Canonical Forms and Conservation Laws in Linear Elastostatics

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Abstract

In this paper, we shall review earlier work on canonical forms in linear elasticity, and applications to the classification of conservation laws (path-independent integrals).

1 Introduction

The detailed investigation of complex mathematical objects can often be simplified through the use of specially adapted coordinate systems in which the object takes a simple "canonical form". Elementary examples include the Jordan canonical form of a square matrix, Sylvester's Theorem on the representation of a quadratic form as a sum of squares, and Darboux' Theorem on the canonical form of Hamiltonian structures. In elasticity, the determination of canonical forms for elastic materials, either linear or nonlinear, is more recent. Lekhnitskii, [14], and Ting, [33], discuss canonical forms and invariants for elastic moduli under the physically based class of rotations. In [22] canonical forms for two-dimensional materials under general linear transformations were found. These were extended to planar displacements of three-dimensional materials in [25] with further refinements in [9]. The classification of canonical forms in fully three-dimensional materials is, however, not known. Applications of these results to the classification of conservation laws or path-independent integrals appear in [8, 9, 10, 20, 23]. Using a remarkable mathematical correspondence between planar elastic and dielectric media, Milton and Movchan, [16], have applied the planar canonical forms to study waves in an anisotropic dielectric medium.

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2   The Equations of Elasticity.

The equations of hyper-elasticity constitute a self-adjoint, quasi-linear system of second-order partial differential equations for the deformation (or, in the linear case, displacement) $\mathbf{u} = \mathbf{f}(\mathbf{x})$. Here $\mathbf{u} = (u^1, \ldots, u^q) \in \mathbb{R}^q$, and $\mathbf{x} = (x_1, \ldots, x_p)$ are the material coordinates in the elastic body $\Omega \subset \mathbb{R}^p$. For planar elasticity, $p = q = 2$, while $p = q = 3$ for fully three-dimensional elastic media. The Stroh formalism discussed below applies to a hybrid case, that of planar displacements of three-dimensional bodies, where $p = 2$, while $q = 3$. When $p = 2$, we will sometimes use the notation $\mathbf{x} = (x_1, x_2) = (x, y)$ and $\mathbf{u} = (u^1, u^2, u^3) = (u, v, w)$. The equilibrium equations are the Euler-Lagrange equations for the stored energy functional

$$\mathcal{W}[u] = \int_\Omega W(\mathbf{x}, \nabla \mathbf{u}) \, d\mathbf{x}. \tag{2.1}$$

The physical conditions of frame indifference, strong ellipticity, etc., will restrict stored energy functions which are of relevance to elasticity, although our initial remarks apply to quite general variational problems. The stored energy is not uniquely determined by its Euler-Lagrange equations, since we can add any divergence, replacing W by $W + \text{Div } \mathbf{P}$, although this will, in general, alter the natural boundary conditions associated with the problem.

At a fixed material point $\mathbf{x} = \mathbf{b}$ and a fixed value of deformation gradient $\mathbf{u} = F$, we define the *symbol* of the variational problem (2.1) to be the "biquadratic" polynomial

$$Q_{\mathbf{b}, F}(\mathbf{x}, \mathbf{u}) = \sum_{i,j,k,l} \frac{\partial^2 W}{\partial u_j^i \partial u_l^k}(\mathbf{b}, F) \, u^i u^k x_j x_l, \qquad \mathbf{x} \in \mathbb{R}^p, \quad \mathbf{u} \in \mathbb{R}^q. \tag{2.2}$$

The symbol is unaffected by the addition of a null Lagrangian to W. The Legendre–Hadamard condition requires that the symbol Q be *positive definite* in the sense that

$$Q_{\mathbf{b}, F}(\mathbf{x}, \mathbf{u}) > 0 \qquad \text{whenever} \qquad \mathbf{x} \neq 0, \quad \mathbf{u} \neq 0. \tag{2.3}$$

for all $\mathbf{b} \in \Omega$, and F such that $\det F > 0$. For simplicity, we will restrict to homogeneous materials, whereby the stored energy function $W(\nabla \mathbf{u})$ depends only on the deformation gradient.

In linear elasticity, the stored energy is a symmetric quadratic function of the displacement gradient

$$\mathcal{W}[\mathbf{u}] = \int_\Omega \sum_{i,j,k,l} a_{ijkl} \frac{\partial u^j}{\partial x_i} \frac{\partial u^k}{\partial x_l} \, d\mathbf{x}. \tag{2.4}$$

The equilibrium equations are the associated Euler-Lagrange equations:

$$\sum_{j,k,l} a_{ijkl} \frac{\partial^2 u^k}{\partial x_i \partial x_l} = 0, \qquad i = 1, \ldots, q. \tag{2.5}$$

The values a_{ijkl} are the *variational moduli*. For a general quadratic variational problem (2.4), the symbol Q is independent of the value of the displacement gradient F, and also the

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material point $\mathbf{b}$ provided the body is homogeneous. It can be found directly by replacing $\nabla\mathbf{u}$ in W by the rank one tensor $\mathbf{x} \otimes \mathbf{u} = \mathbf{u} \cdot \mathbf{x}^T$:

$$Q(\mathbf{x}, \mathbf{u}) = W(\mathbf{x} \otimes \mathbf{u}) = \sum_{i,j,k,l} a_{ijkl} \, u^i u^k x_j x_l. \tag{2.6}$$

Since every quadratic divergence is a linear combination of the 2×2 Jacobian determinants $\partial(u^i, u^k)/\partial(x_j, x_l)$, a homogeneous quadratic stored energy function is uniquely determined by its symbol up to a divergence.

In the case $p = q$, the assumption of frame indifference requires that the stored energy (2.4) depends only on the strain tensor $\varepsilon = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$, so

$$\mathcal{W}[\mathbf{u}] = \int_\Omega \sum_{i,j} \sigma_{ij}\,\varepsilon_{ij}\,d\mathbf{x} = \int_\Omega \sum_{i,j,k,l} c_{ijkl}\,\varepsilon_{ij}\varepsilon_{kl}\,d\mathbf{x}, \tag{2.7}$$

where $\sigma = \mathbf{C}(\varepsilon)$ is the associated stress tensor. In this case, the equilibrium equations (2.5) take the form

$$\mathrm{Div}\ \sigma = 0. \tag{2.8}$$

The fourth rank tensor $\mathbf{C} = (c_{ijkl})$ is the *elasticity tensor*. The components of $\mathbf{C}$ are called the *elastic moduli* or *elasticities* of the material, and have the following symmetries

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \tag{2.9}$$

There are 21 independent elasticities in three-dimensional problems and 6 independent moduli in two-dimensional problems. The symmetry restrictions (2.9) imply that the symbol is *symmetric*, i.e. $Q(\mathbf{x}, \mathbf{u}) = Q(\mathbf{u}, \mathbf{x})$.

3   The Stroh formalism

The Stroh formalism, [5, 31, 32, 33] deals with the hybrid case of planar displacements, $p = 2$, of a three-dimensional body, $q = 3$. For any displacement $\mathbf{u}$ that satisfies (2.8), there exist three *stress potentials* $\Phi = (\phi^1(\mathbf{x}), \phi^2(\mathbf{x}), \phi^3(\mathbf{x}))$ such that

$$\sigma_{i1} = -\frac{\partial \phi^i}{\partial y}, \qquad \sigma_{i2} = \frac{\partial \phi^i}{\partial x}, \qquad i = 1, 2, 3. \tag{3.1}$$

We define the following 3×3 matrices:

$$Q = Q_{ij} = c_{i1j1}, \qquad T = T_{ij} = c_{i2j2}, \qquad R = R_{ij} = c_{i1j2}. \tag{3.2}$$

We note that Q and T are symmetric and positive definite. We then construct the following 6×6 matrices:

$$M_1 = \begin{bmatrix} -R^T & I \\ -Q & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} T & 0 \\ R & I \end{bmatrix}. \tag{3.3}$$

The relations (2.8), (3.1), can be rewritten as

$$M_1 \frac{\partial \mathbf{Y}}{\partial x} = M_2 \frac{\partial \mathbf{Y}}{\partial y} \qquad \text{where} \qquad \mathbf{Y} = \begin{bmatrix} \mathbf{u} \\ \Phi \end{bmatrix} \in \mathbb{R}^6. \tag{3.4}$$

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The 6×6 matrix

$$N = M_2^{-1} M_1 = \begin{bmatrix} -T^{-1} R^T & T^{-1} \\ R T^{-1} R^T - Q & -R T^{-1} \end{bmatrix} \tag{3.5}$$

is called the *fundamental elasticity matrix*.

The characteristic polynomial $s(\lambda)$ of N is a sixth-degree polynomial known as the "Stroh sextic". Since the strain energy is positive definite, so is $s(\lambda)$, [5], and hence its roots come in three complex conjugate pairs. If the matrix N is diagonalizable, the material is called *nondegenerate*. Let p_1, p_2, p_3 be the eigenvalues with positive imaginary part, and $\mathbf{V}_\alpha \in \mathbb{C}^6$ the corresponding complex eigenvectors. In the nondegenerate case, the general solution to (3.4) has the form

$$\mathbf{Y} = \sum_{\alpha=1}^3 \left[\mathbf{V}_\alpha \cdot f_\alpha(z_\alpha) + \overline{\mathbf{V}_\alpha} \cdot \overline{f_\alpha(z_\alpha)} \right]. \tag{3.6}$$

Here the *Stroh functions* f_α are arbitrary complex-analytic functions of their respective argument $z_\alpha = x + p_\alpha y$; see [33] for details. If we write $\mathbf{V}_\alpha = \begin{bmatrix} \mathbf{A}_\alpha \\ \mathbf{B}_\alpha \end{bmatrix}$ with $\mathbf{A}_\alpha, \mathbf{B}_\alpha \in \mathbb{C}^3$, the vectors $\mathbf{A}_\alpha$ are called the *Stroh eigenvectors*. It is easy to show that they satisfy (no sum):

$$\begin{aligned} \left(Q + p_\alpha (R + R^T) + p_\alpha^2 T \right) \mathbf{A}_\alpha &= 0, \\ \mathbf{B}_\alpha = \left(R^T + p_\alpha T \right) \mathbf{A}_\alpha &= -\frac{1}{p_\alpha} \left(Q + p_\alpha R \right) \mathbf{A}_\alpha, \end{aligned} \tag{3.7}$$

where p_α is the corresponding root of the Stroh sextic. The solution for degenerate materials, which includes the isotropic case, is similar, but more complicated; see [33] or [9] for explicit formulations.

The Stroh formalism reduces to the well-known Muskhelishvili, [17], approach in the planar case. The Stroh sextic now reduces to a quartic polynomial with two complex-conjugate pairs of roots. The solution (3.6) is similar, but the sum only goes from 1 to 2. The Airy stress function is $U(x, y) = 2 \, Re \left[U_1(z_1) + U_2(z_2) \right]$ where U_α are arbitrary analytic functions of z_α, with $f_\alpha(z_\alpha) = U_\alpha'(z_\alpha)$ giving the solution (3.6).

4 Changes of Variable.

In the calculus of variations, the basic equivalence problem is to determine when two variational problems can be transformed into each other by a suitable change of variables. For nonlinear variational problems in several independent and dependent variables, there is as yet no solution, although preliminary analysis based on the powerful Cartan equivalence method, [27, 28], has been done. The first of the invariants arising from the Cartan method is the symbol (2.2), and so we must understand canonical forms for linear variational problems before further progress on the general nonlinear problem is possible.

In the case of linear elasticity, we may restrict our attention to linear changes of variables:

$$\mathbf{x} \longmapsto A \mathbf{x}, \qquad \mathbf{u} \longmapsto B \mathbf{u}. \tag{4.1}$$

In the elastic case, we choose B to be a scalar multiple of A^{-T} to preserve the symmetry constraints. Under the change of variables (4.1), the stored energy is transformed according to

$$W(\nabla \mathbf{u}) \longmapsto \widetilde{W}(\nabla \mathbf{u}) = W(B \, \nabla \mathbf{u} A^{-1}) \, |\det A|.$$

Our fundamental problem, then, is to determine matrices A and B which will simplify the elastic moduli c_{ijkl} (or variational moduli a_{ijkl} in the general case) as much as possible. Stated in this form, the question appears to be quite natural from a mathematical point of view, even though it may not have an immediate physical motivation. Indeed, the linear maps determined by the matrices A and B will not in general have any direct physical interpretation, except in the special case of orthogonal transformations (rotations), when they represent a physical change of frame, [14, 33].

The linear change of variables (4.1) acts on the symbol via $Q(\mathbf{x}, \mathbf{u}) \longmapsto Q(A\mathbf{x}, B\mathbf{u})$. (Actually, one should replace A by a multiple of A^{-T}, but this does not affect the discussion.) Thus we are led to the purely algebraic problem of determining canonical forms for biquadratic polynomials under the change of variables.

We now discuss the relevant algebraic properties of biquadratic symbols, concentrating on the cases $p = q = 2$ and $p = 2$, $q = 3$. (Note that if either $p = 1$ or $q = 1$, the symbol is an ordinary quadratic polynomial, whose canonical forms, determined by Sylvester's law of inertia, are well known. In particular, only the rank and signature are invariants, and there are no canonical moduli in these special cases.) Let us write the symbol in the matrix form

$$Q(\mathbf{x}, \mathbf{u}) = \mathbf{u}^T R(\mathbf{x})\mathbf{u}, \tag{4.2}$$

where, assuming strong ellipticity, $R(\mathbf{x})$ is a real $q \times q$ symmetric, positive-definite matrix of homogeneous quadratic polynomials of the variables $\mathbf{x}$. Just as the analysis of ordinary real polynomials requires an understanding of their complex roots, and so we may regard $\mathbf{x}$ and $\mathbf{u}$ as complex vectors, and Q as a complex-valued biquadratic polynomial. By the strong ellipticity assumption, for generic vectors $\mathbf{x} \in \mathbb{C}^p$, the matrix $R(\mathbf{x})$ has full rank; it is important to distinguish the exceptional points where R has less than maximal rank. Define the *discriminant*

$$\Delta_{\mathbf{u}}(\mathbf{x}) = \det R(\mathbf{x}), \tag{4.3}$$

which is a homogeneous polynomial of degree $2q$. A *root* of $\Delta_{\mathbf{u}}(\mathbf{x})$ is a nonzero vector $0 \neq \mathbf{x} \in \mathbb{C}^p$ satisfying $\Delta_{\mathbf{u}}(\mathbf{x}) = 0$. Homogeneity of the symbol polynomial implies that we should identify roots that are complex scalar multiples of each other. The roots of the discriminant play a crucial role in the classification of these biquadratic polynomials, and hence of quadratic variational problems. Strong ellipticity implies that the roots always appear in complex conjugate pairs. Interestingly, in cases covered by the Stroh formalism, the discriminant is the *same* as the Stroh quartic or sextic: $s(\lambda) = \Delta_{\mathbf{u}}(\lambda, 1)$.

Clearly, one can interchange the roles of $\mathbf{x}$ and $\mathbf{u}$ in the above discussion, producing a corresponding discriminant $\Delta_{\mathbf{x}}(\mathbf{u})$. Except in the symmetric elastic case with $p = q$, these two polynomials are not the same (indeed, if $p \neq q$, they do not even depend on the same number of variables), nor are their roots easily compared. Nevertheless, there are subtle and remarkable relations between the roots of the two discriminants. For example, in the planar case $p = q = 2$, the discriminant $\Delta_{\mathbf{u}}(\mathbf{x})$ has simple roots if and only if $\Delta_{\mathbf{x}}(\mathbf{u})$ does. However,

it is not true that if $\Delta_{\mathbf{u}}(\mathbf{x})$ has a double root then $\Delta_{\mathbf{x}}(\mathbf{u})$ has a double root, although it does have a root of multiplicity at least two; see [24].

5   Canonical Elastic Moduli.

The number of canonical moduli can be determined directly by a simple dimension count. A general biquadratic polynomial or symbol $Q(\mathbf{x}, \mathbf{u})$ depending on $\mathbf{x} \in \mathbb{R}^p$, and $\mathbf{u} \in \mathbb{R}^q$ has a total of $\frac{1}{4}p(p+1)q(q+1)$ independent variational moduli. The possible changes of variables (4.1) will involve $p^2 + q^2$ arbitrary parameters, but the transformation just rescaling $\mathbf{x}$ (where A is a multiple of the identity) has the same effect as that rescaling $\mathbf{u}$, so there are $p^2 + q^2 - 1$ independent parameters at our disposal. Thus, in general, we expect the canonical quadratic variational problem to depend on

$$\frac{p(p+1)q(q+1)}{4} - p^2 - q^2 + 1$$

canonical moduli. For planar elasticity, $p = q = 2$, so we will find just 2 canonical elastic moduli. In three dimensions, we should obtain 19 canonical elastic moduli; however, imposing the symmetry conditions (2.9) reduces the count to 12. In the case $p = 2$, $q = 3$ covered by the Stroh formalism, we expect 6 independent canonical elastic moduli.

5.1   The Planar Case

In the case of planar elasticity, $p = q = 2$, the discriminant $\Delta_{\mathbf{u}}(\mathbf{x})$ is a homogeneous quartic polynomial of the two variables $\mathbf{x} = (x, y)$, which has either two complex conjugate pairs of simple roots, or a complex conjugate pair of double roots. In the former case, we can find a real linear change of variables which moves the roots onto the imaginary axis, to $(1, \pm \tau i), (1, \pm \tau^{-1} i)$, for some $\tau > 1$. (The constant τ is an invariant associated with the roots of the quartic.) In the latter case, we move the roots to $(1, \pm i)$. Performing the same change of variables on the other discriminant $\Delta_{\mathbf{x}}(\mathbf{u})$ (where, according to theory, the value of τ is necessarily the same), it can be proved, [22], that the symbol thereby reduces to one of "strongly orthotropic" form

$$x^2 u^2 + y^2 v^2 + \alpha(y^2 u^2 + x^2 v^2) + 2\beta xyuv, \tag{5.1}$$

where the canonical moduli α, β satisfy the inequalities

$$\alpha > 0, \qquad \beta \geq 0, \qquad |\alpha - 1| > \beta, \tag{5.2}$$

in the case when the discriminant has simple roots, or

$$0 < \alpha \leq 1, \qquad \beta = 1 - \alpha, \tag{5.3}$$

in the case of double roots. The corresponding stored energy function is given by the *orthotropic Lagrangian*

$$u_x^2 + \alpha u_y^2 + 2\beta u_x v_y + \alpha v_x^2 + v_y^2. \tag{5.4}$$

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In fact, the Lagrangian (5.4) is, modulo a null Lagrangian, just a rescaled version of the standard stored energy of a linear, planar orthotropic elastic material

$$c_{1111}u_x^2 + c_{1212}(u_y + v_x)^2 + 2c_{1122}u_xv_y + c_{2222}v_y^2. \tag{5.5}$$

Indeed, after adding the null Lagrangian $c_{1212}(u_xv_y - u_yv_x)$, a simple rescaling will place this stored energy into the form (5.4), where

$$\alpha = \frac{c_{1212}}{\sqrt{c_{1111}c_{2222}}}, \qquad \beta = \frac{c_{1122}}{\sqrt{c_{1111}c_{2222}}}. \tag{5.6}$$

The discriminant has a complex conjugate pair of double roots if and only if the material is equivalent to an isotropic material, with $\alpha = \mu/(2\mu + \lambda)$, $\beta = (\mu + \lambda)/(2\mu + \lambda)$, where λ and μ are the classical Lamé moduli. Moreover, the isotropic stored energies are distinguished by the presence of a one-parameter symmetry group corresponding to the rotational invariance of (5.4) when $\alpha + \beta = 1$. Two isotropic Lagrangians determine the same orthotropic Lagrangian if and only if they have the same value for Poisson's ratio. The cases when the discriminant has simple roots, and the Lagrangian has at most discrete symmetries, correspond to "truly" anisotropic materials.

Theorem 5.1 *Let $W(\nabla\mathbf{u})$ be a homogeneous first order planar quadratic Lagrangian which satisfies the Legendre-Hadamard strong ellipticity condition. Then W is equivalent to a orthotropic Lagrangian (5.4), where the canonical elastic moduli α and β satisfy the strong ellipticity inequalities $\alpha > 0$, $|\beta| < \alpha + 1$. The corresponding Euler-Lagrange equations are thus equivalent to the "orthotropic Navier equations"*

$$u_{xx} + \alpha u_{yy} + \beta v_{xy} = 0, \qquad \beta u_{xy} + \alpha v_{xx} + v_{yy} = 0. \tag{5.7}$$

The six components of the elastic tensor $\mathbf{C}$ can be summarized in matrix form by

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{16} \\ & c_{22} & c_{26} \\ & & c_{66} \end{bmatrix},$$

where c_{ij} represents the standard contracted notation, [7]. In this version, the canonical form is found to be

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ & c_{11} & 0 \\ & & c_{66} \end{bmatrix}.$$

Note that it is possible to rescale so that $c_{11} = 1$.

See [22] for the explicit formulas for the change of variables taking a given stored energy function into its canonical orthotropic form. One can reduce a general strongly elliptic orthotropic stored energy (5.4) to a unique strongly orthotropic Lagrangian satisfying the more restrictive inequalities (5.2) or (5.3) using one or more of the three basic discrete equivalences taking the moduli (α, β) to either

$$(\alpha, -\beta), \qquad \text{or} \qquad \left(\frac{1}{\alpha}, \frac{1}{\beta}\right), \qquad \text{or} \qquad \left(\frac{1 + \alpha - \beta}{1 + \alpha + \beta}, \frac{2 - 2\alpha}{1 + \alpha + \beta}\right). \tag{5.8}$$

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Therefore, except in a few "exceptional" cases, each orthotropic Lagrangian is equivalent to seven different orthotropic Lagrangians.

Remark: A complete set of canonical forms for general quadratic variational problems in the case $p = q = 2$ is known; see [24] for details.

5.2   The Stroh Case

Turning to the case of planar deformations of a three-dimensional material, i.e. $p = 2$, $q = 3$, we must determine canonical forms for a positive definite "bi-ternary quadratic"

$$Q(x, y; u, v, w) > 0, \qquad (x, y) \neq 0, \quad (u, v, w) \neq 0. \tag{5.9}$$

Such a symbol will be the planar restriction of a three-dimensional elastic stored energy function W provided it satisfies

$$Q(x, y; u, v, 0) = Q(u, v; x, y, 0). \tag{5.10}$$

The discriminant $\Delta_{\mathbf{u}}(\mathbf{x})$ is a homogeneous sextic polynomial in $\mathbf{x} = (x, y)$, which, according to the strong ellipticity assumption, has three complex conjugate pairs of roots.

A stored energy function is called *separable* if there exist coordinates $\mathbf{x}$, $\mathbf{u}$ such that its symbol takes the form

$$Q(x, y; u, v, w) = R(x, y; u, v) + s(x, y)w^2. \tag{5.11}$$

Note that in this case, the Euler–Lagrange equations separate into a linear system for u, v, and a single separate second order elliptic equation for w, so that the problem essentially reduces to a problem for purely planar elasticity (with a separate anti-planar problem). If R is isotropic, then the rotational symmetry group can be used to diagonalize the quadratic polynomial $s(\mathbf{x})$, but, in general, we are left with the 4 parameter class of separable canonical forms

$$u_x^2 + \alpha u_y^2 + 2\beta u_x v_y + \alpha v_x^2 + v_y^2 + \gamma w_x^2 + 2\delta w_x w_y + \varepsilon w_y^2. \tag{5.12}$$

(One of the parameters $\gamma, \delta, \varepsilon$ can be eliminated by rescaling w.) Thus, the equilibrium equations reduce to the orthotropic Navier equations (5.7) together with a second order elliptic equation for w, which can be easily transformed into Laplace's equation, although not without changing the orthotropic form of the planar part.

In the Stroh formalism, the essential elasticities can be summarized with the three symmetric matrices Q, $R + R^T$, and T, given by

$$\begin{bmatrix} c_{11} & c_{16} & c_{15} \\ & c_{66} & c_{65} \\ & & c_{55} \end{bmatrix} \quad \begin{bmatrix} 2c_{16} & c_{12} + c_{66} & c_{14} + c_{56} \\ & 2c_{26} & c_{46} + c_{25} \\ & & 2c_{45} \end{bmatrix} \quad \begin{bmatrix} c_{66} & c_{26} & c_{46} \\ & c_{22} & c_{24} \\ & & c_{44} \end{bmatrix}$$

One can see that there are 15 independent moduli. The material is separable when the third columns of these matrices are of the form $[0 \; 0 \; *]^T$. In [9] it was shown that a material is separable if and only if one of the Stroh eigenvectors is a real vector.

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If the material if not separable, we have the canonical form, [25],

$$
\begin{bmatrix} c_{11} & 0 & c_{15} \\ & c_{66} & c_{65} \\ & & c_{55} \end{bmatrix}
\begin{bmatrix} 0 & c_{12}+c_{66} & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}
\begin{bmatrix} c_{66} & 0 & c_{46} \\ & c_{11} & c_{24} \\ & & c_{44} \end{bmatrix}
$$

Note again that we may scale so that $c_{11}=1$. These can further be refined if the material is degenerate. It turns out that there are two inequivalent classes of degenerate materials, [8, 9],

$$
\begin{bmatrix} c_{11} & 0 & c_{15} \\ & \dfrac{c_{11}-c_{12}}{2} & c_{56} \\ & & c_{44}+2c_{11} \end{bmatrix}
\begin{bmatrix} 0 & \dfrac{c_{11}+c_{12}}{2} & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}
\begin{bmatrix} \dfrac{c_{11}-c_{12}}{2} & 0 & c_{15} \\ & c_{11} & c_{56} \\ & & c_{44} \end{bmatrix}
\quad (5.13)
$$

$$
\begin{bmatrix} c_{11} & 0 & c_{46}+\dfrac{1}{\sqrt{c_{11}}} \\ & c_{11} & 0 \\ & & (c_{12}+c_{11})^{-2} \end{bmatrix}
\begin{bmatrix} 0 & c_{12}+c_{11} & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}
\begin{bmatrix} c_{11} & 0 & c_{46} \\ & c_{11} & 0 \\ & & (c_{12}+c_{11})^{-2} \end{bmatrix}
\quad (5.14)
$$

If, in addition, there is only one Stroh eigenvalue, so N is "extraordinarily degenerate", then the canonical forms are either (5.13) with $c_{15}=0$ and $c_{12}=\dfrac{c_{11}\left(2c_{11}^{2}-c_{56}^{2}\right)}{2c_{11}^{2}+c_{56}^{2}}$, or (5.14) with $c_{46}=\dfrac{-1}{2\sqrt{c_{11}}}$. For a discussion of inseparable, degenerate materials, see [34].

5.3 Three-Dimensional Case

Complete canonical forms for the fully three dimensional elastic tensor under the action of the general linear group remains an open problem. Lodge [15] has provided a set of necessary a sufficient conditions on the elasticities that guarantee that the tensor can transformed into an isotropic form. The general problem is difficult due to the complexity of the expressions involved. A promising approach seems to lie in constructing the invariants under the action of the general linear group. However, this too is an open problem. For corresponding work with respect to the action of the orthogonal group, see [11, 36] and the references therein.

6 Conservation Laws

By a *conservation law* for a system of partial differential equations, we mean a divergence expression Div $\mathbf{P}=0$ which vanishes on all solutions. The conservation law is called *trivial* if either $\mathbf{P}=0$ vanishes on all solutions, or Div $\mathbf{P}\equiv 0$ vanishes for *all* $\mathbf{u}$. Two conservation laws are *equivalent* if and only if they differ by a trivial conservation law. So far, only first order conservation laws, meaning that $\mathbf{P}(\mathbf{x},\mathbf{u},\nabla\mathbf{u})$ depends on at most first order derivatives, have been classified up to equivalence.

The most well-known example of a conservation law or path-independent integral in elasticity is the celebrated Eshelby energy-momentum tensor, [4, 29], which governs the

energy release rate at a singularity, [1]. In general, Noether's Theorem, [18, 26], provides a one-to-one correspondence between conservation laws and symmetries of the variational problem. Surprisingly, this fundamental result was not systematically applied in elasticity until the work of Günther, [6], and Knowles and Sternberg, [13]. The latter claimed to have a complete classification of all possible elastic conservation laws, but they failed to take into account more general types of symmetries as well as particular constitutive relations which can increase the number of laws. This prompted Edelen, [3], to propose that "...a detailed cataloging of *all* invariance transformations and conservation laws in linear elasticity would seem a worthy task". This served to motivate us to initiate a systematic classification program for conservation laws, [19, 20, 23, 9]. In this section we review what is known to date.

6.1   Betti Reciprocity

Any self-adjoint linear system admits a special class of conservation laws that arise from Betti's reciprocal theorem [7, Sect. 30]. A Betti-reciprocity law takes the explicit form

$$\text{Div } P = 0, \qquad \text{where} \qquad P_i = \sigma_{ij} \cdot \tilde{u}^j - \tilde{\sigma}_{ij} \cdot u^j. \tag{6.1}$$

Here $\mathbf{u}, \tilde{\mathbf{u}}$ are any two solutions of the equilibrium equations, with corresponding stress tensors $\sigma, \tilde{\sigma}$.

   In the linear case, each symmetry provides a recursion operator that produces symmetries and conservation laws of arbitrarily high order. We conjecture that, like the Laplace equation, [30], every higher order symmetry and conservation law is generated by the first order ones. The higher order conservation laws have not been investigated so far.

6.2   Two Independent Variables

The approaches and results for the planar case and the Stroh formalism are similar and we will present them together. We consider first order conservation laws Div $\mathbf{P} = D_x P_1 + D_y P_2 = 0$. The conditions can be presented in the convenient matrix form

$$\nabla P_1 = M \ \nabla P_2 \qquad \text{where} \qquad M = \begin{bmatrix} 0 & QT^{-1} \\ -I & (R + R^T)T^{-1} \end{bmatrix}, \tag{6.2}$$

and ∇ denotes the gradient with respect to to the derivative variables $u^i_j = \partial u^i / \partial x_j$. The matrix in (6.2) is similar to $-N$, cf. (3.5), and hence its characteristic polynomial is essentially same as the Stroh polynomial. The general solution to the equation (6.2) depends on the Jordan structure of the matrix M and can be found in [12]. For each complex conjugate pair of eigenvalues, we define the complex variables

$$\eta_\alpha = \mathbf{b}^T_\alpha \cdot \nabla \mathbf{u}, \qquad \text{where} \qquad \mathbf{b}_\alpha = \begin{bmatrix} -\dfrac{1}{p_\alpha} Q \mathbf{A}_\alpha \\ T \mathbf{A}_\alpha \end{bmatrix}$$

is the corresponding eigenvector of M, and $\mathbf{A}_\alpha$ is the corresponding Stroh eigenvector. The index α ranges from 1 to either 2 in the planar case, or 3 in the Stroh case. If the matrix is

not semisimple, we use the generalized eigenvectors to similarly define variables ξ (and ζ if there are two generalized eigenvectors). The first result appears in [9, 23, 37]; see also [35].

Theorem 6.1 *Every nontrivial first order conservation law for a nondegenerate material is a linear combination of the Betti reciprocal laws and the laws $\mathbf{P}^\alpha$ corresponding to the eigenvalues p_α of the elasticity matrix, where $F_\alpha(z_\alpha, \eta_\alpha) = P_1^\alpha + \overline{p_\alpha} P_2^\alpha$ are complex analytic functions of their arguments $z_\alpha = x + p_\alpha y$, $\eta_\alpha = \eta_{\alpha 1} + i\eta_{\alpha 2} = f'_\alpha(z_\alpha)$.*

If the material is degenerate (i.e. the matrix in (6.2) is irreducible), there are several different cases.

6.3   Planar, Irreducible

In the planar case, the material is equivalent to an isotropic material. If the material is isotropic, one can solve for the eigenvectors explicitly. The explicit form of the conservation laws in isotropic materials was first given in [20].

Theorem 6.2 *Every degenerate planar material is equivalent to an isotropic material. Every nontrivial conservation law is a combination of Betti reciprocity and the laws given by*

$$P_1 + \bar{p}P_2 = F(z, \eta) - \frac{i}{2Im\,p}\left(z\frac{\partial G}{\partial z} + \overline{G(z, \eta)}\right) + \xi\frac{\partial G}{\partial \eta} + c\bar{z} \cdot \eta^2 + \omega \cdot \eta. \qquad (6.3)$$

Here F and G are analytic functions of $\eta = \eta_1 + i\eta_2$, and $z = x + py$, c is a complex scalar, ω is a certain linear combination of the displacements u, v.

6.4   Stroh, Irreducible

In the Stroh formalism, if the matrix is irreducible, one can have either two distinct pairs (one pair doubled) or one tripled pairs of roots. The following results appear in [9].

Theorem 6.3 *Suppose M is irreducible with two real Jordan blocks. Let p_1 be the double root and p_2 the simple root with positive imaginary parts. Then every nontrivial conservation law is a combination of Betti reciprocity and the laws $\mathbf{P}, \widetilde{\mathbf{P}}$, where*

$$P_1 + \bar{p}_1 P_2 = F_1(z_1, \eta_1) - \frac{i}{2Im\,p_1}\left(\bar{z}_1\frac{\partial G}{\partial z_1} + \overline{G(z_1, \eta_1)}\right) + \xi\frac{\partial G}{\partial \eta_1} + c\bar{z}_1 \cdot \eta_1^2 + \omega \cdot \eta_1$$
$$\widetilde{P}_1 + \overline{p_2}\widetilde{P}_2 = F_2(z_2, \eta_2).$$

Here F_i and G are analytic in $\eta_\alpha = \mathbf{b}_\alpha^T \cdot \nabla\mathbf{u}$ and $z_\alpha = x + p_\alpha y$, c is a complex scalar, and ω is a certain linear combination of the displacements $\mathbf{u}$. The terms $c\bar{z}_1 \cdot \eta_1^2 + \omega \cdot \eta_1$ are nontrivial if and only if the material is separable.

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Theorem 6.4 *Suppose that M is irreducible with one real Jordan block, and let $p = p_1 + ip_2$ be the corresponding triple root with positive imaginary part. Every nontrivial conservation law is a combination of Betti reciprocity and the laws $\mathbf{P}$ with components*

$$P_1 + \overline{p}P_2 = F + \xi\frac{\partial G}{\partial \eta} - \frac{i}{2p_2}\overline{G} + \xi^2\frac{\partial^2 H}{\partial \eta^2} + 2\zeta\frac{\partial H}{\partial \eta} - \frac{i}{p_2}\cdot\xi\overline{\frac{\partial H}{\partial \eta}} + \frac{1}{2p_2^2}\overline{H}$$

where

$$F = F_0(z, \eta) - \frac{i}{2p_2}\overline{z}\frac{\partial G_0}{\partial z} - \frac{1}{2p_2^2}\left(\frac{\overline{z}^2}{2}\frac{\partial^2 H}{\partial z^2} + \overline{z}\frac{\partial H}{\partial z}\right) - c_1\overline{z}\cdot\eta^2 + \omega\cdot\eta$$

$$G = G_0(z, \eta) - \frac{i}{p_2}\overline{z}\frac{\partial H}{\partial z} - \frac{1}{2}c_2\overline{z}\cdot\eta^2$$

$$H = H(z, \eta).$$

The functions F_0, G_0 and H complex analytic in their arguments, the c_i are complex scalars, and ω is a certain linear combination of the displacements u^i.

There is some subtlety between repeated roots and irreducible matrices. In general, distinct eigenvalues implies that a given matrix is semisimple but not the other way around. If the Stroh sextic has one tripled pair of complex conjugate roots, then the matrix N will have one real Jordan block if and only if the material is inseparable. If the material is separable, there are exactly two real Jordan blocks. On the other hand, if the Stroh sextic has one doubled pair of roots and a second distinct pair, then the matrix N is semisimple if and only if the material is separable and the two root pairs corresponding to the planar part are distinct. See [9] for details.

6.5 Three independent variables

Finally, let us consider the full three-dimensional case. In general, the number of conservation laws a material may have depends whether or not the elasticities satisfy certain nondegeneracy conditions; these are closely related to the symmetry class of the elastic tensor. In particular, certain materials have more conservation laws than a generic material. We first state all the conservation laws which exist in all three-dimensional materials, regardless of its symmetry, [2].

Theorem 6.5 *Every nontrivial conservation law which exists for all three dimensional materials is a linear combination of the following laws: i) the stress $\sigma_{ij} = C_{ijkl}\cdot u_l^k$, ii) the Eshelby energy-momentum tensor $P_m^i = C_{jkli}\cdot u_k^j\cdot u_m^l - \frac{1}{2}\delta_m^i C_{jkrs}\cdot u_k^j\cdot u_s^r$ where δ_m^i is the Kronecker δ, iii) the "scaling" density $Y^i = x^j P_j^i + \frac{1}{2}u^j\sigma_{ij}$, and iv) the Betti reciprocal relations.*

Theorem 6.6 *All conservation laws of a three-dimensional isotropic material are linear combinations of those listed in Theorem 6.5 along with the densities*

$$R_j^i = \varepsilon_{jkl}\left(x^k P_l^i - u^k\sigma_{ik}\right)$$

$$Q_j^i = c_{11}c_{66}u_i^j u_k^k + c_{66}^2 u_l^j\left(u_i^l - u_l^i\right) + \frac{1}{2}\left(c_{66} + c_{12}\right)c_{11}\delta_j^i u_k^k u_l^l$$

$$T_j^i = \varepsilon_{jkl}\left((c_{66} + c_{12})x^k Q_l^i + c_{66}(c_{11} + c_{66})u^k\sigma_{il}\right) +$$

$$+ \frac{1}{2}c_{66}^2\left(c_{66} + c_{12}\right)\left(\varepsilon_{jkl}u^k u_j^l + \delta_j^i\varepsilon_{klm}u^l u_m^k\right),$$

where ε_{jkl} is the alternating tensor.

The isotropic classification appears in [20]. If the material is transversely isotropic, the most general conservation laws are those listed in Theorem 6.5, and R_3^i and a generalization of Q_3^i in 6.6; see [10]. In this reference, it is also shown that under certain degeneracy conditions, a transversely isotropic material may have generalizations of Q_1^i and Q_2^i as well. There are no more laws unless the material is equivalent to an isotropic material. An open problem is determining precisely when an anisotropic material has conservation laws beyond those of Theorem 6.5. This problem is closely related to the canonical forms problem mentioned earlier.

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