Conservation Laws in Elasticity

II. Linear Homogeneous Isotropic Elastostatics

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1. Introduction

In 1956 ESHELBY, [5], introduced his celebrated energy-momentum tensor in his study of lattice defects. This tensor had the useful property of providing nontrivial path-independent integrals for the equations of finite elasticity, or, in other words, providing densities of nontrivial conservation laws. The importance of this quantity was demonstrated in its rediscovery and use by RICE [19], in the study of the propagation of cracks in linearly elastic materials. The path-independence of the "J-integral" enables one to deduce properties of the material near the tip of a crack from its behavior some distance away, [6], [11].

The belated discovery of these integrals is curious, in that NOETHER's theorem relating symmetries and conservation laws had been available since 1918, [14]. Be that as it may, the elementary derivation of ESHELBY's tensor, as well as an angular momentum analogue and one other general conservation law using group-theoretic techniques remained unnoticed until KNOWLES & STERNBERG, [10], systematically applied a restricted version of NOETHER's theorem to the equations of elasticity. KNOWLES & STERNBERG further asserted that these constituted the only (quadratic) conservation laws derivable from NOETHER's theorem, a point echoed in more recent treatments of the subject, [3], [6], [7]. However, EDELEN, [4], has correctly noted that the restrictive notion of variational symmetry in these references precludes any claims of complete classification of conservation laws. Indeed, the last sentence of [4] reads "a detailed cataloguing of all invariance transformations and conservation laws in linear elasticity would seem a worthy task". It is the express purpose of this paper to complete this program for the most basic case of linear, homogeneous, isotropic elasticity.

As remarked in the first paper of this series, [18] (hereafter referred to as I, so that formulas or sections preceded by Roman I refer therein) the general version of NOETHER's original theorem allows the possibility of many more conservation laws being found by the same group-theoretic techniques. (See also [1], [16].) Here the general techniques developed in I are applied to the specific problems of two-dimensional and three-dimensional linear, homogeneous, isotropic elastostatics in the absence of body forces (Navier's equations—see section 2A). The nondegeneracy assumption $\mu(\mu+\lambda)(2\mu+\lambda) \neq 0$ on the Lamé moduli λ, μ is made throughout.

In three dimensions, the variational problem admits, in addition to the seven parameter group $E(3) \times \mathbb{R}$ of Euclidean motions and scaling found by KNOWLES & STERNBERG, an additional six parameter group, isomorphic to $E(3)$, of generalized first order symmetries, and hence six additional quadratic conservation laws. However, if the unexpected (and somewhat unphysical) restriction $7\mu + 3\lambda = 0$ holds, the underlying group is the full conformal group $O(3, 1)$ of three-dimensional space together with a second conformal group of generalized symmetries, leading to twenty independent quadratic conservation laws. (Although perhaps not physically motivated, a mathematical study of the geometrical properties of this case, which we call the *conformal Navier equations*, could prove very interesting.) Further, it will be shown that together with the linear conservation laws arising from Betti's reciprocal theorem, these constitute the only nontrivial conservation laws, hence invariant integrals, depending on position x, deformation u, and deformation gradient ∇u. Of course, as the underlying equations are linear, infinite families of conservation laws depending on higher order derivatives of the deformation can, as outlined at the end of section 2, be easily constructed.

The seven conservation laws valid in the conformal case $7\mu + 3\lambda = 0$ still lead to interesting divergence expressions of the form

$$\nabla \cdot A = B,$$

where A is quadratic in ∇u, and B is positive definite, more specifically a linear combination of $\|\nabla u\|^2$, $(\nabla \cdot u)^2$ and their moments. These may prove of use in the study of crack propagation, since although $\oint_S A \cdot dS$ no longer vanishes for closed surfaces S as it would if A were a conserved density ($B = 0$), we still have

$$\oint_S A \cdot dS \geqq 0$$

for any closed surface S. Thus measurement of A away from the tip of the crack leads to bounds on A near the crack. However, we shall not pursue this here.

For two-dimensional elasticity, the situation is even more surprising. Now infinite families of independent symmetries and conservation laws, no longer restricted to being at most quadratic in ∇u, appear. In the special *conformal case*, which now corresponds to $3\mu + \lambda = 0$ (when Navier's equations have an elementary explicit solution) even more symmetries and conservation laws apply. The situation is very reminiscent of the classification of conformal symmetries and conservation laws of Laplace's equation in flat space, [12], [16]. In three or more dimensions, the conformal symmetries form only a finite-dimensional group, whereas in two dimensions any analytic function gives rise to a conformal transformation. As might be expected, analytic function techniques play a key role here; the results are described in detail in sections 4, 5C. In particular, an infinite number of nontrivial path-independent integrals, of which RICE's J-integral is the most elementary example, exist. See also [21].

Besides the potential applications to crack problems, there are a number of other directions in which the results here can be applied and extended. FLETCHER,

[7], has shown how the conservation laws of KNOWLES & STERNBERG can be suitably modified to provide conservation laws for the equations of elastodynamics. Presumably, the general Hamiltonian techniques of [15] apply to give dynamical laws for the additional conservation laws derived here. The appearance of conformal symmetries is of interest. STRAUSS & MORAWETZ, [20], have shown the importance of conservation laws derived from the full conformal group of the wave equation for decay and scattering properties of certain types of nonlinear wave equations. Our laws and identities may have similar applications in elasticity. However, this paper will only be concerned with the systematic classification of conservation laws and symmetries for linear isotropic elastostatics. Investigation of these areas of application must be deferred to subsequent papers in this series.

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2. Discussion of Results

Except for the following innovations, standard tensor notation, as in [8], is used throughout. Vectors are 2-dimensional or 3-dimensional column vectors. The summation convention is used unless noted otherwise, the indices running from 1 to 3 (except in sections 4, 5C, where they run from 1 to 2). If v is a (three-dimensional) vector, $\hat{v}$ denotes the skew tensor with entries $\varepsilon_{ijk}v^k$. The cross products between vectors and tensors are defined by

$$v \wedge A = \hat{v} \cdot A, \qquad A \wedge v = A \cdot \hat{v}.$$

If $A(x)$ is a tensor-valued function, its divergence $\nabla \cdot A$ is the vector-valued function with entries $\partial_j A^i_j$, $\partial_j = \partial/\partial x^j$, where i, j are the row, column indices respectively of A. The transpose of a vector or tensor is denoted by a superscript ^{T}.

A. The Equations of Linear Isotropic Elasticity

As detailed by GURTIN, [8], the equilibrium equations for a homogeneous, isotropic linearly elastic medium in the absence of body forces arise from the variational principle

$$\mathscr{I} = \int_\Omega \mu \, \|\nabla u + \nabla u^T\|^2 + \tfrac{1}{2}\lambda(\nabla \cdot u)^2 \, dx. \tag{2.1}$$

The corresponding Euler-Lagrange equations are known as *Navier's equations*

$$\mathscr{E} = \mu \, \Delta u + (\mu + \lambda) \, \nabla(\nabla \cdot u) = 0. \tag{2.2}$$

The constants μ, λ are the Lamé moduli, and are usually subjected either to the restriction

$$\mu > 0, \qquad 2\mu + \lambda > 0 \tag{2.3}$$

ensuring strong ellipticity, or, more restrictively

$$\mu > 0, \quad 2\mu + 3\lambda > 0 \tag{2.4}$$

ensuring positive definiteness of the underlying elasticity tensor. We note here that strong ellipticity is just the Legendre-Hadamard condition that the matrix

$$Q(\xi) = \mu \|\xi\|^2 \mathbf{1} + (\mu + \lambda)\, \xi \otimes \xi \tag{2.5}$$

of quadratic functions of $\xi \in \mathbb{R}^3$ is positive definite (*cf.* I.3) and hence the general results discussed in I apply. We also make the restriction that $\mu + \lambda \neq 0$ throughout the paper; otherwise the Euler-Lagrange equations decouple into separate Laplace equations for the components of u. The structure of the symmetries and conservation laws for Laplace's equation are well known (*cf.* [12], [16]), and we need not further elaborate on this case here.

B. Conservation Laws in Three Dimensions

The complete classification of all first order conservation laws

$$\operatorname{Div} B = \nabla \cdot B(x, u, \nabla u) = 0 \tag{2.6}$$

for Navier's equations is presented here in tensor notation. Note that if $A(x, u, \nabla u)$ is a tensor density, then

$$\operatorname{Div} A = \nabla \cdot A = 0$$

gives three different conservation laws.

For $\mu(\mu + \lambda)(2\mu + \lambda) \neq 0$ and $7\mu + 3\lambda \neq 0$, any first order conservation law is a linear combination of the 16 conservation laws with densities

$$S = \mu \nabla u + (\mu + \lambda)(\nabla \cdot u)\,\mathbf{1},$$

$$P = \mu \nabla u^T \nabla u + (\mu + \lambda) \nabla u^T (\nabla \cdot u) - \tfrac{1}{2}[\mu \|\nabla u\|^2 + (\mu + \lambda)(\nabla \cdot u)^2]\,\mathbf{1},$$

$$R = x \wedge P + u \wedge S,$$

$$Y = x^T P + \tfrac{1}{2} u^T S,$$

$$Q = \mu(2\mu + \lambda) \nabla u (\nabla \cdot u) + \mu^2 \nabla u (\nabla u - \nabla u^T) + \tfrac{1}{2}(\mu + \lambda)(2\mu + \lambda)(\nabla \cdot u)^2\,\mathbf{1},$$

$$T = (\mu + \lambda)\, x \wedge Q + \mu(3\mu + \lambda)\, u \wedge S + \tfrac{1}{2}\mu^2(\mu + \lambda)[(u \wedge \nabla u)^T - tr(u \wedge \nabla u)\,\mathbf{1}],$$

and the infinity of densities

$$K_\varepsilon = \varepsilon^T S - u^T(\mu \nabla \varepsilon + (\mu + \lambda)(\nabla \cdot \varepsilon)\,\mathbf{1})$$

parametrized by solutions $\varepsilon(x)$ of Navier's equations.

If $7\mu + 3\lambda = 0$, then the additional seven densities

$$I = 2(x \otimes x) P - |x|^2 P + (x \otimes u - 2u \otimes x) S + 2(x \cdot u) S - 2\mu u \otimes u - \tfrac{1}{2}\mu |u|^2 \mathbf{1},$$

$$Z = x^T Q + \mu u^T S + \mu^2 [(\nabla \cdot u) u - (\nabla u) u]^T,$$

$$J = 2(x \otimes x) Q - |x|^2 Q + \mu(2x \otimes u - u \otimes x) S + \mu(x \cdot u) S + \mu^2 [2x \otimes (u \nabla \cdot u - (\nabla u) u) + u^T(\nabla u) x \mathbf{1} - (\nabla u^T u) \otimes x + x \wedge (\nabla u^T \wedge u) - tr(u \wedge \nabla u) \hat{x}],$$

arise. (Expressions for these densities using indices can be found in theorems 3.1, 3.2.)

Here S is the stress tensor associated with the deformation u, and reflects the fact that Navier's equation are in divergence form. P is the energy-momentum tensor of ESHELBY, [5], and R the corresponding angular momentum tensor. The vector Y arises from scale invariance; P, R and Y were found by KNOWLES & STERNBERG, [10], from elementary invariance properties of (2.1). The conservation laws K_ε are manifestations of Betti's reciprocal theorem; *cf.* [8, § 30]. (This contradicts a statement of CHEN & SHIELD, [2], that Betti's theorem cannot be derived from NOETHER's theorem.) The remaining densities are new.

C. Symmetry Groups in Three Dimensions

Let ∂_x, ∂_u be "vectors" with entries $\partial/\partial x^i$, $\partial/\partial u^i$ respectively, so that we may perform tensorial operations to derive new triples of tangent vectors. Such a triple will be a symmetry of (2.2) if each component is.

For $\mu(\mu + \lambda)(2\mu + \lambda) \neq 0$ and $(7\mu + 3\lambda) \neq 0$, the Lie algebra of infinitesimal symmetries of Navier's equations is spanned by the Lie symmetries

$$\vec{p} = \partial_x \text{ (translations)},$$

$$\vec{r} = x \wedge \partial_x - u \wedge \partial_u \quad \text{(rotations)},$$

$$\left.\begin{aligned} \vec{s}_1 &= x^T \partial_x \\ \vec{s}_2 &= u^T \partial_u \end{aligned}\right\} \quad \text{(scalings)},$$

and

$$\vec{k}_\varepsilon = \varepsilon(x)\, \partial_u \quad \text{(addition of solutions)}$$

for ε an arbitrary solution of (2.2), together with the six generalized symmetries

$$\vec{q} = \mu \nabla u \cdot \partial_u + (2\mu + \lambda)(\nabla \cdot u)\, \partial_u,$$

$$\vec{t} = (\mu + \lambda)\, x \wedge \vec{q} + \mu(3\mu + \lambda)\, u \wedge \partial_u,$$

$$\vec{w} = \nabla u \wedge \partial_u.$$

If $7\mu + 3\lambda = 0$, then the additional vector fields

$$\vec{i} = |x|^2\, \partial_x - 2(x \otimes x)\, \partial_x + (x \otimes u - 2u \otimes x)\, \partial_u + 2(x \cdot u)\, \partial_u,$$
$$\vec{z} = x^T \vec{q} + \mu u^T\, \partial_u,$$
$$\vec{j} = 2(x \otimes x)\, \vec{q} - |x|^2\, \vec{q} + \mu(2x \otimes u - u \otimes x)\, \partial_u + \mu(x \cdot u)\, \partial_u,$$
$$\vec{a} = 2x\vec{w} + u \wedge \partial_u,$$
$$\vec{b} = |x|^2\, \vec{w} + x^T(u \wedge \partial_u),$$

are also symmetries. (See theorems 4.1, 4.2 for the corresponding expressions with indices.)

Of these vector fields, $\vec{p}, \vec{r}, \vec{y} = \vec{s}_1 + \vec{s}_2 \vec{k}_\varepsilon, \vec{p}, \vec{t}, \vec{i}, \vec{z}, \vec{j}$ are variational symmetries and yield, via NOETHER's theorem, the conservation laws, $P, R, Y, K_\varepsilon, Q, T, I, Z, J$ respectively. The remaining symmetries are non-variational, and will only yield "conserved one-forms"; *cf.* [15].

The vector fields $\vec{p}, \vec{r}, \vec{y}, \vec{i}$ exponentiate to form a 10-dimensional conformal subgroup of the full symmetry group. It is interesting that only in the special case $7\mu + 3\lambda = 0$ are the inversions admissible.

For a generalized symmetry

$$\vec{v}_\psi = \psi_i \frac{\partial}{\partial u^i}$$

the corresponding one parameter group arises as a solution of the evolutionary system

$$\frac{\partial u^i}{\partial \varepsilon} = \psi_i(x, u, \nabla u).$$

If the initial data $u^i(x, 0)$ is a solution of (2.2), then so is $u(x, \varepsilon)$ for any ε. In particular, for the vector field $\vec{w}$, we have

$$\frac{\partial u^i}{\partial \varepsilon} = \varepsilon_{ijk} \frac{\partial u^j}{\partial x^k}.$$

This system can be easily solved in Fourier transform space

$$\hat{u}^i(k, \varepsilon) = \frac{1}{2\pi} \int e^{ik\cdot x}\, u^i(x, \varepsilon)\, dx,$$

so

$$\frac{\partial \hat{u}}{\partial \varepsilon} = k \wedge \hat{u}(k, \varepsilon).$$

This is just the infinitesimal version of a rotation with axis k, so we have the "solution"

$$\hat{u}(k, \varepsilon) = R_k(\varepsilon)\, \hat{u}(k, 0),$$

where $R_k(\varepsilon)$ denotes the rotation through angle ε with axis k. I am unsure as to the precise interpretation of this group in the physical variables $u(x)$, or its geometrical meaning for elasticity. The other generalized symmetries can be treated similarly, but the formulae are more complicated.

D. Further Identities

Although the extra conservation laws valid in the special case $7\mu + 3\lambda = 0$ do not remain conserved in general, the densities do provide interesting divergence identities. Specifically, we can express both $\|\nabla u\|^2$ and $(\nabla \cdot u)^2$, and their moments, in divergence form.

To obtain this result, first note that the densities Z, I, J have divergences

$$\nabla \cdot Z = \tfrac{1}{2}(7\mu + 3\lambda)(2\mu + \lambda)(\nabla \cdot u)^2,$$

$$\nabla \cdot I = -(7\mu + 3\lambda)(\nabla \cdot u)\, u,$$

$$\nabla \cdot J = (7\mu + 3\lambda)\left[(2\mu + \lambda)(\nabla \cdot u)^2 x + \mu(\nabla \cdot u)(\nabla u^T - \nabla u)\, x\right],$$

as can easily be verified. Next note that

$$\nabla \cdot (u^T S) = \mu \|\nabla u\|^2 + (\mu + \lambda)(\nabla \cdot u)^2;$$

hence, in conjunction with Z, we can express both $\|\nabla u\|^2$ and $(\nabla \cdot u)^2$ in divergence form (provided $7\mu + 3\lambda \neq 0$.)

Let

$$U = u \otimes S^T x - (u \cdot x)\, S + \mu u \otimes u,$$

$$V = x \otimes S^T u - \tfrac{1}{2}\mu |u|^2\, \mathbf{1}.$$

Then

$$\nabla \cdot U = (\mu + \lambda)(\nabla \cdot u)(\nabla u - \nabla u^T)\, x + 2(2\mu + \lambda)(\nabla \cdot u)\, u,$$

$$\nabla \cdot V = \left[\mu \|\nabla u\|^2 + (\mu + \lambda)(\nabla \cdot u)^2\right] x + (\mu + \lambda)(\nabla \cdot u)\, u.$$

From these it is easy to express the moments $x^i \|\nabla u\|^2$ and $x^i(\nabla \cdot u)^2$ in divergence form. Let

$$A = 2\mu(2\mu + \lambda)(\mu + \lambda)\, I + (\mu + \lambda)^2 J + \mu(\mu + \lambda)(7\mu + 3\lambda)\, U,$$

$$B = (2\mu + \lambda)(\lambda^2 - \mu^2)\, I - (\mu + \lambda)^2 J - \mu(\mu + \lambda)(7\mu + 3\lambda)\, U$$
$$+ (\mu + \lambda)(7\mu + 3\lambda)(2\mu + \lambda)\, V.$$

Then

$$\nabla \cdot A = (7\mu + 3\lambda)(2\mu + \lambda)(\mu + \lambda)^2 (\nabla \cdot u)^2 x,$$

$$\nabla \cdot B = (7\mu + 3\lambda)(2\mu + \lambda)\mu(\mu + \lambda) \|\nabla u\|^2 x.$$

Applications of these interesting identities will be discussed elsewhere.

E. Higher Order Symmetries and Conservation Laws

For systems of linear partial differential equations, it is well known that symmetries and conservation laws appear not singly but in infinite families, [1], [9], [16]. Specifically, a *linear symmetry* is one of the (standard) form $\vec{v}_\psi$ with

$$\psi = \mathscr{D}u,$$

where $\mathscr{D}$ is a matrix of differential operators. $\vec{v}_\psi$ is a symmetry if and only if $\mathscr{D}$ is a symmetry operator for the linear system in the sense of MILLER, [12]. If $\mathscr{D}$, $\mathscr{D}'$ are two such operators, so is $\mathscr{D} \cdot \mathscr{D}'$; hence any linear combination of power products of symmetry operators is again a symmetry operator.

The first order symmetries of the equations of linear isotropic elasticity thus provide a number of first order symmetry operators. Any operator in the enveloping algebra of these operators is a generalized linear symmetry, although some of these may be trivial, *i.e.* vanish on all solutions of the system. The problem of whether or not all symmetries arise this way is open (as is the same question for Laplace's equation!)

For conservation laws, it may be shown that if $\mathscr{D}$ is a skew-adjoint matrix of differential operators, then $\vec{v}_\psi$ gives rise via NOETHER's theorem to a conservation law, [16]. A second mechanism of finding conservation laws is that if A is a conserved density and $\vec{v}_\psi$ a symmetry, then $\vec{v}_\psi(A)$ (evaluated component-wise) is conserved, [1, p. 83]. Thus there are an infinite number of conservation laws for Navier's equations, whose densities depend on higher and higher derivatives of u. We shall not attempt to classify these laws here, but refer the interested reader to [16] for the basic techniques used to compute them.

3. Derivation of Three-Dimensional Conservation Laws

The derivation of the conservation laws discussed in the previous section is, basically, an involved calculation using the general formulae of paper I. For reasons discussed there, it is easier to work with the conservation laws to begin with, and, subsequently, use these to find the corresponding symmetries, inverting the usual methods in NOETHER's theorem.

First we find all x, u-independent conservation laws. We begin by deriving those which are quadratic in ∇u, but subsequently prove that (in three dimensions) all such laws are quadratic (or lower order) polynomials. These can be separated into trivial and nontrivial laws. The conservation laws which depend explicitly on x and/or u are then found using theorem I.4.5. For the convenience of the reader, we begin by summarizing the relevant constructions, specialized to the current equations of linear isotropic homogeneous elasticity. In this section all calculations and results are for the three-dimensional case.

A. Equations for Conservation Laws

As discussed in full detail in I, a (first order) conservation law for (2.2) will, in general, be equivalent to one of the form

$$\operatorname{Div} A = \psi \cdot \mathscr{E}, \tag{3.1}$$

where A, ψ depend on x, u, ∇u. As stated in theorem I.4.5, if we know all the x, u independent conservation laws $A^1, \ldots, A^N$, then general conservation laws can be found as linear combinations

$$A = \Sigma \omega^i(x, u)\, A^i + \Sigma \theta^i(x, u)\, B^i, \tag{3.2}$$

where $B^1, \ldots, B^M$ form a "complete set" of trivial conservation laws depending on ∇u; *cf.* [17]. Moreover, the conditions on the coefficient functions ω^i, θ^i are

$$\Sigma(D_j \omega^i)\, A^i_j + \Sigma(D_j \theta^i)\, B^i_j = 0, \tag{3.3}$$

where D_j denotes total derivative with respect to x^j, so $D_j = \partial/\partial x^j + u^k_j\, \partial/\partial u^k$, and A^i_j, $j = 1, 2, 3$ are the components of A^i.

B. x, u-Independent Conservation Laws

The first step is the calculation of all x, u-independent conservation laws. The main result to be proved is the following:

Proposition 3.1. Let $\mu(\mu + \lambda)(2\mu + \lambda) \neq 0$. Suppose $\operatorname{Div} A = 0$ is a conservation law for (2.2) depending only on ∇u. Then A is a linear, constant coefficient combination of the following laws.

a) *Nontrivial, quadratic*

$$P^i_j = \mu u^k_i u^k_j + (\mu + \lambda)\, u^j_i u^k_k - \tfrac{1}{2}\, \delta^i_j(\mu u^k_l u^k_l + (\mu + \lambda)\, u^k_k u^l_l), \tag{3.4}$$

$$Q^i_j = (2\mu + \lambda)\, \mu u^i_j u^k_k + \mu^2 u^i_l(u^l_j - u^j_l) + \tfrac{1}{2}(\mu + \lambda)(2\mu + \lambda)\, \delta^i_j u^k_k u^l_l, \tag{3.5}$$

b) *Nontrivial, linear*

$$S^i_j = \mu u^i_j + (\mu + \lambda)\, \delta^i_j u^k_k, \tag{3.6}$$

c) *Trivial, quadratic*

$$A^i_j = \varepsilon_{ikn} \varepsilon_{jlm} u^k_l u^n_m, \tag{3.7}$$

d) *Trivial, linear*

$$B^{ik}_j = \varepsilon_{ijl} u^k_l, \tag{3.8}$$

e) *Constants.*

Note that (3.4–6) are just the conservation laws P, Q, S described in Section 2, but now written out in full detail without the aid of tensor notation. The trivial quadratic laws are all forms of the Jacobian identity

$$D_x \frac{\partial(u, v)}{\partial(y, z)} + D_y \frac{\partial(u, v)}{\partial(z, x)} + D_z \frac{\partial(u, v)}{\partial(x, y)} = 0,$$

while the trivial linear laws are identities of the form

$$D_x(u_y) + D_y(-u_x) = 0.$$

The form of these trivial laws follows from the general characterization given in [17].

The basic equations for an x, u independent law A for the equations of linear isotropic elasticity can be written as follows:

$$\overset{i}{A}{}^i_i = (2\mu + \lambda)\, \psi_i, \tag{3.9a}$$

$$\overset{j}{A}{}^i_j = \mu \psi_i, \quad i \neq j, \tag{3.9b}$$

$$\overset{i}{A}{}^j_j + \overset{j}{A}{}^j_i = (\mu + \lambda)\, \psi_i, \quad i \neq j, \tag{3.9c}$$

$$\overset{i}{A}{}^j_k + \overset{k}{A}{}^j_i = 0, \quad i, j, k \text{ distinct}. \tag{3.9d}$$

In these equations, and for the remainder of this subsection we have dropped the summation convention. The components of the conservation law A have now been written $\overset{j}{A}$, and the notation $\overset{j}{A}{}^i_k = \partial \overset{j}{A} / \partial u^i_k$ has been used. The equations (3.9) come from the basic equations (3.1), or, more specifically, equations (I.4.10) for our specific stored energy function.

The proof of proposition 3.1 amounts to finding the general solution to equations (3.9). This proceeds in three steps: First, all linear solutions are easily computed and shown to be linear combinations of (3.6,8); this we leave to the reader. Second, the quadratic solutions will be computed; this is the hardest computational step. Finally, we show that all third order derivatives of the $\overset{i}{A}$ with respect to u^j_k vanish, and so there are no other solutions to the equations.

To analyze quadratic solutions, we differentiate (3.9) to obtain relations among the second derivatives $\overset{i}{A}{}^{jl}_{km} = \partial^2 \overset{i}{A} / \partial u^j_k \partial u^l_m$ of the $\overset{i}{A}$'s. These fall naturally into two classes, those in which only two distinct indices appear and those in which all three indices appear at least once.

Eliminating derivatives of the ψ_i, for the former class we have the following relations: for any $i \neq j$.

$$\begin{aligned}
\mu(\mu + \lambda)\, \overset{i}{A}{}^{ii}_{ii} &= (2\mu + \lambda)(\mu + \lambda)\, \overset{j}{A}{}^{ii}_{ij} = \mu(2\mu + \lambda)\left(\overset{i}{A}{}^{ij}_{ij} + \overset{j}{A}{}^{ij}_{ii}\right), \\
\mu(\mu + \lambda)\, \overset{i}{A}{}^{ij}_{ij} &= (2\mu + \lambda)(\mu + \lambda)\, \overset{j}{A}{}^{ij}_{jj} = \mu(2\mu + \lambda)\left(\overset{i}{A}{}^{jj}_{jj} + \overset{j}{A}{}^{jj}_{ij}\right), \\
\mu(\mu + \lambda)\, \overset{j}{A}{}^{ij}_{jj} &= (2\mu + \lambda)(\mu + \lambda)\, \overset{i}{A}{}^{ij}_{ji} = \mu(2\mu + \lambda)\left(\overset{i}{A}{}^{ii}_{jj} + \overset{j}{A}{}^{ii}_{ij}\right), \\
\mu(\mu + \lambda)\, \overset{j}{A}{}^{jj}_{ij} &= (2\mu + \lambda)(\mu + \lambda)\, \overset{i}{A}{}^{jj}_{ii} = \mu(2\mu + \lambda)\left(\overset{i}{A}{}^{ij}_{ji} + \overset{j}{A}{}^{ij}_{ii}\right).
\end{aligned} \tag{3.10}$$

The first two sets of equations result from eliminating ψ_i from (3.9a–c) and differentiating with respect to u_i^i and u_j^j respectively, the second two from interchanging i and j in (3.9a–c), eliminating ψ_j and differentiating with respect to u_j^i and u_i^j respectively. For each $i \neq j$ there are (assuming λ, μ and $2\mu + \lambda$ do not vanish) eight different equations in ten unknowns, resulting in two independent solutions. These are:

$$
\begin{aligned}
\overset{i}{A}{}^{ii}_{ii} &= (2\mu + \lambda)(p_i + (3\mu + \lambda)\, q_i),\\
\overset{i}{A}{}^{ii}_{jj} &= -\mu(p_i + 2\mu q_i),\\
\overset{i}{A}{}^{ij}_{ji} &= \mu^2 q_i,\\
\overset{i}{A}{}^{ij}_{ii} &= \mu p_i,\\
\overset{i}{A}{}^{ij}_{ij} &= (2\mu + \lambda)^2\, q_i,\\
\overset{i}{A}{}^{jj}_{jj} &= -(2\mu + \lambda)(p_i - (\mu + \lambda)\, q_i),\\
\overset{j}{A}{}^{ii}_{ij} &= \mu(p_i + (3\mu + \lambda)\, q_i),\\
\overset{j}{A}{}^{ij}_{ii} &= (\mu + \lambda)\, p_i - \mu^2 q_i,\\
\overset{j}{A}{}^{ij}_{jj} &= (2\mu + \lambda)\, \mu q_i,\\
\overset{j}{A}{}^{ij}_{ij} &= (2\mu + \lambda)\, p_i,
\end{aligned}
\tag{3.11}
$$

where p_i, q_i are arbitrary, independent of j.

The second derivatives $\overset{i}{A}{}^{jl}_{km}$ for which all three indices 1, 2, 3 appear fall into two subclasses.: First consider the case when two indices appear twice. The relevant equations are:

$$
\begin{aligned}
&\mu\left(\overset{i}{A}{}^{ij}_{kj} + \overset{k}{A}{}^{ij}_{ij}\right) = (\mu + \lambda)\, \overset{j}{A}{}^{jk}_{jj} = \mu(\mu + \lambda)(2\mu + \lambda)\, q_k,\\
&(2\mu + \lambda)\left(\overset{i}{A}{}^{ij}_{kj} + \overset{j}{A}{}^{ij}_{ki}\right) = (\mu + \lambda)\, \overset{i}{A}{}^{ii}_{ik} = (\mu + \lambda)(2\mu + \lambda)\, p_k,\\
&\overset{i}{A}{}^{ij}_{jk} = -\overset{k}{A}{}^{ij}_{ji} = \overset{j}{A}{}^{ij}_{ki} \equiv t_k.
\end{aligned}
\tag{3.12}
$$

The first two equations follow from (3.9b, c) (differentiated with respect to u_j^j), (3.9a, c) (differentiated with respect to u_k^i) and (3.11). The third equation comes from (3.9d), and serves as the definition of t_k. The general solution of (3.12) is easily seen to be

$$
\begin{aligned}
\overset{i}{A}{}^{ij}_{kj} &= (\mu + \lambda)\, p_k - t_k,\\
\overset{k}{A}{}^{ij}_{ij} &= (\mu + \lambda)(2\mu + \lambda)\, q_k - (\mu + \lambda)\, p_k + t_k.
\end{aligned}
$$

The last case is when all three indices appear, but one index appears three times. The claim is that all such second derivatives necessarily vanish. To prove this, first we compute, using (3.9),

$$\begin{aligned}
\mu(\mu+\lambda)\,\overset{i}{A}{}^{ij}_{ik} &= \mu(2\mu+\lambda)\left(\overset{i}{A}{}^{kj}_{kk} + \overset{k}{A}{}^{kj}_{ik}\right), \\
&= -\mu(2\mu+\lambda)\,\overset{k}{A}{}^{jk}_{ik} + \mu(2\mu+\lambda)\,\overset{i}{A}{}^{jk}_{ii}, \\
&= -(2\mu+\lambda)^2\,\overset{i}{A}{}^{jk}_{ii} + \mu(2\mu+\lambda)\,\overset{i}{A}{}^{jk}_{ii}, \\
&= -(2\mu+\lambda)(\mu+\lambda)\,\overset{i}{A}{}^{jk}_{ii}.
\end{aligned}$$

The last term is symmetric in j, k, hence (assuming $\mu(\mu+\lambda) \neq 0$)

$$\overset{i}{A}{}^{ij}_{ik} = \overset{i}{A}{}^{ik}_{ij}.$$

On the other hand,

$$\mu\,\overset{i}{A}{}^{ij}_{ik} = (2\mu+\lambda)\,\overset{k}{A}{}^{ij}_{kk}$$

so

$$\begin{aligned}
\mu(\mu+\lambda)\,\overset{i}{A}{}^{ij}_{ik} &= (2\mu+\lambda)(\mu+\lambda)\,\overset{k}{A}{}^{ij}_{kk}, \\
&= \mu(2\mu+\lambda)\left(\overset{j}{A}{}^{ik}_{kk} + \overset{k}{A}{}^{ik}_{kj}\right), \\
&= -\mu(2\mu+\lambda)\,\overset{k}{A}{}^{ik}_{jk} + \mu^2\,\overset{i}{A}{}^{ik}_{ij}, \\
&= -(2\mu+\lambda)^2\,\overset{j}{A}{}^{ik}_{jj} + \mu^2\,\overset{i}{A}{}^{ik}_{ij}, \\
&= -\mu(2\mu+\lambda)\,\overset{i}{A}{}^{ik}_{ij} + \mu^2\,\overset{i}{A}{}^{ik}_{ij}, \\
&= -\mu(\mu+\lambda)\,\overset{i}{A}{}^{ik}_{ij}.
\end{aligned}$$

Therefore $\overset{i}{A}{}^{ij}_{ik} = 0$, and all other second derivatives in which i occurs three times, j, k once, can, by the above calculations, be seen to vanish.

Thus the second order derivatives of any conservation law A are given by (3.11–13) with p_i, q_i, t_i arbitrary. Setting $p_i = \delta_j^i$, $q_i = 0$, $t_i = 0$ we recover the momenta (3.4). Similarly $q_i = \delta_i^j$, $p_i = t_i = 0$ gives (3.5), while $t_i = \delta_i^j$, $p_i = q_i = 0$ recovers the Jacobian identities (3.7).

It remains to show that A can be at most a quadratic function of ∇u, in other words show that all the third order derivatives, denoted $\overset{i}{A}{}^{jln}_{kmp}$, vanish. There are a large number of possibilities for the seven indices i, j, k, l, m, n, p, but these can be reduced as follows. If only two distinct indices occur, then the third derivative must by (3.11) be a linear combination of derivatives of the p_i and q_i. However by (3.12) these can in turn be replaced by linear combinations of third order derivatives in which all three different indices 1, 2, 3 occur at least once, so we restrict our attention to such derivatives.

Next note that if we choose five indices consisting of two of the columns (j, k) or (l, m) or (n, p) plus the index i, and among these five indices one number appears thrice, the other two appearing once each (*e.g.* 1, 1, 1, 2, 3) then by the above results for second order derivatives, the corresponding third order derivative vanishes. For instance, if among the seven indices $i, \ldots, p$ one number occurs five times, the other two occurring each once, then, by the preceding remark, the derivative must vanish.

To investigate the remaining cases, we use the notation

$$\left(i \middle| \begin{matrix} j & l & n \\ k & m & p \end{matrix}\right) \tag{3.14}$$

to denote the above derivative. To save space, this notation may also mean third order derivatives with the columns reversed, *e.g.* $\overset{i}{A}{}^{klp}_{jmn}$. Equivalence $\simeq$ between such symbols will mean equality up to a nonzero multiple between the corresponding derivatives.

If one number occurs four times amongst $i, \ldots, p$, then the only derivatives not covered by the remark in the preceding paragraph are

$$\left(i \middle| \begin{matrix} i & i & j \\ i & k & j \end{matrix}\right) \simeq \left(j \middle| \begin{matrix} i & j & j \\ j & k & j \end{matrix}\right) = 0,$$

by (3.9a, b) and the remark. Thus the only third order derivatives standing any chance of not being zero are those in which all numbers occur at least once, but no more than thrice.

The following types of derivatives are not covered by the above remark:

$$\left(i \middle| \begin{matrix} i & i & j \\ j & j & k \end{matrix}\right) \simeq \left(j \middle| \begin{matrix} i & j & j \\ j & j & k \end{matrix}\right) = 0,$$

since j occurs 5 times. The equivalence is based on the fact that second order derivatives with only two indices occurring can all be expressed in terms of p_i and q_i, which in turn by (3.11) can be expressed in terms of $\overset{j}{A}{}^{ij}_{jj}$ and $\overset{j}{A}{}^{ij}_{ij}$. (Thus in the above equation we are implicitly using the ambiguity of our symbol in the ordering of the indices of the first column; the right hand side actually denotes a linear combination of two different third order derivatives, both of which vanish for the same reason.) Also

$$\left(k \middle| \begin{matrix} i & i & i \\ j & j & j \end{matrix}\right) \simeq \left(j \middle| \begin{matrix} i & i & i \\ j & j & k \end{matrix}\right)$$

by (3.9d), hence this reduces to the previous case. Several other types of third derivatives are also covered by an application of this result.

The only remaining derivatives are those given by symbols of the type

$$\left(j \middle| \begin{matrix} i & i & i \\ j & k & k \end{matrix}\right)$$

and those obtained by switching the column entries. Now we need to be careful as to which order the column entries are written. The easiest such derivative to treat is

$$\left(j\middle|\begin{matrix} i\,i\,k \\ j\,k\,i \end{matrix}\right) \simeq \left(i\middle|\begin{matrix} i\,i\,k \\ j\,k\,j \end{matrix}\right) = 0,$$

by (3.9d). The same argument holds if the first or second column has its indices reversed. Next

$$\left(j\middle|\begin{matrix} j\,i\,i \\ i\,k\,k \end{matrix}\right) \simeq \left(k\middle|\begin{matrix} j\,i\,i \\ i\,k\,j \end{matrix}\right)$$

by (3.9d), which reduces to the previous case (with j and k interchanged). Here we still have freedom in the order of the second column. Finally, by (3.9b, c),

$$\left(j\middle|\begin{matrix} i\,i\,i \\ j\,k\,k \end{matrix}\right)$$

can be written as a linear combination of

$$\left(j\middle|\begin{matrix} j\,i\,i \\ i\,k\,k \end{matrix}\right) \quad \text{and} \quad \left(i\middle|\begin{matrix} j\,i\,i \\ j\,k\,k \end{matrix}\right)$$

both of which have been shown to vanish. This completes the verification that all third order derivatives with three different numbers occurring in the indices, and hence all third order derivatives, vanish.

C. General Conservation Laws

We now turn to the investigation of conservation laws which are allowed to depend explicitly on x and u.

Proposition 3.2. Suppose $\operatorname{Div} A = 0$ is a *nontrivial* conservation law for Navier's equations (2.2). Suppose $\mu(\mu+\lambda)(2\mu+\lambda) \neq 0$. If, in addition, $7\mu + 3\lambda \neq 0$, then A is a linear combination of the laws (3.4–6) in proposition 3.1 and the following additional laws:

$$R_j^i = \varepsilon_{ikl}(x^k P_j^l + u^k S_j^l), \tag{3.15}$$

$$Y_j = x^k P_j^k + \tfrac{1}{2} u^k S_j^k, \tag{3.16}$$

$$T_j^i = \varepsilon_{ikl}\{(\mu+\lambda)\, x^k Q_j^l + \mu(3\mu+\lambda)\, u^k S_j^l\} + \tfrac{1}{2}\mu^2(\mu+\lambda)\{\varepsilon_{jkl} u^k u_i^l + \delta_j^i \varepsilon_{klm} u^l u_m^k\}, \tag{3.17}$$

$$K_j^\varepsilon = \varepsilon^k(x)\, S_j^k - u^k\left(\mu \frac{\partial \varepsilon^k}{\partial x^j} + (\mu+\lambda)\, \delta_j^k \frac{\partial \varepsilon^l}{\partial x^l}\right), \tag{3.18}$$

where $\varepsilon(x)$ is an arbitrary solution of (2.2).

If $7\mu + 3\lambda = 0$ then A is a linear combination of all the preceding laws as well as the additional 7 laws

$$I_j^i = 2x^i x^k P_j^k - x^k x^k P_j^i + (x^i u^k - 2x^k u^i)\, S_j^k + 2x^k u^k S_j^i - 2\mu u^i u^j - \tfrac{1}{2}\mu\, \delta_j^i u^k u^k, \tag{3.19}$$

$$Z_j = x^k Q_j^k + \mu u^k S_j^k + \mu^2(u^j u_k^k - u^k u_k^j), \tag{3.20}$$

$$J_j^i = 2x^i x^k Q_j^k - x^k x^k Q_j^i + \mu(2x^i u^k - x^k u^i)\, S_j^k + \mu x^k u^k S_j^i + 2\mu^2 x^i(u^j u_k^k - u^k u_k^j) + \mu^2(\varepsilon_{ijk}\varepsilon_{lmn} + \varepsilon_{ikn}\varepsilon_{lmj})\, x^k u^l u_n^m + \mu^2(\delta_j^i x^k u^l u_k^l - x^j u^l u_i^l). \tag{3.21}$$

To prove the proposition, we use the representation (3.2) in the form

$$A = \alpha^i P^i + \beta^i Q^i + \gamma^i A^i + \varepsilon^i S^i + \theta^{ik} B^{ik} + \omega, \tag{3.22}$$

where α, β, γ, ε, θ, ω are functions of x, u, subject to conditions (3.3). To find the general solution, we must look at the coefficients of the various monomials in ∇u.

Lemma 3.3. *If* (3.22) *is a conservation law, then* α, β *are independent of* u, *and*

$$\Sigma\, \partial\gamma^i/\partial u^i = 0. \tag{3.23}$$

Proof. By use of the formulae in proposition 3.1, the coefficient of $(u_i^i)^3$ in (3.3) can be written as.

$$\tfrac{1}{2}(2\mu + \lambda)\left[\frac{\partial\alpha^i}{\partial u^i} + (3\mu + \lambda)\frac{\partial\beta^i}{\partial u^i}\right] = 0. \quad \text{(no sum)}$$

The coefficient of $u_i^i(u_j^i)^2$ is

$$\mu\left\{\tfrac{1}{2}\frac{\partial\alpha^i}{\partial u^i} + (2\mu + \lambda)\frac{\partial\beta^i}{\partial u^i}\right\} = 0. \quad \text{(no sum)}$$

Therefore α^i and β^i do not depend on u^i. The coefficients of $(u_i^j)^3$ and $u_i^j(u_j^j)^2$ are, respectively,

$$\tfrac{1}{2}\mu\frac{\partial\alpha^i}{\partial u^j} = 0,$$

$$\tfrac{1}{2}(2\mu + \lambda)\left\{\frac{\partial\alpha^i}{\partial u^j} + (\mu + \lambda)\frac{\partial\beta^i}{\partial u^j}\right\} = 0;$$

hence α^i, β^i cannot depend on u^j. Finally, the coefficient of $u_i^i u_j^j u_k^k$ (i, j, k distinct) yields the condition on γ.

Lemma 2.4. *If γ satisfies* (3.23), *then there is locally a trivial conservation law C such that*

$$\operatorname{Div}(\gamma^i A^i + C) = \operatorname{Div}(\tilde{\theta}^{ik} B^{ik}) \tag{3.24}$$

for functions $\tilde{\theta}^{ik}$.

Proof. (Sketch). If $f(x, u)$ is a function, define the differentials $d_x f = \Sigma\, \partial f/\partial x^i \cdot dx^i$, $d_u f = \Sigma\, \partial f/\partial u^i \cdot du^i$ and $df = d_x f + d_u f = \Sigma\, D_i f\, dx^i$. Let

$$\omega = \varepsilon_{ijk} \gamma^i\, du^j \wedge du^k,$$

so that

$$d\omega = \operatorname{Div}(\gamma^i A^i)\, dx^1 \wedge dx^2 \wedge dx^3 .$$

The condition (3.23) implies that

$$d_u \omega = 0;$$

hence, by Poincaré's lemma, *cf.* [22], there exists (locally) a one-form

$$\theta = \theta^i\ du^i$$

with

$$d_u \theta = \omega .$$

Then

$$d\omega = dd_u\theta = d(d\theta - d_x\theta) = -d(d_x\theta),$$

so

$$d_x\theta = \frac{\partial \theta^i}{\partial x^j} dx^j \wedge du^i$$

is of the required form.

Thus for nontrivial conservation laws we can let $\gamma = 0$ in (3.22) without loss of generality.

Lemma 2.5 *If* (3.22) *is a conservation law, then the vector field*

$$\vec{v} = \alpha^i \frac{\partial}{\partial x^i} \tag{3.25}$$

generates a one parameter group of conformal transformations in x. If $7\mu + 3\lambda \neq 0$, then

$$\vec{w} = \beta^i\, \partial/\partial x^i \tag{3.26}$$

generates a one-parameter group of Euclidean motions. If $7\mu + 3\lambda = 0$, then $\vec{w}$ generates a conformal group.

Before proving the lemma, first recall, [16], that the conditions on (3.25) that it generate a conformal group are

$$\frac{\partial \alpha^i}{\partial x^j} + \frac{\partial \alpha^j}{\partial x^i} = \delta^i_j \varphi, \tag{3.27}$$

for some function φ. In $\mathbb{R}^3$, the Lie algebra of conformal vector fields is spanned by the ten vector fields

$$\frac{\partial}{\partial x^i}, \varepsilon_{ijk} x^j \frac{\partial}{\partial x^k}, \quad x^k \frac{\partial}{\partial x^k}, \quad 2x^i x^k \frac{\partial}{\partial x^k} - x^k x^k \frac{\partial}{\partial x^i}, \tag{3.28}$$

which generate groups of translations, rotations, dilatations, and inversions respectively. The subalgebra of Euclidean vector fields is spanned by the translations and rotations only; it is specified by (3.27), but with the additional constraint that $\varphi = 0$.

To prove lemma 2.5, we look at the coefficients of quadratic monomials in ∇u. For $i \neq j$, the coefficient of $(u_j^i)^2$ is

$$\mu \left\{ \sum_k (\delta_k^j - \tfrac{1}{2}) \frac{\partial \alpha^k}{\partial x^k} - \mu \frac{\partial \beta^i}{\partial x^i} + \frac{\partial \varepsilon^i}{\partial u^i} \right\} = 0 \quad (\text{no sum on } i)$$

from which we conclude

$$\frac{\partial \alpha^i}{\partial x^i} = \varphi, \quad \frac{\partial \varepsilon^i}{\partial u^i} = \tfrac{1}{2}\varphi + \mu \frac{\partial \beta^i}{\partial x^i} \quad (\text{no sum}). \tag{3.29}$$

The coefficient of $u_j^i u_k^i$ (i, j, k distinct) is

$$\mu \left\{ \frac{\partial \alpha^j}{\partial x^k} + \frac{\partial \alpha^k}{\partial x^j} \right\} = 0; \tag{3.30}$$

hence (3.27) is satisfied, and the α^i are conformal. The coefficient of $(u_i^i)^2$ is (using (3.29))

$$(\mu + \tfrac{1}{2}\lambda) \left\{ 2\mu \frac{\partial \beta^i}{\partial x^i} + (\mu + \lambda) \sum_k \frac{\partial \beta^k}{\partial x^k} + 2 \frac{\partial \varepsilon^i}{\partial u^i} - \varphi \right\} = 0 \quad (\text{no sum}).$$

Comparing this with (3.29), we see that

$$4\mu \frac{\partial \beta^i}{\partial x^i} + (\mu + \lambda) \sum_k \frac{\partial \beta^k}{\partial x^k} = 0 \quad (\text{no sum on } i);$$

hence, since $\eta = 0$,

$$\frac{\partial \beta^i}{\partial x^i} = \psi \quad (\text{no sum})$$

with $\psi = 0$ unless $7\mu + 3\lambda = 0$. Moreover

$$\frac{\partial \varepsilon^i}{\partial u^i} = \tfrac{1}{2}\varphi + \mu\psi \quad (\text{no sum}). \tag{3.31}$$

Next, the coefficient of $u_k^i u_k^j$ shows that

$$\mu \left\{ \frac{\partial \beta^i}{\partial x^j} + \frac{\partial \beta^j}{\partial x^i} \right\} = \frac{\partial \varepsilon^i}{\partial u^k} + \frac{\partial \varepsilon^k}{\partial u^i}, \tag{3.32}$$

while $u_i^i u_j^j$ shows

$$(\mu+\lambda)\frac{\partial\alpha^i}{\partial x^j}+\mu(2\mu+\lambda)\frac{\partial\beta}{\partial x^i}-\mu^2\frac{\partial\beta^i}{\partial x^j}+(2\mu+\lambda)\frac{\partial\varepsilon^i}{\partial u^j}+\mu\frac{\partial\varepsilon^j}{\partial u^i}=0.$$

Combining this, (3.30) and (3.32), we see that both sides of (3.32) must vanish, so β is Euclidean (or conformal) and ε satisfies (3.31) and

$$(\mu+\lambda)\left(\frac{\partial\varepsilon^i}{\partial u^j}+\frac{\partial\alpha^i}{\partial x^j}\right)+\mu(3\mu+\lambda)\frac{\partial\beta^j}{\partial x^i}=0. \tag{3.33}$$

Finally, the remaining monomials, *i.e.* $u_i^i u_j^j$, $u_j^i u_i^j$, $u_i^i u_k^j$ and $u_k^i u_i^j$ yield the further relations

$$\begin{aligned}\sum_{i,j,k}\varepsilon_{ikj}\frac{\partial\theta^{ik}}{\partial u^j}&=2\mu^2\psi,\\ \sum_{k,l}\varepsilon_{jkl}\frac{\partial\theta^{ik}}{\partial u^l}&=\mu^2\frac{\partial\beta^i}{\partial x^j}.\end{aligned} \tag{3.34}$$

To complete the equations that must be satisfied by α, β, ε, θ, we look at the linear and constant monomials in ∇u, giving the further conditions:

$$\begin{aligned}\mu\frac{\partial\varepsilon^i}{\partial x^i}+(\mu+\lambda)\sum_k\frac{\partial\varepsilon^k}{\partial x^k}+\sum_{k,l}\varepsilon_{ikl}\frac{\partial\theta^{ki}}{\partial x^l}+\frac{\partial\omega^i}{\partial u^i}&=0,\quad\text{(no sum on } i)\\ \mu\frac{\partial\varepsilon^i}{\partial x^j}+\sum_{k,l}\varepsilon_{jkl}\frac{\partial\theta^{ki}}{\partial x^l}+\frac{\partial\omega^j}{\partial u^i}&=0,\\ \sum_k\frac{\partial\omega^k}{\partial x^k}&=0.\end{aligned} \tag{3.35}$$

Note that differentiating the first two with respect to x^i and x^j respectively and adding, we find that for each fixed u, $\varepsilon(x,u)$ must be a solution to the original equations (2.2).

Given a conformal vector field $\alpha^i\,\partial/\partial x^i$, or an orthogonal (conformal) vector field $\beta^i\,\partial/\partial x^i$, it is a straightforward computational exercise to find specific solutions ε, θ, ω to (3.31, 33, 34, 35). For brevity, we omit all of the intervening computations, except for the case when

$$\alpha^i=2x^ix^j-\delta_j^ix^kx^k\ (j\text{ fixed})$$

is an inversion. Then (3.31, 33) implies that

$$\varepsilon^i=x^ju^i-2x^iu^j+2\delta_j^ix^ku^k.$$

However, since $\beta=\psi=0$, (3.34) implies

$$\frac{\partial\theta^{ik}}{\partial u^l}=\frac{\partial\theta^{il}}{\partial u^k}.$$

Differentiating (3.35a) with respect to u^j, and (3.35b) (interchanging i, j) with respect to u^i and subtracting, we find that the above equations reduce to

$$7\mu + 3\lambda = 0,$$

hence if α is an inversional symmetry, a consistent conservation law can be found only when $7\mu + 3\lambda = 0$.

We summarize our results as follows:

1) *α-rotations*

$$\alpha^i = \varepsilon_{ijk}x^k, \quad \varepsilon^i = \varepsilon_{ijk}u^k, \quad \beta = \theta = \omega = 0.$$

2) *α-dilatation*

$$\alpha^i = x^i, \quad \varepsilon^i = \tfrac{1}{2}u^i, \quad \beta = \theta = \omega = 0.$$

3) *β-rotations*

$$\beta^i = (\mu + \lambda)\,\varepsilon_{ijk}x^k, \quad \varepsilon^i = \mu(3\mu + \lambda)\,\varepsilon_{ijk}u^k,$$

$$\theta^{ik} = \tfrac{1}{2}\mu^2(\mu + \lambda)\,(\delta_j^k u^k - \delta_i^k u^j), \quad \alpha = \omega = 0.$$

For $7\mu + 3\lambda = 0$, we further have

4) *α-inversion*

$$\alpha^i = 2x^ix^j - \delta_j^i x^kx^k, \quad \varepsilon^i = x^iu^j - 2x^ju^i + 2\delta_j^i x^kx^k, \quad \beta = \theta = 0,$$

$$\omega^i = -2\mu u^iu^j - \mu\,\delta_j^i u^ku^k.$$

5) *β-dilatation*

$$\beta^i = x^i, \quad \varepsilon^i = \mu u^i, \quad \theta^{ik} = \mu^2\varepsilon_{ijk}u^j, \quad \alpha = \omega = 0.$$

6) *β-inversion*

$$\beta^i = 2x^ix^j - \delta_j^i x^kx^k, \quad \varepsilon^i = \mu(2x^ju^i - x^iu^j + \delta_j^i x^ku^k), \quad \alpha = \omega = 0,$$

$$\theta^{ik} = \mu^2[\varepsilon_{jkl}((1 + \delta_j^i)\,x^iu^l - x^lu^i) + \varepsilon_{ijk}(1 + \delta_j^l)\,x^iu^l + \delta_j^k\varepsilon_{ilm}(1 + \delta_j^l)\,x^lu^m].$$

It is also straightforward to check that these formulae give rise to the conservation laws (3.15–21).

Finally, if $\alpha = \beta = 0$, then from (3.20, 22), ε must be independent of u, so $\varepsilon(x)$ is an arbitrary solution of (2.2), with corresponding conservation law K^ε. For $\varepsilon = 0$, any further solutions in θ, ω yields only trivial conservation laws. This completes the proof of proposition 3.2.

4. The Two-Dimensional Case

The procedures for classifying conservation laws for the two-dimensional Navier equations is the same as in three dimensions, but the results and intervening computations are of a completely different character. Here analytic function techniques are essential, and infinite families of nontrivial conservation laws arise. As before, though, the basic strategy is to first compute x, u-independent conservation laws, and then proceed to the general case. The notation is the same as before, only now indices just have values 1, 2.

Define the complex variables

$$z = x^1 + ix^2, \qquad w = u^1 + iu^2,$$

with corresponding complex derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

etc. We further define the following complex combinations of deformation gradients:

$$\xi = \xi^1 + i\xi^2 = 2\frac{\partial w}{\partial \bar{z}} = (u_1^1 - u_2^2) + i(u_2^1 + u_1^2),$$

$$\eta = \eta^1 + i\eta^2 = \mu(u_1^2 - u_2^1) + i(2\mu + \lambda)(u_1^1 + u_2^2).$$

We call η the *complex stress*, since Navier's equations take the compact form

$$D_z\eta = 0 \tag{4.1}$$

in this notation (D_z being the complex total derivative).

In terms of MUSKHELISHVIKI's complex potentials, [13],

$$\sigma = -\Theta_{\bar{z}} = -z\,\overline{\Phi'(z)} + \overline{\psi(z)}, \qquad \eta = 2i(\lambda + 2\mu)\,\overline{\Phi(z)}/(\lambda + \mu).$$

A conservation law

$$D_1A^1 + D_2A^2 = 0$$

has *complex conserved density*

$$A = A^1 + iA^2,$$

and hence can be rewritten as

$$\operatorname{Re}(2D_zA) = 0, \tag{4.2}$$

whenever (4.1) holds. With this notation, the x, u independent laws can be derscibed as follows:

Theorem 4.1. *Suppose* $\mu(\mu + \lambda)(2\mu + \lambda) \neq 0$. *The complex function* $A(\nabla u)$ *is a complex density for the two dimensional Navier equations if and only if there exist analytic functions* B, C *of* η *with*

$$A = 2\mu(2\mu + \lambda)\,\xi\frac{\partial B}{\partial \eta} + (\mu + \lambda)\,i\overline{B} + C. \tag{4.3}$$

Thus, in contrast to the three-dimensional case, there are infinitely many x, u independent conservation laws, which can be as nonlinear in the deformation gradient as desired. (Compare TSAMASPHYROS & THEOCARIS, [21].)

Proof. First note that although the basic equations (3.9) for a conservation law remain the same, there are no conditions (3.9d) as only two different indices are involved. Change variables from u_j^i to ξ^i, η^j $(i, j = 1, 2)$ in (3.9a–c) and eliminate ψ_i. This leads to the system

$$(\mu + \lambda)\frac{\partial A^1}{\partial \xi^1} = (\mu + \lambda)\frac{\partial A^2}{\partial \xi^2} = \mu(2\mu + \lambda)\left(\frac{\partial A^1}{\partial \eta^2} + \frac{\partial A^2}{\partial \eta^1}\right),$$

$$-(\mu + \lambda)\frac{\partial A^1}{\partial \xi^2} = (\mu + \lambda)\frac{\partial A^2}{\partial \xi^1} = \mu(2\mu + \lambda)\left(\frac{\partial A^1}{\partial \eta^1} - \frac{\partial A^2}{\partial \eta^2}\right),$$

or, equivalently, in complex notation,

$$\frac{\partial A}{\partial \bar{\xi}} = 0,$$

$$2\mu(2\mu + \lambda)\frac{\partial A}{\partial \bar{\eta}} = i(\mu + \lambda)\frac{\partial \bar{A}}{\partial \xi}.$$

Differentiating the latter with respect to $\bar{\xi}$ shows that

$$\frac{\partial^2 A}{\partial \xi^2} = 0,$$

hence

$$A = B'\xi + C',$$

with

$$\frac{\partial B'}{\partial \bar{\eta}} = 0, \quad 2\mu(2\mu + \lambda)\frac{\partial C'}{\partial \bar{\eta}} = i(\mu + \lambda)\,\bar{B}'.$$

Thus B' is analytic in η. Set

$$B' = 2\mu(2\mu + \lambda)\frac{\partial B}{\partial \eta},$$

with B analytic. Then

$$C' = i(\mu + \lambda)\,\bar{B} + C$$

for some analytic C, and the theorem is proven.

Turning to the more general case, the special "conformal" restriction $3\mu + \lambda = 0$, when many more conservation laws arise, is left aside to begin with.

Theorem 4.2. *Let* $\mu(\mu + \lambda)(2\mu + \lambda) \neq 0$, *and assume also that* $3\mu + \lambda \neq 0$. *Then any complex conserved density* $A(x, u, \nabla u)$ *of the two-dimensional Navier* *equations is equivalent to a linear combination of the following conserved densities:*

a) $$2\mu(2\mu+\lambda)\,\xi\,\frac{\partial B}{\partial\eta}+(\mu+\lambda)\,i\bar{B},$$

where $B(\bar{z},\eta)$ is analytic in both variables

b_1) $$C(\bar{z},\eta),$$

where C is analytic

b_2) $$[4\mu(2\mu+\lambda)\,w-(\mu+\lambda)\,iz\eta]\,\eta,$$

b_3) $$i(\tilde{w}\eta-w\tilde{\eta}),\qquad \text{(Betti reciprocity)}$$

where $\tilde{w}=\tilde{u}^1+i\tilde{u}^2$ is an arbitrary solution of Navier's equations with corresponding complex stress $\tilde{\eta}$.

Proof. In accordance with the results of I, all such densities must be of the form (4.3), with B, C allowed to depend on x^1, x^2, u^1, u^2. (Note that (4.3) includes all trivial x, u-independent densities, which in this case are all linear in ∇u, [17].) This ensures that the coefficients of all second order derivatives in (4.1) vanish, so we are left with the condition

$$\operatorname{Re}\left\{2\frac{\partial A}{\partial z}+\zeta\frac{\partial A}{\partial w}+\bar{\xi}\frac{\partial A}{\partial\bar{w}}\right\}=0,$$

where

$$\zeta=2\,\partial w/\partial z=(2\mu+\lambda)^{-1}\eta^2+i\mu^{-1}\eta^1.$$

Substituting (4.3), we find

$$\begin{aligned}\operatorname{Re}\Bigg\{&2\mu(2\mu+\lambda)\,|\xi|^2\frac{\partial^2 B}{\partial\eta\,\partial\bar{w}}+2\mu(2\mu+\lambda)\,\xi\left(2\frac{\partial^2 B}{\partial\eta\,\partial z}+\zeta\frac{\partial^2 B}{\partial\eta\,\partial w}\right)\\&+\bar{\xi}\left((\mu+\lambda)\,i\frac{\partial\bar{B}}{\partial w}+\frac{\partial C}{\partial\bar{w}}\right)+2(\mu+\lambda)\,i\frac{\partial\bar{B}}{\partial z}\\&+2\frac{\partial C}{\partial z}+\zeta\left((\mu+\lambda)\,i\frac{\partial\bar{B}}{\partial w}+\frac{\partial C}{\partial w}\right)\Bigg\}=0.\end{aligned}$$

Separating the coefficients of the various powers of ξ leads to the basic equations

$$\operatorname{Re}\frac{\partial^2 B}{\partial\eta\,\partial\bar{w}}=0,\tag{4.4}$$

$$(\mu+\lambda)\,i\frac{\partial B}{\partial\bar{w}}+\frac{\partial\bar{C}}{\partial w}+2\mu(2\mu+\lambda)\left(2\frac{\partial^2 B}{\partial\eta\,\partial z}+\zeta\frac{\partial^2 B}{\partial\eta\,\partial w}\right)=0,\tag{4.5}$$

$$\operatorname{Re}\left\{2(\mu+\lambda)\,i\frac{\partial\bar{B}}{\partial z}+2\frac{\partial C}{\partial z}+\zeta\left((\mu+\lambda)\,i\frac{\partial\bar{B}}{\partial w}+\frac{\partial C}{\partial w}\right)\right\}=0.\tag{4.6}$$

At this point it is helpful to consider how trivial conservation laws look in this complex notation. If

$$A^1 = -D_2 T, \quad A^2 = D_1 T,$$

where $T = T(x, u)$ is *real*, then

$$A = 2iD_{\bar{z}} T = i\left(2\frac{\partial T}{\partial \bar{z}} + \xi \frac{\partial T}{\partial w} + \bar{\zeta}\frac{\partial T}{\partial \bar{w}}\right). \tag{4.7}$$

Using the formula

$$\zeta = \frac{i[(\mu + \lambda)\eta + (3\mu + \lambda)\bar{\eta}]}{2\mu(2\mu + \lambda)}, \tag{4.8}$$

we see that (4.7) has the representation (4.3) with

$$2\mu(2\mu + \lambda) B = i\eta \frac{\partial T}{\partial w},$$

$$C = 2i\frac{\partial T}{\partial \bar{z}} + \frac{3\mu + \lambda}{2\mu(2\mu + \lambda)}\eta \frac{\partial T}{\partial \bar{w}}. \tag{4.9}$$

Also, the following easy lemma is crucial.

Lemma 4.3. *Let z, w be independent complex variables and suppose f is analytic in w. Then*

$$\operatorname{Re}\left(\frac{\partial f}{\partial \bar{z}}\right) = 0 \tag{4.10}$$

if and only if

$$f = g(z, w) + i\partial_z h,$$

where g is analytic in both z and w, $g(z, 0) = 0$, and $h(z, \bar{z})$ is real-valued, independent of w.

Proof. Condition (4.10) is equivalent to

$$\frac{\partial f}{\partial \bar{z}} + \frac{\partial \bar{f}}{\partial z} = 0.$$

Differentiate with respect to w:

$$\frac{\partial^2 f}{\partial \bar{z}\,\partial w} = 0,$$

hence $\partial f/\partial w$ is analytic in z, w, from which the lemma follows easily.

Returning to the proof of theorem 4.2, we see by the lemma, (4.4) implies

$$\frac{\partial B}{\partial \eta} = \tilde{B}(x, y, w, \eta) + i\frac{\partial T}{\partial w}$$

for $\tilde{B}$ analytic in w, η, $\tilde{B}|_{\eta=0} = 0$, and $T(x, u)$ real valued. But T corresponds to a trivial conservation law, so can be ignored. Moreover, without loss of generality $B|_{\eta=0} = 0$, hence B itself is analytic in w, η, satisfying

$$B|_{\eta=0} = 0 = \left.\frac{\partial B}{\partial \eta}\right|_{\eta=0}. \tag{4.11}$$

Next, differentiate (4.5) with respect to η and then $\bar{\eta}$ to find

$$(3\mu + \lambda)\, i\, \frac{\partial^3 B}{\partial \eta^2\, \partial w} = 0,$$

hence, as $3\mu + \lambda \neq 0$, B can be assumed independent of w, analytic in η, and hence, again from (4.5), C is analytic in w, η.

Next differentiating (4.6) with respect to η and $\bar{\eta}$, using (4.8), we see

$$\operatorname{Im} \frac{\partial^2 C}{\partial w\, \partial \eta} = 0;$$

hence

$$C = k(x)\, w\eta + D(x, \eta) + E(x, w)$$

with k real and $D|_{\eta=0} = 0$. Set $\eta = 0$ in (4.6), to find $\operatorname{Re} \partial E/\partial z = 0$, so by the lemma and the expressions (4.9) for trivial densities, $E = E(\bar{z}, w)$ is analytic in both $\bar{z}, w$.

Next set

$$D = \tilde{D}(x, \eta) + i\tilde{w}(x)\, \eta,$$

with

$$\tilde{D}|_{\eta=0} = 0 = \left.\frac{\partial \tilde{D}}{\partial \eta}\right|_{\eta=0}.$$

Differentiate (4.6) with respect to η and set $\eta = 0$:

$$4\mu(2\mu + \lambda)\frac{\partial \tilde{w}}{\partial z} + (\mu + \lambda)\frac{\partial E}{\partial w} - (3\mu + \lambda)\frac{\partial \overline{E}}{\partial w} = 0,$$

from which we conclude

$$E = -i\tilde{\eta}(x)\, w,$$

where $\tilde{\eta}$ is the complex stress associated with $\tilde{w}(x)$. (The terms in E independent of w are easily seen to be trivial densities.) Furthermore, since E is analytic in $\bar{z}$,

$$\partial\tilde{\eta}/\partial z = 0,$$

hence $\tilde{w}$ is a solution of Navier's equations, cf. (4.1).

The remaining terms in (4.6) are

$$\operatorname{Re}\left\{2(\mu + \lambda)\, i\, \frac{\partial \overline{B}}{\partial z} + 2\frac{\partial \tilde{D}}{\partial z} + 2w\eta\frac{\partial k}{\partial z} + \frac{(\mu + \lambda)}{2\mu(2\mu + \lambda)}\, i\eta^2 k\right\} = 0,$$

since

$$\operatorname{Re} \zeta\eta = \operatorname{Re}\left\{\frac{\mu+\lambda}{2\mu(2\mu+\lambda)} i\eta^2\right\}.$$

The only term with w gives $\partial k/\partial z = 0$, hence k is a constant. Set

$$F = 2(\mu+\lambda)\, i\bar{B} + 2\tilde{D} + \tfrac{1}{2}(\mu+\lambda)\,[\mu(2\mu+\lambda)]^{-1}\, ikz\eta^2,$$

so the above condition is just

$$\operatorname{Re}\frac{\partial F}{\partial z} = 0;$$

hence by the lemma F is analytic in $\bar{z}$, η (modulo trivial densities). Putting together the above information completes the proof of the theorem.

Now suppose $3\mu + \lambda = 0$. The complex stress is now $\eta = i\mu\, \partial w/\partial z$, so Navier's equations have the elementary form

$$\frac{\partial^2 w}{\partial z^2} = 0,$$

with general solution

$$w = f(\bar{z})\, z + g(\bar{z}),$$

for arbitrary analytic functions f, g. The conservation laws are then given as follows:

Theorem 4.4. *Suppose* $3\mu + \lambda = 0$, $\mu \neq 0$. *Then every complex conserved density of Navier's equations is equivalent to a linear combination of the following densities*:

a) $$w_{\bar{z}} \frac{\partial}{\partial w_z}\left(\frac{\partial K}{\partial z} + w_z \frac{\partial K}{\partial w}\right) - \overline{\left(\frac{\partial K}{\partial z} + w_z \frac{\partial K}{\partial w}\right)} + \frac{\partial K}{\partial \bar{z}},$$

where

$$K(z, \bar{z}, w, w_z) = z \int F(\bar{z}, w - zw_z, w_z)\, dw_z + \int C(\bar{z}, w - zw_z, w_z)\, dw_z, \tag{4.12}$$

where F, G *are analytic in their arguments.*

b) $$C(\bar{z}, w - zw_z, w_z),$$

where C *is analytic.*

Proof. We use the representation (4.3), in the slightly different form

$$A = w_{\bar{z}} \frac{\partial B}{\partial w_z} - \bar{B} + C,$$

where B, C are analytic in w_z. As above, (4.2) leads to the conditions

$$\operatorname{Re}\frac{\partial^2 B}{\partial w_z\,\partial \overline{w}} = 0, \tag{4.13}$$

$$\frac{\partial^2 B}{\partial z\,\partial w_z} + w_z\frac{\partial^2 B}{\partial w\,\partial w_z} - \frac{\partial B}{\partial w} + \frac{\partial \overline{C}}{\partial w} = 0, \tag{4.14}$$

$$\operatorname{Re}\left\{\frac{\partial C}{\partial z} - \frac{\partial \overline{B}}{\partial z} + w_z\left(\frac{\partial C}{\partial w} - \frac{\partial \overline{B}}{\partial w}\right)\right\} = 0. \tag{4.15}$$

Also, as in (4.11),

$$B|_{w_z=0} = 0 = \left.\frac{\partial B}{\partial w_z}\right|_{w_z=0}$$

without loss of generality. Thus from (4.12) $B = B(z, \overline{z}, w, w_z)$, analytic in w, w_z, hence (4.14) implies $\partial C/\partial \overline{w}$ is independent of w_z, and hence, on setting $w_z = 0$, $C = C(z, \overline{z}, w, w_z)$.

Let $v = w - zw_z$, $B = \tilde{B}(z, \overline{z}, v, w_z) = \partial\tilde{K}(z, \overline{z}, v, w_z)/\partial z$. Then (4.14) is

$$\frac{\partial^2}{\partial z^2}\left(\frac{\partial \tilde{K}}{\partial w_z} - z\frac{\partial \tilde{K}}{\partial v}\right) = 0.$$

Thus $\tilde{K} = K(z, \overline{z}, w, w_z)$ is of the form (4.12). Also (4.15) implies

$$\frac{\partial \tilde{C}}{\partial z} = \frac{\partial^2 \tilde{K}}{\partial z \partial \overline{z}},$$

$C = \tilde{C}(z, \overline{z}, v, w_z)$. The theorem now follows from elementary manipulations on these latter two equations.

The conservation laws in theorem 4.4 provide divergence identities in case $3\mu + \lambda \neq 0$, but these appear to be of little value, and are not discussed here.

5. Symmetry Groups

This section provides a brief discussion of the symmetry groups of the equations of linear isotropic elasticity. Applications of these symmetries to finding group-invariant solution, separation of variables, *etc.* are, for reasons of space, not treated here (but are certainly well worth investigating).

Recall first from section I.2 that a vector field

$$\vec{v}_\psi = \psi_i\frac{\partial}{\partial u^i}$$

is a generalized symmetry of Navier's equations (2.2) if and only if the basic symmetry equations, which in our case are

$$\mu\,\Delta\psi + (\mu + \lambda)\,\nabla(\nabla\cdot\psi) = 0, \tag{5.1}$$

hold whenever u is a solution of the original system, *cf.* theorem I.2.2, lemma I.2.3. Here, for simplicity, we restrict our attention to first order symmetries, so ψ depends only on x, u, ∇u. (Higher order symmetries can be generated by the methods of section 2E.)

If ψ has the special form

$$\psi_i = \varphi_i - u^i_j \xi^j, \tag{5.2}$$

where φ, ξ depend only on x, u, then $\vec{v}_\psi$ is the standard form of the Lie symmetry

$$\tilde{v} = \xi^j \frac{\partial}{\partial x_j} + \varphi_i \frac{\partial}{\partial u^i}, \tag{5.3}$$

whose associated group transformations can be realized geometrically on x, u space, as detailed in section I.2A.

Finally, recall that if A is the conserved density of a first order conservation law for (2.2), then $\vec{v}_\psi$ with ψ_i given by (3.9) is automatically a variational symmetry. We can thus immediately write down all variational symmetries. Nonvariational symmetries require the detailed solution of (5.1). For the three-dimensional case, these are discussed in subsection 5B.

A. Three-dimensional Variational Symmetries

Proposition 5.1. Consider the system (2.2) with $\mu(\mu + \lambda)(2\mu + \lambda) \neq 0$. If $7\mu + 3\lambda \neq 0$, then the Lie algebra of variational symmetries is spanned by the following vector fields:

a) *Lie symmetries*

$$\frac{\partial}{\partial x^i} \quad \text{— translations,}$$

$$\varepsilon_{ijk}\left(x^j \frac{\partial}{\partial x^k} - u^j \frac{\partial}{\partial u^k}\right) \quad \text{— rotations,} \tag{5.5}$$

$$x^i \frac{\partial}{\partial x^i} - \frac{1}{2} u^i \frac{\partial}{\partial u^i} \quad \text{— dilatations,}$$

b) *Linear symmetries*

$$\varepsilon^i(x) \frac{\partial}{\partial u^i} \quad \text{— addition of solutions,} \tag{5.6}$$

where $\varepsilon(x)$ is an arbitrary solution to (2.2),

c) *Generalized Symmetries*

$$\vec{q}_i = \left(\mu \frac{\partial u^i}{\partial x^j} + (2\mu + \lambda)\, \delta^i_j \frac{\partial u^k}{\partial x^k}\right) \frac{\partial}{\partial u^j}, \tag{5.7}$$

$$\varepsilon_{ijk}\left((\mu + \lambda)\, x^j \vec{q}_k + \mu(3\mu + \lambda)\, u^j \frac{\partial}{\partial u^k}\right).$$

If $7\mu + 3\lambda = 0$, then the symmetry algebra is spanned by the above vector fields and the following additional vector fields,

d) *Inversions*

$$x^k x^k \frac{\partial}{\partial x^i} - 2x^i x^k \frac{\partial}{\partial x^k} + (x^i u^j - 2x^j u^i)\frac{\partial}{\partial u^j} + 2x^k x^k \frac{\partial}{\partial u^i}, \tag{5.8}$$

e) *Further generalized symmetries*

$$x^k \vec{q}_k + \mu u^k \frac{\partial}{\partial u^k},$$

$$2x^i x^k \vec{q}_k - x^k x^k \vec{q}_i + \mu(2x^i u^k - x^k u^i)\frac{\partial}{\partial u^k} + \mu x^k u^k \frac{\partial}{\partial u^i}. \tag{5.9}$$

The proof of this theorem follows directly from the application of (3.9) to the conservation laws derived in propositions 3.1 and 3.2. For brevity we omit the details. Before discussing the symmetry group generated by the above vector fields, we turn to a discussion of non-variational symmetries.

B. Three-Dimensional Non Variational Symmetries

Proposition 5.2. For $\mu(\mu + \lambda)(2\mu + \lambda) \neq 0$, if $\vec{v}_\psi$ is a symmetry of (2.2) with $\psi = \psi(x, u, \nabla u)$, then $\vec{v}_\psi$ is a linear combination of the (standard forms of) the variational symmetries listed in proposition 4.1 and the nonvariational symmetries

$$\vec{s} = \Sigma u^i \frac{\partial}{\partial u^i}, \quad \text{(scaling)} \tag{5.10}$$

$$\vec{w} = \varepsilon_{ijk} u^i_j \frac{\partial}{\partial u^k}, \tag{5.11}$$

unless $7\mu + 3\lambda = 0$, in which case the following

$$2x^i \vec{w} + \varepsilon_{ijk} u^j \frac{\partial}{\partial u^k}, \tag{5.12}$$

$$x^k x^k \vec{w} + \varepsilon_{ijk} x^i u^j \frac{\partial}{\partial u^k} \tag{5.13}$$

are also included.

Proof. We need to solve the symmetry equations (5.1) for ψ. This is simplified by the techniques in section I.4, especially proposition I.4.1 and (I.4.6). Thus, the first step is to find all x, u independent symmetries, of which the translations $\vec{p}_i = u^j_i\, \partial/\partial u^j$ (now taken in standard form) and the generalized symmetries $\vec{q}_i$ constitute the variational ones. According to theorem I.5.2, in view of the form

(2.5) of $Q(\xi)$ in our case, the nonvariational x, u independent symmetries $\vec{v}_\psi$ are all found as solutions to

$$\frac{\partial \psi_i}{\partial u_k^j} = c\varepsilon_{ijk},$$

where $c = c(\nabla u)$. It is easy to show c is constant; hence all such symmetries are multiples of $\vec{w}$.

The second step is to substitute the general form

$$\vec{v}_\psi = \alpha^i(x, u)\,\vec{p}_i + \beta^i(x, u)\,\vec{q}_i + \gamma(x, u)\,\vec{w} + \delta^i(x, u)\frac{\partial}{\partial u^i}$$

into the symmetry equations (5.1). The solution of the resulting system of equations for α^i, β^i, γ, δ^i is another tedious computation in the spirit of those in section 3. For brevity, we omit the details, noting that the proposition gives the final results. (Our prior knowledge of all the variational symmetries does help here.)

C. Two-Dimensional Symmetries

The calculations here are analogous. If $\vec{v}_\psi = \psi^1\,\partial_{u^1} + \psi^2\,\partial_{u^2}$, we write $\psi = \psi^1 + i\psi^2$ in complex form. Then from (3.9) all variational symmetries are given by

$$\psi = \frac{1}{2\mu + \lambda}\frac{\partial \bar{A}}{\partial \xi^1} + \frac{\partial A}{\partial \eta^2},$$

where A is a complex conserved density. From (4.3), we have

$$\psi = 2\mu(2\mu + \lambda)\,i\,\xi\frac{\partial^2 B}{\partial \eta^2} + (3\mu + \lambda)\frac{\overline{\partial B}}{\partial \eta} + i\frac{\partial C}{\partial \eta},$$

where B, C are given in theorem 4.2, or, in case $3\mu + \lambda = 0$, theorem 4.4. (There are also symmetries corresponding to the othen cases b_2), b_3) in theorem 4.2.) The geometry of these symmetries looks very complicated in general.

Nonvariational symmetries will not be treated for lack of space.

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