(x) = |\,f(x)\,| - g_1(x) f(x), \qquad x \in \operatorname{dom} f, \tag{3.11}$$

is generalized copntinuous, and so has a continuous extension to all of $\mathbb{R}$, which we call $g_0 \in \mathcal{C}(\mathbb{R})$. This implies that $|\, f \,| = g_0 + g_1 f$ is (linearly) absolutely polynomial. Indeed, the only points where $\widetilde{g}_0$ is not continuously defined are the $x_j \in \mathcal{D}(f)$; at such points we have

$$g_0(x_j^+) - g_0(x_j^-) = \big|\, f(x_j^+) \,\big| - \big|\, f(x_j^-) \,\big| - g_1(x_j)\big[f(x_j^+) - f(x_j^-) \big] = \beta_j - \rho_j \alpha_j = 0,$$

so that $\widetilde{g}_0$ can be continuously extended to x_j, which completes the proof in this case. Adapting the construction of the functions g_0, g_1 for one-sided discontinuities, e.g., where $f(x) \neq f(x^+)$, while $f(x^-)$ is not defined, is straightforward. Therefore, we have proven that f is linearly absolutely polynomial provided $\mathcal{D}(f) \cap (\operatorname{dom} f)^0 = \varnothing$.

Now suppose that there exist one or more points $x_j \in \mathcal{D}(f) \cap (\operatorname{dom} f)^0$, so that $f(x_j^+), f(x_j^-), f(x_j)$ are all defined, but not all equal. In order that $|\, f \,| = g_0 + g_1 f$ be linearly absolutely polynomial, we must be able to simultaneously satisfy all three conditions

$$\begin{aligned} |\, f(x_j) \,| &= g_0(x_j) + g_1(x_j) f(x_j), \\ |\, f(x_j^+) \,| &= g_0(x_j) + g_1(x_j) f(x_j^+), \\ |\, f(x_j^-) \,| &= g_0(x_j) + g_1(x_j) f(x_j^-), \end{aligned} \tag{3.12}$$

at the discontinuity point x_j. The possibilities listed in case (b) are necessary and sufficient for the solvability of the linear system (3.12) for $g_0(x_j),\ g_1(x_j)$. If this holds at every $x_j \in \mathcal{D}(f) \cap (\operatorname{dom} f)^0$, then the preceding proof — where we interpolate the values of g_1 at the discontinuities and then use (3.11) to determine g_0 — works, thereby proving that f is linearly absolutely polynomial. On the other hand, if there exists $x_j \in \mathcal{D}(f) \cap (\operatorname{dom} f)^0$ where (3.12) cannot be simultaneously satisfied, then f is *not* linearly absolutely polynomial. However, we can represent $|\, f \,| = g_0 + g_1 f + g_2 f^2$ as a quadratic polynomial in f. Indeed, the corresponding quadratic conditions

$$\begin{aligned} |\, f(x_j) \,| &= g_0(x_j) + g_1(x_j) f(x_j) + g_2(x_j) f(x_j)^2, \\ |\, f(x_j^+) \,| &= g_0(x_j) + g_1(x_j) f(x_j^+) + g_2(x_j) f(x_j^+)^2, \\ |\, f(x_j^-) \,| &= g_0(x_j) + g_1(x_j) f(x_j^-) + g_2(x_j) f(x_j^-)^2, \end{aligned} \tag{3.13}$$

do have a solution $g_0(x_j), g_1(x_j), g_2(x_j)$, since the (Vandermonde) determinant of the coefficient matrix is nonzero when $f(x_j^+), f(x_j^-), f(x_j)$ are all different. We then continuously interpolate the values of $g_1(x_j)$ and $g_2(x_j)$ to produce $g_1, g_2 \in \mathcal{C}(\mathbb{R})$. The final function g_0 is defined so as to continuously interpolate $\widetilde{g}_0 = |\, f \,| - g_1 f - g_2 f^2$, which is defined on $\operatorname{dom} f$. The generalized continuity of $\widetilde{g}_0$ relies on the fact that $\widetilde{g}_0(x_j) = \widetilde{g}_0(x_j^+) = \widetilde{g}_0(x_j^-)$ as a consequence of (3.13). This completes the proof of the Theorem. *Q.E.D.*

Remark: Note that even though the discontinuities of $\tan x$ are discrete, it is quadratically, and not linearly absolutely polynomial, because it has infinite one-sided limits and so Theorem 3.17 does not apply.

An alternative approach to study the discontinuities of absolutely polynomial functions is to work locally, in a neighborhood of each individual singularity. We first prove that a function with a single jump discontinuity is absolutely polynomial and then appeal to the following local version of our basic definition, which we generalize to arbitrary topological spaces in the obvious manner.

Definition 3.18. A function is *locally absolutely polynomial* of *degree* n on a topological space X if for each point $x \in X$ there is a neighborhood $x \in U \subset X$ such that either f is not defined on U, $U \cap \operatorname{dom} f = \varnothing$, or the restriction $f \mid U$ is absolutely polynomial of degree n on U, meaning that we can find continuous functions $g_0, \ldots, g_n \in \mathcal{C}(U)$ such that (3.1) holds for all $x \in U \cap \operatorname{dom} f$.

The next result says that, for reasonable topological spaces, we only need understand the local behavior of absolutely polynomial functions.

Theorem 3.19. *If X is a paracompact topological space, then every locally absolutely polynomial function of degree $\leq n$ is absolutely polynomial of degree $\leq n$.*

Proof: The hypothesis means that the collection $\mathcal{U} = \{U_\alpha\}$ of open subsets where f is locally absolutely polynomial forms an open cover of X. Given $U_\alpha \in \mathcal{U}$, let $f_\alpha = f \mid U_\alpha$. We can then write

$$| \, f_\alpha(x) \, | = \sum_{k=0}^{n} g_{\alpha,k}(x) \, f_\alpha(x)^k, \qquad x \in \operatorname{dom} f_\alpha = U_\alpha \cap \operatorname{dom} f,$$

where $g_{\alpha,k} \in \mathcal{C}(U_\alpha)$. The one requirement is that the maximal degree n of the sum does not depend on U_α. According to Dugundji, [**3**; p. 170], there exists a locally finite partition of unity subordinate to this cover, so we can find functions $\kappa_\alpha \in \mathcal{C}(X)$ such that $\sum \kappa_\alpha = 1$ where $\operatorname{supp} \kappa_\alpha \subset U_\alpha$ contained in one of the neighborhoods. We can then write

$$| \, f \, | = \sum_\alpha \kappa_\alpha \, | \, f \, | = \sum_\alpha \kappa_\alpha \, | \, f_\alpha \, | = \sum_{k=0}^{n} \sum_\alpha \, \kappa_\alpha \, g_{\alpha,k} \, (f_\alpha)^k.$$

Since $\operatorname{supp} \kappa_\alpha \subset U_\alpha$, the function $g_k = \sum_\alpha \kappa_\alpha \, g_{\alpha,k}$ is well defined and continuous on all of X, and hence $| \, f \, | = \sum_{k=0}^{n} g_k \, f^k$, proving the result. *Q.E.D.*

Theorem 3.19 does not hold if one removes the bound on the local absolutely polynomial degree — an example appears at the end of this section. One can now reprove Theorem 3.17 utilizing Theorem 3.19 by first showing that any function with a single jump discontinuity is absolutely polynomial, which gives the local result since we are assuming that the discontinuities have no accumulation point. Another application is that a function which has a fixed sign in a neighborhood of any of its discontinuities is automatically absolutely polynomial. This result explains why the jump ratio limit trivially exists when the limiting value is nonzero.

Theorem 3.20. *Suppose that a function f has the property that, for every point of discontinuity $x \in \mathcal{D}(f)$, there exists a neighborhood $x \in U \subset \mathbb{R}$ such that either $f \mid U \geq 0$ or $f \mid U \leq 0$. Then f is absolutely polynomial of degree at most* 1.

If we allow the discontinuities of a regulated function to have an accumulation point, then things become more interesting, and we may actually step outside the class of absolutely polynomial functions.

Example 3.21. Consider the function

$$f(x) = \begin{cases} x - \dfrac{2}{4i+1}, & \dfrac{1}{2i+1} < x < \dfrac{1}{2i}, \\ 0, & \text{otherwise,} \end{cases} \tag{3.14}$$

where $i = 1, 2, 3, \ldots$. Thus f has discontinuities at $x_j = \frac{1}{j}$, $j = 1, 2, 3, \ldots$, and

$$f(x_{2i}^-) = \frac{1}{2i(4i+1)}, \qquad f(x_{2i}^+) = 0, \qquad f(x_{2i+1}^-) = 0, \qquad f(x_{2i+1}^+) = \frac{-1}{(2i+1)(4i+1)}.$$

Therefore the jumps (3.8) and ratios (3.9) are

$$\alpha_{2i} = \beta_{2i} = \frac{-1}{2i(4i+1)}, \quad \rho_{2i} = 1, \quad \alpha_{2i+1} = -\beta_{2i+1} = \frac{-1}{(2i+1)(4i+1)}, \quad \rho_{2i+1} = -1.$$

Note that the ratios ρ_j do not have a limit as $x_j \to 0$ and so one cannot construct a continuous extension satisfying (3.10).

To actually prove that (3.14) is not absolutely polynomial, suppose on the contrary that we can find continuous functions $g_0, \ldots, g_n$ satisfying (3.1). Evaluating the right and left hand limits at the discontinuities x_j leads to the following. At x_{2i}^- and x_{2i}^+, we find

$$\frac{1}{2i(4i+1)} = \sum_{k=0}^{n} g_k\left(\frac{1}{2i}\right) \cdot \left(\frac{1}{2i(4i+1)}\right)^k, \qquad 0 = g_0\left(\frac{1}{2i}\right). \tag{3.15}$$

At x_{2i+1}^- and x_{2i+1}^+, we find

$$0 = g_0\left(\frac{1}{2i+1}\right), \qquad \frac{1}{(2i+1)(4i+1)} = \sum_{k=0}^{n} g_k\left(\frac{1}{2i+1}\right) \cdot \left(\frac{-1}{(2i+1)(4i+1)}\right)^k. \tag{3.16}$$

Conditions (3.15), (3.16) imply, respectively,

$$\begin{aligned} 1 &= g_1\left(\frac{1}{2i}\right) + \sum_{k=2}^{n} g_k\left(\frac{1}{2i}\right) \cdot \left(\frac{1}{2i(4i+1)}\right)^{k-1}, \\ 1 &= -g_1\left(\frac{1}{2i+1}\right) - \sum_{k=2}^{n} g_k\left(\frac{1}{2i+1}\right) \cdot \left(\frac{-1}{(2i+1)(4i+1)}\right)^{k-1}. \end{aligned} \tag{3.17}$$

Now we let $i \to \infty$. In view of the continuity of g_k, both summations in (3.17) go to 0. This leads to the final contradiction: $g_1(0) = 1 = -g_1(0)$.

The function (3.14) is continuous at the accumulation point $x = 0$, and even has bounded variation, since

$$\int_0^1 |\, f'(x) \,|\, dx = \sum_{i=0}^{\infty} \frac{1}{2i(2i+1)} < 1.$$

We have therefore shown that not every function of bounded variation is absolutely polynomial. Interestingly, if we move the function (3.14) slightly, replacing f by $f+k$ where k is any nonzero constant, or more generally any nonzero continuous function with $k(0) \neq 0$, then the resulting function is absolutely polynomial because it will have a fixed sign in some small neighborhood of 0, piecewise continuous everywhere else, and so our local Theorem 3.19 applies. In other words, unlike $\sin \frac{1}{x}$, any slight perturbation of this non-absolutely polynomial function is absolutely polynomial.

Remark: The product of two piecewise continuous functions is also piecewise continuous, and hence by Theorem 3.17 absolutely polynomial. On the other hand, we already saw that the product of a continuous and an absolutely polynomial function need not be absolutely polynomial. The function in Example 3.21 provides a counterexample to the conjecture that the product of a continuous and a regulated absolutely polynomial function must be absolutely polynomial. Indeed, consider the function

$$h(x) = \begin{cases} \left(1 - \dfrac{2}{(4i+1)x}\right) \sec \dfrac{\pi}{x}, & \dfrac{1}{2i+1} < x < \dfrac{1}{2i}, \quad x \neq \dfrac{2}{4i+1}, \\ \dfrac{4i+1}{2\pi}, & x = \dfrac{2}{4i+1}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.18}$$

where $i = 1, 2, 3, \ldots$. A straightforward application of l'Hôspital's Rule proves that h is a regulated function; in fact, h is continuous at the points 0 and $\frac{2}{4i+1}$, $i = 1, 2, \ldots$. Moreover, $h(x) \geq 0$ for all x, and hence h is trivially absolutely polynomial (only a single point of accumulation prevents it from being piecewise continuous). Let

$$g(x) = \begin{cases} x \cos \frac{\pi}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{3.19}$$

Then g is continuous, but the product $g \cdot h = f$ is our non-absolutely polynomial regulated function (3.14).

Let us now return to a more detailed discussion of general regulated functions, which will eventually lead us to the complete characterization of those that are absolutely polynomial (at least in the case $\mathcal{D}(f) \cap \operatorname{dom} f = \varnothing$). While (3.14) provides an example of how things can go wrong, the proof of Theorem 3.17 can be adapted to certain types of regulated functions.

Theorem 3.22. *A regulated function is absolutely polynomial of degree ≤ 1 provided every accumulation point $x_* = \lim x_j \in \mathcal{D}(f)^*$ of its discontinuity set is a point of continuity of f, and the jump ratios ρ_j, as defined in* (3.8), *have a limit as $x_j \to x_*$.*

The fact that the jump ratios have a limit at each accumulation point allows us to construct the continuous interpolating function g_1 as before; the rest of the proof is similar. If $f(x_*) > 0$, then, as remarked above, $\rho_j = +1$ for j sufficiently large, while if $f(x_*) < 0$, then $\rho_j = -1$ for j sufficiently large. Therefore, the jump ratios ρ_j will certainly have a limit except, possibly, when $f(x_*) = 0$, as was the case in Example 3.21. This is a reflection of the "misbehavior" of the absolute function at the origin.

Remark: One can easily construct a function whose (left and right) jump ratios have any limit $-1 \leq \rho_* \leq 1$. The cases $\rho_* = \pm 1$ are trivial; otherwise, set $\sigma = (1+\rho_*)/(1-\rho_*)$, so $\rho_* = (\sigma - 1)/(\sigma + 1)$ and $\sigma \geq 0$. Let the right and left hand limits of f at a point of discontinuity x_j be

$$f(x_j^-) = -c_j, \qquad f(x_j^+) = \sigma c_j,$$

where the $c_j > 0$ are arbitrary. Then $\rho_j = \rho_*$ for all j. If $x_j \to x_*^+$, say, and $f(x_*^+)$ exists, then, necessarily, $f(x_*^+) = 0$ unless $\rho_* = \pm 1$, so that we must let $c_j \to 0$ in order to keep f regulated. Furthermore, one can use different sequences on the right and left of the limit point x_* so as to make the jump ratio itself have any prescribed right and left hand limits ρ_*^- and ρ_*^+ at x_*, as long as they are both between -1 and 1. However, if f is regulated, and both $-1 < \rho_*^-, \rho_*^+ < +1$, then, by the preceding remark, $f(x_*^+) = 0 = f(x_*^-)$, and so f can be continuously extended to x_* by setting $f(x_*) = 0$.

On the other hand, if f is not continuous at an accumulation point of its discontinuity set $\mathcal{D}(f)$, then the argument in Theorem 3.17 does not work, and we cannot in general write $|\, f \,|$ as a linear polynomial in f. However, it may be possible to write $|\, f \,|$ as a quadratic or higher degree polynomial. For example, suppose $x_j \to x_*^-$ and the jump ratios (3.9) have a limit $\rho_j \to \rho_*^-$ at x_*, while $f(x_*^+) \neq f(x_*^-)$. If the limiting jump ratio ρ_*^- equals the jump ratio ρ_* at x_*, so

$$\lim_{x_j \to x_*^-} \frac{|\, f(x_j^+) \,| - |\, f(x_j^-) \,|}{f(x_j^+) - f(x_j^-)} = \lim_{j \to \infty} \rho_j = \rho_*^- = \rho_* = \frac{|\, f(x_*^+) \,| - |\, f(x_*^-) \,|}{f(x_*^+) - f(x_*^-)},$$

then the argument used in Theorem 3.17 demonstrates that f is still linearly absolutely polynomial. For instance, this will automatically hold if $f(x_*^-) f(x_*^+) > 0$, so that the right and left hand limits have the same sign at x_*, although this case is trivially covered by Theorem 3.20. However, if the limiting jump ratio does not agree with the jump ratio at x_*, then it is not possible to write $|\, f \,|$ as a linear polynomial $g_0 + g_1 f$ with g_0, g_1 continuous at x_*. In this case, let us try to represent $|\, f \,| = g_0 + g_1 f + g_2 f^2$ as a quadratic polynomial. Evaluating at x_j^+ and x_j^- and subtracting the resulting equations, we find

$$\rho_j = g_1(x_j) + g_2(x_j)\big[f(x_j^-) + f(x_j^+)\big]. \tag{3.20}$$

On the other hand, at x_* itself,

$$\begin{aligned} |\, f(x_*^-) \,| &= g_0(x_*) + g_1(x_*) f(x_*^-) + g_2(x_*) f(x_*^-)^2, \\ |\, f(x_*^+) \,| &= g_0(x_*) + g_1(x_*) f(x_*^+) + g_2(x_*) f(x_*^+)^2. \end{aligned} \tag{3.21}$$

Letting $x_j \to x_*^-$ in (3.20) and subtracting the equations in (3.21), we find

$$\begin{aligned} \rho_*^- &= g_1(x_*) + 2 g_2(x_*) f(x_*^-), \\ \rho_* &= g_1(x_*) + g_2(x_*)\big[f(x_*^-) + f(x_*^+)\big]. \end{aligned} \tag{3.22}$$

If $f(x_*^-) \neq f(x_*^+)$, then we can uniquely solve (3.22) for $g_1(x_*)$, $g_2(x_*)$. We can now specify $g_2(x) = g_2(x_*)$ to be constant, and use (3.20) to define interpolation values for $g_1(x_j)$; these will have the proper limit $g_1(x_j) \to g_1(x_*)$ because of (3.22), and hence allow us to define

g_1 as a continuous function. The final step is to verify that $\widetilde{g}_0 = |\, f\,| - g_1 f - g_2 f^2$ is generalized continuous and hence can be extended to a continuous function $g_0 \in \mathcal{C}(\mathbb{R})$.

If x_* is also an accumulation point from the right, then one also needs to follow through the limiting procedure in that direction. The result will be another equation of the form

$$\rho_*^+ = g_1(x_*) + 2g_2(x_*)f(x_*^+), \tag{3.23}$$

which must be satisfied along with (3.22). As long as $f(x_*^+) \neq f(x_*^-)$, the full system (3.22), (3.23) has a solution if and only if

$$\rho_* = \tfrac{1}{2}\left[\, \rho_*^- + \rho_*^+ \,\right]. \tag{3.24}$$

Note that, according to an earlier remark, we must have either $\rho_*^- = \pm 1$ or $\rho_*^+ = \pm 1$, since otherwise $f(x_*^+) = 0 = f(x_*^-)$. If (3.24) does not hold, then f will not be quadratically absolutely polynomial, and one must then look at representing $|\, f\,| = g_0 + g_1 f + g_2 f^2 + g_3 f^3$ as a cubic in f, which requires

$$\begin{aligned} \left|\, f(x_*^-)\,\right| &= g_0(x_*) + g_1(x_*)f(x_*^-) + g_2(x_*)f(x_*^-)^2 + g_3(x_*)f(x_*^-)^3, \\ \left|\, f(x_*^+)\,\right| &= g_0(x_*) + g_1(x_*)f(x_*^+) + g_2(x_*)f(x_*^+)^2 + g_3(x_*)f(x_*^+)^3. \end{aligned} \tag{3.25}$$

Performing the same limiting procedure in this case, we find the conditions

$$\begin{aligned} \rho_*^- &= g_1(x_*) + 2g_2(x_*)f(x_*^-) + 3g_3(x_*)f(x_*^-)^2, \\ \rho_* &= g_1(x_*) + g_2(x_*)\left[f(x_*^-) + f(x_*^+)\right] + g_3(x_*)\left[f(x_*^-)^2 + f(x_*^-)f(x_*^+) + f(x_*^+)^2\right], \\ \rho_*^+ &= g_1(x_*) + 2g_2(x_*)f(x_*^+) + 3g_3(x_*)f(x_*^+)^2. \end{aligned} \tag{3.26}$$

Treating (3.26) as a system of three linear equations for $g_1(x_*)$, $g_2(x_*)$, $g_3(x_*)$, we note that its coefficient matrix has determinant $\bigl(f(x_*^+) - f(x_*^-)\bigr)^3$, and hence, when x_* is a two-sided point of discontinuity, there is a unique solution. A similar argument as above allows us to conclude that in such cases f is cubically absolutely polynomial.

So far, we have not allowed f to be defined at its limiting discontinuity. If $f(x_*)$ exists, then there is another condition that must be satisfied, which, in the cubic case under consideration, is

$$|\, f(x_*)\,| = g_0(x_*) + g_1(x_*)f(x_*) + g_2(x_*)f(x_*)^2 + g_3(x_*)f(x_*)^3. \tag{3.27}$$

It may not be possible to simultaneously satisfy the complete system of conditions (3.25), (3.26), (3.27), which would mean that f cannot be absolutely polynomial of degree ≤ 3. Indeed, if we treat[†] (3.25), (3.26), (3.27) as a linear system for the 4 unknowns $g_0(x_*)$, $g_1(x_*)$, $g_2(x_*)$, $g_3(x_*)$, then, by basic linear algebra, we can find a solution if and only if the vector formed by the left hand sides lies in the column space of the resulting coefficient matrix. With the aid of MATHEMATICA, the following conclusion was reached: such a function f is cubically absolutely polynomial if and only if either

[†] Actually, there are only 5 independent equations, since the middle equation for ρ_* in (3.26) is a consequence of (3.25).

(a) any two of the three values $f(x_*), f(x_*^+), f(x_*^-)$ are equal, or

(b) the limiting jump ratios satisfy the complicated condition

$$\begin{aligned}
&(f(x_*) - f(x_*^-))\,\rho_*^+ + (f(x_*) - f(x_*^+))\,\rho_*^- \\
&\qquad = \frac{(f(x_*^-) - f(x_*))(2f(x_*) + f(x_*^-) - 3f(x_*^+))}{(f(x_*^+) - f(x_*))(f(x_*^+) - f(x_*^-))}\,\big|\, f(x_*^+) \,\big| + \\
&\qquad\quad + \frac{(f(x_*^-) - f(x_*^+))^2}{(f(x_*) - f(x_*^+))(f(x_*) - f(x_*^-))}\,\big|\, f(x_*) \,\big| + \\
&\qquad\quad + \frac{(f(x_*^+) - f(x_*))(2f(x_*) + f(x_*^+) - 3f(x_*^-))}{(f(x_*^-) - f(x_*))(f(x_*^-) - f(x_*^+))}\,\big|\, f(x_*^-) \,\big|.
\end{aligned} \tag{3.28}$$

If neither condition holds, then the function f is not cubically absolutely polynomial. However, a further analysis proves that it will, in such cases, always be absolutely polynomial of degree 4. Indeed, the required conditions,

$$\begin{aligned}
\big|\, f(x_*^-) \,\big| &= g_0(x_*) + g_1(x_*) f(x_*^-) + g_2(x_*) f(x_*^-)^2 + g_3(x_*) f(x_*^-)^3 + g_4(x_*) f(x_*^-)^4, \\
\big|\, f(x_*^+) \,\big| &= g_0(x_*) + g_1(x_*) f(x_*^+) + g_2(x_*) f(x_*^+)^2 + g_3(x_*) f(x_*^+)^3 + g_4(x_*) f(x_*^+)^4, \\
\big|\, f(x_*) \,\big| &= g_0(x_*) + g_1(x_*) f(x_*) + g_2(x_*) f(x_*)^2 + g_3(x_*) f(x_*)^3 + g_4(x_*) f(x_*)^4, \\
\rho_*^- &= g_1(x_*) + 2g_2(x_*) f(x_*^-) + 3g_3(x_*) f(x_*^-)^2 + 4g_4(x_*) f(x_*^-)^3, \\
\rho_*^+ &= g_1(x_*) + 2g_2(x_*) f(x_*^+) + 3g_3(x_*) f(x_*^+)^2 + 4g_4(x_*) f(x_*^+)^3,
\end{aligned} \tag{3.29}$$

form a linear system of 5 equations for the 5 unknowns $g_0(x_*)$, $g_1(x_*)$, $g_2(x_*)$, $g_3(x_*)$, $g_4(x_*)$. The determinant of the coefficient matrix is

$$(f(x_*^-) - f(x_*^+))^4\,(f(x_*^-) - f(x_*))^2\,(f(x_*^+) - f(x_*))^2,$$

which does not vanish when all three values $f(x_*), f(x_*^+), f(x_*^-)$ are distinct.

The complete generalization of the method indicated by this discussion will be presented shortly, but it is worth stating a particularly important case here. If f is monotone, then the right and left hand jump ratio limits always exist, since they are just ± 1, and hence the preceding argument proves the following:

Proposition 3.23. *Any monotone function is absolutely polynomial of degree at most* 4.

Most monotone functions are absolutely polynomial of degree ≤ 2. The only discontinuity that can cause the degree to be higher is a point $x_* \in \mathcal{D}(f)^*$ where the right and left hand limits have opposite signs, $f(x_*^-) f(x_*^+) \leq 0$. Note that by monotonicity, there can be at most one such point that is both a right and left hand accumulation point for the discontinuity set $\mathcal{D}(f)$. Let us assume for definiteness that f is increasing. Since $f(x) < 0$ for $x < x_*$, and $f(x) > 0$ for $x > x^*$, we have $\rho_*^+ = 1$, $\rho_*^- = -1$. Therefore, condition (3.24) implies that a monotone function that has such a discontinuity is quadratically absolutely polynomial if and only if $\rho_* = 0$, which requires $f(x_*^+) = -f(x_*^-) \neq 0$. Otherwise, the function f is cubically absolutely polynomial if and only if one of the following holds:

(a) $f(x_*)$ is not defined,
(b) $f(x_*) = f(x_*^-)$,
(c) $f(x_*) = f(x_*^+)$,
(d) $f(x_*) > 0$ and $f(x_*^-)^2 = \frac{1}{2} f(x_*)[f(x_*^-) + f(x_*^+)]$, or
(e) $f(x_*) < 0$ and $f(x_*^+)^2 = \frac{1}{2} f(x_*)[f(x_*^-) + f(x_*^+)]$.

The latter two conditions follow from (3.28). If none of these conditions is satisfied, then f is absolutely polynomial of degree 4.

Remark: Proposition 3.23 implies that any function of bounded variation, although not necessarily absolutely polynomial itself, is always the sum of two absolutely polynomial functions of degree at most 4. Thus Example 3.21 shows that the sum of two absolutely polynomial functions of bounded variation can fail to be absolutely polynomial. Multiplying each of the functions g and h in (3.18), (3.19) by an appropriate function, e.g., $k(x) = e^{-1/|\,x\,|}$, results in an example of two absolutely polynomial functions of bounded variation whose product is not absolutely polynomial — this is because the first jump ratios of f, cf. (3.14) and $f \cdot k^2$ are the same. It is also interesting to note just how quickly things can go askew: the function (3.14), while not absolutely polynomial, can be written as $f = (f + a) - a$, where $f + a$ is monotone increasing, and a is monotone increasing and continuous — simply let a be any such function that interpolates the values $[1/(2i+2)] + [1/(2i+1)(4i+1)]$ at the points x_{2i+1}.

Example 3.24. We construct examples of monotone functions that are cubically and quartically absolutely polynomial. The points

$$x_{m,n} = -\frac{1}{n} + \frac{1}{(m+1)n(n+1)}, \qquad y_{m,n} = -\frac{1}{n+1} - \frac{1}{(m+2)n(n+1)}, \qquad z_n = -\frac{1}{n},$$

defined for all positive integers m, n, satisfy

$$x_{m,n} < y_{m',n}, \quad \text{for all} \quad m, m', \qquad x_{m,n} \to z_n^+, \quad y_{m,n} \to z_{n+1}^-, \quad \text{as} \quad m \to \infty.$$

Define

$$g(x) = \begin{cases} -1 - \dfrac{1}{n} + \dfrac{1}{3n(n+1)} + \dfrac{1}{12mn(n+1)}, & x_{m,n} < x < x_{m+1,n}, \\ -1 - \dfrac{1}{n+1} - \dfrac{1}{12mn(n+1)}, & y_{m,n} < x < y_{m+1,n}, \\ 0, & x > 0. \end{cases}$$

The jump ratios at $x_{m,n}, y_{m,n}, z_n$ are all equal to -1, and have a limit at 0^- equal to the jump ratio of g at 0, so that g is linearly absolutely polynomial.

Now set $f(x) = g(x) - g(-x)$ where defined, so that f is monotone increasing, and the discontinuities of f lie on both sides of the origin. The jump ratios are still -1 at the negative discontinuities and $+1$ at positive discontinuities. If $f(0)$ is not defined, or $f(0) = \pm 1$, then f is cubically, but not quadratically, absolutely polynomial. On the other hand, if we define the value of f at 0 so that $|\,f(0)\,| < 1$, then (3.28) does not hold, and the resulting function is quartically absolutely polynomial.

Remark: One can build more complicated absolutely polynomial functions of degree 3 and 4 by putting monotone increasing ones on certain intervals and monotone decreasing ones on others.

Let us now formalize the preceding discussions to cover yet more general situations. Given the discontinuity set $\mathcal{D}^{(0)}(f) = \mathcal{D}(f)$ of a regulated function, we define the set of "k^{th} order discontinuities" $\mathcal{D}^{(k)}(f)$, inductively, so that $\mathcal{D}^{(k+1)}(f)$ equals the intersection of the accumulation points of $\mathcal{D}^{(k)}(f)$ with $\mathcal{D}(f)$. One can construct functions (of bounded variation) which have discontinuities up to some finite order k, but with $\mathcal{D}^{(k+1)}(f) = \varnothing$, as well as examples with infinite order discontinuities.

Example 3.25. Take the rationals $\mathbb{Q}_1 = \mathbb{Q} \cap [0,1]$ between 0 and 1, and put them in bijective correspondence with the natural numbers. For the n^{th} rational x_n define the step function

$$s_n(x) = \begin{cases} 0, & x < x_n, \\ 2^{-n}, & x \geq x_n. \end{cases}$$

Let $f = \sum s_n$. The function f is well-defined and monotone increasing because, at each point x, the series $\sum s_n(x)$ is a sub-series of the convergent geometric series $\sum 2^{-k} = 1$. Since f is discontinuous at each rational, its higher derived sets are all the same, $\mathcal{D}^{(n)}(f) = \mathbb{Q}_1$. Note that this particular function is linearly absolutely polynomial since it is always positive. Subtracting a small constant, e.g., $\frac{1}{2}$, from it will change it to a higher degree absolutely polynomial function.

Definition 3.26. Let f be a regulated function. The *average n^{th} power* of f is the function

$$\overline{f^n}(x) = \frac{1}{n+1} \left(\frac{f(x^+)^{n+1} - f(x^-)^{n+1}}{f(x^+) - f(x^-)} \right) = \frac{1}{n+1} \sum_{i=0}^{n} f(x^+)^i f(x^-)^{n-i}, \quad x \in (\operatorname{dom} f)^{\pm}. \tag{3.30}$$

Note that if f is defined and continuous at x, then $\overline{f^n}(x) = f(x)^n$. Note also that

$$\overline{f^n}(x^+) = f(x^+)^n, \qquad \overline{f^n}(x^-) = f(x^-)^n, \tag{3.31}$$

for every point $x \in (\operatorname{dom} f)^{\pm}$. Consequently, $\overline{f^n}$ is regulated.

Definition 3.27. The *f–jump ratio function* of a regulated function h is

$$\rho_h(x) = \frac{h(x^+) - h(x^-)}{f(x^+) - f(x^-)}, \qquad x \in \operatorname{dom} \rho_h = \mathcal{D}(f)^* \cap (\operatorname{dom} h)^{\pm}. \tag{3.32}$$

If $h = |\, f \,|$, then (3.32) recovers our original jump ratio function (3.9).

Definition 3.28. Let f be a regulated function. We define $\mathcal{P}^{(n)}(f)$ to be the set of regulated functions h such that $\operatorname{dom} h \subset (\operatorname{dom} f)^{\pm}$ and there exist continuous function $g_0, \ldots, g_n \in \mathcal{C}(\mathbb{R})$ such that

$$h(x) = \sum_{k=0}^{n} g_k(x)\, \overline{f^k}(x), \qquad \text{for all} \qquad x \in \operatorname{dom} h. \tag{3.33}$$

Each $\mathcal{P}^{(n)}(f)$ is a real vector space, and their union $\mathcal{P}^*(f) = \bigcup \mathcal{P}^{(n)}(f)$ is an algebra. Now, $h \in \mathcal{P}^{(0)}(f)$ if and only if h is generalized continuous, as in Proposition 3.14. More generally, if $h \in \mathcal{P}^{(n)}(f)$, then $\mathcal{D}(h) \subset \mathcal{D}(f)$. If $\operatorname{dom} h \cap \mathcal{D}(f) = \varnothing$, then (3.33) reduces to our usual polynomial condition

$$h(x) = \sum_{k=0}^{n} g_k(x)\, f(x)^k, \qquad \text{for all} \qquad x \in \operatorname{dom} h. \tag{3.34}$$

In particular, suppose f is a regulated function which is not defined at its points of discontinuity: $\mathcal{D}(f) \cap \operatorname{dom} f = \varnothing$. Then f is absolutely polynomial of degree n if and only if $|\, f\,| \in \mathcal{P}^{(n)}(f) \setminus \mathcal{P}^{(n-1)}(f)$.

Lemma 3.29. *Let f be a regulated function with $\mathcal{D}(f) \cap \operatorname{dom} f = \varnothing$. A regulated function $h \in \mathcal{P}^{(n)}(f)$, $n \geq 1$, if and only if its f–jump ratio function ρ_h is regulated and $\rho_h \in \mathcal{P}^{(n-1)}(f)$.*

Proof: Suppose $h \in \mathcal{P}^{(n)}(f)$. Consider a point $x \in \mathcal{D}(h) \subset \mathcal{D}(f) = \mathcal{D}(f)^*$. We are able to compute the right and left hand limits of (3.33) at x using (3.31). Subtracting the resulting formulae, we find that

$$h(x^+) - h(x^-) = \sum_{k=1}^{n} g_k(x)\left[f(x^+)^k - f(x^-)^k \right].$$

Dividing by $f(x^+) - f(x^-)$ produces the formula for the jump ratio function:

$$\rho_h(x) = \sum_{k=1}^{n} k\, g_k(x)\, \overline{f^{k-1}}(x), \qquad x \in \mathcal{D}(f). \tag{3.35}$$

Clearly ρ_h is regulated, and therefore $\rho_h \in \mathcal{P}^{(n-1)}(f)$.

To prove the converse, suppose that (3.35) holds, so that $g_1, \ldots, g_n$ are known. We can then use (3.33) to define $g_0(x)$ for $x \in \operatorname{dom} h$. Comparing its right and left hand limits, it is not hard to see that g_0 can be continuously interpolated to define $g_0 \in \mathcal{C}(\mathbb{R})$. *Q.E.D.*

Since the case $n = 0$ is trivial, Lemma 3.29 provides an inductive mechanism for determining when a given regulated function belongs to $\mathcal{P}^{(n)}(f)$. One merely checks whether the successive f–jump ratio functions are regulated, up until the n^{th} order one, which must be generalized continuous. In particular, this solves the problem of determining which regulated functions, that are not defined on their domain of discontinuity, are absolutely polynomial.

Theorem 3.30. *Let f be regulated with $\mathcal{D}(f) \cap \operatorname{dom} f = \varnothing$. Set $h_0 = |\, f\,|$. Then f is absolutely polynomial of degree n if and only if for each $k = 1, \ldots, n-1$, the k^{th} order f–jump ratio function $h_k = \rho_{h_{k-1}}$, defined on $\mathcal{D}(f)^*$, is regulated, and $\mathcal{D}(h_k) \neq \varnothing$, while $h_n = \rho_{h_{n-1}}$ is generalized continuous.*

Remark: The k^{th} order jump function h_k has the property that $\mathcal{D}(h_k) \subset \mathcal{D}^{(k)}(f)$ for each k. In particular, if $\mathcal{D}^{(n)}(f) = \varnothing$ and the jump ratios are all regulated, then f is absolutely polynomial of degree $\leq n$. Example 3.21 illustrates the fact that the jump ratios for a regulated function need not be regulated. Indeed, for (3.14), $\mathcal{D}^{(1)}(f)$ is empty, and f is not absolutely polynomial.

Example 3.31. For the function f in Example 3.24, the discontinuity set is $\mathcal{D}(f) = \{0, \pm x_{m,n}, \pm y_{m,n}, \pm z_n\}$. Leaving f not defined on $\mathcal{D}(f)$, the points $\pm x_{m,n}$ and $\pm y_{m,n}$ are isolated discontinuities, while the points $\pm z_n$ are in the first order discontinuity set $\mathcal{D}^{(1)}(f)$. The first jump ratio function has values

$$h_1(x_{m,n}) = h_1(y_{m,n}) = h_1(z_n) = -1, \qquad h_1(-x_{m,n}) = h_1(-y_{m,n}) = h_1(-z_n) = 1,$$

and is discontinuous at 0, which is the only point in $\mathcal{D}^{(2)}(f)$. The second order jump ratio then has $h_2(\pm z_n) = 0$, and thus $h_2(0^+) = 0 = h_2(0^-)$, while $h_2(0) = 1$, and so h_2 is not continuous at 0. The third jump ratio has $h_3(0) = 0$, reconfirming our earlier conclusion that f is cubically absolutely polynomial. On the other hand, Theorem 3.30 does not apply to the cases (both cubic and quartic) when $f(0)$ is defined. A general theorem covering regulated functions that are defined at their points of discontinuity is more technical, and we shall not attempt to state it here.

Example 3.32. In this example, we construct regulated functions which are absolutely polynomial of arbitrarily high degree. We begin with a variant of the function considered in Example 3.21. Let $0 < \varepsilon < 1$, $0 < r < 1$, and define

$$g^{(1)}_{\varepsilon,r}(x) = \begin{cases} \dfrac{(4 - \varepsilon r + 4j)x - 4\varepsilon}{4 - \varepsilon r}, & \dfrac{\varepsilon}{j+1} < x < \dfrac{\varepsilon}{j}, \\ 0, & x \leq 0, \quad \text{or} \quad x > \varepsilon, \end{cases} \tag{3.36}$$

where $j = 1, 2, 3, \ldots$. Note that

$$\operatorname{supp} g^{(1)}_{\varepsilon,r} \subset (0, \varepsilon), \qquad \left| \, g^{(1)}_{\varepsilon,r} \right| < \varepsilon. \tag{3.37}$$

The function (3.36) has discontinuities at $x_j = \varepsilon / j$, $j = 1, 2, 3, \ldots$, and

$$g^{(1)}_{\varepsilon,r}(x_j^-) = \frac{\varepsilon}{j}, \quad j \geq 1, \qquad g^{(1)}_{\varepsilon,r}(x_j^+) = \frac{-\varepsilon^2 r}{j(4 - \varepsilon r)}, \quad j \geq 2.$$

Therefore, ignoring x_1, the jump ratios for (3.36) are all the same:

$$h_1(x_j) = \rho_j = 1 - \tfrac{1}{2}\varepsilon r, \qquad j \geq 2. \tag{3.38}$$

Since the ratios ρ_j have a limit as $x_j \to 0^+$, and $g^{(1)}_{\varepsilon,r}$ is continuous at 0, we conclude that (3.36) is linearly absolutely polynomial.

Now define

$$f^{(2)}_{\varepsilon,r}(x) = g^{(1)}_{\varepsilon,r}(-x) + \tfrac{1}{2}\left[\, g^{(1)}_{\varepsilon,r}(x) + k_\varepsilon(x) \,\right]. \tag{3.39}$$

where

$$k_\varepsilon(x) = \begin{cases} \varepsilon - x, & 0 < x < \varepsilon, \\ 0, & x < 0, \quad \text{or} \quad x \geq \varepsilon. \end{cases} \tag{3.40}$$

In view of (3.37),

$$\operatorname{supp} f^{(2)}_{\varepsilon,r} \subset (-\varepsilon,\varepsilon), \qquad \left| f^{(2)}_{\varepsilon,r} \right| < \varepsilon.$$

Note that the discontinuity set of $f^{(2)}_{\varepsilon,r}$ equals $\{0, \pm\varepsilon, \pm\frac{1}{2}\varepsilon, \pm\frac{1}{3}\varepsilon, \ldots\}$; in particular

$$f^{(2)}_{\varepsilon,r}(0^+) = \tfrac{1}{2}\varepsilon, \qquad f^{(2)}_{\varepsilon,r}(0^-) = 0.$$

Since $f^{(2)}_{\varepsilon,r}(x)$ agrees with $g^{(1)}_{\varepsilon,r}(-x)$ when $x < 0$, its jump ratios at the points $-\varepsilon/j$ are the same as those of $g^{(1)}_{\varepsilon,r}$ at $x_j = \varepsilon/j$, as given in (3.38). Therefore, the first jump ratio function for $f^{(2)}_{\varepsilon,r}$ satisfies

$$h_1\left(-\frac{\varepsilon}{j}\right) = 1 - \frac{\varepsilon r}{2}, \quad j \geq 2, \qquad \text{hence} \qquad h_1(0^-) = 1 - \frac{\varepsilon r}{2}. \tag{3.41}$$

On the other hand, since $f^{(2)}_{\varepsilon,r}(x) > 0$ for all $0 < x < \varepsilon$, the first jump ratios at the positive discontinuity points ε/j are all equal to $+1$, and hence

$$h_1\left(\frac{\varepsilon}{j}\right) = 1, \quad j \geq 1, \qquad \text{hence} \qquad h_1(0^+) = 1. \tag{3.42}$$

Since h_1 is not continuous at 0, the function $f^{(2)}_{\varepsilon,r}$ is not linearly absolutely polynomial. But h_1 is regulated, and the first order discontinuity set $\mathcal{D}^{(1)}(f^{(2)}_{\varepsilon,r}) = \{0\}$ has no accumulation points, so by Theorem 3.30 $f^{(2)}_{\varepsilon,r}$ is quadratically absolutely polynomial. The second order jump ratio at 0 is

$$h_2(0) = \frac{h_1(0^+) - h_1(0^-)}{f^{(2)}_{\varepsilon,r}(0^+) - f^{(2)}_{\varepsilon,r}(0^-)} = r. \tag{3.43}$$

Using the functions $f^{(2)}_{\varepsilon,r}$ as our building blocks, we now proceed to inductively construct a sequence of regulated functions $f^{(n)}_{\varepsilon,r}$ which are absolutely polynomial of degrees $n = 3, 4, \ldots$, and satisfy

$$\operatorname{supp} f^{(n)}_{\varepsilon,r} \subset (-\varepsilon,\varepsilon), \qquad \left| f^{(n)}_{\varepsilon,r} \right| < \varepsilon. \tag{3.44}$$

The key idea is to build up $f^{(n)}_{\varepsilon,r}$ by adjoining suitably "compressed" copies of the preceding function $f^{(n-1)}_{\varepsilon,r}$ on intervals that converge to the origin. We therefore first define

$$g^{(n-1)}_{\varepsilon,r}(x) = \sum_{j=2}^{\infty} f^{(n-1)}_{\varepsilon/4j^2,\,\varepsilon r/2}\left(x - \frac{\varepsilon}{j}\right). \tag{3.45}$$

Note that, according to (3.44), the supports of the summands are all mutually disjoint, and so there are no convergence issues to discuss. The inductive hypothesis (3.44) (with n replaced by $n-1$) implies that

$$\operatorname{supp} g^{(n-1)}_{\varepsilon,r} \subset \left(0, \tfrac{9}{16}\varepsilon\right), \qquad \left| g^{(n-1)}_{\varepsilon,r} \right| < \frac{\varepsilon}{16}. \tag{3.46}$$

Moreover, $g_{\varepsilon,r}^{(n-1)}$ is continuous at 0 with $g_{\varepsilon,r}^{(n-1)}(0) = 0$. We then define

$$f_{\varepsilon,r}^{(n)}(x) = g_{\varepsilon,r}^{(n-1)}(-x) + \tfrac{1}{2}\left[\, g_{\varepsilon,r}^{(n-1)}(x) + k_\varepsilon(x) \,\right]. \tag{3.47}$$

The inductive verification of (3.44) is immediate from (3.46). Moreover, (3.46) and (3.40) imply that $f_{\varepsilon,r}^{(n)}(x) > 0$ for all $0 < x < \varepsilon$ where defined, and

$$f_{\varepsilon,r}^{(n)}(0^-) = 0, \qquad f_{\varepsilon,r}^{(n)}(0^+) = \frac{\varepsilon}{2}. \tag{3.48}$$

We claim that for $n \geq 2$, all the relevant jump ratio functions h_k for $f_{\varepsilon,r}^{(n)}$ are regulated, and, moreover, the n^{th} order one satisfies

$$h_n(0) = (-1)^n r. \tag{3.49}$$

Because the supports of the summands in (3.45) are disjoint, the only new limit point of concern is at 0. It can be approached in a variety of ways, i.e., through discontinuities of varying orders. Positivity of $f_{\varepsilon,r}^{(n)}$ for $0 < x < \varepsilon$ implies that the jump ratios at the positive discontinuity points are trivial:

$$h_k(x) = \begin{cases} 1, & k = 1, \\ 0, & k > 1, \end{cases} \qquad \text{for any} \qquad 0 < x \in \mathcal{D}^{(k)}(f_{\varepsilon,r}^{(n)}). \tag{3.50}$$

Therefore, $h_1(0^+) = 1$, while $h_k(0^+) = 0$ for $k \geq 2$.

To understand the proof of the claims, the case of $f_{\varepsilon,r}^{(3)}$ is instructive. Its first jump ratio function h_1 is defined at all discontinuities and $h_1(0^-) = 1$, either because by (3.50) its value is actually 1, or because by (3.41) its value has the form $1 - \varepsilon^2 r/4j^2$ which tends to 1 as the points converge to 0. Therefore, the jump ratio function h_1 for $f_{\varepsilon,r}^{(3)}$ is continuous at 0. At the next order, $f_{\varepsilon,r}^{(4)}$ is (essentially) built from repeated compressed copies of $f_{\varepsilon,r}^{(3)}$ on intervals that approach 0 from the left. From the definition of limit, the fact that h_1 approaches 1 in $f_{\varepsilon,r}^{(3)}$ means that for $f_{\varepsilon,r}^{(4)}$ the function h_1 can be made arbitrarily close to 1 on all but a finite number of intervals, and hence for all but finitely many discontinuities in any sequence approaching 0 from the left. In short, the first jump ratio function h_1 for $f_{\varepsilon,r}^{(4)}$, and indeed, inductively for all higher $f_{\varepsilon,r}^{(n)}$, is continuous at 0 and takes the value $h_1(0) = 1$ there.

Now consider the second order jump ratio function h_2. For the function $f_{\varepsilon,r}^{(3)}$, we can only approach 0 through points in $\mathcal{D}^{(1)}(f_{\varepsilon,r}^{(3)}) = \{0, \pm\varepsilon, \pm\frac{1}{2}\varepsilon, \ldots\}$. From the right we get $h_2(0^+) = 0$ and from the left, using (3.43), $h_2(0^-) = \frac{1}{2}\varepsilon r$. Thus h_2 is regulated, though not continuous at 0, and $h_3(0) = -r$, in accordance with (3.49). Proceeding to the second order jump ratio h_2 for $f_{\varepsilon,r}^{(4)}$, the preceding discussion implies that the limit from the right at $-\varepsilon/j$ is $\varepsilon^2 r/4j^2$, while from the left it is 0. As the points converge to 0^- both values tend to 0, and hence $h_2(0^-) = 0$, while $h_2(0^+) = 0$ by (3.50). Again the same will hold for h_2 for all higher $f_{\varepsilon,r}^{(n)}$. For $f_{\varepsilon,r}^{(4)}$, the jump function h_3 will be discontinuous at 0, with $h_3(0^-) = -\varepsilon r/2$, $h_3(0^+) = 0$, and thus, using (3.50), $h_4(0) = r$.

By a similar procedure, we prove that all jump ratio functions $h_1, \ldots, h_n$ for $f^{(n)}_{\varepsilon,r}$ are regulated. Moreover, $h_1, \ldots, h_{n-2}$ are all continuous at zero. But, using our inductive hypothesis (3.49),

$$h_{n-1}(0^-) = \lim_{j\to\infty} h_{n-1}\left(-\frac{\varepsilon}{j}\right) = (-1)^{n-1}\varepsilon r, \qquad h_{n-1}(0^+) = 0.$$

The second equality (which holds for all j) is because the value of the order $n-1$ jump ratio function for $f^{(n)}_{\varepsilon,r}$ at the point $x = -\varepsilon/j$ is, according to the definitions (3.45), (3.47), the same as the value of the $(n-1)^{\text{st}}$ jump ratio function for $f^{(n-1)}_{\varepsilon/4j^2,\varepsilon r/2}$ at $x=0$, which, by the inductive hypothesis (3.49) equals $(-1)^{n-1}\varepsilon r/2$. Therefore, by (3.48),

$$h_n(0) = \frac{h_{n-1}(0^+) - h_{n-1}(0^-)}{f^{(n)}_{\varepsilon,r}(0^+) - f^{(n)}_{\varepsilon,r}(0^-)} = (-1)^n r,$$

which completes the inductive proof of (3.49).

Theorem 3.30 implies that $f^{(n)}_{\varepsilon,r}$ is absolutely polynomial of degree n, and *not* absolutely polynomial of any degree less than n. In this way, we have explicitly constructed a regulated function of any desired absolutely polynomial degree. Its higher order discontinuity sets have the form

$$\mathcal{D}^{(n-2)}(f^{(n)}_{\varepsilon,r}) = \left\{0, \pm\varepsilon, \pm\tfrac{1}{2}\varepsilon, \pm\tfrac{1}{3}\varepsilon, \ldots\right\}, \qquad \mathcal{D}^{(n-1)}(f^{(n)}_{\varepsilon,r}) = \{0\}, \qquad \mathcal{D}^{(n)}(f^{(n)}_{\varepsilon,r}) = \varnothing.$$

Finally, an adaptation of the Cantor diagonal argument produces a regulated function which is not absolutely polynomial even though all of its jump ratio functions are also regulated. We set

$$g^{(\infty)}_{\varepsilon,r}(x) = \sum_{n=2}^{\infty} f^{(n)}_{\varepsilon/4n^2,\varepsilon r/2}\left(x - \frac{\varepsilon}{n}\right), \tag{3.51}$$

and

$$f^{(\infty)}_{\varepsilon,r}(x) = g^{(\infty)}_{\varepsilon,r}(-x) + \tfrac{1}{2}\left[\, g^{(\infty)}_{\varepsilon,r}(x) + k_\varepsilon(x)\,\right]. \tag{3.52}$$

Since $\varepsilon/n \in \mathcal{D}^{(n-2)}(g^{(\infty)}_{\varepsilon,r})$, and the jump ratio h_{n-1} is discontinuous there, the functions $g^{(\infty)}_{\varepsilon,r}$ and $f^{(\infty)}_{\varepsilon,r}$ cannot satisfy the final condition of Theorem 3.30 and are not absolutely polynomial of any degree. Note that 0 is a point of continuity of $g^{(\infty)}_{\varepsilon,r}$, but is an "infinite order discontinuity point" of $f^{(\infty)}_{\varepsilon,r}$. A related example is the function

$$\widetilde{g}^{(\infty)}_{\varepsilon,r}(x) = \sum_{n=2}^{\infty} f^{(n)}_{\varepsilon/2,r}(x-n). \tag{3.53}$$

Note that $\widetilde{g}^{(\infty)}_{\varepsilon,r}$ is locally absolutely polynomial at each point of $\mathbb{R}$, of degree n at $x=n$, but is not globally absolutely polynomial.

Remark: One can straightforwardly modify these examples (without altering their validity) so that the functions are of bounded variation. For instance, this can be accomplished by multiplying the original function (3.36) by any sufficiently rapidly decreasing function, e.g., $e^{-1/x}$ for $x > 0$.

4. Topological Spaces.

In this section, we investigate how far the theory of absolutely polynomial functions defined on the reals can be generalized to other types of topological spaces. We let $\mathcal{C}(X)$ denote the ring of all continuous real-valued functions defined on *all* of X, so $g \in \mathcal{C}(X)$ requires $\operatorname{dom} g = X$. Replacing the domain space $\mathbb{R}$ by X in Definition 3.1 leads to the general concept of a absolutely polynomial function on a topological space.

Definition 4.1. A topological space X will be called *absolutely closed* if every continuous function defined on a subset of X is absolutely polynomial.

Theorem 4.2. *A real topological manifold is not absolutely closed.*

Proof: Locally we can identify X with an open subset of $\mathbb{R}^n$ for some n. The basic argument given in Proposition 2.1 can be used to prove that the function $\sin(1/\rho(x))$, where $\rho(x) = \| x \|$ is the usual Euclidean metric on $\mathbb{R}^n$, is not absolutely polynomial. *Q.E.D.*

The question now is to determine which topological spaces are absolutely closed. A discrete space is immediately seen to be absolutely closed. However, as the next result shows, spaces containing accumulation points of sequences are not absolutely closed.

Lemma 4.3. *The space* $X = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\right\} \subset \mathbb{R}$, *which is the one point compactification of the dense subspace* $N = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} = X \setminus \{0\}$, *is not absolutely closed.*

Proof: The set N is discrete and countable, hence any function $f\colon N \to \mathbb{R}$ will be continuous. Let $\mathbb{Q}_1 = \mathbb{Q} \cap [-1, 1]$ denote the set of rational numbers that lie between -1 and $+1$. We can construct a function $f\colon N \to \mathbb{Q}_1$ so that every $w \in \mathbb{Q}_1$ has an infinite preimage. This can be done (in many ways) because N and $\mathbb{Q}_1$ have the same cardinality.

Now follow the idea of the proof of Proposition 2.1. Again assume that the g_k exist, defined on X and let $c_k = g_k(0)$. The function $r(z)$, which was independent of the function $\sin \frac{1}{x}$, is defined as in equation (2.2). Note also that z_0 can be chosen to be in $\mathbb{Q}_1$, i.e., it is rational. This is possible because $r(z)$ is continuous — were it zero on the rationals it would be zero everywhere. We then find $\delta > 0$ to satisfy (2.4). Since $|\, f(x) \,| \leq 1$, we conclude that

$$\left|\; |\, f(x) \,| - \sum_{k=0}^{n} c_k f(x)^k \right| < \varepsilon, \qquad |\, x \,| < \delta. \tag{4.1}$$

Finally, note that there are only finitely many elements of the set N that lie outside any delta neighborhood of 0, but there are infinitely many elements in $x \in N$ such that $f(x) = z_0$. Therefore one of them lies inside the δ neighborhood of 0 giving the same conclusion. *Q.E.D.*

Theorem 4.4. *Any first countable topological space with an accumulation point is not absolutely closed. Consequently, every absolutely closed first countable topological space is necessarily discrete.*

Proof: This follows immediately from Lemma 4.3 and the fact, [**3**; Theorem X.6.2], that any accumulation point of a first countable space has a sequence that converges to it. *Q.E.D.*

An interesting open question is whether every absolutely closed topological space is necessarily discrete.

Definition 4.5. A point $a \in X$ in a topological space X will be called *absolutely isolated* if every continuous function defined on $X \setminus \{a\}$ is absolutely polynomial.

Clearly, an isolated point is absolutely isolated. On the other hand, no point of a topological manifold (of dimension 1 or more) is absolutely isolated. Another important example is provided by the Cantor space.

Proposition 4.6. *No point in the Cantor space $C \subset [0,1]$ is absolutely isolated.*

Proof: We only need to show that $0 \in C$ is not absolutely isolated. We use the fact that each open set in the Cantor set is the *disjoint* union of clopen (closed and open) sets, and that the number of sets used in the union must be *countable*. In particular, the set $U = C \setminus \{0\}$ is the disjoint union of clopen sets D_n, indexed by $n \in \mathbb{N}$. The sets D_n are closed in the reals. Let $a_n = \inf D_n$ be the smallest number that belongs to D_n, in other words, the "left-hand beginning point" of D_n. Suppose that $\delta > 0$ is a positive number, and consider a δ neighborhood $N \subset C$ of 0. Since $C \setminus N$ is closed in C, it is compact. Since the subsets D_n cover $C \setminus N$, we can choose a finite subcover. This implies that only a finite number of the endpoints a_n lie in $C \setminus N$. In other words any δ neighborhood of $0 \in C$ contains all but finitely many a_n.

This idea can be used to define a continuous function $f\colon U \to \mathbb{Q}_1 = \mathbb{Q} \cap [-1,1]$ just as was done in the case of the discrete set N in the proof of Lemma 4.3. Let f be defined by making it constant on each of the D_n, and picking the values from $\mathbb{Q}_1$ as before — making sure that every element of $\mathbb{Q}_1$ occurs as the image of infinitely many different D_n. That ensures that an arbitrary δ neighborhood of 0 will contain a point of C assuming the value z_0. Thus f is defined on $C \setminus \{0\}$, and its absolute value is not a polynomial in f with coefficients g_i defined on the Cantor space C. *Q.E.D.*

Theorem 4.7. *Let k be an infinite cardinal. Suppose that X is a topological space that has a point a with the properties:*
(i) $X \setminus \{a\}$ is the disjoint union of k clopen subsets D_n,
(ii) for every neighborhood N of a, the number of D_n from which N is disjoint has cardinality less than k.
Then a is not absolutely isolated.

Proof: Define the function $f\colon D_n \to \mathbb{Q}_1$ as before, keeping its value constant on each D_n, so that f is continuous. Use a to replace 0 when defining the constants c_k as in (4.1). Given the ε of the proof, replace the δ with a neighborhood N of a. *Q.E.D.*

The argument for the Cantor space will hold for any space X that contains, as a subspace, a copy of C that is also a retract of X. That is to say, a space X for which there is a continuous function $g\colon X \to C$ such that g restricts to the identity on C. The argument is similar — write $C \setminus \{0\} = \bigcup D_n$ as a disjoint union of clopen sets. The union of their preimages $U = \bigcup V_n = \bigcup g^{-1}(D_n)$, is an open set. If U is not dense we replace it by the dense set $V = U \cup (X \setminus \overline{U})$. We obtain a continuous function $f\colon V \to \mathbb{Q}_1$ as follows:

make $f = 0$ on $X \setminus \bar{U}$. On the countably many V_n assign constant values in $\mathbb{Q}_1$ as before, i.e., make sure that every non-zero number in $\mathbb{Q}_1$ is assigned to infinitely many V_n. Now use as the "kicking off point" for the definition of the c_k, the copy of 0 inside the copy of C in X. Each neighborhood of 0 meets all but finitely many V_n because it meets all but finitely many D_n in the Cantor set.

According to [**12**; p. 42], every compact totally disconnected metrizable space has the property that its closed subsets are retracts of itself. A space which is totally disconnected, compact, and Hausdorff is called *Boolean*. The moment that a Boolean space X contains a copy of the Cantor space C, then C is a retract of X. The reason for this is that the Cantor space is injective in the category of Boolean spaces, cf. [**8**].

Theorem 4.8. *If X is a Boolean space that contains a copy of the Cantor space, then there is a dense open set $U \subset X$, excluding 0, and a continuous function $f: U \to \mathbb{R}$ which is not absolutely polynomial.*

The same result holds for any space X containing C with the property that its Stone–Čech compactification is totally disconnected and metrizable, or for that matter, any space X containing C that lies inside a compact totally disconnected metrizable space.

We further note that our discussion of real rational functions in Theorem 3.5 admits the following generalization.

Proposition 4.9. *Let $p, q \in \mathcal{C}(X)$, and suppose that there exist e, u, v, with $e^2 = e$, such that*

$$|\, pq \,| = |\, pq \,|\, e, \qquad e = up + vq.$$

Then the ratio p/q is absolutely polynomial.

Proof: We adapt the proof of Theorem 3.5 as follows:

$$\left| \frac{p}{q} \right| = \frac{|\, pq \,|}{q^2} = \frac{|\, pq \,|\, e}{q^2} = \frac{|\, pq \,|\, e^2}{q^2} = |\, pq \,| \left(u \left(\frac{p}{q} \right) + v \right)^2,$$

which is a quadratic polynomial in p/q. *Q.E.D.*

Remark: The function $e = e^2$ forms an idempotent in $\mathcal{C}(X)$, which means that it only assumes the values 0 or 1. Its support satisfies

$$(\operatorname{supp} p) \cap (\operatorname{supp} q) \subset \operatorname{supp} e \subset (\operatorname{supp} p) \cup (\operatorname{supp} q).$$

Example 4.10. Let C be the Cantor space. Consider the following disjoint clopen subsets:

$$C_1 = C \cap \left(0, \tfrac{1}{27}\right), \qquad C_2 = C \cap \left(\tfrac{2}{27}, \tfrac{1}{9}\right), \qquad C_3 = C \cap \left(\tfrac{2}{9}, \tfrac{7}{27}\right),$$
$$C_4 = C \cap \left(\tfrac{8}{27}, \tfrac{1}{3}\right), \qquad C_5 = C \cap \left(\tfrac{2}{3}, \tfrac{19}{27}\right)$$

Define $p, q, u, v \in \mathcal{C}(C)$ by

$$\begin{aligned}
p(x) &= x + 1, && x \in \operatorname{supp} p = C_1 \cup C_2 \cup C_3, \\
q(x) &= x^2, && x \in \operatorname{supp} q = C_3 \cup C_4 \cup C_5, \\
u(x) &= \frac{1}{x+1}, && x \in \operatorname{supp} u = C_2 \cup C_3, \\
v(x) &= \frac{1}{x^2}, && x \in \operatorname{supp} v = C_4,
\end{aligned}$$

where we have only indicated the nonzero values. Thus $e = up + vq$ is an idempotent with $\operatorname{supp} e = C_2 \cup C_3 \cup C_4$, and hence $p/q = x + (1/x)$ for $x \in \operatorname{supp} q = C_3 \cup C_4 \cup C_5$ is absolutely polynomial with respect to the Cantor space.

Recall that a topological space is called a P–space, cf. [**5**], if every finitely generated ideal in $\mathcal{C}(X)$ is principal and generated by an idempotent. If e is the idempotent generating the ideal $p\,\mathcal{C}(X) + q\,\mathcal{C}(X)$, then the hypotheses of Proposition 4.9 are easily seen to hold.

Corollary 4.11. *Let X be a P–space. If $p, q \in \mathcal{C}(X)$, then p/q is absolutely polynomial.*

This result can be further generalized to the class of F–spaces, [**5**]. For example, any basically disconnected space is an F–space, [**5**; 14N4].

Theorem 4.12. *Let X be a F–space. If $p, q \in \mathcal{C}(X)$, then p/q is absolutely polynomial.*

Proof: According to Gillman and Jerison, [**5**; Theorem 14.25 (6), p. 208], in an F–space, any cozero set is C^* embedded. We write

$$\left| \frac{p}{q} \right| = \frac{h(p^2+q^2)}{q^2}, \qquad \text{where} \qquad h = \frac{|\,p\,|\,|\,q\,|}{p^2+q^2},$$

which holds over the set $(\operatorname{supp} p) \cup (\operatorname{supp} q) = \operatorname{supp}(\,|\,p\,| + |\,q\,|\,)$. Furthermore, $0 \le h \le \frac{1}{2}$ is a bounded function. Since X is an F–space, h lifts to a continuous function $g \in \mathcal{C}(X)$. Clearly, the identity

$$\left| \frac{p}{q} \right| = g \left(\frac{p}{q} \right)^2 + g$$

holds on all of X because both sides are zero on the set

$$\mathcal{Z}(\,|\,p\,| + |\,q\,|\,) = X \setminus \operatorname{supp}(\,|\,p\,| + |\,q\,|\,).$$

This completes the proof *Q.E.D.*

5. Applications to the Theory of Φ–Algebras.

Finally, we discuss some interesting implications of our theory of absolutely polynomial functions for the general theory of rings of continuous functions and Φ–algebras. Suppose that R is a commutative ring with identity that has no non-zero nilpotent elements. An over ring S of R is called a *ring of quotients* of R if for every $s \in S$, $s \neq 0$, there exist $a, b \in R$, $b \neq 0$, so that $sa = b$. There exists a largest ring of quotients that is unique up to isomorphism over R, called the complete ring of quotients of R, [**9**]. For example, the complete ring of quotients of $\mathbb{Z}$, the ring of integers, is the field of rational numbers $\mathbb{Q}$.

Now consider the case when $R = \mathcal{C}(X)$, the ring of all continuous real-valued functions on a completely regular space. This ring was the subject of the classic work by Gillman and Jerison, [**5**]. The later book by Fine, Gillman and Lambek, [**4**], was devoted to the study of its rings of quotients. The complete ring of quotients of $\mathcal{C}(X)$, denoted $Q(X)$, [**4**], is the limit of rings of continuous functions $\mathcal{C}(V)$, where V ranges over all dense open subspaces of X, and two functions defined on dense open subspaces are identified if they agree on the intersection of their domains.

An important generalization of $\mathcal{C}(X)$ exists in the literature, and this is the notion of a Φ–algebra, [**6**]. These structures abstract the key properties of $\mathcal{C}(X)$ as an algebra, and as a partially ordered set.

Definition 5.1. A Φ*–algebra* is an Archimedean lattice-ordered algebra over the field of real numbers which has an identity element that is a weak order unit.

Although any $\mathcal{C}(X)$ is a Φ–algebra, many Φ–algebras are not isomorphic to any ring of the form $\mathcal{C}(X)$, cf. [**6**]. For example, the complete ring of quotients of $\mathcal{C}(X)$ is easily seen to always be a Φ–algebra, but is almost never of the form $\mathcal{C}(X)$, [**7**]. A similar result holds for the epimorphic hull of $\mathcal{C}(X)$, [**10**], which is a particular ring of quotients of $\mathcal{C}(X)$.

In this context, the significance of our examples is as follows. Let f be any non-absolutely polynomial function whose domain is a dense open subset of $\mathbb{R}$, e.g., $f(x) = \sin \frac{1}{x}$. Then the polynomial ring $\mathcal{C}(\mathbb{R})[f]$ is not a Φ–algebra because it is not closed under the formation of absolute values. (A Φ–algebra is easily seen to be closed under this operation, [**6**].) But it is a ring of quotients of $\mathcal{C}(\mathbb{R})$ because f is defined on a dense open subset $\operatorname{dom} f$, cf. [**4**]. Thus the result shows that a ring of quotients of a ring of functions need not be a Φ–algebra. On the other hand, this condition does hold trivially for discrete spaces. By the same argument, one has:

Theorem 5.2. *Suppose that X is a topological space with a non-absolutely isolated point a, and a function f defined on $X \setminus \{a\}$ that is not absolutely polynomial. Then the polynomial ring $\mathcal{C}(X)[f]$ is not a Φ–algebra.*

Remark: The condition that a real function f is absolutely polynomial is necessary, but not sufficient, to make the ring $\mathcal{C}(\mathbb{R})[f]$ a Φ–algebra. The function $f = 2 + \sin \frac{1}{x}$ is absolutely polynomial, but the ring $\mathcal{C}(\mathbb{R})[f]$ is not a Φ–algebra — it contains $\sin \frac{1}{x}$, but not its absolute value, since the latter is not a polynomial in f with continuous coefficients. However, oddly, it is easy to check that if p and q satisfy condition (3.5) of Theorem 3.5, then $\mathcal{C}(\mathbb{R})[p/q]$ is indeed a Φ–algebra. Thus there are many such functions.

6. Open Questions.

In this paper, we have established the basic theory of absolutely polynomial functions and their topological ramifications. As our examples make clear, this theory is nontrivial, and leads to a number of interesting developments.

We shall conclude this paper with a list of open questions that arose during our investigations. Let X be a topological space.

(1) If a point $a \in X$ is absolutely isolated, is a isolated in X?

(2) If X is absolutely closed is it discrete?

(3) If X is not absolutely closed, does it have a continuous function on a dense open subset $U \subset X$ that is not absolutely polynomial? If not, then an elementary argument shows that X admits a function f that is not absolutely polynomial, whose domain is not contained in the closure of its interior.

(4) Which subsets S of a topological space X have the property that a continuous function $f\colon S \to \mathbb{R}$ with $\operatorname{dom} f = S$ is algebraically expressible as a sum of products of functions on S that are absolutely polynomial with respect to X? For example, this holds for compact subspaces because any continuous function whose domain is a compact subset is bounded, and hence the sum of two absolutely polynomial functions.

(5) A related question is whether the absolutely polynomial functions on S could be dense (in the usual metric) in the algebra of continuous functions on S.

(6) Does Proposition 3.3 have a generalization to absolutely polynomial functions of degree greater than 1? (See also the Remark following Theorem 3.6.) Must f^n even be absolutely polynomial when f is absolutely polynomial and n is odd? Note that by Proposition 3.23 a monotone function that is quadratically absolutely polynomial has all of its odd powers also quadratically absolutely polynomial. But a monotone function can be cubically absolutely polynomial and have a cube that is only quartically so. For example, this will occur if, in condition (d) of Proposition 3.23, we let $f(x_*) = 2$, $f(x_*^-) = [1 - \sqrt{13})]/2$, and $f(x_*^+) = 3$. In fact, all higher odd powers of such a function are also only quartically absolutely polynomial.

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