Differential Invariants of Sub-submanifolds

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The following question was raised.

Suppose G is a Lie group acting on $\mathbb{R}^n$. Let $S \subset \mathbb{R}^n$ be a submanifold with differential invariant algebra generated by I_ν. Suppose we know $S \subset N$ where $N \subset \mathbb{R}^n$ is a submanifold of higher dimension. What can we say about the differential invariants of S?

Let's restrict attention to curves $C \subset \mathbb{R}^n$, although the methods should work in general. To avoid having to deal with the reparametrization invariance, let's assume the curve is (locally) given by the graph of a function $C = \{ u = f(x) \}$ where $(x, u) = (x, u^1, \ldots, u^{n-1})$ are the coordinates on $\mathbb{R}^n$.

Now let's assume we have constructed a moving frame for the jets of curves. This is a situation where it helps to think of the moving frame normalizations or choice of cross-section as placing the submanifold (curve) C into normal form $\widehat{C} = g \cdot C$, as discussed in [**4**]. In other words, we use group transformation to normalize certain coefficients in the Taylor expansion of the curve. The unnormalized Taylor coefficients are then the differential invariants, and their general formulas can be obtained using the "all-powerful" recurrence formulae, [**1**, **4**].

Now suppose the larger submanifold N containing C is given implicitly by a system of equations

$$F_\nu(x, u) = 0. \tag{1}$$

Normalizing C as above using the group transformation g specified by the moving frame effectively "unnormalizes" N to produce the transformed submanifold $\widehat{N} = g \cdot N$, which is given by a system of equations of the form

$$\widehat{F}_\nu(g, x, u) = 0, \tag{2}$$

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that the normal form of the curve must satisfy since $\widehat{C} \subset \widehat{N}$. (Here it is important that we not substitute the moving frame formulas for g just yet. In fact, in the example, we don't even need to know them, and this procedure will work in general.) Now we can successively differentiate (2) to obtain a series of equations in the jet space that the jets of the curve must satisfy; for example at order 1 we have

$$\frac{\partial \widehat{F}_\nu}{\partial x} + \sum_{i=1}^{n-1} \frac{\partial \widehat{F}_\nu}{\partial u^i}\, u^i_x = 0, \tag{3}$$

and so on for the higher order derivatives (jets). Substituting the normal form (or normalized jet) of the curve into these produces a system of equations involving the group parameters g and the differential invariants appearing in the normalized Taylor expansion. Eliminating the group parameters g from these equations produces a set of equations relating the differential invariants of the curve C, which thus answers the original question.

Example 1. Let's see how this works in the case of a curve $C \subset \mathbb{R}^3$ contained in a sphere of radius r under the standard action of the (special) Euclidean group SE(3). We will use the notation and moving frame computations in [**4**; Example 3.1]. The classical moving frame, [**3**], relies on the normalization equations

$$x = 0, \qquad u = 0, \qquad v = 0, \qquad u_x = 0, \qquad v_x = 0, \qquad v_{xx} = 0, \tag{4}$$

which define a valid cross-section provided $u_{xx} \neq 0$, i.e., we are not at an inflection point. Transforming the curve into produces the resulting normal form Taylor expansions

$$u = \tfrac{1}{2}\kappa\, x^2 + \tfrac{1}{6}\kappa_s\, x^3 + \ \cdots\ , \qquad v = \tfrac{1}{6}\kappa\,\tau\, x^3 + \ \cdots\ ,$$

where additional higher order terms can be found in [**4**].

The transformed curve $\widehat{C}$ is contained in the transformed sphere $\widehat{S}$ of radius r. Since $\widehat{C}$ passes though the origin $x = u = v = 0$, which is the order 0 moving frame normalization obtained by translation, the same is true of $\widehat{S}$, which thus satisfies an equation of the form

$$(x-a)^2 + (u-b)^2 + (v-c)^2 = r^2, \quad \text{where} \quad a^2 + b^2 + c^2 = r^2. \tag{5}$$

Observe that there are only three parameters a, b, c here, which could be interpreted as the translation parameters for $g \in \mathrm{SE}(3)$. This is because a sphere has a three-dimensional rotational isotropy group $\mathrm{SO}(3) \subset \mathrm{SE}(3)$. Indeed, for more general submanifolds, one would have to figure out how the Euclidean group element acts on it, and there could be as many as 6 independent Euclidean group parameters for a general submanifold. Now we successively differentiate (5):

$$\begin{aligned} x - a + (u-b)\,u_x + (v-c)\,v_x &= 0,\\ 1 + (u-b)\,u_{xx} + u_x^2 + (v-c)\,v_{xx} + v_x^2 &= 0,\\ (u-b)\,u_{xxx} + 3\,u_x u_{xx} + (v-c)\,v_{xxx} + 3\,v_x v_{xx} &= 0, \end{aligned} \tag{6}$$

and so on, although owing to the isotropy of the sphere, this suffices for our purposes. (For more general submanifolds, one would need to prolong to higher order in order to generate enough equations so as to eliminate all the group parameters that appear.) Substituting the moving frame normalizations into (6), we find

$$-a = 0, \qquad 1 - b\,\kappa = 0, \qquad -b\,\kappa_s - c\,\kappa\,\tau = 0.$$

The second equation implies $\kappa \neq 0$. The third equation then implies that if $\tau = 0$, then $\kappa_s = 0$. If this holds at all points on the curve, it must be a plane curve, which implies that it must be a circle. In this case the z coordinate of the center of the sphere, namely c, is arbitrary, and its radius is bounded from below by $r = \sqrt{\kappa^{-2} + c^2} \geq 1/\kappa$. Otherwise, we can solve for

$$a = 0, \qquad b = \frac{1}{\kappa}, \qquad c = -\,\frac{\kappa_s}{\kappa^2\,\tau}. \tag{7}$$

Substituting back into (5) produces

$$r^2 = \frac{\kappa_s^2 + \kappa^2\,\tau^2}{\kappa^4\,\tau^2} \geq \frac{1}{\kappa^2}, \tag{8}$$

which serves to define the radius r of the sphere S in terms of the curvature and torsion of the curve $C \subset S$, thus reproducing the formula in [**3**; Problem 9, p. 161]. Further note that (7) gives a formula for the center of the normalized sphere in terms of the differential invariants. The inequality $r \geq 1/\kappa$ implies that the radius of the sphere must be greater than or equal to that of the osculating sphere at each point on the curve. Vice versa, the curvature at each point must be greater than or equal to the reciprocal of the radius of the sphere.

On the other hand, we can regard (8) as imposing a constraint on the signature curve $\mathcal{S} = \left\{ \, (\kappa, \tau, \kappa_s) \, \right\}$ when the curve is contained in a sphere of radius r. Conversely, if (8) is satisfied at all points in the curve, then the curve is contained in a sphere of radius r. (Proof?)

Remark: Formulae (7–8) also determine the normalized osculating sphere for a more general curve. One can then adapt the formulas at a general point by using the fact that the moving frame vectors, consisting of the unit tangent $\mathbf{t}$, unit normal $\mathbf{n}$ and unit binormal $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ — for the normalized curve are the standard basis vectors e_1, e_2, e_3 and use equivariance to write the result in terms of the moving frame vectors at the original point on the curve. Thus, the osculating sphere at a point $z \in C$ has radius r given by (8) and center

$$\mathbf{c} = z + \frac{1}{\kappa}\,\mathbf{n} - \frac{\kappa_s}{\kappa^2\,\tau}\,\mathbf{b}, \tag{9}$$

reproducing the formula in [**2**; Corollary 8.13].

The procedure illustrated in this example will clearly work in general. It would be worth going through additional examples, including higher dimensional sub-submanifolds.

References

[**1**] Fels, M., and Olver, P.J., Moving coframes. II. Regularization and theoretical foundations, *Acta Appl. Math.* **55** (1999), 127–208.

[**2**] Gray, A., Abbena, E., and Salamon, S., *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 3rd ed., Chapman & Hall/CRC, Boca Raton, Fl., 2006.

[**3**] Guggenheimer, H.W., *Differential Geometry*, McGraw–Hill, New York, 1963.

[**4**] Olver, P.J., Normal forms for submanifolds under group actions, *in*: *Symmetries, Differential Equations and Applications*, V. Kac, P.J. Olver, P. Winternitz, and T. Özer, eds., Proceedings in Mathematics & Statistics, Springer, New York, 2018, pp. 3–27.