

*Errata*  
 “*Conservation Laws in Elasticity*”  
 II. *Linear Homogeneous Isotropic Elastostatics*”

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1. On page 133, formula (2.1) should be

$$\mathcal{E} = \int_{\Omega} \left\{ \mu \|\frac{1}{2}(\nabla u + \nabla u^T)\|^2 + \frac{1}{2} \lambda (\nabla \cdot u)^2 \right\} dx,$$

*i.e.* a factor of  $\frac{1}{2}$  was left out of the first term in the stored energy.

2. On page 134–135, and subsequently, there is an unfortunate error in the identification of the tensor

$$S = \mu \nabla u + (\mu + \lambda) (\nabla \cdot u) \mathbf{1} \quad (*)$$

with the stress tensor  $\sigma$ , whose correct expression is

$$\sigma = \mu(\nabla u + \nabla u^T) + \lambda(\nabla \cdot u) \mathbf{1}.$$

The confusion arises because both  $S$  and  $\sigma$  determine the self-same system of Navier’s equations:

$$\mathcal{E}[u] = \nabla \cdot S = \nabla \cdot \sigma = \mu \Delta u + (\mu + \lambda) \nabla(\nabla \cdot u) = 0.$$

However,  $S$  and  $\sigma$  are *not* the same tensor. They differ by a trivial conservation law

$$\tau \equiv \sigma - S = \mu(\nabla u^T - (\nabla \cdot u) \mathbf{1}),$$

where “trivial” means that the divergence of  $\tau$  vanishes identically for *all* functions  $u(x)$ :

$$\nabla \cdot \tau \equiv 0.$$

(In terms of the notation for trivial conservation laws introduced on page 139,

$$\tau_j^i = \varepsilon_{ilm} B_j^{lm}.) \quad (**)$$

How does this affect the results in the paper? From a mathematical standpoint, the results as stated are all correct; each of the densities listed on pages 134–5 (or, equivalently, on pages 139 and 144–5) *does* give a mathematical conservation

law of Navier's equations in the sense that each of their divergences vanishes on solutions. However, from the point of view of physical interpretation, one should be using the stress  $\sigma$  to represent the linear conservation law equivalent to the system, rather than the unphysical tensor  $S$ . It is important to note that *each of the conservation laws which explicitly involves  $S$  is equivalent to a conservation law which involves  $\sigma$  instead.* The two laws will merely differ by a trivial conservation law or null divergence.

For example, consider the reciprocity relations which correspond to the conserved density  $K_s$  on page 134. Suppose  $u$  and  $\tilde{u}$  are displacements with corresponding stress tensors  $\sigma$  and  $\tilde{\sigma}$ . Let  $S$  and  $\tilde{S}$  be the corresponding tensors defined by fomula (\*). The standard Betti reciprocal theorem states that

$$\nabla \cdot (u \cdot \tilde{\sigma} - \tilde{u} \cdot \sigma) = u \cdot \mathcal{E}[\tilde{u}] - \tilde{u} \cdot \mathcal{E}[u],$$

where  $\mathcal{E}[u]$  denotes the left hand side of Navier's equations. The conserved density  $K_s$  leads to the equivalent reciprocity relation

$$\nabla \cdot (u \cdot \tilde{S} - \tilde{u} \cdot S) = u \cdot \mathcal{E}[\tilde{u}] - \tilde{u} \cdot \mathcal{E}[u].$$

Both of these laws are mathematically valid identities, but the former has the more immediate physical interpretation. These two reciprocity relations are *equivalent* in the sense that they differ only by a "trivial reciprocity relation", which takes the form

$$\nabla \cdot (u \cdot (\tilde{\sigma} - \tilde{S}) - \tilde{u} \cdot (\sigma - S)) \equiv 0,$$

and which vanishes for all displacement fields  $u$  and  $\tilde{u}$ .

Note also that the formula for the energy-momentum tensor  $P$  on page 134 is more usually written in terms of the stress  $\sigma$  as

$$P = \nabla u^T \cdot \sigma - W \cdot \mathbf{1},$$

where  $W = \mu \|\frac{1}{2}(\nabla u + \nabla u^T)\|^2 + \frac{1}{2} \lambda (\nabla \cdot u)^2$  is the stored energy function. The reader can verify that the two expressions are, in this case, identical. (Here, there is no need to add a null divergence!)

In general, then, to pass to an equivalent conservation law involving  $\sigma$  rather than  $S$ , we can replace the expression (3.22) for the general form of a conserved density by an equivalent expression

$$A = \alpha^i P^i + \beta^i Q^i + \gamma^i A^i + \varepsilon^i \sigma^i + \tilde{\theta}^{ik} B^{ik} + \omega,$$

which is related to (3.22) by

$$\tilde{\theta}^{ik} = \theta^{ik} - \varepsilon_{ikm} \varepsilon^m, \tag{+}$$

cf. (\*\*). (Unfortunately, the notation here has become slightly confusing:  $\varepsilon_{ikm}$  represents the alternating symbol, while  $\varepsilon^m$  is the coefficient of  $\sigma^m$  in the density  $A$ .) Thus the coefficients of  $S$  and  $\sigma$  are the same, and only the lower order terms in the two conserved densities differ. The explicit formulas for these equivalent conservation laws can easily be determined from the formulae on page 149 using (+).

The important point, then, is that for any conservation law or reciprocity relation, there are many *different*, but equivalent forms which it can take. But, as was pointed out to me by Professor R. K. KAUL, there are often good physical reasons for preferring one particular formulation over the others. For example, for a body in equilibrium, it is natural to impose the condition that the resulting moment of the conserved tensor density be zero. However, these physical restrictions on the forms of allowable conserved densities must be imposed after the mathematical analysis has been completed, and do not restrict the mathematical form that they may take. Presumably, each of the conservation laws in the paper has a preferred "physical" form, but I have not pursued this question. An interesting open problem is which physical restrictions serve to define uniquely conserved densities and how one incorporates these restrictions into the original search for conservation laws.

3. On page 150, in the formula on line 21, the  $\sigma$  should be a  $\xi$ , conforming to the usual Muskhelishvili complex notation.

4. On page 152, line 2 should be read

$$a) \quad 2\mu(2\mu + \lambda) \xi \frac{\partial B}{\partial \eta} + (\mu + \lambda) i \left[ \bar{B} + z \frac{\partial B}{\partial \bar{z}} \right],$$

*i.e.* the last term was inadvertently omitted. Also, on line 1 it should perhaps be emphasized that the complex conserved density is a *real* linear combination of the conserved densities a) (as corrected above),  $b_1$ ,  $b_2$ ) and  $b_3$ ).

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