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Graph Theory and Classical Invariant Theory

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This paper presents a simple graphical method, closely related to the "algebrochemical method" of Clifford and Sylvester, for computations in the classical invariant theory of binary forms. Applications to syzygies and transvectants of covariants, and the determination of a Hilbert basis of covariants using Gordan's method are presented.

1. INTRODUCTION

Classical invariant theory has died and been resurrected many times. Its golden age in the last century was marked with the flowering of unsurpassed computational ability, and the explicit determination of the invariants of most of the elementary polynomials. The computational approach is commonly acknowledged to have been dealt a death-blow by Hilbert's celebrated Basis Theorem, an existential result *par excellence*. However, recent years have witnessed a reflowering of interest in classical invariant theory, both as a mathematical subject in its own right [6] and, perhaps more significantly, in important applications, including dynamical systems [1], existence results for the solution of nonconvex variational problems [7], and elasticity [8], among others. The applications have required a revivial of the computational approach, a task that is somewhat ameliorated by the current availability of symbolic manipulation computer programs. (Who knows where the subject would have gone if such powerful tools had been in the hands of the great computational mathematicians like Cayley, Clebsch, Gordan, Sylvester, etc.!)

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One of the barriers awaiting any serious student of the subject is the algebraic complexity of many of the constructions in the classical theory. With a view to rendering these complicated algebraic manipulations more manageable and more motivated, Clifford [2] began developing a graphical method for the description of the invariants and covariants of binary forms (polynomials), although he died before he could publish his findings to any significant extent. Contemporaneously, Sylvester [10] unveiled his "algebro-chemical theory," whose aim was to apply the methods of classical invariant theory to the then rapidly developing science of molecular chemistry. As far as we can tell, his theory was never taken very seriously by chemists, and not developed any further by mathematicians, and so died a perhaps well-deserved death. (However, this theory may not have been altogether misguided, as we note modern books on atomic and molecular physics, e.g., [11], that are essentially treatises in representation theory, a subject not so far removed from classical invariant theory.) The present graphical treatment of invariant theory is closest to that of Kempe [5] which builds on Clifford's posthumous notes.

Although Sylvester envisioned his theory as the future of chemistry, it is Clifford's graph theory that, with one slight but important modification, could have become a useful tool in computational invariant theory. The algebro-chemical theory reduces computations of invariants to methods of graph theory. Our thesis is that the correct framework for the subject is to use *digraphs* or "directed molecules" as the fundamental objects. One can ascribe both a graph theoretical as well as a chemical interpretation to these objects; both are useful for motivating the method. The fundamental relations or syzygies of invariant theory then translate into certain operations which can be performed on digraphs, or, equivalently, certain allowable reactions which can occur among directed molecules. The determination of a basis of irreducible digraphs or "atomic molecules" is the same problem as the determination of the Hilbert basis for the covariants of a binary form. All the computations in the symbolic calculus of invariant theory have elementary pictorial analogues using the graphical counterparts. Thus, the many complex algebraic constructions and computations appearing in the classical literature on the subject can all be reinterpreted simply and graphically. In other words, our basic guiding principle is that "a picture is worth a thousand algebraic manipulations."

An outline of the paper follows: In Section 2 we review the basic concepts of invariants and covariants of binary forms, and discuss some elementary examples. Section 3 introduces the powerful symbolic method of Aronhold, which is the key to the computation developments in the subject. Certain particular symbolic polynomials, known as bracket polynomials, play a fundamental role in these symbolic computations, and these are discussed in Section 4. Section 5 begins the heart of the paper,

and presents the molecular/graphical representation of invariants and covariants based on their symbolic bracket expressions. The syzygies or relations among bracket polynomials translate into operations with their graphical equivalents, leading to on "algebra of digraphs"; these are discussed in detail in Section 6. The remaining two sections are devoted to a simplified explanation of the constructive method of Gordan for the determination of the Hilbert basis of covariants for binary forms of a given degree. This method is based on the idea of "transvection," which can be reinterpreted as a way of reacting two different molecules or digraphs together to produce more complicated molecules/digraphs. The construction of the basis of covariants is illustrated in the final section by the simplest cases of a binary quadratic, cubic, and quartic polynomial. These last computations, we believe, amply illustrate the power and efficacy of the graphical method for treating complicated calculations in classical invariant theory.

2. Invariants and Covariants

By a *form* we mean a homogeneous polynomial. The most important case is that of a *binary form*

$$Q(\mathbf{x}) = Q(x, y) = \sum_{i=0}^{n} \binom{n}{i} a_i x^i y^{n-i}, \tag{1}$$

which is a homogeneous polynomial function of the variables $\mathbf{x} = (x, y)$, which can be either real or complex, depending on one's interests. The coefficients a_i are accordingly either real or complex. (The binomial coefficients $\binom{n}{i}$ are introduced for later convenience.) The integer n is the *degree* of the form.

One of the principal goals of classical invariant theory is to elucidate the fundamental geometric properties of forms, meaning those properties which do not depend on the introduction of a particular coordinate system (x, y). We thus consider the effect of general linear changes of variables

$$(x, y) \to (a\tilde{x} + b\tilde{y}, c\tilde{x} + d\tilde{y}), \tag{2}$$

in which the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is nonsingular, i.e., an element of the general linear group $GL(2)$ (either real or complex, depending on the type of form under consideration). Under such a linear transformation, the polynomial $Q(\mathbf{x})$ gets mapped to a new polynomial $\tilde{Q}(\tilde{\mathbf{x}})$, given by

$$\tilde{Q}(\tilde{x}, \tilde{y}) = Q(a\tilde{x} + b\tilde{y}, c\tilde{x} + d\tilde{y}). \tag{3}$$

Thus, the matrix A induces a transformation on the coefficients a_i of Q, mapping them into the corresponding new coefficients $\tilde{a}_i$ of $\tilde{Q}$. It is not difficult to write down the explicit formulas for the new coefficients $\tilde{a}_i$, but they are not overly instructive or helpful.

The key concept in classical invariant theory is the notion of an invariant, which is a function of the coefficients a_i of a form Q whose value does not change (apart from a multiplicative factor) under such changes of variables.

DEFINITION 1. An *invariant* of *weight* g of a binary form $Q(x, y)$, of degree n, is a function $I(\mathbf{a}) = I(a_0, ..., a_n)$, depending on the coefficients of Q, which, up to a determinantal factor, does not change under the action (3) of the general linear group:

$$I(\tilde{\mathbf{a}}) = (\det A)^g \cdot I(\mathbf{a}), \qquad A \in GL(2).$$

A *covariant* of *weight* g is a function $J(\mathbf{a}, \mathbf{x})$ depending both on the coefficients a_i and on the independent variables $\mathbf{x} = (x, y)$ which, up to a determinantal factor, is unchanged under the group action:

$$J(\tilde{\mathbf{a}}, \tilde{\mathbf{x}}) = (\det A)^g \cdot J(\mathbf{a}, \mathbf{x}), \qquad A \in GL(2).$$

(Note that invariants are just covariants that do not explicitly depend on $\mathbf{x}$.)

EXAMPLE 2. The most familiar example of an invariant is the *discriminant* $\Delta = a_1^2 - a_0 a_2$ of a quadratic polynomial

$$Q(\mathbf{x}) = a_0 x^2 + 2a_1 xy + a_2 y^2. \tag{4}$$

Under the linear change of variables (2), the quadratic polynomial Q is changed into the quadratic polynomial

$$\tilde{Q}(\tilde{\mathbf{x}}) = \tilde{a}_0 \tilde{x}^2 + 2\tilde{a}_1 \tilde{x}\tilde{y} + \tilde{a}_2 \tilde{y}^2,$$

with transformed coefficients

$$\tilde{a}_0 = a_0 a^2 + 2a_1 ac + a_2 c^2, \qquad \tilde{a}_1 = a_0 ab + a_1(ad + bc) + a_2 cd,$$
$$\tilde{a}_2 = a_0 b^2 + 2a_1 bd + a_2 d^2.$$

Thus the new discriminant is

$$\tilde{\Delta} = \tilde{a}_1^2 - \tilde{a}_0 \tilde{a}_2 = (ad - bc)^2 \cdot (a_1^2 - a_0 a_2) = (\det A)^2 \cdot \Delta;$$

hence Δ is an invariant of weight 2.

EXAMPLE 3. An important classical example is the case of a binary quartic

$$Q(\mathbf{x}) = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 x y^3 + a_4 y^4. \tag{5}$$

There are two fundamental invariants (cf. [3, p. 205]), namely a quadratic one

$$i = 2a_0 a_4 - 8a_1 a_3 + 6a_2^2, \tag{6}$$

which is of weight 4, and a cubic one

$$j = 6 \det \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}, \tag{7}$$

which is of weight 6. The reader might enjoy verifying that these expressions really are invariants. Any homogeneous combination of invariants is also an invariant. This we find the *discriminant* of the quartic Q, which can be identified with the product of the squares of the differences of the roots [3, p. 198],

$$\Delta = \tfrac{1}{27}(i^3 - 6j^2)$$

to be a sixth order invariant of weight 12.

The most important covariant of a quartic, or, indeed, of any binary form Q is the Hessian

$$H(\mathbf{x}) = \frac{2}{n^2(n-1)^2}(Q_{xx}Q_{yy} - Q_{xy}^2), \tag{8}$$

which is a polynomial of degree $2n-4$, and is covariant of weight 2. (The subscripts on Q indicate partial derivatives.) If Q is a quartic polynomial in $\mathbf{x}$, then its Hessian is also a quartic, and is given explicitly by

$$\begin{aligned} H = \tfrac{1}{72}(Q_{xx}Q_{yy} - Q_{xy}^2) &= 2(a_0 a_2 - a_1^2)\, x^4 + 4(a_0 a_3 - a_1 a_2)\, x^3 y \\ &+ (2a_0 a_4 + 4a_1 a_3 - 6a_2^2)\, x^2 y^2 + 4(a_1 a_4 - a_2 a_3)\, x y^3 + 2(a_2 a_4 - a_3^2)\, y^4. \end{aligned} \tag{9}$$

Besides the form Q itself, there is only one other independent covariant of the quartic, which is the Jacobian of Q and H,

$$T = \tfrac{1}{16}(Q_x H_y - Q_y H_x).$$

A classical result, which we shall prove later, states that any other polynomial invariant or covariant of a binary quartic can be written in terms of the covariants Q, H, i, j, and T (cf. [4, p. 286]).

As can be expected, the invariants and covariants of a form play a key role in understanding its geometric properties since, apart from the determinantial factor, their values do not depend on which coordinate system one is working in. In particular, if a covariant vanishes in one coordinate system, it vanishes in all coordinate systems, which indicates some important underlying property of the form. For instance, it can be shown that the Hessian of a binary form vanishes if and only if the form is the nth power of a linear form [6, 9]. For this reason, in the last century classical invariant theory tended to concentrate on the construction and investigation of explicit covariants of forms. However, the precise relationship between covariants and geometric properties of the form remains a poorly understood part of the subject.

A central result in the theory is Hilbert's Basis Theorem, which states that in all cases there are at most a finite number of fundamentally different invariants and covariants [4, Theorem 21.1; 6, Theorem 6.1].

THEOREM 4. *Let Q be a binary form of degree n. Then there are a finite number of covariants $C_1, ..., C_N$ with the property that any other covariant C can be written as a polynomial in these* basis covariants: $C = P(C_1, ..., C_N)$.

Thus the construction of the *Hilbert basis* of covariants for the form of a given degree has the net effect of describing all covariants, and hence all intrinsic properties of such forms. Gordan, in his constructive proof of the Basis Theorem for binary forms, gives a reasonably straightforward procedure, which we will review in Section 8. However, Gordan's method has only been successfully carried out for binary forms of degrees 2, 3, 4, 5, 6 and 8. (However, very recent results [9] indicate that it is not really necessary to know all the covariants to completely understand the various geometric properties of forms!)

3. THE SYMBOLIC METHOD

Although at the outset the determination of the covariants and invariants of a binary form of a given degree might appear to be a daunting task, there is a powerful constructive technique, introduced by Aronhold, called the *symbolic method*, which will readily provide a complete list of all the polynomial invariants and covariants. The motivating idea behind the symbolic method is that the theory of binary forms would be extremely simple if our binary form $Q(x, y)$ were just the nth power of a linear form

$$(\alpha \mathbf{x})^n = (\alpha_1 x + \alpha_2 y)^n = \sum_{i=0}^{n} \binom{n}{i} \alpha_1^i \alpha_2^{n-i} x^i y^{n-i}.$$

Comparing with the general expression (1), we see that in this special case the coefficients have the simple form

$$a_i = \alpha_1^i \alpha_2^{n-i}. \tag{10}$$

In the symbolic method, one effectively "pretends" that the general form Q is a power of a linear form. Each polynomial $J(\mathbf{a}, \mathbf{x})$ depending on the coefficients a_i of the form Q and the variables $\mathbf{x} = (x, y)$ will have a corresponding symbolic form, which is essentially found by replacing each occurrence of a coefficient a_i by the "symbolic power" $\alpha_1^i \alpha_2^{n-i}$, where $\alpha = (\alpha_1, \alpha_2)$ is a "symbolic letter." Symbolic letters by themselves have no real meaning; it is only when they appear in the particular power products (10) of degree n that they acquire a meaning in terms of the coefficients of our binary form.

However, simple examples reveal that this naïve approach must be immediately modified so as to avoid ambiguities. For instance, in the case of a binary quadratic (4), this approach would not distinguish between the monomials $a_0 a_2$ and a_1^2; if we were to replace both factors by the same symbolic letter, they would both degenerate to the same symbolic form $\alpha_1^2 \cdot \alpha_2^2$. (This is just another way of stating that a quadratic form is a perfect square if and only if its discriminant vanishes.) The way to resolve this ambiguity is to use a *different* symbolic letter for each occurrence of a coefficient a_i. Thus, for the binary quadratic, in the monomial $a_0 a_2$ we replace a_0 by α_1^2 and a_2 by β_2^2, where $\beta = (\beta_1, \beta_2)$ is a *second* symbolic letter, leading to the symbolic form $\alpha_1^2 \beta_2^2$; on the other hand, for the monomial a_1^2, we replace one factor a_1 by the product $\alpha_1 \alpha_2$ and the second factor by $\beta_1 \beta_2$, leading to the different symbolic form $\alpha_1 \alpha_2 \beta_1 \beta_2$. Note that we can readily pass back and forth between the explicit formula for a polynomial depending on the coefficients of the binary form and its symbolic forms; for instance, in the case of a quadratic form, the symbolic polynomial $\alpha_1^2 \beta_1 \beta_2 \gamma_2^2 x y^2$ depending on three symbolic letters would represent the monomial $a_0 a_1 a_2 x y^2$. (The x's and y's are not affected by the symbolic method.) Clearly, the number of symbolic letters required to write out an unambiguous symbolic form of a homogeneous polynomial functions of the coefficients a_i of the form is the same as the degree of the polynomial in the a_i.

Since we can rearrange the factors in any product of the coefficients a_i, there is *not* a uniquely determined symbolic form for a given polynomial $J(\mathbf{a}, \mathbf{x})$. For instance, if we write $a_0 a_2$ in reverse order as $a_2 a_0$, we obtain the symbolic form $\alpha_2^2 \beta_1^2$, not $\alpha_1^2 \beta_2^2$. However, all the different symbolic forms can be obtained one from the other merely by *interchanging the symbolic letters*. Thus, besides the previous example $\alpha_1^2 \beta_1 \beta_2 \gamma_2^2 x y^2$, the symbolic forms $\alpha_2^2 \beta_1 \beta_2 \gamma_1^2 x y^2$, $\beta_2^2 \alpha_1 \alpha_2 \gamma_1^2 x y^2$, $\beta_2^2 \gamma_1 \gamma_2 \alpha_1^2 x y^2$, etc., all represent the same

monomial $a_0a_1a_2xy^2$. It is not difficult to see that there is a unique *symmetric* symbolic form for any given polynomial, obtained by symmetrizing any given representative over all the symbolic letters occurring in it. For instance, the discriminant $a_0a_2 - a_1^2$ of a quadratic has symmetric symbolic form

$$\begin{aligned}&\tfrac{1}{2}\{(\alpha_1^2\beta_2^2 - \alpha_1\alpha_2\beta_1\beta_2) + (\alpha_2^2\beta_1^2 - \alpha_1\alpha_2\beta_1\beta_2)\}\\ &\quad = \tfrac{1}{2}\{\alpha_1^2\beta_2^2 - 2\alpha_1\alpha_2\beta_1\beta_2 + \alpha_2^2\beta_1^2\} = \tfrac{1}{2}(\alpha_1\beta_2 - \alpha_2\beta_1)^2.\end{aligned}$$

The last factorization, which at the outset appears to be merely coincidental, turns out to be an important clue to the general result on the symbolic forms of covariants.

In general, we introduce an *alphabet* $\mathscr{A} = \{\alpha, \beta, \gamma, \ldots\}$, which is an *ordered* infinite collection of symbols called *symbolic letters*, so α is the first, β the second, γ the third symbolic letter, and so on. Each symbolic letter $\alpha, \beta, \ldots$ represents a vector in $\mathbb{R}^2$ or $\mathbb{C}^2$, so we write $\alpha = (\alpha_1, \alpha_2)$, etc. Let $\mathscr{P}^n_{mk}$ denote the space of all polynomials $J(\mathbf{a}, \mathbf{x})$ depending on the coefficients of a binary form of degree n, which are homogeneous of degree m in the coefficients $\mathbf{a} = (a_0, \ldots, a_n)$ and of degree k in the variables $\mathbf{x} = (x, y)$. Note that each $J \in \mathscr{P}^n_{mk}$ is a sum of monomials of the form

$$c \cdot a_{i_1} \cdot a_{i_2} \cdot \cdots \cdot a_{i_m} \cdot x^j \cdot y^{k-j}, \tag{11}$$

where the coefficient c is a constant. The corresponding symbolic form of such a polynomial $J(\mathbf{a}, \mathbf{x})$ will be an element of the *umbral space* $\mathscr{U}^n_{mk}$, which consists of all homogeneous polynomials $P(\alpha, \beta, \ldots, \omega, \mathbf{x})$ depending on the first m symbolic letters $\alpha, \beta, \ldots, \omega$ and the variables $\mathbf{x} = (x, y)$, and which are homogeneous of degree n in each of the symbolic letters and of degree k in $\mathbf{x}$. (The words "symbolic" and "umbral" are used interchangeably in the literature.) Given a monomial as in (11), the corresponding symbolic monomial is obtained by replacing each coefficient a_{i_ν} by the corresponding power $\lambda_1^{i_\nu}\lambda_2^{n-i_\nu}$, as in (10), where λ is the νth symbolic letter in the alphabet $\mathscr{A}$. Thus, the symbolic form of the above monomial is

$$c\alpha_1^{i_1}\alpha_2^{n-i_1}\beta_1^{i_2}\beta_2^{n-i_2}\cdots\omega_1^{i_m}\omega_2^{n-i_m}x^j y^{k-j}.$$

Summing all the resulting symbolic monomials together, we obtain a symbolic representative $P(\alpha, \beta, \ldots, \omega, \mathbf{x}) \in \mathscr{U}^n_{mk}$ for J. The ambiguity stemming from the interchange of factors in the monomials of J can be resolved by symmetrizing the symbolic polynomial, i.e., averaging over the symmetric group consisting of all possible permutations of the m symbolic letters $\alpha, \beta, \ldots, \omega$.

THEOREM 5. *Each polynomial* $J(\mathbf{a}, \mathbf{x}) \in \mathscr{P}^n_{mk}$ *has a unique symmetric symbolic form* $P(\alpha, \beta, \ldots, \omega, \mathbf{x}) \in \mathscr{U}^n_{mk}$.

Although the symmetric symbolic forms are unique determined, it is nevertheless useful to allow more general symbolic polynomials, as long as we remember that these are not uniquely determined by the polynomial J.

One important comment: if we are given a polynomial in the symbolic letters, then it will represent a polynomial in the coefficients of a binary form of degree n if and only if each symbolic letter occurs precisely n times in each term of the symbolic polynomial; i.e., the symbolic polynomial is homogeneous of degree n in each of its symbolic letters. Note also that the number of different symbolic letters in a symbolic polynomial represents the degree of the polynomial in the coefficients a_i of the form.

EXAMPLE 6. For the invariant

$$i = 2a_0a_4 - 8a_1a_3 + 6a_2^2,$$

of the binary quartic, we obtain one symbolic form immediately:

$$2\alpha_1^4\beta_2^4 - 8\alpha_1^3\alpha_2\beta_1\beta_2^3 + 6\alpha_1^2\alpha_2^2\beta_1^2\beta_2^2.$$

To obtain the symmetrized form, we interchange α and β and average the two expressions, leading to

$$\alpha_1^4\beta_2^4 - 4\alpha_1^3\alpha_2\beta_1\beta_2^3 + 6\alpha_1^2\alpha_2^2\beta_1^2\beta_2^2 - 4\alpha_1\alpha_2^3\beta_1^3\beta_2 + \alpha_2^4\beta_1^4 = (\alpha_1\beta_2 - \alpha_2\beta_1)^4.$$

Again, we see a similar factorization as with the discriminant of the binary quadratic.

As an example of a covariant, consider the Hessian of the quartic. Replacing each coefficient in the explicit expression (9) by its symbolic form, we find that H has the symbolic form

$$\begin{aligned} &2(\alpha_1^4\beta_1^2\beta_2^2 - \alpha_1^3\alpha_2\beta_1^3\beta_2)\, x^4 + 4(\alpha_1^4\beta_1\beta_2^3 - \alpha_1\alpha_2^3\beta_1^2\beta_2^2)\, x^3y \\ &\quad + (2\alpha_1^4\beta_2^4 + 4\alpha_1^3\alpha_2\beta_1\beta_2^3 - 6\alpha_1^2\alpha_2^2\beta_1^2\beta_2^2)\, x^2y^2 \\ &\quad + 4(\alpha_1^3\alpha_2\beta_2^4 - \alpha_1^2\alpha_2^2\beta_1\beta_2^3)\, xy^3 + 2(\alpha_1^2\alpha_2^2\beta_2^4 - \alpha_1\alpha_2^3\beta_1\beta_2^3)\, y^4. \end{aligned}$$

Interchanging α and β, and averaging, we obtain the unique symmetric symbolic form for the Hessian. Remarkably, this symbolic covariant also factors:

$$(\alpha_1\beta_2 - \alpha_2\beta_1)^2 \cdot (\alpha_1 x + \alpha_2 y)^2 \cdot (\beta_1 x + \beta_2 y)^2.$$

4. Bracket Polynomials

The preceding examples have indicated that certain particular symbolic polynomials play a distinguished role in the theory of covariants.

Definition 7. (a) A *bracket factor of the first kind* is a linear monomial

$$(\alpha \mathbf{x}) = \alpha_1 x + \alpha_2 y,$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathscr{A}$ is any symbolic letter.

(b) A *bracket factor of the second kind* is the 2×2 determinant

$$[\alpha\beta] = \det \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = \alpha_1\beta_2 - \alpha_2\beta_1,$$

where $\alpha, \beta \in \mathscr{A}$ are distinct symbolic letters.

If, following Kung and Rota [6], we were to write $u_2 = x$, $u_1 = -y$, then the bracket factors of the first kind could be written as a bracket factors of the second kind involving α and $\mathbf{u}$. However, this rather artificial approach is special to the case of binary forms, and does not immediately extend to forms in three or more variables. Besides, there are other good reasons for keeping the two types of bracket factors distinct (see below).

A *bracket polynomial* is a symbolic expression which can be written as a polynomial in the bracket factors of the first and second kinds. All of the symmetric symbolic covariants encountered so far have been bracket polynomials. The First Fundamental Theorem of Invariant Theory states that every *covariant* of a binary form can be written in symbolic form as a bracket polynomial (cf. [4, p. 206; 6, Theorem 3.1]).

Theorem 8. *If $J(\mathbf{a}, \mathbf{x})$ is a covariant of the form Q, then the symmetric symbolic form of J can be written as a bracket polynomial. Conversely, if $P(\alpha, ..., \omega, \mathbf{x})$ is any homogeneous bracket polynomial (not necessarily symmetric), then P is a symbolic form of a covariant J.*

The *degree* of the covariant in the coefficients a_i is equal to the number of distinct symbolic letters occurring in the bracket polynomial representative. Similarly, the *weight* of the covariant is equal to the number of bracket factors of the second kind in any monomial of P. (By homogeneity, this does not change among the monomials.) Finally, the degree of the covariant in the variables $\mathbf{x}$ is equal to the number of bracket factors of the first kind in any monomial of P.

EXAMPLE 9. In the case of a quartic form, as in Example 3 above, the invariants i and j have symbolic bracket expressions

$$[\alpha\beta]^4,$$

and

$$[\alpha\beta]^2[\alpha\gamma]^2[\beta\gamma]^2,$$

respectively. The Hessian has symbolic bracket expression

$$[\alpha\beta]^2(\alpha\mathbf{x})^2(\beta\mathbf{x})^2,$$

while the other covariant T has the symbolic form

$$[\alpha\beta]^2[\beta\gamma](\alpha\mathbf{x})^2(\beta\mathbf{x})(\gamma\mathbf{x})^3.$$

(See [7] for a direct method of determining the partial derivative (hyperjacobian) formulas for covariants directly from their symbolic bracket expressions.)

Thus, we can explicitly write down all the invariants and covariants of a binary form merely by writing down all the bracket polynomials. However, there is a lot of redundancy in this procedure. First, since any homogeneous polynomial function of a collection of covariants of a binary form is also a covariant, we should eliminate these reducible covariants from our "minimal" list of covariants. We shall subsequently see how to effect this. Even more fundamentally is the fact that the symbolic form of a given covariant does *not* have a unique bracket polynomial representative, owing to the presence of certain relations or *syzygies* among the bracket factors themselves. There are three of these fundamental syzygies, from which all the others can be deduced (cf. [4, p. 211; 6, Corollary 3.1]):

$$[\alpha\beta] = -[\beta\alpha], \tag{12}$$

$$[\alpha\beta](\gamma x) = [\alpha\gamma](\beta x) + [\gamma\beta](\alpha x), \tag{13}$$

$$[\alpha\beta][\gamma\delta] = [\alpha\gamma][\beta\delta] + [\alpha\delta][\gamma\beta]. \tag{14}$$

Here α, β, γ, δ are distinct symbolic letters. The reader can easily verify each of these identities directly from the formulas for the bracket factors. Below we shall see how each of these syzygies can be applied to simplify bracket polynomials, and, ultimately, derive the Hilbert basis for the covariants of a binary form of a given degree.

One further remark on bracket polynomials: If we know the degree of a covariant, and are given just the bracket factors of the second kind occurring in any homogeneous bracket polynomial representative, we can

readily reconstruct the bracket factors of the first kind. Let α be any symbolic letter occurring in the symbolic polynomial. In order that the polynomial be the symbolic form of some covariant, it must be homogeneous of degree n, the degree of the underlying form, in each symbolic letter. Therefore, in any bracket monomial, if α occurs k times in the bracket factors of the second kind, then it must occur precisely $n-k$ times in the bracket factors of the first kind so that α will occur exactly n times in all. Thus we need to multiply the given factors of the second kind by $(\alpha\mathbf{x})^{n-k}$ in order to get the degree of homogeneity right. For example, if we have a symbolic monomial of degree 3 in the coefficients a_i of the form whose bracket factors of the second kind are

$$[\alpha\beta][\beta\gamma]^2,$$

then we know that the full bracket monomial must be

$$[\alpha\beta][\beta\gamma]^2(\alpha\mathbf{x})^{n-1}(\beta\mathbf{x})^{n-3}(\gamma\mathbf{x})^{n-2},$$

since α occurs once, β three times, and γ twice in the second factors. (If the monomial were of degree 4 in the a_i's, then we would have

$$[\alpha\beta][\beta\gamma]^2(a\mathbf{x})^{n-1}(\beta\mathbf{x})^{n-3}(\gamma\mathbf{x})^{n-2}(\delta\mathbf{x})^n,$$

since the fourth symbolic letter δ would not occur at all in the second factors, but must still be accounted for in the full monomial.) Since from now on we will primarily concentrate on the bracket factors of the second kind, we will call them just *brackets* for short.

5. Digraphs and Molecules

We are now in a position to present the graphical method used to both represent and calculate with invariants and covariants of binary forms. Consider a binary form of degree n, and let P be a bracket polynomial representing the symbolic form of some covariant. To each *monomial* in P we will associate a "molecule," or, more mathematically, a *digraph.* It is easiest to first explain our procedure from a chemical point of view.

Let M be any unit bracket monomial (i.e., with coefficient 1). To each distinct symbolic letter in M we associate an *atom.* For a binary form of degree n, the atoms will all have "valence" n; i.e., there are n or fewer possible bonds that can be made with other atoms. (In Sylvester's somewhat fanciful terminology, the atoms can be named after real atoms of

the same valency, so that "oxygen" represents a quadratic form, "carbon" a quartic form, etc. In our simple exposition, all the atoms have the same valence; in the more general theory of covariants of several binary forms, one runs into molecules with atoms of different valences.) The bonds in our molecule will correspond to all the bracket factors of the second kind occurring in M. Thus, if $[\alpha\beta]$ is a bracket in M, then we have a bond between the atom labelled α and the atom labelled β. If a bracket occurs to the kth power—$[\alpha\beta]^k$—in M, then there will be k bonds between atom α and atom β. So far, this is the algebro-chemical theory as proposed by Sylvester [10]. Our key departure is to make use of *directed* (or *polarized*) *bonds*, which will enable us to distinguish between the bracket factors $[\alpha\beta]$ and $[\beta\alpha]$. (For the moment let us ignore the fact that these just differ by a sign.) Thus $[\alpha\beta]$ will be represented by a bond *from* atom α *to* atom β, whereas $[\beta\alpha]$ will be represented by a bond *from* atom β *to* atom α. Note that since each symbolic letter can occur at most n times in the monomial M, our *directed molecule* representing M observes the valence restrictions that each atom has at most n bonds connecting it to any other atom. In such a directed molecule, the *valence* of each constituent atom is defined as the number of unused bond sites, so if atom α has k bonds connecting it to other atoms in M, then its valence is $n-k$. In this case, in the corresponding bracket monomial there will be k bracket factors of the second kind with α as one of the two symbolic letters, and the bracket factor of the first kind $(\alpha\mathbf{x})$ will occur to the $(n-k)$th power. The *valence* of the entire molecule M is just the sum of the valences of the constitutive atoms, and indicates the total number of bracket factors of the first kind in the symbolic monomial, which is the same as the degree of the corresponding covariant in $\mathbf{x}$. We can therefore distinguish between *ions*, in which there are one or more atoms with unused free bonding sites, and so the valence is strictly positive, and *neutral molecules*, in which each atom has exactly n bonds, and the entire molecule has valence 0. Neutral molecules correspond to invariants, while ions correspond to more general covariants.

As a simple example, consider the Hessian (8) of a binary form of degree n. As with the binary quartic, it can be shown to have the symbolic form

$$[\alpha\beta]^2(\alpha x)^{n-2}(\beta x)^{n-2}.$$

There are two distinct symbolic letters, α and β, and so the corresponding molecule will consist of two atoms, labelled accordingly. Moreover, since the bracket factor $[\alpha\beta]$ occurs twice, there will be two directed bonds from atom α to atom β. Thus the directed molecule representing the Hessian is

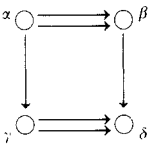
In particular, the Hessian is an invariant for a binary form ($n=2$), but a covariant for $n>2$, since the two atoms each still have valence $n-2$. The total valence of H, namely $2n-4$, is the same as its degree in $\mathbf{x}$. Similarly, the discriminant of the binary cubic,

$$\Delta = 2a_0^2a_3^2 - 6a_1^2a_2^2 - 12a_0a_1a_2a_3 + 8a_0a_2^3 + 8a_1^3a_3$$

[4, p. 154], has symbolic bracket expression

$$[\alpha\beta]^2[\alpha\gamma][\beta\delta][\gamma\delta]^2$$

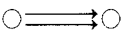
[3, p. 194] and so is represented by the neutral four-atom molecule



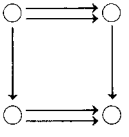
Now, an important point is that *since the symbolic letters are all interchangeable, the molecular representation does not depend on how we label the constituent atoms.* Thus, we can represent the discriminant of the binary cubic by any of the equivalent forms

$$[\alpha\delta]^2[\alpha\beta][\delta\gamma][\beta\gamma]^2, \qquad [\beta\gamma]^2[\beta\delta][\gamma\alpha][\delta\alpha]^2,$$

etc. All of these equivalent bracket representatives have the *same* molecular representation, modulo a relabelling of the atoms. Thus, once we determine the appropriate molecule for a given bracket monomial, we can drop the labels for the individual atoms, and concentrate on the pure "chemistry" of our molecule. For example, we will say that



is the molecular representation of the Hessian, while



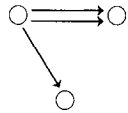
will be the molecular representation of the discriminant of the binary cubic.

More generally, if we are given a bracket polynomial P, which is the symbolic form of a covariant of a binary form of degree n, there is a

corresponding "linear combination of molecules" which represents it. If $P = \sum c_i M_i$, where the M_i are the unit bracket monomials occurring in P, then we define its molecular representation to be the formal sum of digraphs $D = \sum c_i D_i$, where D_i is the molecular representation of the monomial M_i. For example, in the case of a binary cubic, the bracket monomial

$$M_1 = [\alpha\beta]^2[\alpha\gamma](\beta\mathbf{x})(\gamma\mathbf{x})^2$$

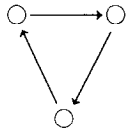
has molecular representation

$D_1 =$ 

while

$$M_2 = [\alpha\beta][\beta\gamma][\gamma\alpha](\alpha\mathbf{x})(\beta\mathbf{x})(\gamma\mathbf{x})$$

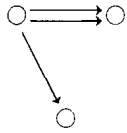
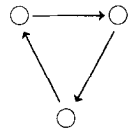
has molecular representation

$D_2 =$ 

Therefore, the bracket polynomial

$$P = 2[\alpha\beta]^2[\alpha\gamma](\beta\mathbf{x})(\gamma\mathbf{x})^2 - \tfrac{1}{2}[\alpha\beta][\beta\gamma][\gamma\alpha](\alpha\mathbf{x})(\beta\mathbf{x})(\gamma\mathbf{x})$$

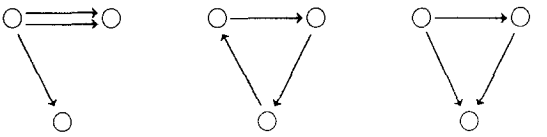
has molecular representation

$2D_1 - \frac{1}{2}D_2 = 2$  $- \frac{1}{2}$ 

which can be interpreted to mean twice digraph D_1 plus $-\frac{1}{2}$ times the digraph D_2. (Chemically, these linear combinations of molecules might be interpreted as "mixtures" of molecular substances, although the admission of negative coefficients stretches this analogy rather thin.)

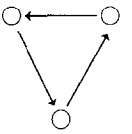
Mathematically, we are replacing each unit bracket monomial with a digraph. Recall that a *graph* is pictorially represented by a collection of vertices and line segments connecting the vertices. A *digraph* (or *directed*

graph) is a graph in which the line segments, now called *darts*, are directed, and so can be represented by line segments with arrows. Thus



D_1 D_2 D_3

represent distinct digraphs. However, note that the digraph



while ostensibly the mirror image of D_2, is really the same as D_2. In a digraph, the vertices correspond to the atoms in the molecular representation, and the darts correspond to the directed bonds.

Any bracket monomial will have a unique digraph representation. It is easy to see which digraphs or molecules correspond to bracket monomials representing covariants of a form of degree n. Since each symbolic letter α can occur at most n times in the bracket factors of a monomial representing a covariant, there are at most n darts attached to any given vertex. If a vertex has exactly k darts attached to it, the corresponding symbolic letter α will appear k times in the bracket factors of the second kind, and there will be $n-k$ additional bracket factors of the first kind $(\alpha\mathbf{x})$ in the monomial. Thus, we define an *n-digraph* to be a digraph with the property that there are at most n darts originating or terminating at any vertex. Note that any n-digraph is automatically an m-digraph whenever $n \leqslant m$. For instance, in the above examples, D_1 is a 3-digraph (and also a 4- or 5-digraph), but not a 2-digraph, whereas D_2 and D_3 are 2-digraphs, as well as 3-digraphs, etc. The number of vertices in the digraph equals the degree of the covariant in the coefficients a_i; the number of darts equals the weight of the covariant, and the valence or total number of remaining free bond sites equals the degree of the covariant in $\mathbf{x}$. A *saturated digraph* (i.e., a neutral molecule) is one in which every vertex has k darts attached to it, and represents an invariant of the binary form.

More generally, to represent polynomials, we need to pass to the space of "linear combinations of digraphs." Thus, we let $\mathscr{D}$ denote the free module (over $\mathbb{C}$ or $\mathbb{R}$) generated by all possible digraphs. Furthermore, let $\mathscr{D}_n$ denote the submodule generated by all possible n-digraphs. Note that

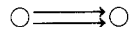
$\mathscr{D}_n \subset \mathscr{D}_m$ whenever $n \leqslant m$. Therefore, $2D_1 - \frac{1}{2}D_2$, as illustrated above, is a typical member of $\mathscr{D}_3$.

THEOREM 10. *Let Q be a binary form of degree n. Then there is a one-to-one correspondence between bracket polynomials representing covariants of Q and elements of the space $\mathscr{D}_n$ of linear combinations of n-digraphs.*

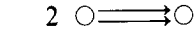
By the *trivial digraph*, we mean the zero element of $\mathscr{D}$, which corresponds to the trivial covariant 0. A digraph is *reducible* if it is the disjoint union of two subdigraphs or *components*, which we write as $D = D_1 \vee D_2$, meaning there are no darts in D connecting a vertex of D_1 to a vertex of D_2. It is easy to see that a reducible digraph corresponds to a reducible bracket monomial.

LEMMA 11. *Let D be a digraph corresponding to the bracket monomial M which in turn is the symbolic form of a covariant C. Then D is reducible into the disjoint union of the digraphs D_1 and D_2 if and only if M is the product of the corresponding bracket monomials M_1 and M_2, or, equivalently, C can be written as the product of two lower order covariants $C_1 \cdot C_2$.*

For example, the reducible digraph on four vertices



represents the square of the Hessian H of a form: H^2. (*Important*: this is not the same as the multiple



which represents twice the Hessian: $2H$.)

6. SYZYGIES AND THE ALGEBRA OF DIGRAPHS

Although there is one-to-one correspondence between linear combinations of digraphs and bracket polynomials, there is not a one-to-one correspondence between digraphs and covariants owing to the syzygies among the bracket factors themselves. Thus there are certain equivalence relations among digraphs which mirror the basic syzygies (12), (13), (14). The implementation of these leads to an "algebra of digraphs" which corresponds to the algebra of covariants of a binary form. From the chemical point of view, there are certain allowable "reactions" which a

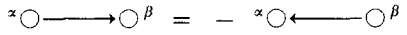
given molecule may entertain. The goal is then to utilize the allowable reactions to simplify a molecule or digraph as much as possible. It is this theory, we maintain, that constitutes the proper form of the chemico-algebraic theory that Sylvester and Clifford were aiming for.

There are three basic rules in the algebra of digraphs:

Rule #1. From the first syzygy (12),

$$[\alpha\beta] = -[\beta\alpha],$$

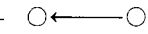
we conclude that reversing any dart in a digraph changes the sign of the digraph. We represent this rule pictorially by



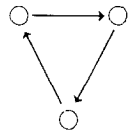
where we are just indicating the relevant vertices and darts in the digraph; all other vertices and darts are left unchanged.

For instance, dropping the inessential symbolic labels for the vertices, Rule #1 shows that the elementary digraph

equals its own negative, which is



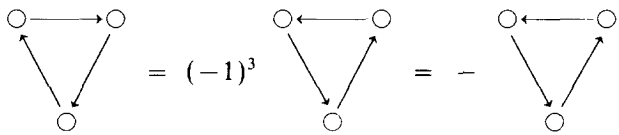
and hence represents the trivial covariant 0. (This rule also implies that if a dart connects a vertex to itself, then the digraph is automatically 0.) As another application of Rule #1, consider the digraph



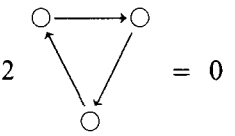
It corresponds to the bracket monomial

$$[\alpha\beta][\beta\gamma][\gamma\alpha](\alpha\mathbf{x})^{n-2}(\beta\mathbf{x})^{n-2}(\gamma\mathbf{x})^{n-2}.$$

However, this monomial is a symbolic form of the trivial (zero) covariant. (Verify!) Indeed, if we reverse the direction of all three darts in the digraph, we see that

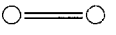


But, as remarked above, the two digraphs on each side of this equality are really the same, and so

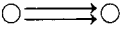


is equivalent to the trivial digraph.

Note that Rule #1 implies that the directions of single bonds or darts make a difference in the sign of a digraph, but double bonds can be simultaneously reversed without changing it. To simplify the pictures, we will often denote double bonds which point in the same direction by plain line segments, so



will be an alternative way of representing the Hessian

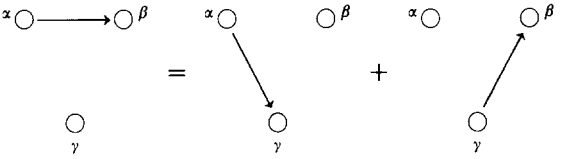


Note that this is the negative of the digraph

Rule #2. The syzygy (13), namely

$$[\alpha\beta](\gamma x) = [\alpha\gamma](\beta x) + [\gamma\beta](\alpha x),$$

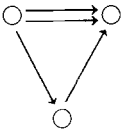
translates into the digraph rule



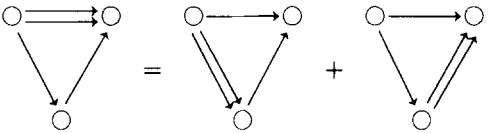
This means that any digraph with a dart between vertices α and β can be transformed into the sum of two digraphs, one with a dart from α and γ and one with a dart from γ to β. To remember this rule, note that either the head or the tail of the moving dart is fixed in each of the summands on the right hand side, while the other end of the dart has attached itself to the new vertex γ. Of course, if we are dealing with a form of degree n, we cannot have a digraph with more than n darts at a single vertex, so this operation is allowed *only* if the vertex corresponding to γ in the original

digraph has fewer than n darts connected to it; i.e., the atom γ has valence at least one.

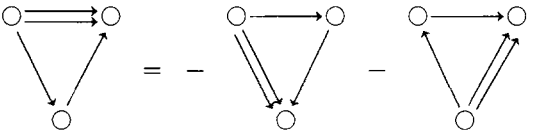
For example, consider the digraph



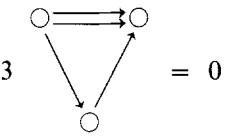
If we use Rule 2, we find that



However, if we now use Rule #1 to reverse one of the single darts in each of the digraphs on the right hand side of this equation, we find



But all three of these digraphs are exactly the same, hence

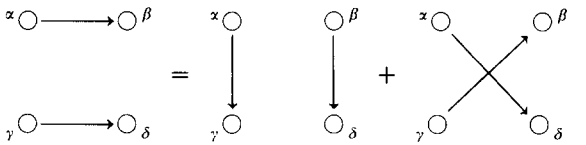


and our original digraph represents a trivial covariant.

Rule #3. The remaining syzygy (14) has the form

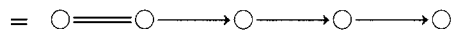
$$[\alpha\beta][\gamma\delta] = [\alpha\gamma][\beta\delta] + [\alpha\delta][\gamma\beta].$$

This says that we can "switch" darts in digraph to the following rule:

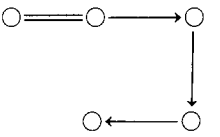


Thus a digraph with darts from α to β and γ to δ is the same as the sum of two digraphs, one with darts from α to γ and from β to δ and the other with darts from α to δ and from γ to β. To memorize this rule, note that in each of the digraphs on the right hand side one end of each of the relevant darts has remained fixed, whereas the other ends of the darts have switched vertices. Note that each vertex has the same number of darts connected to it in each of the resulting digraphs, so there are no restrictions with Rule #3 as there were with Rule #2. Also note that Rule #3 only gives nontrivial results when the vertices corresponding to α, β, γ, and δ are all *distinct*.

For example, we can show that the digraph

$D =$ 

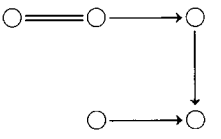
or, equivalently,

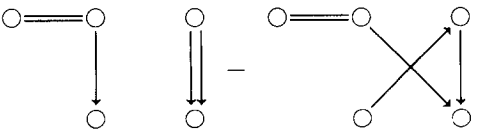
$D =$ 

corresponding to the bracket monomial

$$[\alpha\beta]^2[\beta\gamma][\gamma\delta][\delta\varepsilon](\alpha\mathbf{x})^{n-2}(\beta\mathbf{x})^{n-3}(\gamma\mathbf{x})^{n-2}(\delta\mathbf{x})^{n-2}(\varepsilon\mathbf{x})^{n-1}$$

is equivalent to a reducible digraph, so this bracket monomial corresponds to a covariant which is the product of two simpler covariants. First applying Rule #1 to the bottom dart, and then Rule #3 to the top and bottom darts, we get

$D = -$ 

$= -$ 

On the right hand side, the first digraph is reducible. Untangling the second digraph, and using Rule #1 to reverse the directions of two darts,

we see that it is exactly the same as the original digraph. Thus the preceding digraph equation takes the form

$$D = -R - D,$$

where R is reducible, hence $D = -\frac{1}{2}R$ is also reducible. The reader might find it revealing to compare this elementary "graphic proof" with the more cumbersome algebraic proof it represents.

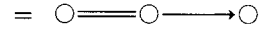
7. Transvectants

Given a molecular ion representing a covariant of a binary form, we can obtain new, more complicated molecules by "reacting" with other ions, in particular with free atoms. The invariant theoretic name for this reaction is *transvection*, and it provides a ready mechanism for constructing new covariants from old ones. Note that for a given pair of ions, there will usually be a number of different ways of connecting them together by a prescribed number of bonds. The transvectant between the two digraphs is just the sum of all such possible reactions.

Definition 12. Let $k \geqslant 0$. Let $D, E \in \mathscr{D}_n$ be digraphs, each of valence at least k. Then the kth *transvectant* of D and E is the digraph $(D, E)^{(k)}$ formed by summing all possible digraphs in $\mathscr{D}_n$ obtained by connecting D to E by k darts. In forming the sum, one treats all of the free bond sites on both D and E, and all of the k darts as distinct, so that there are certain combinatorial multiplicities associated with the digraphs appearing in $(D, E)^{(k)}$.

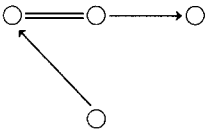
In particular, the 0th tranvectant $(D, E)^{(0)}$ is just the reducible digraph representing the product of the covariants corresponding to D and E. The first transvectant $(D, E)^{(1)}$ is just the graphical form of the covariant representing the Jacobian of the covariants corresponding to D and E. If Q denotes the digraph consisting of a single vertex, corresponding to the form itself, then we write $(Q, D)^{(k)}$ simply as $D^{(k)}$, and call it the kth *transvectant* of D. In this paper, for simplicity, we treat only these simple transvectants.

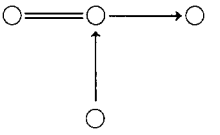
Example 13. Consider the digraph

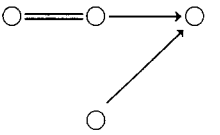
T = 

which represents the covariant T of the binary quartic. Consider the first transvectant $(Q, T)^{(1)} = T^{(1)}$. This will be a linear combination of all possible digraphs in $\mathscr{D}_4$ which can be obtained by joining a single atom or

vertex, representing the quartic Q itself, to the digraph for T with a single dart. There are three possible such digraphs:

$D_1 =$ 

$D_2 =$ 

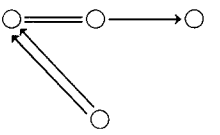
$D_3 =$ 

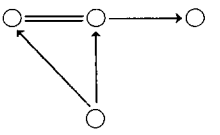
For the first transvectant, the combinatorial multiples that are associated with each of these digraphs just equals the valence of the atom being connected up. For D_1, the left hand atom of T has two free bond sites remaining, so D_1 gets multiple 2; by the same reasoning, D_2 gets multiple 1, and D_3 gets multiple 3. Therefore

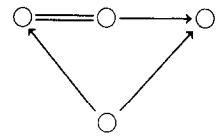
$$(Q, T)^{(1)} = 2D_1 + D_2 + 3D_3.$$

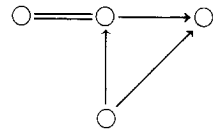
(Actually, to agree with the classical formulas, we should divide the right hand side by 6 so that the sum of the coefficients is 1.)

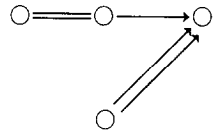
The second transvectant $(Q, T)^{(2)} = T^{(2)}$ will be linear combination of all possible digraphs in $\mathscr{D}_4$ which can be obtained by joining a single atom by two darts to the digraph for T. Since we are working with quartic polynomials, there are five such possible digraphs:

$D_1 =$ 

$D_2 =$ 

$D_3 =$ 

$D_4 =$ 

$D_5 =$ 

(Note that we cannot attach two darts to the middle vertex in T since then there would be five darts at that vertex, which is not allowed for a quartic; if we were dealing with a quintic or even higher degree polynomial, then we would have yet another component to the transvectant. Thus, technically speaking, we should indicate the degree n of the underlying form when writing down a transvectant.) We find the combinatorial multiples to be

$$(Q, T)^{(2)} = 2D_1 + 4D_2 + 12D_3 + 6D_4 + 6D_5.$$

For instance, to obtain the coefficient of D_5, we find that we need to attach two distinct darts to the three free bond sites of the right hand atom, and there are $3 \cdot 2 = 6$ ways in which this can be done.

8. Gordan's Method

A Hilbert basis for the covariants of a binary form of a given degree corresponds to the determination of a complete set of "atomic molecules" ("atomicules" in Sylvester's terminology), or *irreducible digraphs*. Indeed, the content of the Basis Theorem is that any more complicated molecule or digraph is equivalent, under the various digraph rules, to some reducible combination (or mixture) of irreducible digraphs. Gordan devised an efficient, constructive recursive procedure for generating the Hilbert basis for the covariants of a form of a given degree. In outline, the method begins with the covariant represented by a single atom, i.e., the form Q itself. We then successively construct all nontrivial transvectants of it, all transvectants of the nontrivial transvectants, etc. At each stage, we only need to append one further vertex to the digraphs from the previous stage. The

main points to be clarified are (a) which transvectants need to be considered at each stage and (b) when do we stop the procedure? We begin by stating several elementary but important lemmas.

LEMMA 14. *If a digraph D is reducible, and one component of D is an invariant, then each summand in $D^{(k)}$ is also reducible, with one component the same invariant.*

Proof. This is more or less obvious. We cannot attach any more darts to a saturated digraph (neutral molecule), so the same saturated digraph will appear as a component in all summands of the transvectant $D^{(k)}$.

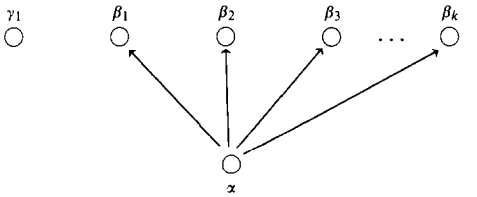
As we have seen, when we write down a transvectant $D^{(k)}$, there will usually be many different summands to be taken into account. The key result underlying the construction is that we really only need to consider *one* of these summands in our construction of the Hilbert basis.

LEMMA 15. *Let D be a digraph with m vertices. Suppose E_1 and E_2 are two digraphs appearing in the kth transvectant $D^{(k)}$. Then E_1 is equivalent to E_2 modulo lower order transvectants, meaning that*

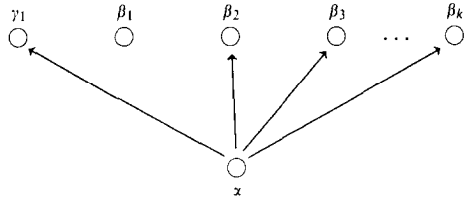
$$E_1 = E_2 + \sum F_\mu,$$

where each digraph F_μ occurs as a summand in a transvectant $\hat{D}^{(j)}$ for some $0 \leqslant j < k$, where $\hat{D}$ is some other digraph with m vertices.

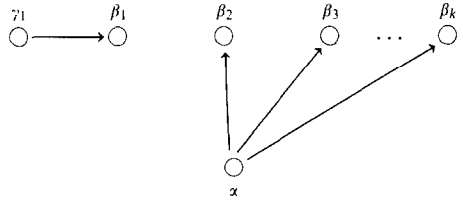
Proof. Let us label the new vertex in each summand in $D^{(k)}$ by α. Let $\beta_1, \ldots, \beta_k$ be labels for the vertices in D which are connected to α in E_1, and $\gamma_1, \ldots, \gamma_k$ be labels for the vertices in D which are connected to α in E_2. (Some of these vertices might coincide.) It clearly suffices to consider the case when only one of the β's and γ's are different, since we can then proceed by an obvious induction to prove the general case. Thus we assume that $\beta_i = \gamma_i$ for $i = 2, \ldots, k$. We draw the relevant parts of first of the digraph E_1:



and, second, of the digraph E_2:



Here the upper vertices are all in D itself, and α is the new vertex in the transvectant $D^{(k)}$. Now apply Rule #2 to the dart connecting α to β_1 in E_1. We immediately deduce that $E_1 = E_2 + F$, where F is the digraph obtained from D by attaching α to $\beta_2, ..., \beta_k$ by darts, and also connecting γ_1 to β_1 by a dart, which we can represent pictorially by



Thus, F is obtained as a $(k-1)$st transvectant $\hat{D}^{(k-1)}$, where $\hat{D}$ is the graph obtained from D by connecting γ_1 to β_1. (Note that the application of Rule #2 is allowed, since both β_1 and γ_1 must have valence at least one in D in order that E_1 and E_2 be well-defined n-digraphs.) This proves the lemma in this special case; a straightforward induction will complete the proof in general.

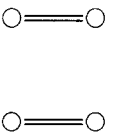
COROLLARY 16. *If D is any nonzero digraph, then the kth transvectant $D^{(k)}$ is equivalent modulo lower order transvectants to a positive numerical multiple of any one of its constituent digraphs.*

LEMMA 17. *If D is reducible, and one of the components of D has valence at least k, then $D^{(k)}$ is equivalent modulo lower order transvectants to a reducible digraph.*

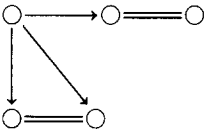
Proof. It suffices to note that at least one summand in $D^{(k)}$ has all the new darts connected to the indicated component of D, and that by Corollary 16, $D^{(k)}$ is equivalent to a nonzero multiple of each one of its summands.

However, Lemmas 15 and 17 do *not* imply that if D is any reducible digraph, then $D^{(k)}$ is equivalent modulo lower order transvectants to a reducible digraph, because all the summands might be forced to interconnect the two disconnected components of D. For example, in the case of a

binary cubic, consider the third transvectant of the square of the Hessian: $(H^2)^{(3)}$. The digraph representing H^2 is



Each component of this reducible digraph has valence two, so there is essentially only one summand in the transvectant $(H^2)^{(3)}$:



This digraph turns out to be reducible (see below), but it cannot be immediately ruled out on the basis of any of the preceding lemmas.

We can now outline Gordan's Method. At each step, we recursively construct a complete set $\mathscr{I}_m$ of "irreducible" digraphs with exactly m vertices whose corresponding covariants appear in a minimal Hilbert basis for the covariants of a binary form of degree n. The method is recursive, and can actually be fashioned into a proof of the Basis Theorem by demonstrating that the method terminates in finitely many steps (cf. [3, Chap. 6]). However, we will not complete the final details of the proof here, although we can rest assured that the method must terminate.

Step $\#1$. Let $\mathscr{I}_1$ consist of the single monatomic digraph, corresponding to the form itself.

Step $\#m$. To construct $\mathscr{I}_m$ knowing $\mathscr{I}_1, \ldots, \mathscr{I}_{m-1}$, we proceed as follows: Let $\mathscr{B}_{m-1}$ be the set of all digraphs D with $m-1$ vertices constructed by one of the following two rules:

(a) D is an ionic digraph in $\mathscr{I}_{m-1}$; i.e., D has positive valence, and so does not represent an invariant, or

(b) D is a reducible digraph, with exactly $m-1$ vertices, taking the form $D_1 \vee \cdots \vee D_l$ (disjoint union), where each irreducible component D_v is an ion of valence $0 < k_v < n$, lying in some $\mathscr{I}_{j_v}$, $2 \leqslant j_v \leqslant m-3$, with $j_1 + \cdots + j_k = m-1$. (In particular, $m \geqslant 5$ for this rule to be applicable.)

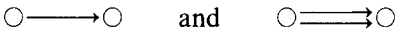
From the set $\mathscr{B}_{m-1}$ we construct a set $\mathscr{C}_m$ of digraphs on m vertices by taking *one* term in each possible transvectant $D^{(k)}$, $0 < k \leqslant n$, for each $D \in \mathscr{B}_{m-1}$, i.e., just one of all the possible digraphs which can be constructed by attaching a single vertex to D by k darts. After constructing $\mathscr{C}_m$,

one then uses the digraph rules to determine a subset $\mathscr{I}_m \subset \mathscr{C}_m$ of digraphs which do not differ by a reducible digraph. (It is at this stage that complications may arise, since it is sometimes quite complicated to recognize a reducible digraph!) In particular, using Lemma 17, one can immediately rule out transvectants $D^{(k)}$ of reducible digraphs constructed using rule (b) if any subcomponent $D_{j_1} \vee \cdots \vee D_{j_\mu}$, $\mu < l$, has valence k or more, i.e., $k_{j_1} + \cdots + k_{j_\mu} > k$. In particular, if a component has valence $k_{j_1} + \cdots + k_{j_\mu} \geqslant n$, then we can exclude D from consideration entirely. The method terminates when $\mathscr{I}_m$ consists only of saturated digraphs (invariants), and moreover, rule (b) does not lead to any irreducible transvectants for any higher m. The Hilbert Basis of covariants will consist of all the covariants corresponding to all the digraphs appearing in the sets $\mathscr{I}_j$, $j = 1, 2, \ldots, m$.

To illustrate Gordan's method and demonstrate the power of our "graphical algebra," we show how to construct a complete system of covariants for the binary quadratic, cubic, and quartic.

EXAMPLE 18. For a quadratic, we are working in $\mathscr{D}_2$, the space of 2-digraphs, so we can attach at most two darts to any given vertex. We begin with the digraph

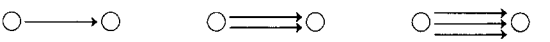
which represents the form Q itself. There are only two possible transvectants:



The first is trivial by Rule #1, and the second is the Hessian or twice the discriminant, which is an invariant. Lemmas 14 and 17 imply that we cannot get anything further by transvecting again, so we have shown that the only covariants of a binary quadratic are the form itself and its discriminant.

EXAMPLE 19. Turning to the binary cubic, we begin with Q, i.e.,

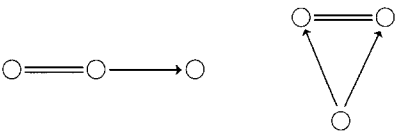
from which we can form three transvectants:



Two of these are trivial by Rule #1, the only nontrivial one being the Hessian

$H =$ 

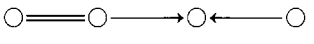
Therefore $\mathscr{I}_1 = \{Q\}$, while $\mathscr{I}_2 = \{H\}$. The only digraph in $\mathscr{B}_2$ is the Hessian, which has valence two, and so we can form the two further transvectants:

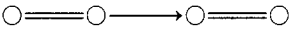


which are the digraphs in $\mathscr{C}_3$. The first digraph represents the covariant

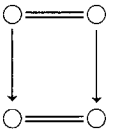
$$T = (Q, H) = \tfrac{1}{6}(Q_x \cdot H_y - Q_y \cdot H_x),$$

while the second is trivial by Rule #2. Thus $\mathscr{I}_3 = \{T\}$. Rule (b) in Gordan's Method does not apply, so $\mathscr{B}_3 = \{T\}$ also. Now T has valence three, so we can form three further transvectants. Representative summands are

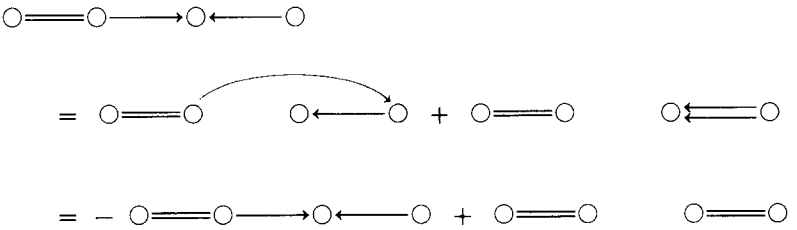




and



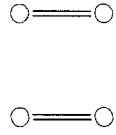
which are the digraphs in $\mathscr{C}_4$. The first is equivalent to $\frac{1}{2}H^2$ by Rule #2:



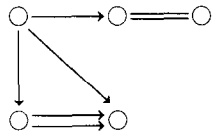
The second is obviously zero by Rule #1, while the third is the discriminant

$$\Delta = (Q, T)^{(3)}$$

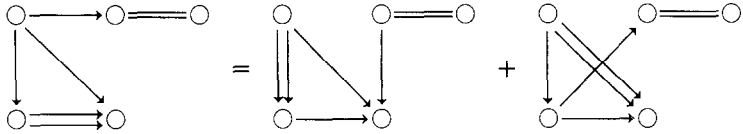
discussed in Section 5. Therefore $\mathscr{I}_4 = \{\Delta\}$. To form the next step, we see that rule (a) is no longer applicable since $\mathscr{I}_4$ consists only of an invariant. However, there is one further case from rule (b) which needs to be taken into account, namely the reducible digraph



corresponding to H^2. Note that each component has valence two, so we can form the possibly irreducible transvectant $(H^2)^{(3)}$:



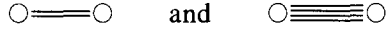
However, this covariant is trivial, since using Rule #3,



The two digraphs on the right each differ from the original digraph by one reversed arrow, so the equation has the form $D = -D - D$, hence $D = 0$. There are no more possible irreducible transvectants, and so Gordan's Method has terminated. A complete system of covariants for the binary cubic thus consists of the form Q, the covariants T and H, and the invariant Δ.

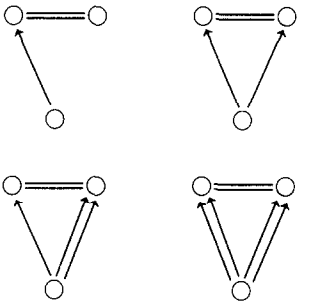
EXAMPLE 20. Finally, we outline how the same method produces the Hilbert basis of covariants for the binary quartic. We begin with $\mathscr{I}_1$, which, as always, consists of the only monatomic digraph

We can now form four transvectants, two of which are trivial, so $\mathscr{I}_2$ consists of

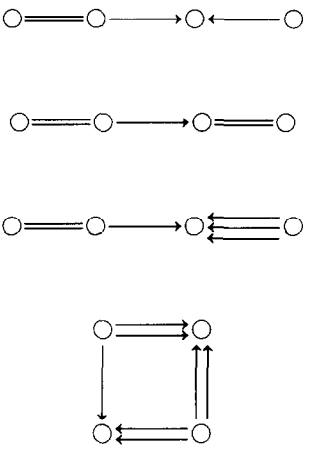


which correspond to the Hessian H and the invariant i of the quartic. Since we cannot get nontrivial transvectants from an invariant, we are left with

only the Hessian to work with, i.e., $\mathscr{B}_2 = \{H\}$. The possible transvectants are

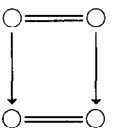


The first digraph represents the covariant T, the second and third are trivial, and the fourth represents the invariant j. Thus $\mathscr{I}_3$ consists of the digraphs for T and j. As j is an invariant, we can only get nontrivial transvectants from T. There are four possibilities:

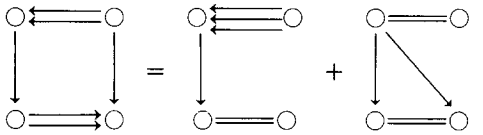


It is not difficult to see that all four are either trivial, or equivalent to reducible digraphs. There are no other possibilities for getting irreducible transvectants using either rule (a) or rule (b), so Gordan's method is finished, and we have proved that a basis for the covariants of the binary quartic consists of Q, H, T, i, and j.

Note the interesting phenomenon that

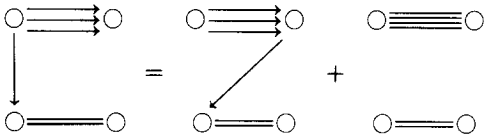


is an irreducible invariant for a cubic, i.e., considered as an element of $\mathscr{D}_3$, but is reducible for a quartic, i.e., when considered as an element of $\mathscr{D}_4$. This can be seen by the following: We first decompose



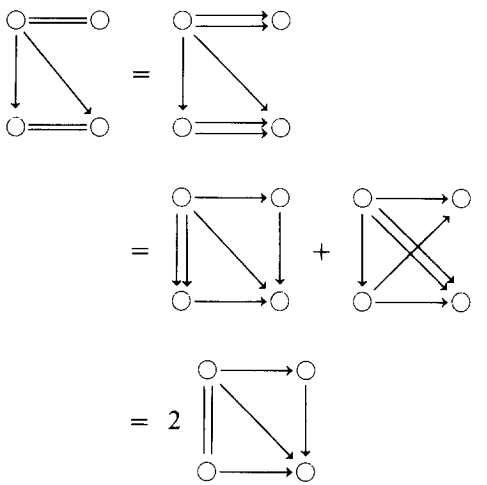
$$A = B + C$$

using Rule #2. (This step is *not* allowed for a cubic, since each vertex can have at most three darts terminating at it.) The first of these digraphs is easily seen to be reducible since



$$-B = B + i \cdot H$$

hence $B = -\frac{1}{2}iH$. As for the second, we first use Rule #3 on the two double bonds:

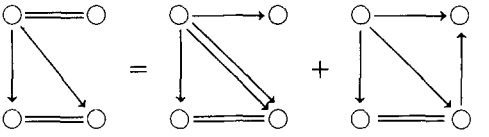


$$= 2$$

which we write as

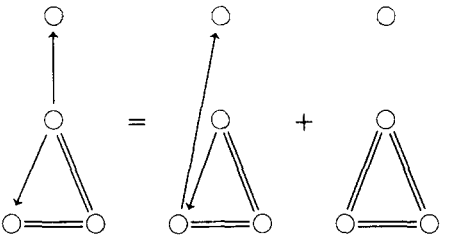
$$C = 2D.$$

On the other hand, using Rule #2, we see that



$$C = E - D$$

and the digraph E is reducible since



$$E = -E + Q \cdot j$$

so $E = \frac{1}{2} Q \cdot j$. Since

$$2D = C = E - D,$$

we see that

$$D = \tfrac{1}{3} E = \tfrac{1}{6} Q \cdot j,$$

and hence

$$C = \tfrac{1}{3} Q \cdot j.$$

Therefore, our original covariant can be written in terms of the basis covariants as

$$A = B + C = -\tfrac{1}{2} i \cdot H + \tfrac{1}{3} Q \cdot j.$$

We thus see that if a digraph D represents a reducible covariant for a binary form of degree n, it also represents a reducible covariant for any binary form of degree $m \geqslant n$; however, it may not remain reducible (even if it is defined) if $m < n$.

The examples presented here have, we hope, convinced the reader that the graphical representation of covariants is a powerful tool for effecting complicated algebraic manipulations in classical invariant theory. Exercises and results in the classic textbooks can now be recovered graphically with

a minimum of computational difficulty, and, we believe, much added insight. Finally, we remark that the method can be extended to the invariant theory of ternary or higher degree forms using hypergraph theory, although it loses some of its power and simplicity in the translation.

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