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Differential Invariants

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Abstract. This paper summarizes recent results on the number and characterization of differential invariants of transformation groups. Generalizations of theorems due to Ovsiannikov and to M. Green are presented, as well as a new approach to finding bounds on the number of independent differential invariants.

Consider a group of transformations acting on a jet space coordinatized by the independent variables, the dependent variables, and their derivatives. Scalar functions which are not affected by the group transformations are known as differential invariants. Their importance was emphasized by Sophus Lie, [**9**], who showed that every invariant system of differential equations, [**10**], and every invariant variational problem, [**11**], could be directly expressed in terms of the differential invariants. As such they form the basic building blocks of many physical theories, where one begins by postulating the invariance of the equations or the variational principle under a prescribed symmetry group. Lie also demonstrated, [**10**], how differential invariants could be used to integrate invariant ordinary differential equations, and succeeded in completely classifying all the differential invariants for all possible finite-dimensional Lie groups of point transformations in the case of one independent and one dependent variable. Lie's results were pursued by Tresse, [**18**], and, much later, Ovsiannikov, [**17**]. In this paper, I will summarize some recent, new results extending these earlier classification theorems, which were discovered in the course of writing the forthcoming book, [**16**]. Space considerations preclude the inclusion of proofs and significant examples here. It is worth remarking that, surprisingly, the complete classification of differential invariants for many of the groups of physical importance, including the general linear, affine, conformal, and Poincaré groups, does not yet seem to be known!

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Consider the space† $M = X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$ whose coordinates represent our independent variables $x = (x^1, \ldots, x^p) \in X$ and dependent variables $u = (u^1, \ldots, u^q) \in U$. Let J^n denote the associated jet bundle of order n, whose coordinates $(x, u^{(n)})$ represent the independent variables and the derivatives $u^\alpha_I = \partial^k u / \partial x^{i_1} \cdots \partial x^{i_k}$, $\alpha = 1, \ldots, q$, $1 \leq i_\nu \leq p$, of the dependent variables of orders $0 \leq k = \#I \leq n$. Thus, $\dim \mathrm{J}^n = p + q^{(n)}$, where $q^{(n)} = q\binom{p+n}{n}$. The number of derivative coordinates of order exactly n is denoted by

$$q_n = \dim \mathrm{J}^n - \dim \mathrm{J}^{n-1} = q^{(n)} - q^{(n-1)} = q p_n = q\binom{p+n-1}{n}. \tag{1}$$

A smooth function (or section) $u = f(x)$ from X to U has n^{th} *prolongation* (or n-jet) $u^{(n)} = \mathrm{pr}^{(n)} f(x)$, which is the section of J^n given by $u^\alpha_I = \partial_I f^\alpha(x)$.

A differential one-form θ on the jet space J^n is called a *contact form* if it is annihilated by all prolonged functions. It is easy to prove that every contact form on J^n is a linear combination of the basic contact forms

$$\theta^\alpha_I = du^\alpha_I - \sum_{k=1}^{p} u^\alpha_{I,k}\, dx^k, \qquad \alpha = 1, \ldots, q, \quad 0 \leq \#I < n. \tag{2}$$

We call $\#I$ the *order* of the contact form θ^α_I, so the contact forms on J^n have orders at most $n-1$. A one-form on J^n is called *horizontal* if it annihilates all vertical tangent directions, *i.e.*, just involves the dx^i's.

A smooth, real-valued function $F\colon \mathrm{J}^n \to \mathbb{R}$ is called a *differential function* of order n. Note that any differential function $F(x, u^{(n)})$ of order n automatically defines a differential function on any higher order jet space merely by treating the coordinates $(x, u^{(n)})$ of J^n as a subset of the coordinates $(x, u^{(n+k)})$ of J^{n+k} — this is the same as composing F with the natural projection $\pi^{n+k}_n\colon \mathrm{J}^{n+k} \to \mathrm{J}^n$. In the sequel, we will not distinguish between F and $F \circ \pi^{n+k}_n$. Given a differential function $F\colon \mathrm{J}^n \to \mathbb{R}$, its differential is the one -form

$$dF = \sum_{i=1}^{p} \frac{\partial F}{\partial x^i}\, dx^i + \sum_{\alpha=1}^{q} \sum_{\#I \leq n} \frac{\partial F}{\partial u^\alpha_I}\, du^\alpha_I.$$

On the next higher order jet space J^{n+1}, we can uniquely decompose dF into a horizontal one-form plus a contact form. The horizontal component is called the *total differential* of F, and given by $DF = \sum_{i=1}^{p} D_i F\, dx^i$, where D_i denotes the total derivative with respect to x^i.

Definition 1. A local diffeomorphism $\Psi\colon \mathrm{J}^n \to \mathrm{J}^n$ defines a *contact transformation* of order n if it preserves the space of contact forms, meaning that if θ is any contact form on J^n, then $\Psi^* \theta$ is also a contact form.

† More generally, M can be a vector or fiber bundle, or even an arbitrary smooth manifold, [**14**]. However, as all our considerations are local, there is no loss in generality in restricting our attention to open subsets of Euclidean space.

In particular, any point transformation $\Phi: M \to M$ defines a zero$^{\text{th}}$ order contact transformation. Contact transformations act on functions by point-wise transforming their prolonged graphs. If $\Psi: \mathrm{J}^n \to \mathrm{J}^n$ defines an n^{th} order contact transformation, then, for any $k \geq 0$, there is a uniquely defined $(n+k)^{\text{th}}$ order contact transformation $\mathrm{pr}^{(n+k)}\Psi: \mathrm{J}^{n+k} \to \mathrm{J}^{n+k}$, the k^{th} *prolongation* of Ψ, which projects back down to Ψ. Bäcklund's Theorem, [**3**], demonstrates that, except in the case of a single dependent variable, all contact transformations are merely prolonged point transformations.

Theorem 2. *If the number of dependent variables is greater than one, $q > 1$, then every contact transformation is the prolongation of a point transformation $\Phi: M \to M$. If $q = 1$, then there are first order contact transformations which do not come from point transformations, but every n^{th} order contact transformation is the $(n-1)^{\text{st}}$ prolongation of a first order contact transformation $\Psi: \mathrm{J}^1 \to \mathrm{J}^1$.*

By a *transformation group* G, then, we mean either a local Lie group of point transformations acting on (an open subset of) the space M of independent and dependent variables, or, in the single dependent variable case, a local Lie group of contact transformations acting on (an open subset of) the first jet space J^1. (In this paper, a transformation group is always a finite-dimensional Lie group.) To keep the notation uniform, we let $G^{(n)}$ denote the associated prolonged group action (by contact transformations) on the jet space J^n, so that $G^{(n)} = \mathrm{pr}^{(n)}G$ in the case of point transformations, whereas $G^{(n)} = \mathrm{pr}^{(n-1)}G$ in the case of first order contact transformations. (In the latter case we assume that $n \geq 1$.) The (prolonged) infinitesimal generators of $G^{(n)}$ form a Lie algebra $\mathfrak{g}^{(n)}$ of vector fields $\mathbf{v}^{(n)}$ on J^n satisfying the same commutation relations as the Lie algebra $\mathfrak{g}$ of G, and determined by the standard prolongation formula, *cf.* [**15**].

Definition 3. Let G be a group of point or contact transformations. A *differential invariant* is a real-valued function $I: \mathrm{J}^n \to \mathbb{R}$ which satisfies $I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$ for all $g^{(n)} \in G^{(n)}$, and all $(x, u^{(n)}) \in \mathrm{J}^n$ where the prolonged transformation $g^{(n)} \cdot (x, u^{(n)})$ is defined.

As usual, the differential invariant I may only be defined on an open subset of the jet space, although we shall still write $I: \mathrm{J}^n \to \mathbb{R}$ to indicate its jet space of definition. Differential invariants (of connected groups) are most easily determined using infinitesimal methods.

Proposition 4. *A function $I: \mathrm{J}^n \to \mathbb{R}$ is a differential invariant for a connected transformation group G if and only if $\mathbf{v}^{(n)}(I) = 0$ for every prolonged infinitesimal generator $\mathbf{v}^{(n)} \in \mathfrak{g}^{(n)}$.*

In applications, the determination of a complete set of functionally independent differential invariants of a given group action is of significant importance. Functional independence is guaranteed if their differentials are linearly independent: $dF_1 \wedge \cdots \wedge dF_k \neq 0$. Since, as noted above, any lower order differential invariant $I(x, u^{(k)})$, $k < n$, is automatically an n^{th} order differential invariant, it will be important to distinguish differential invariants of order exactly n from lower order differential invariants. We will call a set of differential functions on J^n *strictly independent* if, as functions of the n^{th} order derivative coordinates alone, they are functionally independent. The functions $F_1, \ldots, F_k$ are *strictly independent* if their n^{th} order differentials, given (intrinsically) by

$$d_n F = \sum_{\alpha=1}^{q} \sum_{\# I = n} \frac{\partial F}{\partial u_I^\alpha} \, du_I^\alpha, \tag{3}$$

are linearly independent at each point: $d_n F_1 \wedge d_n F_2 \wedge \cdots \wedge d_n F_r \neq 0$. Note that, in particular, this implies that none of the F's, or any function thereof, can be of order strictly less than n.

In order to study the differential invariants of a transformation group, a more detailed knowledge of the structure of the prolonged group actions is required. The following remarks all follow directly from Frobenius' Theorem, *cf.* [**15**]. Let G be an r-dimensional Lie group of transformations. Let s_n denote the maximal (generic) orbit dimension of the prolonged action $G^{(n)}$, so that $G^{(n)}$ acts semi-regularly on the open subset $V^{(n)} \subset \mathrm{J}^n$ consisting of all points contained in the orbits of maximal dimension. (If G acts analytically, then the subset $V^{(n)}$ is dense in J^n.) In the remainder of this paper, we shall restrict our attention to the subset $V^{(n)}$, thereby avoiding more delicate questions concerning singularities of the prolonged group actions. Let h_n denote the dimension of any isotropy subgroup $H_z^{(n)} = \{ g \mid g^{(n)} \cdot z = z \}$ for $z \in V^{(n)}$. (These isotropy subgroups are, in general, different, but all have the same dimension.) Then the orbit dimensions satisfy $s_n = r - h_n$. Locally, near any point $z \in V^{(n)}$, there are

$$i_n = p + q^{(n)} - s_n = p + q^{(n)} - r + h_n \tag{4}$$

functionally independent differential invariants of order at most n. Since each differential invariant of order less than n is included in this count, the integers i_n form a non-decreasing sequence: $i_0 \leq i_1 \leq i_2 \leq \cdots$. The difference

$$j_n = i_n - i_{n-1} = q_n - s_n + s_{n-1} = q_n + h_n - h_{n-1}, \tag{5}$$

cf. (1), will count the number of strictly independent n^{th} order differential invariants. For groups of point transformations, we set $j_0 = i_0$ to be the number of ordinary invariants; for contact transformation groups, where i_0 and s_0 are not defined, we set $j_1 = i_1$. Note that j_n cannot exceed q_n, the number of independent derivative coordinates of order n; this implies the elementary inequalities

$$i_{n-1} \leq i_n \leq i_{n-1} + q_n. \tag{6}$$

If $\mathcal{O}^{(n)} \subset \mathrm{J}^n$ is any orbit of $G^{(n)}$, then, for any $k < n$, its projection $\pi_k^n(\mathcal{O}^{(n)}) \subset \mathrm{J}^k$ is an orbit of the k^{th} prolongation $G^{(k)}$. Therefore, the maximal orbit dimension s_n of $G^{(n)}$ is a *nondecreasing* function of n, bounded by $r = \dim G$:

$$s_0 \leq s_1 \leq s_2 \leq \cdots \leq r. \tag{7}$$

On the other hand, since the orbits cannot increase in dimension any more than the increase in dimension of the jet spaces themselves, we have the the elementary inequalities

$$s_{n-1} \leq s_n \leq s_{n-1} + q_n, \tag{8}$$

governing the orbit dimensions. Note that, in view of equations (1) and (4), the inequalities (8) are equivalent to those in (6). Condition (7) implies that the maximal orbit dimension eventually stabilizes, so that there exists an integer s such that $s_m = s$ for all m sufficiently large. In particular, if the orbit dimension is ever the same as that of G, meaning $s_n = r$ for some n, then $s_m = r$ for all $m \geq n$. We will call s the stable orbit dimension, and the minimal order n for which $s_n = s$ the *order of stabilization* of the group.

Example 5. Consider the three-parameter group action $(x,u) \mapsto (\lambda x + a, \lambda u + b)$, $(x,u) \in M \simeq \mathbb{R}^2$, generated by the vector fields $\partial_x, \partial_u, x\partial_x + u\partial_u$. There are no ordinary invariants since the group is transitive on $M = \mathbb{R}^2$. Furthermore, all three vector fields happen to coincide with their first prolongations, and hence there is one independent first order differential invariant, namely u_x. The second prolongations are $\partial_x, \partial_u, x\partial_x + u\partial_u - u_{xx}\partial_{u_{xx}}$, and hence there are no differential invariants of (strictly) second order. There is a single third order differential invariant, namely $u_{xx}^{-2} u_{xxx}$, a single fourth order invariant, $u_{xx}^{-3} u_{xxxx}$, and, in general, a single n^{th} order differential invariant $u_{xx}^{-n-1} D_x^n u$. Therefore, the number of strictly independent differential invariants is given by $j_0 = j_2 = 0$, $j_1 = j_3 = \cdots = j_n = 1$, $n \geq 3$. Therefore, $i_0 = 0$, $i_1 = i_2 = 1$, $i_3 = 2, \ldots, i_n = n-1$, which implies that the maximal orbit dimensions are $s_0 = s_1 = 2$, $s_2 = s_3 = \cdots = 3 = \dim G$. We observe that the orbit dimensions "pseudo-stabilized" at order 0 since $s_0 = s_1$, but the correct order of stabilization is $n = 2$. More generally, the r-dimensional group generated by $\partial_x, \partial_u, x\partial_u, \ldots, x^{r-3}\partial_u, x\partial_x + (r-2)u\partial_u$ has orbit dimensions $s_0 = 1$, $s_1 = 2, \ldots, s_{r-3} = s_{r-2} = r-1$, $s_{r-1} = s_r = \cdots = r$, so the orbit dimensions pseudo-stabilize at order $r-3$.

A transformation group acts *effectively* if different group elements have different actions, so that $g \cdot x = h \cdot x$ for all $x \in M$ if and only if $g = h$. The *global isotropy subgroup* $G_M = \{ g \mid g \cdot x = x \text{ for all } x \in M \}$, which is a closed normal subgroup of G, measures the "effectiveness" of the action of G in the sense that G acts effectively if and only if $G_M = \{e\}$ is trivial. If G does not act effectively, we can replace it by the quotient group G/G_M, which does act effectively on M in essentially the same way as G itself. Thus, there is no loss in generality in assuming that all our group actions are (locally) effective. A Lie group G is said to act *locally effectively* if the global isotropy group G_M is a discrete subgroup of G, in which case G/G_M has the same dimension (and the same Lie algebra) as G. Remarkably, the local effectiveness of a group action is characterized by its stable orbit dimension, [**17**].

Theorem 6. *A transformation group G acts locally effectively if and only if its dimension is the same as its stable orbit dimension, so that $s_m = r = \dim G$ for all m sufficiently large.*

The basic method for constructing a complete system of differential invariants of a given transformation group is to use invariant differential operators. A differential operator is said to be G-invariant if it maps differential invariants to higher order differential invariants, and thus, by iteration, produces hierarchies of differential invariants of arbitrarily large order. For n sufficiently large, we can guarantee the existence of sufficiently many such differential operators so as to completely generate all the higher order independent differential invariants of the group by successively differentiating lower order differential invariants. Thus, a complete description of all the differential invariants is provided by a collection of low order "fundamental" differential invariants along with the requisite invariant differential operators.

Definition 7. A differential one-form ω on J^n is called *contact-invariant* under a transformation group G if and only if, for every $g \in G$, we have $(g^{(n)})^*\omega = \omega + \theta$ for some contact form $\theta = \theta_g$.

The infinitesimal criterion for contact-invariance is that the Lie derivative of the form with respect to the prolonged infinitesimal generators vanish: $\mathbf{v}^{(n)}(\omega) = 0$ for all $\mathbf{v}^{(n)} \in \mathfrak{g}^{(n)}$. Contact forms are trivially contact-invariant, so only the horizontal contact-invariant forms are of interest. If $I(x, u^{(n)})$ is any n^{th} order differential invariant, its total differential $DI = \sum D_j I \, dx^j$ is a contact-invariant one-form on J^{n+1}. The construction of high order differential invariants is facilitated by the existence of enough horizontal contact-invariant forms.

Definition 8. Let G be a transformation group acting on a space having p independent variables. An n^{th} *order differential invariant coframe* is a collection of $p = \dim X$ independent contact-invariant horizontal one-forms $\omega^1, \ldots, \omega^p$, defined locally on the jet space J^n.

Differential invariant coframes are the jet space counterparts of the coframes from differential geometry that form the foundation of the Cartan equivalence method, [**4**], [**5**]. If $F(x, u^{(m)})$ is any differential function, we can rewrite its total differential in terms of the coframe,

$$DF = \sum_{k=1}^{p} \mathcal{D}_k F \, \omega^k. \tag{9}$$

The resulting "coframe differential operators" $\mathcal{D}_k$ are G-invariant differential operators:

Proposition 9. *If $I(x, u^{(m)})$ is any differential invariant of order m, then $\mathcal{D}_k I$ is a differential invariant of order $\leq \max\{n, m+1\}$.*

In local coordinates, if $\omega^i = \sum_k P^i_k(x, u^{(n)}) dx^k$, then $\mathcal{D}_k = \sum_i Q^i_k(x, u^{(n)}) D_i$, where $\mathbf{Q} = \big(Q^i_j(x, u^{(n)})\big) = \mathbf{P}^{-T}$. In particular, if the $\omega^i = DI_i$ are obtained from functionally independent differential invariants $I_1, \ldots, I_p$, then $\mathbf{P} = \big(D_j I_i(x, u^{(n)})\big)$ is their total Jacobian matrix.

Let us first consider the case $p = 1$, so there is a single independent variable x. A differential invariant coframe is just a nonvanishing contact-invariant one-form $\omega = P(x, u^{(n)})\, dx$. The associated invariant differential operator $\mathcal{D} = (1/P) D_x$ maps a differential invariant J to the differential invariant $\mathcal{D}J = D_x J / P$. In particular, if $I(x, u^{(n)})$ is any (non-constant) differential invariant, the corresponding invariant differential operator is $\mathcal{D} = (D_x I)^{-1} D_x$, which maps J to $dJ/dI = D_x J / D_x I$. Therefore, starting from a pair of differential invariants (or, more generally, a single differential invariant and a contact-invariant horizontal one-form) we construct an infinite sequence of higher and higher order differential invariants $\mathcal{D}^k J$, $k = 0, 1, 2, \ldots$. The functional independence of the resulting differential invariants is guaranteed by the following lemma.

Lemma 10. *Suppose $J_1, \ldots, J_r$ are strictly independent n^{th} order differential invariants, and I is either a differential invariant of order strictly less than n, or an n^{th} order differential invariant which is strictly independent of the J_ν's. Let $\mathcal{D} = (D_x I)^{-1} D_x$ denote the invariant differential operator associated with DI. Then the differential functions $\mathcal{D}J_1, \ldots, \mathcal{D}J_r$ are strictly independent $(n+1)^{\text{st}}$ order differential invariants.*

Theorem 11. *Suppose that G is a group of point or contact transformations acting on a space M having one independent variable and q dependent variables. Then, locally, there exist $q + 1$ fundamental, independent differential invariants $I, J_1, \ldots, J_q$, such that every differential invariant can be written as a function of these differential invariants and their derivatives $\mathcal{D}^m J_\nu$, where $\mathcal{D} = (D_x I)^{-1} D_x$ is the invariant differential operator associated with the first differential invariant I.*

Both results can be readily generalized by using a contact-invariant one-form instead of DI.

Theorem 11 has some important consequences governing the order of stabilization n of a transformation group. As we saw in Example 5, it is possible for the orbit dimension to pseudo-stabilize at some lower order, meaning that $s_k = s_{k+1} < s_{k+2}$ for some $k < n$. First, we note that a pseudo-stabilization of the orbit dimensions can only occur if the orbit dimension is rather high.

Theorem 12. *Suppose that, for some $n \geq 0$, the maximal orbit dimensions of the prolonged group actions satisfy $s_{n-1} < s_n = s_{n+1} \leq q^{(n)}$. Then n is the order of stabilization of G.*

The next result, which follows directly from Theorem 12, demonstrates that there can be at most one such pseudo-stabilization.

Theorem 13. *Suppose that the maximal orbit dimensions of the prolonged group actions satisfy $s_k = s_{k+1}$ and, also, $s_n = s_{n+1}$ for some $n > k$. Then $s_m = s_n$ for all $m \geq n$.*

Both of these results are valid as stated in the general case of several independent variables and several dependent variables — see below. Theorem 13 provides a significant strengthening of Ovsiannikov's stabilization theorem, [**17**; p. 313], which states that if $s_{n-1} < s_n = s_{n+1} = s_{n+2}$, then the orbit dimension stabilizes at order n. Indeed, even in this case, the proof of Theorem 13 in [**16**] is new and much more direct than that of Ovsiannikov.

As a consequence of Theorems 12 and 13, there are essentially only two possibilities for the orders of the fundamental differential invariants, as described in Theorem 11, of a group of transformations acting on a space with just one independent variable. Assume G is an r-dimensional group acting locally effectively, and let n denote the order of stabilization. Then either

a) The fundamental differential invariants have order at most $n+1$. In this case, there exist $q+1$ differential invariants, I, of order $\le n$, $J_1, \ldots, J_{q-1}$ of order $\le n+1$, and J_q of order $= n+1$. In this case $\dim G = r \le 1 + (n+1)q$.

b) The fundamental differential invariants have order at most $n+2$. In this case, there exist $q+1$ differential invariants $I, J_1, \ldots, J_{q-1}$, all of order $= n+1$, and J_q of order $= n+2$. This case can *only* occur if the dimension of G equals $r = 1 + (n+1)q$.

Consequently, in the case of a single independent variable, the order of stabilization n of an r-dimensional locally effective group action obeys the inequalities

$$\frac{r-1}{q} - 1 \le n \le r - 1. \tag{10}$$

In his study of the differential invariants of curves in a homogeneous space, [**6**], M. Green discovered a striking formula relating the number of fundamental differential invariants to the dimensions of the isotropy subgroups of the prolonged group action. Our Theorem 11 implies that Green's results are, in fact, valid for completely general transformation groups on spaces with one independent variable! Let k_n denote the number of strictly independent fundamental differential invariants of order n, *i.e.*, those differential invariants which are not expressed in terms of differentiated invariants of any lower order. Since, according to Lemma 10, the differentiated invariants coming from strictly independent invariants are themselves strictly independent, these numbers satisfy $k_n = j_n - j_{n-1}$ provided either $i_{n-2} \ge 1$, so there is at least one lower order invariant to provide the require invariant differential operator, or, more generally, there exists a contact-invariant one-form of order at most $n-1$.

Theorem 14. *Let G be a transformation group. Then the number k_n of fundamental differential invariants of order n is given in terms of the minimal dimension h_n of the isotropy subgroups of $G^{(n)}$ according to*

$$k_n = \begin{cases} h_n - 2h_{n-1} + h_{n-2} + 1, & \text{if } i_{n-2} = 0,\ i_{n-1} > 0, \\ h_n - 2h_{n-1} + h_{n-2}, & \text{otherwise.} \end{cases} \tag{11}$$

Equation (11) *is valid for all $n \ge 0$ provided we set $i_n = 0$ and $h_n = r - 1 - (n+1)q$ whenever the action $G^{(n)}$ is not defined, i.e., for $n = -2, -1$, and, in the case of a contact transformation group, $n = 0$.*

Example 15. Let G be an r-dimensional Lie group and $H \subset G$ a closed subgroup of dimension s. Let $M = \mathbb{R} \times G/H$, so that the functions $u = f(x)$ are described by curves in the homogeneous space G/H. The group G acts on M by the Cartesian product of the trivial action on the independent variable $x \in \mathbb{R}$ and its usual action via left multiplication on G/H. In this case, $h_0 = s = \dim H$, and $i_0 = j_0 = k_0 = 1$, since there is a single ordinary invariant x, with consequential invariant differential operator D_x. Formula (11) implies M. Green's main result that there are $k_n = h_n - 2h_{n-1} + h_{n-2}$ fundamental differential invariants of order $n \geq 2$. For $n = 1$, we have $k_1 = h_1 - 2h_0 + h_{-1} + 1 = h_1 - 2s + r$ since $i_{-1} = 0$, $i_0 = 1$, while $h_{-1} = r - 1$ according to our convention. (Green sets $h_{-1} = r$, but this does not conform with our general formula.) See [**6**] for a wide variety of applications and explicit examples, including affine, projective, and conformal geometry.

Let us now specialize even further, to the case $p = q = 1$, so there is just one independent and one dependent variable. Here $q_n = 1$ for all n, so (6) implies that, for each $n \geq 1$, there is at most one independent differential invariant of order $= n$, *i.e.*, $j_n \leq 1$. Theorem 11 implies that there are precisely two fundamental differential invariants $I(x, u^{(s)})$ and $J(x, u^{(t)})$, having orders $0 \leq s < t$ respectively. (Here we are leaving aside the trivial case when $G = \{e\}$ acts trivially on M, where there are two independent zero$^{\text{th}}$ order invariants, namely x and u, and every differential function is a differential invariant.) Every other differential invariant can be written in terms of I, J, and the differentiated invariants $\mathcal{D}^m J = d^m J/dI^m$, so that $j_s = 1$, and $j_m = 1$ for every $m \geq t$. Equation (5) implies that the orbit dimensions and number of invariants are given by

$$\begin{array}{lll} s_k = k+2, & i_k = 0, & k \leq s-1, \\ s_k = k+1, & i_k = 1, & s \leq k \leq t-1, \\ s_k = t, & i_k = k-t+2, & k \geq t. \end{array} \tag{12}$$

Assuming G acts locally effectively, we conclude that the second fundamental differential invariant J necessarily has order $t = r = \dim G$. There are three subcases: If the first fundamental invariant I has order $s = 0$, the group acts intransitively on M, the orbit dimension stabilizes at order $r - 1$, and there is one ordinary invariant (of order 0) and one r^{th} order differential invariant. At the other extreme, if $s = r - 1$, then the group has fundamental differential invariants of orders $r - 1$ and r, and the orbit dimension stabilizes at order $r - 2$. The intermediate cases $0 < s < r - 1$ are when the orbit dimension pseudo-stabilizes at order s, and finally stabilizes at order $r - 1$. Thus, our methods provide rather detailed information on the possible orbit dimensions of prolonged group actions in the single variable case. However, even this is not as detailed as possible. Lie, [**8**], (see also [**12**]) completely classified the Lie groups of both point and contact transformations acting on a two-dimensional complex manifold. Moreover, in [**10**] he determined the differential invariants for each of the point transformation groups. (Unfortunately, I have not yet found where (if anywhere) Lie classified the differential invariants of the contact transformation groups.) Inspecting Lie's tables, we are led to the following remarkable result.

Theorem 16. *Let G be a locally effective r-dimensional Lie group of point transformations acting on $M \simeq \mathbb{R}^2$. Then G has fundamental differential invariants $I(x, u^{(s)})$ and*

$J(x,u^{(r)})$ having orders $s<r$. Moreover, $s=r-1$ unless either a) G acts intransitively, in which case $s=0$, or b) the prolonged orbit dimensions pseudo-stabilize, in which case $s=r-2$, and the pseudo-stabilization occurs at order $r-3$. In fact, the orbit dimensions pseudo-stabilize if and only if the group action is equivalent, under a change of variables, to the r-dimensional Lie group action described in Example 5!

We now discuss generalizations of these results in the case of several independent variables. We have already seen how to constuct a differential invariant coframe and the consequent invariant differential operators $\mathcal{D}_1,\ldots,\mathcal{D}_p$. To proceed further, we must find an independence result for the differentiated invariants similar to that in Lemma 10. The multi-variable case, though, is more complicated since the strict independence of n^{th} order differential invariants $J_1,\ldots,J_r$ does not necessarily imply the independence of the differentiated invariants $\mathcal{D}_i J_\nu$. A new approach to this problem relies on a combinatorial theorem proved by Macaulay, [**13**], in his study of the Hilbert function of an algebraic polynomial ideal.†

Lemma 17. *Let $p\geq 1$ be an integer. Then any nonnegative integer $r\in\mathbb{N}$ can be uniquely written in the form*

$$r=\binom{k_1+p-1}{p}+\binom{k_2+p-2}{p-1}+\cdots+\binom{k_s+p-s}{p-s+1}, \tag{13}$$

where $k_1\geq k_2\geq\cdots\geq k_s\geq 1$ form a non-increasing sequence of positive integers with $1\leq s\leq p$.

Write $k_p(r)=(k_1,\ldots,k_s)$ for the integer sequence associated with $r\in\mathbb{N}$. Define the function $\mu_p\colon\mathbb{N}\to\mathbb{N}$ which takes an integer r, represented by the sequence $k_p(r)=(k_1,\ldots,k_s)$, to the integer $\mu_p(r)$ which satisfies $k_p(\mu_p(r))=(k_1+1,k_2+1,\ldots,k_s+1)$. Macaulay's Theorem provides lower bounds for the dimensions of the homogeneous§ components of polynomial ideals using the functions μ_p.

Theorem 18. *Let $\mathcal{I}\subset\mathbb{R}[x_1,\ldots,x_p]$ be a homogeneous polynomial ideal in p variables. Let $d_n=\dim\mathcal{I}^{(n)}$ be the dimension of the set $\mathcal{I}^{(n)}=\{P\in\mathcal{I}\mid P(\lambda x)=\lambda^n P(x)\}$ of polynomials of degree n in $\mathcal{I}$. (Note that, by homogeneity, $\mathcal{I}=\bigoplus_{n\geq 0}\mathcal{I}^{(n)}$.) Then $d_{n+1}\geq\mu_p(d_n)$.*

Given $q\geq 1$, we define $\mu^n_{p,q}\colon\mathbb{N}\to\mathbb{N}$ as follows: for $r\in\mathbb{N}$, we write $r=sp_n+t$, where s is the quotient, and t the remainder, when r is divided by $p_n=\binom{p+n-1}{n}$. Then $\mu^n_{p,q}(r)=sp_{n+1}+\mu_p(t)$. Note, in particular, $\mu^n_{p,q}(q_n)=q_{n+1}$. We can now state the basic (new) inequality for the number of differential invariants.

† In more recent years, Macaulay's theorem has been considerably generalized in the combinatorial theory of extremal multi-sets, and forms a special case of the Kruskal-Katona Theorem, *cf.* [**7**].

§ Macaulay also extends this result to nonhomogeneous ideals.

Theorem 19. *Let G be a transformation group acting on a space with p independent variables and q dependent variables. Suppose $\mathcal{D}_1, \ldots, \mathcal{D}_p$ form a complete set of invariant differential operators coming from a differential invariant coframe of order n or less. Suppose $J_1, \ldots, J_r$ are strictly independent n^{th} order differential invariants. Then the set of differentiated invariants $\mathcal{D}_i J_\nu$, $i = 1, \ldots, p$, $\nu = 1, \ldots, r$, contains at least $\mu^n_{p,q}(r)$ strictly independent $(n+1)^{\text{st}}$ order differential invariants. In particular, if there are a maximal number of strictly independent n^{th} order differential invariants, $J_1, \ldots, J_{q_n}$, then the set of differentiated invariants $\mathcal{D}_i J_\nu$, $i = 1, \ldots, p$, $\nu = 1, \ldots, q_n$, contains a complete set of q_{n+1} strictly independent $(n+1)^{\text{st}}$ order differential invariants.*

Theorem 20. *Suppose that G is a group of point or contact transformations. Let n denote the order of stabilization of the group action. Then there exists a differential invariant coframe $\omega^1, \ldots, \omega^p$ on J^{n+2}, with corresponding invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$, and differential invariants $J_1, \ldots, J_m$, of order at most $n+2$, such that, locally, every differential invariant can be written as a function of these differential invariants and their derivatives $\mathcal{D}_{j_1} \cdots \mathcal{D}_{j_\kappa} J_\nu$, $\kappa \geq 0$, $\nu = 1, \ldots, m$.*

Note that it is not asserted (in contrast to the single variable case in Theorem 11) that the differentiated invariants are necessarily functionally independent. Indeed, the classification of syzygies, or functional dependencies, among the differentiated invariants is an interesting problem that, as far as I know, has not been investigated in any degree of generality. Theorem 20 states that if the orbit dimension stabilizes at order n, then all the differential invariants can be obtained from those of order at most $n+2$ by applying the invariant differential operators. Moreover, if the stable orbit dimension satisfies $r = s_n \leq q^{(n)}$, and so there are at least p independent differential invariants $I_1, \ldots, I_p$ of order n, then the differential invariant coframe $DI_1, \ldots, DI_p$ lives on J^{n+1}, and, moreover, the fundamental differential invariants have orders at most $n+1$. Finally, Theorem 20 also implies that our earlier stabilization and pseudo-stabilization Theorems 12 and 13 remain valid in the general multi-dimensional context. I do not know of any interesting examples of multi-dimensional transformation groups whose orbit dimensions pseudo-stabilize. Moreover, I do not know precisely how many fundamental differential invariants are required, although the dimension bounds of Theorem 19 should provide some useful estimates. Indeed, as remarked in [**6**], the generalization of Theorem 14 to the multi-dimensional case would be of great importance for studying, for example, the differential invariants of surfaces and higher dimensional submanifolds of homogeneous spaces.

Finally, for completeness, we recall how differential invariants are used to characterize systems of differential equations and variational problems which admit the transformation group as a symmetry group. The following results go back to Lie, [**9**], [**11**]; see also [**15**].

Theorem 21. *Let G be a transformation group, and let $I_1, \ldots, I_k$, $k = i_n$, be a complete set of functionally independent n^{th} order differential invariants. A system of differential equations admits G as a symmetry group if and only if (restricted to the subset $V^{(n)}$) it can be rewritten in terms of the differential invariants:*

$$\Delta_\nu(x, u^{(n)}) = F_\nu(I_1(x, u^{(n)}), \ldots, I_k(x, u^{(n)})) = 0, \qquad \nu = 1, \ldots, l. \tag{14}$$

Theorem 22. *Let G be a transformation group, and $\omega^1, \ldots, \omega^p$ a differential invariant coframe . A variational problem admits G as a variational symmetry group if and only if it has the form*

$$\mathcal{L}[u] = \int L(x, u^{(n)})\, dx = \int F\big(I_1(x, u^{(n)}), \ldots, I_k(x, u^{(n)})\big)\ \omega^1 \wedge \cdots \wedge \omega^p, \qquad (15)$$

where $I_1, \ldots, I_k$ are functionally independent differential invariants of G.

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