Journal of Elasticity 19: 189–212 (1988)



# Canonical elastic moduli

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Received 5 January 1987; accepted in revised form 8 June 1987

**Abstract.** Every linear planar anisotropic elastic material is equivalent, under a linear change of coordinates, to an orthotropic material. Consequently, up to linear changes of variables, there are just two "canonical" planar elastic moduli which determine the properties of any linearly elastic material. Extensions to three-dimensional elasticity and applications are indicated.

## Preface

The detailed investigaton of complex mathematical objects can often be considerably simplified through the use of specially adapted coordinate systems in which the object takes a simple "canonical form". Elementary examples include the Jordan canonical form of a square matrix, and Sylvester's Theorem on the representation of a quadratic form as a sum of squares. Use of a canonical form results in a great simplification of complicated calculations and often provides extra geometric insight which might otherwise be difficult to extract.

In elasticity, the determination of canonical forms for elastic materials, either linear or nonlinear, does not appear to have been investigated in the literature before. The basic mathematical problem is to determine a coordinate system in which the elastic material has as simple expression as possible. Restricting our attention to linear (hyper)-elasticity, the natural question is to ask for a linear change of coordinates in both the material coordinates $\mathbf{x}$ and the displacement $\mathbf{u}$:

$$\mathbf{x} \to A\mathbf{x}, \quad \mathbf{u} \to B\mathbf{u},$$

which will simplify the general elastic moduli $c_{ijkl}$ as much as possible. Stated in this form, the question appears to be very natural from a mathematical point of view, even though it does not have an obvious physical motivation. (This may account for the lack of interest in this question in the standard elasticity literature.) Lekhnitskii, [6; Chapter 1], perhaps motivated by the changes of frame in finite elasticity, discusses the effect of a general pair of rotations on the elastic moduli, but I am unaware of any discussion of the effect of more general linear changes of variables.

In the case of planar elasticity, an elementary dimension count (cf. Section 2) shows that one should expect the standard 6 independent elastic moduli to reduce to just 2 canonical elastic moduli. Indeed, the main result of this paper verifies this intuitive result by explicitly showing how a general anisotropic planar elastic material can be reduced to an orthotropic elastic material by a suitably clever change of coordinates. One immediate consequence of this result is that many of the complicated computations involving anisotropic materials considerably simplify once they are placed into orthotropic form. For example, the complete determination of conservation laws for planar anisotropic elasticity, [11], relies heavily on these results. Thus, although admitting general linear changes of variables lacks a good physical motivation, the mathematical simplification in the equations more than justifies the method. (One possible physical interpretation – which is not suggested too seriously – is that the change of variables amounts to viewing the elastic body through some kind of weird "prismatic lens" which distorts the body according to the desired linear transformation.)

## 1. Summary of results

The equation of linear hyper-elasticity constitute a self-adjoint, strongly elliptic linear system of second-order partial differential equations for the displacement $\mathbf{u} = \mathbf{f}(\mathbf{x})$. Here the independent variables $\mathbf{x} = (x_1, \ldots, x_p)$ are the material coordinates in the elastic body $\Omega$, which is a domain in $\mathbb{R}^p$, and the dependent variables $\mathbf{u} = (u^1, \ldots, u^p)$ determine the displacement. In the planar case $p = 2$, while $p = 3$ for fully three-dimensional elastic media. The equations themselves are the Euler–Lagrange equations for a variational integral of the form

$$\mathscr{W}[\mathbf{u}] = \int_\Omega W(\nabla\mathbf{u})\,d\mathbf{x}. \tag{1}$$

Under the assumption of material homogeneity, the *stored energy function* $W(\nabla\mathbf{u})$ is a symmetric quadratic function of the deformation gradient $\nabla\mathbf{u}$,

$$W(\nabla\mathbf{u}) = \sum_{i,j,k,l=1}^{p} a_{ijkl} \frac{\partial u^i}{\partial x_j}\frac{\partial u^k}{\partial x_l}, \tag{2}$$

where the constants $a_{ijkl}$ are called the *variational moduli* of the problem.

Without loss of generality we can assume that they have the symmetry $a_{ijkl} = a_{klij}$, the construction resting on the underlying assumption of hyperelasticity. The corresponding Euler–Lagrange equations are a self-adjoint linear, second order system of partial differential equations:

$$\sum_{j,k,l=1}^{p} a_{ijkl} \frac{\partial^2 u^k}{\partial x_j \partial x_l} = 0, \quad i = 1, \ldots, p.$$

Frame indifference requires that the stored energy function be *symmetric*, meaning that it can be written in terms of the strain tensor $\mathbf{e} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$. We have

$$W(\nabla \mathbf{u}) = \sum c_{ijkl} \cdot e_{ij} \cdot e_{kl}, \tag{3}$$

where the constants $c_{ijkl}$ are the *elastic moduli* which describe the physical properties of the elastic material of which the body is composed; the variational moduli $a_{ijkl}$ are certain specific linear combinations of the elastic moduli. The symmetry of the strain tensor implies that we can assume that the elastic moduli obey the symmetry restrictions

$$c_{ijkl} = c_{jikl} = c_{ijlk}, \quad c_{ijkl} = c_{klij}. \tag{4}$$

Thus in planar elasticity there are 6 independent elastic moduli, while in three dimensions 21 independent moduli are required in general. Additional symmetry restrictions stemming from the constitutive properties of the elastic material may place additional constraints on the moduli.

Furthermore, the elastic moduli must satisfy certain inequalities stemming from the Legendre-Hadamard strong ellipticity condition. This states that the quadratic stored energy function $W(\nabla \mathbf{u})$ must be positive definite whenever the deformation gradient $\nabla \mathbf{u}$ is a rank one tensor. Following [10], we define the *symbol* of the quadratic variational problem (1) to be the biquadratic polynomial

$$Q(\mathbf{a}, \mathbf{b}) = W(\mathbf{a} \otimes \mathbf{b})$$

obtained by replacing $\nabla \mathbf{u}$ by the rank one tensor $\mathbf{a} \otimes \mathbf{b}$, where $\mathbf{a}$, $\mathbf{b}$ are vectors in $\mathbb{R}^p$. At the slight risk of confusion, it is convenient to replace the symbols $\mathbf{a}$, $\mathbf{b}$ by $\mathbf{x}$, $\mathbf{u}$, which are still vectors in $\mathbb{R}^p$, and write $Q(\mathbf{x}, \mathbf{u})$ for the symbol of $\mathscr{W}$. For a general quadratic Lagrangian of the form (2), the symbol takes the form

$$Q(\mathbf{x}, \mathbf{u}) = \sum a_{ijkl} x_j x_l u^i u^k.$$

The Legendre–Hadamard condition requires that the symbol $Q$ be *positive definite* in the sense that

$$Q(\mathbf{x}, \mathbf{u}) > 0 \quad \text{whenever} \quad \mathbf{x} \neq \mathbf{0} \quad \text{and} \quad \mathbf{u} \neq \mathbf{0}. \tag{5}$$

We will always assume that our quadratic variational problem (1) satisfies this condition throughout this paper. (See [10] for a treatment of more general quadratic variational problems.)

The symmetry restrictions (4) placed on the elastic moduli are easily seen to be equivalent to the restriction that the symbol $Q$ be *symmetric*, i.e.,

$$Q(\mathbf{x}, \mathbf{u}) = Q(\mathbf{u}, \mathbf{x}) \tag{6}$$

for all $\mathbf{x}, \mathbf{u} \in \mathbb{R}^p$. Some of our constructions will use this symmetry assumption, but the basic method does not really depend upon our starting with a symmetric symbol.

Before discussing changes of variables, we note that in any variational problem, one can always add any *null Lagrangian* or total divergence to the integrand without affecting the Euler-Lagrange equations, cf. [9; Theorem 4.7]. (However, this can affect the associated natural boundary conditions, cf. [4; page 211].) Thus two stored energy functions $W$ and $\hat{W}$ determine the same variational problem and, consequently, the same Euler–Lagrange equations if and only if

$$\hat{W} = W + N,$$

where $N = \operatorname{Div} P$ is a total divergence. For example, in the planar quadratic case we can add in any constant multiple of the Jacobian determinant

$$u_x v_y - u_y v_x = D_x(uv_y) + D_y(-uv_x)$$

to the stored energy $W(u_x, u_y, v_x, v_y)$ without affecting the Euler–Lagrange equations. Thus, the Lagrangians $u_x v_y$ and $u_y v_x$ and $\frac{1}{2}[u_x v_y + u_y v_x]$ all have exactly the same Euler–Lagrange equations. In fact, there is a general theorem, [2], that says that all such quadratic null Lagrangians are given as linear combinations of suitable Jacobian determinants. An easy lemma states that two quadratic Lagrangians have the same symbol if and only if they differ by such a quadratic null Lagrangian, cf. [10].

Since the process of minimization does not depend on any particular coordinate system in use, it makes eminent sense to try to simplify the stored energy function, and hence the associated Euler–Lagrange equations, as much as possible through the introduction of "adapted" coordinates. Since we are restricting our attention to quadratic variational problems, we will only allow linear changes of variables

$$\mathbf{x} = A\tilde{\mathbf{x}}, \quad \mathbf{u} = B\tilde{\mathbf{u}}, \tag{7}$$

in which $A$ and $B$ are arbitrary nonsingular $p \times p$ matrices. In terms of the new variables $\tilde{\mathbf{x}}$, $\tilde{\mathbf{u}}$, the variational problem has an analogous form

$$\tilde{\mathscr{W}}[\tilde{\mathbf{u}}] = \int_{\tilde{\Omega}} \tilde{W}(\tilde{\mathbf{u}})\, d\tilde{\mathbf{x}},$$

where the new stored energy function $\tilde{W}$ has the same form (2), but with new variational moduli $\tilde{a}_{ijkl}$. The minima of the two variational problems $\mathscr{W}$ and $\tilde{\mathscr{W}}$ are in one-to-one correspondence under the change of variables (7), so from a coordinate-free standpoint, they are essentially the same problem. Thus the goal is to find a particular linear change of variables (7) which will simplify the moduli as much as possible, leading to a simple canonical form for the variational problem, and, hence, the *canonical moduli* of the title.

Combining the previous two paragraphs, we will define a general notion of equivalence of two quadratic Lagrangians to mean that the corresponding variational problems are mapped to each other by some linear change of variables (7). In other words, $W$ and $\tilde{W}$ are *equivalent* if there exist nonsingular matrices $A$, $B$ and a null Lagrangian $N$ such that

$$\tilde{W}(\nabla\tilde{\mathbf{u}}) = \{W(\nabla\mathbf{u}) + N(\nabla\mathbf{u})\}|\det A|, \quad \text{where} \quad \mathbf{x} = A\tilde{\mathbf{x}},\ \mathbf{u} = B\tilde{\mathbf{u}}.$$

Thus, given a stored energy function $W$, the goal is to find matrices $A$ and $B$ and a null Lagrangian $N$ such that the resulting stored energy function $\tilde{W}$ is as simple as possible.

When we have a series of such changes of variables, it is often helpful at each stage to drop the tildes and re-express everything in terms of $\mathbf{x}$ and $\mathbf{u}$ rather than $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{u}}$. Thus, it will be helpful to adopt the following "substitutional" notation for changes of variables. We write (7) in the form

$$\mathbf{x} \to A\tilde{\mathbf{x}}, \quad \mathbf{u} \to B\tilde{\mathbf{u}}.$$

Thus a function $f(\mathbf{x}, \mathbf{u})$ gets transformed into the function

$$\tilde{f}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}) = f(A\tilde{\mathbf{x}}, B\tilde{\mathbf{u}}).$$

Now we can omit the tildes and write the change of variables as

$$\mathbf{x} \to A\mathbf{x}, \quad \mathbf{u} \to B\mathbf{u}, \tag{8}$$

and our function $f(\mathbf{x}, \mathbf{u})$ gets transformed into the function

$$\tilde{f}(\mathbf{x}, \mathbf{u}) = f(A\mathbf{x}, B\mathbf{u}).$$

The transformation (8) can thus be read "substitute $A\mathbf{x}$ for $\mathbf{x}$ and $B\mathbf{u}$ for $\mathbf{u}$ where-ever they occur".

EXAMPLE. An orthotropic elastic material is one that has three orthogonal planes of reflected symmetry, cf. [5; page 159]. In two dimensions, it is characterized by the conditions

$$c_{1112} = c_{1222} = 0$$

on the elastic moduli. Thus the stored energy function takes the form

$$c_{1111}u_x^2 + c_{1212}(u_y + v_x)^2 + 2c_{1122}u_x v_y + c_{2222}v_y^2,$$

where we write $(x, y)$ for $\mathbf{x} = (x_1, x_2)$ and $(u, v)$ for $\mathbf{u} = (u^1, u^2)$. When we expand, as remarked above we can replace the term $u_x v_y$ by $u_y v_x$, leading to the *semi-diagonal Lagrangian*

$$pu_x^2 + qu_y^2 + 2ru_x v_y + sv_x^2 + tv_y^2, \tag{9}$$

where

$$p = c_{1111}, \quad q = c_{1212}, \quad r = c_{1212} + c_{1122}, \quad s = c_{1212}, \quad t = c_{2222}.$$

In particular, the strong ellipticity implies that $p$, $q$, $s$, and $t$ are all positive.

We can thus further simplify any semi-diagonal Lagrangian (9) by rescaling both $\mathbf{x}$ and $\mathbf{u}$:

$$x \to \sqrt[4]{ps} \cdot x, \quad y \to \sqrt[4]{qt} \cdot y, \quad u \to \sqrt[8]{\frac{s}{p^3 qt}} \cdot u, \quad v \to \sqrt[8]{\frac{q}{pst^3}} \cdot v.$$

The net effect is a Lagrangian depending on only two parameters,

$$u_x^2 + \alpha u_y^2 + 2\beta u_x v_y + \alpha v_x^2 + v_y^2, \tag{10}$$

where

$$\alpha = \sqrt{\frac{qs}{pt}}, \quad \text{and} \quad \beta = \frac{r}{\sqrt{pt}}.$$

We will call (10) an *orthotropic Lagrangian*, and the parameters $\alpha$ and $\beta$ the corresponding *canonical elastic moduli* for such an orthtropic elastic medium. Thus we have shown that, up to rescaling, there is just a two-parameter family of planar orthotropic elastic media.

In particular, the Lagrangian for linear isotropic elasticity rescales to the special case in which the canonical elastic moduli are related by the equation

$$\alpha + \beta = 1. \tag{11}$$

Indeed, under the above scaling, we find that

$$\alpha = \frac{\mu}{2\mu + \lambda}, \quad \beta = \frac{\mu + \lambda}{2\mu + \lambda},$$

where $\mu$ and $\lambda$ are the classical Lamé moduli, [5; page 161]. Two isotropic Lagrangians determine the same orthotropic Lagrangian if and only if their Lamé moduli are proportional: $\lambda/\mu = \tilde{\lambda}/\tilde{\mu}$, or, equivalently, they have the same value for Poisson's ratio $\nu = \lambda/(\mu + \lambda) = \tilde{\nu} = \tilde{\lambda}/(\tilde{\mu} + \tilde{\lambda})$. Thus there is, up to rescaling, just a one-parameter family of inequivalent isotropic Lagrangians.

The main result of this paper is that the orthotropic Lagrangians actually provide a complete list of canonical forms for planar elastic media. In other words, every planar linear elastic medium is equivalent under a linear change of variables to an orthotropic elastic medium.

THEOREM 1. *Let $W(\nabla \mathbf{u})$ be a first order planar quadratic Lagrangian which satisfies the Legendre–Hadamard strong ellipticity condition. Then $W$ is equivalent to a orthotropic Lagrangian (10), where the canonical elastic moduli $\alpha$ and $\beta$ are constants, satisfying the strong ellipticity inequalities*

$$\alpha > 0, \quad |\beta| < \alpha + 1. \tag{12}$$

*The corresponding Euler–Lagrange equations are thus equivalent, under a linear change of variables, to a "generalized" system of Navier's equations*

$$u_{xx} + \alpha u_{yy} + \beta v_{xy} = 0, \quad \beta u_{xy} + \alpha v_{xx} + v_{yy} = 0. \tag{13}$$

In other words, for planar elasticity, once we allow arbitrary linear changes of variable, there are in reality only two independent elastic moduli. The inequality constraints (12) are readily seen to be equivalent to the Legendre–Hadamard condition (5). Consequently, while the general planar elastic problem in a general coordinate system has 6 independent elastic moduli $c_{ijkl}$, Theorem 1 shows that if we choose a special adapted coordinate system, there are in reality only *two* independent moduli.

Now that we are allowing general linear changes of variables, there is the additional possibility that different *orthotropic* Lagrangians are themselves equivalent under some linear change of variables. In fact, it turns out that, except for the "exceptional" isotropic case, each orthotropic Lagrangian is equivalent to seven other orthotropic Lagrangians.

THEOREM 2. *Let $W$ and $\tilde{W}$ be different orthotropic Lagrangians with moduli $\alpha, \beta$ and $\tilde{\alpha}, \tilde{\beta}$ respectively. Then $W$ is equivalent to $\tilde{W}$ if and only if their moduli are related by one of the following pairs of equations:*

i) $\tilde{\alpha} = \alpha, \qquad \tilde{\beta} = -\beta,$

ii) $\tilde{\alpha} = \dfrac{1}{\alpha}, \qquad \tilde{\beta} = \dfrac{\beta}{\alpha},$

iii) $\tilde{\alpha} = \dfrac{1}{\alpha}, \qquad \tilde{\beta} = -\dfrac{\beta}{\alpha},$

iv) $\tilde{\alpha} = \dfrac{1+\alpha-\beta}{1+\alpha+\beta}, \qquad \tilde{\beta} = \dfrac{2-2\alpha}{1+\alpha+\beta},$

v) $\tilde{\alpha} = \dfrac{1+\alpha-\beta}{1+\alpha+\beta}, \qquad \tilde{\beta} = \dfrac{2\alpha-2}{1+\alpha+\beta},$

vi) $\tilde{\alpha} = \dfrac{1+\alpha+\beta}{1+\alpha-\beta}, \qquad \tilde{\beta} = \dfrac{2-2\alpha}{1+\alpha-\beta},$

vii) $\tilde{\alpha} = \dfrac{1+\alpha+\beta}{1+\alpha-\beta}, \qquad \tilde{\beta} = \dfrac{2\alpha-2}{1+\alpha-\beta},$

Note that transformations i), iv) and v) leave an isotropic Lagrangian unchanged, but ii), iii), vi) and vii) change it into a different orthotropic Lagrangian with

$$\alpha - \beta = 1. \tag{14}$$

In future, we will consider both (11) and (14) as *isotropic Lagrangians*. Their structure is somewhat exceptional. (See especially [10].)

In particular, excluding isotropic Lagrangians, in the strongly elliptic case one can always use one of the above transformations to make the canonical elastic moduli $\alpha$ and $\beta$ satisfy the additional restrictions

$$0 < \alpha \leqslant 1, \quad 0 \leqslant \beta \leqslant 1 - \alpha. \tag{15}$$

In fact we will see that any strongly elliptic Lagrangian is equivalent to a *unique* orthotropic Lagrangian whose moduli satisfy (15). Isotropic materials will also satisfy (11).

An important feature of this result on canonical forms for elastic moduli is that the construction of the linear transformation (8) which places a general strongly elliptic Lagrangian into canonical form is *completely explicit*. Indeed, Theorem 6 will provide an elementary constructive procedure for determining the explicit form of the transformation, and hence of the canonical elastic moduli $\alpha$, $\beta$.

The proofs of these results presented here are basically simplified versions of the general procedure for finding canonical forms of *arbitrary* quadratic Lagrangians in the plane, a problem solved completely in [10], but where we exploit the underlying strong ellipticity assumption to full advantage.

The motivation for the study of these problems was the author's continuing studies on conservation laws in linear elasticity, [7], and, more specifically, attempts to extend the results on linear isotropic elasticity to the anisotropic case [8]. It was found that, without some kind of elementary canonical form, the intervening computations for symmetries and conservation laws are just too complicated to effectively analyze. A complete classification of the conservation laws of planar anisotropic elasticity based on the result of the present paper can be found in [11]. Further applications to the simplification of the equations of anisotropic elasticity, and the complex variable methods associated with the Airy stress function, [5], in planar elasticity are also direct consequences of the main theorem, and will be explored in detail elsewhere.

## 2. Counting canonical elastic moduli

Before we implement the rigorous proof of Theorem 1, it is useful to take a more "hand-waving" view of the subject, which at least will give us a reason to suspect that such a result should be true, even if it does not provide much in the way of rigorous proof. Besides providing a motivation for the two-dimensional result, the following counting methods readily extend to three dimensions, and tell us how many canonical elastic moduli we can reasonably expect there, even though it gives us no idea of what the canonical forms for three-dimensional strongly elliptic Lagrangians are, or how to construct the corresponding canonical elastic moduli.

The basic idea is to effect a *dimension count* comparing the number of independent elastic moduli with the number of available degrees of freedom in the allowable changes of variables. Let us consider a quadratic variational problem (1), in which both $\mathbf{x}$ and $\mathbf{u}$ are in $\mathbb{R}^p$. For a general Lagrangian, as in (2) there are $\frac{1}{2}p^2(p^2 + 1)$ independent variational moduli $a_{ijkl}$ since we can always impose the symmetry condition $a_{ijkl} = a_{klij}$ without loss of generality. However, as mentioned in section 1, we can also add in any quadratic null Lagrangian $N$ to $W$ without affecting the Euler–Lagrange equations. The most general such null Lagrangian is a linear combination of the $2 \times 2$ Jacobian determinants

$$N_{ijkl}(\nabla \mathbf{u}) = \frac{\partial u^i}{\partial x_j} \cdot \frac{\partial u^k}{\partial x_l} - \frac{\partial u^i}{\partial x_l} \cdot \frac{\partial u^k}{\partial x_j}.$$

Again, in $\mathbb{R}^p$, it is not hard to see that there are $\frac{1}{4}p^2(p - 1)^2$ independent quadratic Jacobian determinants, so there are in reality only $\frac{1}{2}p^2(p^2 + 1) - \frac{1}{4}p^2(p - 1)^2 = \frac{1}{4}p^2(p + 1)^2$ independent coefficients in the form of $W$. (This is also, of course, the dimension of the space of symbols $Q(\mathbf{x}, \mathbf{u})$ when $\mathbf{x}$ and $\mathbf{u}$ are in $\mathbb{R}^p$.)

Now, in the general change of variables

$$\mathbf{x} \to A\mathbf{x}, \quad \mathbf{u} \to B\mathbf{u} \tag{16}$$

there are $p^2$ different entries in each $p \times p$ matrix $A$ and $B$, and so we have $2p^2$ different parameters at hand. However, the particular scaling

$$\mathbf{x} \to \lambda\mathbf{x}, \quad \mathbf{u} \to \lambda^{(2-p)/2}\mathbf{u}, \quad \lambda > 0$$

leaves the variational problem unchanged, and so we have one less independent parameter with which to effect nontrivial changes in $W$. Assuming that the remaining $2p^2 - 1$ parameters all act "independently" (an assumption that does hold generically, but might fail at particular Lagrangians, e.g., isotropic Lagrangians) we can expect to eliminate $2p^2 - 1$ independent moduli in the Lagrangian by a suitable change of variables. Thus we have a total of

$$\tfrac{1}{4}p^2(p + 1)^2 - (2p^2 - 1) = \frac{p^4 + 2p^3 - 7p^2 + 4}{4}$$

remaining moduli, and this should provide the number of canonical moduli $a_{ijkl}$ for a general quadratic variational problem in $\mathbb{R}^p$. We summarize the results in tabular form for $p = 1, \ldots, 4$:

| Dimension | # Moduli | # Parameters | # Canonical moduli |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 2 | 9 | 7 | 2 |
| 3 | 36 | 17 | 19 |
| 4 | 100 | 31 | 69 |
| $p$ | $\frac{1}{4}p^2(p+1)^2$ | $2p^2 - 1$ | $\frac{1}{4}(p^4 + 2p^3 - 7p^2 + 4)$ |

This can be contrasted with the situation discussed by Lekhnitskii, [6], in which the matrices $A$ and $B$ in the change of variables (16) are restricted to be orthogonal matrices. Now there are only $p(p-1)$ independent parameters than can be used to effect changes in $W$, and we find the following alternative table for "orthogonal-canonical moduli":

| Dimension | # Moduli | # Parameters | # Canonical moduil |
|---|---|---|---|
| 1 | 1 | 0 | 1 |
| 2 | 9 | 2 | 7 |
| 3 | 36 | 6 | 30 |
| 4 | 100 | 12 | 88 |
| $p$ | $\frac{1}{4}p^2(p+1)^2$ | $p^2 - p$ | $\frac{1}{4}(p^4 + 2p^3 - 3p^2 + 4p)$ |

In the elastic case, the Lagrangian satisfies the additional symmetry restriction (6), and so there are only $\frac{1}{8}p(p+1)(p^2+p+2)$ independent elastic moduli $c_{ijkl}$. (To see this, note that there are $\frac{1}{2}p(p+1)$ different strain components $e_{ij}$, and the Lagrangian is a general quadratic polynomial (3) in these components.) On the other hand, we can no longer perform an arbitrary linear change of variables (16) as this would destroy the symmetry of the Lagrangian. Thus we are restricted to changes of variables of the form

$$\mathbf{x} \to A\mathbf{x}, \quad \mathbf{u} \to \lambda \cdot A^{-1}\mathbf{u}, \quad \lambda \in \mathbb{R} \tag{17}$$

which *do* preserve this symmetry. There are then only $p^2$ independent parameters (as above, one scaling is trivial), and so, for the symmetric case, we have the alternative table:

| Dimension | # Moduli | # Parameters | # Canonical moduli |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 2 | 6 | 4 | 2 |
| 3 | 21 | 9 | 12 |
| 4 | 55 | 16 | 39 |
| $p$ | $\frac{1}{8}p(p+1)(p^2+p+2)$ | $p^2$ | $\frac{1}{8}p(p-1)(p^2+3p-2)$ |

Note that in two dimensions, the number of canonical moduli is the same whether or not we impose the additional symmetry condition (6) on the symbol. This is borne out by the general result of Theorem 1 that every strongly elliptic variational problem is equivalent to an orthotropic, and hence symmetric, one. This is *not* the case in three dimensions, where there are 19 canonical moduli for a general quadratic Lagrangian, but only 12 in the symmetric case. Also, we see that the generalization of Theorem 1 to three-dimensional elasticity *cannot* be true, since there is only a nine-parameter family of orthotropic elastic media, [5: p. 159], even before possible rescalings are taken into account. This is borne out by results of Cowin and Mehrabadi, [3], on three-dimensional linear elasticity, in which they determine additional restrictions on the elastic moduli for the material to admit one (or more) planes of reflectional symmetry.

Again, we can contrast this with the canonical moduli found when the matrix $A$ in (17) is restricted to be orthogonal. In this case, there are only $\frac{1}{2}p(p-1)$ parameters available, and we find the following table:

| Dimension | # Moduli | # Parameters | # Canonical moduli |
|---|---|---|---|
| 1 | 1 | 0 | 1 |
| 2 | 6 | 1 | 5 |
| 3 | 21 | 3 | 18 |
| 4 | 55 | 6 | 49 |
| $p$ | $\frac{1}{8}p(p+1)(p^2+p+2)$ | $\frac{1}{2}(p^2-p)$ | $\frac{1}{8}(p^4+2p^3-p^2+6p)$ |

Note especially the increasing discrepancy between the number of canonical elastic moduli in the two cases as the dimensions of the space gets larger.

These results strongly indicate, but by no means prove, the validity of Theorem 1, that there is a two-parameter family of canonical quadratic Lagrangians, with every other strongly elliptic Lagrangian equivalent to at least one canonical Lagrangian. Similarly, in three dimensions, one suspects the existence of a twelve-parameter family of canonical quadratic Lagrangians, with the property that every strongly elliptic quadratic Lagrangian is equivalent to one of these. However, this naïve dimension count gives no idea of the form that such canonical Lagrangians should take, nor how to actually prove a rigorous result to that effect. Indeed, there must be a note of caution in interpreting these dimension counts, since it is *not* true that *every* planar quadratic Lagrangian is equivalent to an orthotropic one once we drop the strong ellipticity assumption; see [10].

## 3. A proof of Theorem 1

In this section we prove the assertion of Theorem 1 that every planar elastic material is equivalent under a linear change of variables to an orthotropic material. Although the proof is constructive, the resulting change of variables is not the most efficacious to use in practice, and in Section 4 we give a "streamlined" version of the change of variables which can be easily implemented in practice. For some reason, it appears that we need to know the truth of Theorem 1 before we can be sure that the streamlined version really works, which is why we require two sections for the discussion. The method of proof is a simplified version of the general calculations appearing in [10].

For a general planar quadratic variational problem, the symbol has the form

$$Q(\mathbf{x}, \mathbf{u}) = a_1 x^2 u^2 + a_2 xyu^2 + a_3 y^2 u^2 + a_4 x^2 uv + a_5 xyuv + a_6 y^2 uv + a_7 x^2 v^2 + a_8 xyv^2 + a_9 y^2 v^2, \tag{18}$$

where the coefficients $a_i$ are just a simpler notation for the variational moduli $a_{ijkl}$. If we assume strong ellipiticity, then $Q$ must be positive definite as a function of $\mathbf{x} = (x, y)$ and $\mathbf{u} = (u, v)$, which will place certain restrictions on allowable moduli $a_1, \ldots, a_9$. In the symmetric (elastic) case (6), we have the additional restrictions

$$a_2 = a_4, \quad a_3 = a_7, \quad a_6 = a_8,$$

although these are not required for the subsequent argument.

Although $\mathbf{x}$ and $\mathbf{u}$ play essentially interchangeable roles in the discussion we single out one of them, say $\mathbf{x}$, and write $Q$ as a quadratic polynomial in $\mathbf{x}$,

$$Q(\mathbf{x}, \mathbf{u}) = A(\mathbf{u})x^2 + 2B(\mathbf{u})xy + C(\mathbf{u})y^2, \tag{19}$$

where the coefficients $A$, $B$, $C$ are homogeneous quadratic polynomials in $\mathbf{u}$, e.g.,

$$A(\mathbf{u}) = a_1 u^2 + a_4 uv + a_7 v^2.$$

A symbol is called *semi-diagonal* if the coefficients

$$a_2 = a_4 = a_6 = a_8 = 0$$

all vanish, and so $Q$ has the form

$$Q(\mathbf{x}, \mathbf{u}) = px^2u^2 + qy^2u^2 + 2rxyuv + sx^2v^2 + ty^2v^2, \tag{20}$$

which is just the symbol for the semi-diagonal Lagrangian (9). In the example, it was shown how any semi-diagonal Lagrangian could be rescaled to an orthotropic Lagrangian, and the same rescaling obviously works for the corresponding symbols. Thus the goal is to show that any strongly elliptic symbol is equivalent to a semi-diagonal symbol. (Without the strong ellipticity, this *not* true, and there are indefinite biquadratic symbols which are not equivalent to any semi-diagonal form, cf. [10].)

Before starting, note that if we are given a linear change of variables

$$\mathbf{x} \to A\mathbf{x}, \quad \mathbf{u} \to B\mathbf{u}, \tag{21}$$

then, by the change of variables formula for multiple integrals, the symbol of the new stored energy is related to that of the old by the formula

$$\tilde{Q}(\mathbf{x}, \mathbf{u}) = |\det A| \cdot Q(A^{-1}\mathbf{x}, B\mathbf{u}). \tag{22}$$

Since we will be working exclusively with symbols in this section, it helps to simplify matters by replacing the matrix $A$ by the matrix $\sqrt{|\det A|} \cdot A^{-1}$. Thus the change of variables (21) will be assumed to have the simpler effect of changing the symbol to

$$\tilde{Q}(\mathbf{x}, \mathbf{u}) = Q(A\mathbf{x}, B\mathbf{u}). \tag{23}$$

(Of course, once we derive the proper change of variables, we must remember to translate (23) back to the proper Lagrangian picture (22).)

The basic invariant of a quadratic polynomial is its *discriminant*, which, for (19), is the function

$$\Delta_{\mathbf{x}}(\mathbf{u}) = B(\mathbf{u})^2 - A(\mathbf{u}) \cdot C(\mathbf{u}), \tag{24}$$

which is a homogeneous quartic polynomial in $\mathbf{u}$. (There is also a discriminant $\Delta_{\mathbf{u}}(\mathbf{x})$, but we only need consider one of these two polynomials; in the symmetric case they are the same polynomial.) A *root* of $\Delta_{\mathbf{x}}(\mathbf{u})$ is a (possibly complex) nonzero solution $\mathbf{u}_0 = (u_0, v_0)$ to the quartic equation $\Delta_{\mathbf{x}}(\mathbf{u}) = 0$. However, since $\Delta_{\mathbf{x}}(\mathbf{u})$ is homogeneous, any complex scalar multiple $\lambda\mathbf{u}_0$ of a root $\mathbf{u}_0$ is also a root, so we will only distinguish roots if they are not scalar multiples of each other. Thus, by the Fundamental Theorem of

Algebra, counting multiplicities, $\Delta_{\mathbf{x}}$ has precisely four complex roots, which we denote by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

Now, the discriminant of a quadratic polynomial vanishes if and only if the polynomial is a perfect square; therefore, for each root $\mathbf{u}_j$ of $\Delta_{\mathbf{x}}$, the corresponding (complex) quadratic polynomial $Q_j(\mathbf{x}) = Q(\mathbf{x}, \mathbf{u}_j)$ must be a perfect square:

$$Q_j(\mathbf{x}) \;=\; \pm(b_j x + c_j y)^2. \tag{25}$$

Note that since each root $\mathbf{u}_j$ is only defined up to a scalar multiple, the same is true for the perfect square $Q_j$. We assume throughout that some consistent choice of these scalar multiples has been made. (See below.)

At this point, we invoke the strong ellipticity condition (5) to conclude that $\Delta_{\mathbf{x}}(\mathbf{u})$ can *only* have complex roots. Indeed, if $\mathbf{u}_j$ were a real root, then there would be a nonzero real root $\mathbf{x}_j$ of the perfect square $Q_j$, and hence $(\mathbf{x}_j, \mathbf{u}_j)$ would be a nonzero real root of the symbol $Q$, which would violate the positivity assumption (5). Furthermore, by reality, the four complex roots come in complex conjugate pairs, so we can assume that $\mathbf{u}_2 = \overline{\mathbf{u}_1}$, and $\mathbf{u}_4 = \overline{\mathbf{u}_3}$.

Let us take one pair of complex conjugate roots $\mathbf{u}_2 = \overline{\mathbf{u}_1}$ of $\Delta_{\mathbf{x}}$. Since the symbol $Q$ is real, the corresponding polynomials $Q_1$ and $Q_2$, cf. (25), are also complex conjugates

$$Q_2(\bar{\mathbf{x}}) \;=\; \overline{Q_1(\mathbf{x})}.$$

By multiplying by a suitable complex scalar, we can arrange that the root $\mathbf{u}_1$ takes the form $\mathbf{u}_1 = (1, \gamma + \mathrm{i}\delta)$, where $\delta \neq 0$ as otherwise the root would be real. We use the real transformation

$$(u, v) \to (u, \gamma u + \delta v)$$

to take this root to $\tilde{\mathbf{u}}_1 = (1, \mathrm{i})$, and its conjugate to $\tilde{\mathbf{u}}_2 = (1, -\mathrm{i})$.

At this juncture, there are two distinct cases. In the first, we find that the two perfect squares $Q_1$ and $Q_2$ corresponding to our two roots are genuinely complex polynomials. In other words,

$$Q_1(x, y) \;=\; -4[(a + \mathrm{i}b)x + (c + \mathrm{i}d)y]^2,$$

where the complex numbers $a + \mathrm{i}b$ and $c + \mathrm{i}d$ are not real multiples of each other, and $Q_2$ is the complex conjugate square. By replacing

$$(x, y) \to (ax + cy, bx + dy),$$

we can transform $Q_1$ into the elementary square $-4(x + iy)^2$ and hence $Q_2$ to its conjugate $-4(x - iy)^2$. Thus, the net effect of these two transformations is to place the symbol in the form

$$Q = (x - iy)^2(u - iv)^2 + \{\varrho(x - iy)^2 + \sigma(x - iy)(x + iy) + \bar{\varrho}(x + iy)^2\}$$
$$\times (u - iv)(u + iv) + (x + iy)^2(u + iv)^2,$$

where $\varrho$ is complex and $\sigma$ is real. Now, simultaneously rotating $(x, y)$ through an angle $\theta$ and $(u, v)$ through angle $-\theta$ (which is the same as multiplying $x + iy$ by $e^{i\theta}$ and $u + iv$ by $e^{-i\theta}$) leaves $Q$ in the same form, and only has the effect of multiplying $\varrho$ by $e^{2i\theta}$. Thus we can choose the angle $\theta$ so that $\varrho$ is real, and $Q$ takes the semi-diagonal form (20) where

$$p = 2\varrho + \sigma + 2, \quad q = -2\varrho + \sigma - 2, \quad r = -8,$$
$$s = 2\varrho + \sigma - 2, \quad t = -2\varrho + \sigma + 2.$$

Thus such a symbol $Q$ is equivalent to an orthotropic symbol.

In the second case, the two perfect squares $Q_1$ and $Q_2$ are complex conjugate multiples of the same real square. In other words,

$$Q_1 = 4(a + ib)^2(cx + dy)^2, \quad Q_2 = 4(a - ib)^2(cx + dy)^2,$$

where $a, b, c, d$ are real. By the strong ellipticity, neither $Q_1$ nor $Q_2$ is zero, so we can replace $x$ by $cx + dy$ to transform them into multiples of the elementary square $x^2$. In the new coordinates $Q$ has the form

$$Q = x^2\{(a - ib)^2(u - iv)^2 + (a + ib)^2(u + iv)^2\}$$
$$+ (\tilde{\varrho}x^2 + \tilde{\sigma}xy + \tilde{\tau}y^2)(u - iv)(u + iv),$$

for certain real constants $\tilde{\varrho}, \tilde{\sigma}, \tilde{\tau}$. If we replace $(u, v)$ by $1/(a^2 + b^2) \times (au + bv, -bu + av)$, then the symbol becomes

$$Q = x^2\{(u - iv)^2 + (u + iv)^2\} + (\varrho x^2 + \sigma xy + \tau y^2)(u - iv)(u + iv)$$
$$= x^2(u^2 - v^2) + (\varrho x^2 + \sigma xy + \tau y^2)(u^2 + v^2),$$

where $\varrho$, $\sigma$, $\tau$ are respectively equal to $\tilde{\varrho}$, $\tilde{\sigma}$, $\tilde{\tau}$ divided by $a^2 + b^2$. Now by strong ellipticity $\tau \neq 0$, so we can translate

$$(x, y) \to \left(x, -\frac{\sigma}{2\tau}x + y\right),$$

which has the effect of changing $Q$ into the semidiagonal form

$$Q = x^2(u^2 - v^2) + (\omega x^2 + \tau y^2)(u^2 + v^2),$$

where $\omega = \varrho - \sigma^2/2\tau$. Thus, in either case, the symbol is equivalent to a semi-diagonal symbol; this completes the demonstration that any strongly elliptic symbol is equivalent to an orthotropic symbol, and hence the proof of Theorem 1.

## 4. Constructing the change of variables

Now that we know the validity of Theorem 1, we can provide a more practical implementation of the change of variables which is required to reduce a general planar elastic Lagrangian to one in orthotropic form. Rather than working with the "homogeneous roots" $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$ of the $\mathbf{x}$-discriminant $\Delta_{\mathbf{x}}(\mathbf{u})$, we will use their corresponding complex representatives $z_1, z_2, z_3, z_4$, e.g. if $\mathbf{u}_1 = (u_1, v_1)$, then $z_1 = v_1/u_1$. (In particular, a root with $u_1 = 0$ would correspond to $z_1 = \infty$; however, this possibility is ruled out by strong ellipticity.) Note that the roots $z_j$ come in complex conjugate pairs: $z_2 = \overline{z_1}$, $z_4 = \overline{z_3}$. If $\mathbf{u} \to A\mathbf{u}$, $\mathbf{u} = (u, v)$, is a linear change of variables, then $A$ acts on the complex variable $z = v/u$ via a linear fractional transformation, [1, §3.3],

$$z \to \frac{dz + c}{bz + a}, \quad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{26}$$

Finally, if we assume that the symbol $Q$ satisfies the symmetry condition (6), then the $\mathbf{u}$-discriminant $\Delta_{\mathbf{u}}(\mathbf{x})$ is the same quartic polynomial, and has precisely the same complex roots.

LEMMA 3. *Let $z_2 = \overline{z_1}$, $z_4 = \overline{z_3}$ be non-real complex conjugate numbers. Then there is a unique real number $\tau \geqslant 1$ and a linear fractional transformation given by a* ***real*** *matrix $A$ which transforms $z_1, z_2, z_3, z_4$ to the complex numbers $\tau i$, $-\tau i$, $\tau^{-1} i$, $-\tau^{-1} i$. Moreover, if $\tilde{A}$ is any other such matrix, then $\tilde{A} = KA$, where either* *i) if* $\tau \neq 1$, *then K is a scalar multiple of one of the following four elementary matrices*:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \tag{27}$$

*ii) if* $\tau = 1$, *then K is a scalar multiple of an orthogonal* $2 \times 2$ *matrix, i.e., a matrix of the form*

$$\begin{bmatrix} \alpha \cos \phi & -\alpha \sin \phi \\ \alpha \sin \phi & \alpha \cos \phi \end{bmatrix}. \tag{28}$$

*Proof.* First of all, it is easy to translate the first pair of roots to be $\pm$i; if $z_1 = a + ib$, then the real translation matrix

$$T = \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}$$

will accomplish this. The new roots will be denoted by $z'_j$, $j = 1, \ldots, 4$, where $z_j = az'_j + b$. In particular $z'_1 =$ i (and, of course, since $T$ is real, $z'_2 = -$i remains its complex conjugate). Next we perform a second linear transformation which fixes $z'_1 =$ i, but moves the second pair of roots $z'_3$, $z'_4$ onto the imaginary axis. It is not difficult to see that a real linear fractional transformation (26) fixes the complex number i if and only if its matrix is a multiple of a rotation matrix; thus we can take

$$R = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix},$$

where the multiple has been taken to be 1 without loss of generality. Now if $z'_3 = p + iq$, and $p \neq 0$ (otherwise $z'_3$ would already be on the imaginary axis) then the above rotation matrix $R$ will move $z'_3$ to a point on the imaginary axis provided $\phi$ solves the equation

$$\cot 2\phi = \frac{p^2 + q^2 - 1}{2p}. \tag{29}$$

Thus, by composing the linear fractional transformation determined by $T$ with that determined by $R$ we have a linear fractional transformation with matrix $T \cdot R$, which moves all four roots onto the imaginary axis. The new roots are labelled $z_j''$, with $z_1''$, $z_2''$ being $\pm$i, while $z_3''$, $z_4''$ are some other complex conjugate pair of purely imaginary numbers $\pm\sigma$i, where, without loss of generality, $\sigma > 0$. It is then a simple matter to use the scaling determined by the matrix

$$S = \begin{bmatrix} \tau & 0 \\ 0 & 1 \end{bmatrix}$$

to move the roots to the desired positions $\pm\tau$i, $\pm\tau^{-1}$i, where $\tau = \sigma^{-1/2}$. Thus the composition

$$A = T \cdot R \cdot S \tag{30}$$

of the above three linear transformations has the desired effect. Even more explicitly, if we multiply out (30) and use polar coordinates $a = r \cdot \cos\theta$, $b = r \cdot \sin\theta$ for $z_1 = a + ib = r \cdot e^{i\theta}$, then we find that

$$A = \begin{bmatrix} \tau\cos\phi & -\sin\phi \\ r\tau\cos(\theta - \phi) & r\sin(\theta - \phi) \end{bmatrix} \tag{31}$$

is the desired matrix.

The angle $\phi$, defined by (29), has a nice geometrical interpretation. Indeed, if we apply the complex linear fractional transformation

$$w = \frac{z - i}{z + i},$$

to the roots $z_j'$, a simple calculation shows that $-2\phi$ is just the argument of the image

$$w_3' = \frac{z_3' - i}{z_3' + i} = \varrho \cdot e^{-2i\phi}.$$

Using the fact that linear fractional transformations take circles to circles, [1; Theorem 14, p. 80], we conclude that $-2\phi$ is the angle between the imaginary axis and the circle passing through the four points i, $-$i, $z_3'$ and $z_4'$, or, in terms of the original roots, the circle passing through the four roots $z_1$, $z_2$, $z_3$ and $z_4$. See Fig. 1.

The only other quantity in the formula (31) for our linear transformation which does not yet have a geometrical interpretation is the quantity $\tau$. If we

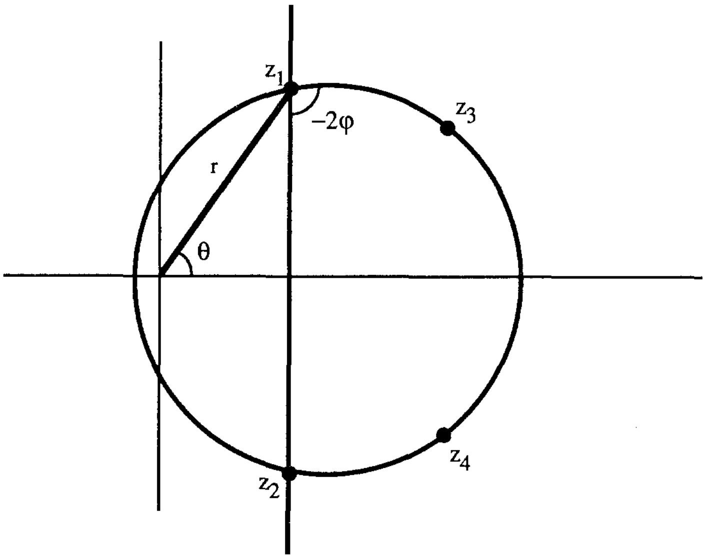


*Fig. 1.*

use the fact, [1; Theorem 12, p. 78], that the cross-ratio

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}$$

is preserved under any linear fractional transformation (26), we see that $\tau$ must be a solution of the equation

$$\frac{(\tau\mathrm{i} - \tau^{-1}\mathrm{i}) \cdot (-\tau\mathrm{i} + \tau^{-1}\mathrm{i})}{2\tau\mathrm{i} \cdot 2\tau^{-1}\mathrm{i}} = \frac{(z_1 - z_3) \cdot (z_2 - z_4)}{(z_1 - z_2) \cdot (z_3 - z_4)},$$

Simplifying, we find that if $z_1 = a + \mathrm{i}b$, $z_3 = c + \mathrm{i}d$, (and $z_2$ and $z_4$ are the complex conjugates), then

$$\tau = \frac{\sqrt{(a-c)^2 + (b-d)^2} + \sqrt{(a-c)^2 + (b+d)^2}}{\sqrt{4bd}} = \frac{s_1 + s_2}{\sqrt{s_3 \cdot s_4}},$$

where $s_1$, $s_2$, $s_3$, $s_4$ are the lengths of the four indicated line segments in Fig. 2.

Once we have transformed the roots to $\pm\tau\mathrm{i}$, $\pm\tau^{-1}\mathrm{i}$, the "uniqueness" part of the lemma is proved by direct determination of which $2 \times 2$ matrices

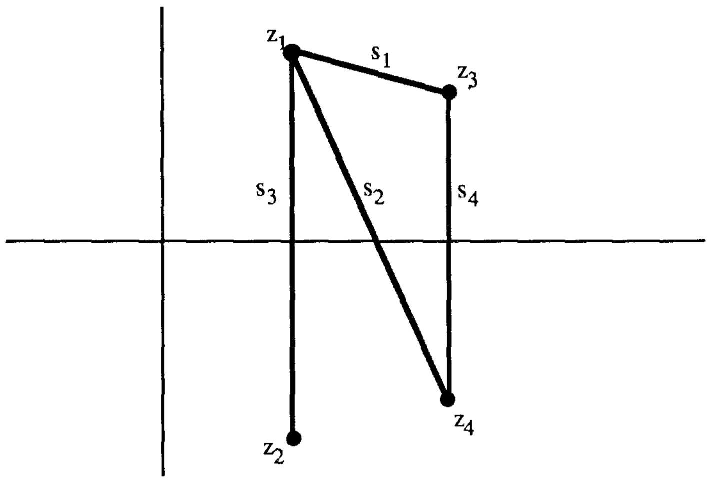


*Fig. 2.*

$K$ correspond to linear fractional transformations which preserve the set of points $S_\tau = \{\pm\tau i, \pm\tau^{-1}i\}$. It is straightforward to verify that a linear fractional transformation with matrix $K$ takes the set $S_\tau$ to the set $S_{\tilde{\tau}}$ if and only if $\tau = \tilde{\tau}$, and either, for $\tau \neq 1$, $K$ is one of the four matrices (27), or for $\tau = 1$, $K$ is a multiple of an orthogonal matrix, (28); the details are left to the reader. This completes the proof of Lemma 3.

An important point is that, once the roots $z_1, z_2, z_3, z_4$ of the discriminant $\Delta_{\mathbf{x}}(\mathbf{u})$ are known, the construction of the linear transformation (31) which maps them to $\pm\tau i$, $\pm\tau^{-1}i$ is completely explicit.

LEMMA 4. *Let $Q(\mathbf{x}, \mathbf{u})$ be a strongly elliptic planar biquadratic symbol. If the two discriminants $\Delta_{\mathbf{x}}(\mathbf{u})$ and $\Delta_{\mathbf{u}}(\mathbf{x})$ have the same roots $\pm\tau i$, $\pm\tau^{-1}i$ as an orthotropic symbol, then $Q$ is a scalar multiple of an orthotropic symbol.*

(In other words, the only positive definite symbols with the same roots as an orthotropic symbol are the orthotropic symbols.)

*Proof.* Suppose $Q$ has the given discriminant roots $S_\tau = \{\pm\tau i, \pm\tau^{-1}i\}$. We know by Theorem 1 that there is a linear change of variables $\mathbf{x} \to A\mathbf{x}$, $\mathbf{u} \to B\mathbf{u}$ which changes $Q$ into an orthotropic symbol, which must have roots $S_{\tilde{\tau}} = \{\pm\tilde{\tau} i, \pm\tilde{\tau}^{-1}i\}$ for some $\tilde{\tau}$. Thus the matrices $A$ and $B$ must map the set of roots $S_\tau$ to the set $S_{\tilde{\tau}}$. However, according to the proof of Lemma 3, this is possible if and only if $\tau = \tilde{\tau}$ and $A$ and $B$ have either the form (27) in the case of simple roots ($\tau \neq 1$; the orthotropic case) or the form (28) in the case of double roots ($\tau = 1$; the isotropic case). Moreover, the inverses

$A^{-1}$ and $B^{-1}$, which map the orthotropic symbol back to $Q$, have the same form. However, it is easy to see that these particular linear transformations map an orthotropic symbol to a scalar multiple of some other orthotropic symbol. Thus $Q$ itself must be an orthotropic symbol, and the lemma is proved.

We can now give an easy, explicit method for transforming an arbitrary strongly elliptic symbol into an orthotropic symbol.

THEOREM 5. *Let $Q(\mathbf{x}, \mathbf{u})$ be a strongly elliptic planar biquadratic symbol. Let $A$ be a linear transformation which maps the roots of the discriminant $\Delta_{\mathbf{x}}(\mathbf{u})$ to $\pm\tau i$, $\pm\tau^{-1} i$ and $B$ be a linear transformation which maps the roots of $\Delta_{\mathbf{u}}(\mathbf{x})$ to the same values $\pm\tau i$, $\pm\tau^{-1} i$. (The fact that $\tau$ is the same for both discriminants is a consequence of the fact that $Q$ is equivalent to an orthotropic symbol, or, more fundamentally, of the fact proved in [12] that the cross-ratios of the four roots of both discriminants are the same.) Then the linear transformation $\mathbf{x} \to A\mathbf{x}$, $\mathbf{u} \to B\mathbf{u}$ changes $Q$ into a multiple of an orthotropic symbol, and so, by possibly one further rescaling $\mathbf{x} \to \lambda\mathbf{x}$, into an orthotropic symbol. In particular, if $Q$ is symmetric, then $A = B$, and we can construct a symmetric change of variables $\mathbf{x} \to \tilde{A}\mathbf{u}$, $\mathbf{u} \to \tilde{A}\mathbf{u}$, where $\tilde{A}$ is a suitable multiple of $A$, which changes $Q$ into an orthotropic symbol.*

This theorem is a simple consequence of the previous two lemmas. The "explicitness' of the construction of the matrices $A$ and $B$ is subject only to the solution of the two quartic polynomial equations $\Delta_{\mathbf{x}}(\mathbf{u}) = 0$ and $\Delta_{\mathbf{u}}(\mathbf{x}) = 0$. In the symmetric case, these are, of course, the same equation.

Returning to the Lagrangian picture, and keeping (22) in mind, we see that we have proved the following explicit result concerning the determination of the canonical orthotropic form of a symmetric strongly elliptic Lagrangian:

THEOREM 6. *Let $W(\nabla\mathbf{u})$ be a symmetric quadratic planar stored energy function, so $W$ is a quadratic polynomial in the strain tensor, (3). Let $Q(\mathbf{x}, \mathbf{u})$ be the symbol of $W$, and let $z_1, z_2, z_3, z_4$ be the complex roots of the discriminant $\Delta_{\mathbf{x}}(\mathbf{u})$ of $Q$, cf. (19), (24). Let $A$ be the matrix determined by (31), and let $B = A^{-1}$. Then the linear transformation*

$$\mathbf{x} \to B \cdot \mathbf{x}, \quad \mathbf{u} \to A \cdot \mathbf{u},$$

*will convert $W$ into a scalar multiple of an orthotropic stored energy function. Moreover, $W$ is equivalent to an isotropic stored energy function if* *and only if the roots $z_1$, $z_2$, $z_3$, $z_4$ form a pair of double complex conjugate roots.*

We can easily determine the explict formulas for the canonical elastic moduli $\alpha$ and $\beta$. Let $Q(x, y; u, v) = Q(\mathbf{x}, \mathbf{u})$ be the symbol, and let $a$, $b$, $c$, $d$ denote the entries of the matrix (31), as in (26). The coefficient of $u_x^2$ in the new Lagrangian under the prescribed change of variables is easily seen to be $Q(a, c, a, c)$, hence rescaling to get a truly orthotropic Lagrangian, we find that

$$\alpha = \frac{Q(a, c, b, d)}{Q(a, c, a, c)},$$

$$\beta = \frac{Q(a, d, a, d) + 2Q(a, d, b, c) + Q(b, c, b, c)}{2Q(a, c, a, c)}.$$

Finally, we prove Theorem 2. It is not difficult to see that if $Q$ is a positive definite orthotropic symbol, then one of three possibilities hold:

1) The roots $z_1$, $z_2$, $z_3$, $z_4$ are simple, and lie on the imaginary axis. This is the type of orthotropic symbol derived in the proof of Theorem 5.
2) The roots $z_1$, $z_2$, $z_3$, $z_4$ are simple, and lie on the unit circle $|z| = 1$. These are the orthotropic symbols whose moduli do *not* satisfy inequality (15).
3) There are a complex conjugate pair of double roots at $\pm\mathrm{i}$. These are the isotropic symbols.

Thus, to prove Theorem 2, we need only deal with case 2. Here the elementary linear fractional transformation

$$z \to \frac{z - 1}{z + 1}$$

transforms the unit circle to the imaginary axis. The corresponding transformation in the physical variables is

$$(x, y) \to (x + y, x - y), \quad (u, v) \to (u + v, u - v). \tag{32}$$

It is easy to check that this has the effect of changing the canonical elastic moduli $\alpha$ and $\beta$ into moduli $\tilde{\alpha}$ and $\tilde{\beta}$ determined by case iv) of Theorem 2. (Vice versa, if the roots start out on the imaginary axis, then (32) transforms them onto the unit circle.) Once the roots $z_j$ are on the imaginary axis, Lemma 3 gives all possible transformations which preserve the orthotropic form of the symbol. Applying these to either $\alpha$, $\beta$ or to the $\tilde{\alpha}$, $\tilde{\beta}$ obtained from (32), produces all the remaining possibilities i)–vii). Note especially that in the isotropic case, the rotations (28) leave the moduli unchanged, and thereby give a one-parameter symmetry group of an isotropic material which has no anisotropic counterpart. (The existence of an extra one-parameter symmetry group is yet another way of recognizing an isotropic material.)

## Acknowledgements

I would like to thank Don Carlson for many helpful comments on an earlier version of this paper.

This research was supported in part by NSF grant DMS 86-02004.

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