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The structure of null Lagrangians

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Abstract. We say that $L(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u})$ is a null Lagrangian if and only if the corresponding integral functional $\mathscr{E}(\boldsymbol{u}) = \int_\Omega L(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u})\,\mathrm{d}x$ has the property that $\mathscr{E}(\boldsymbol{u}+\boldsymbol{\phi}) = \mathscr{E}(\boldsymbol{u})$ $\forall \boldsymbol{\phi} \in C_0^\infty(\Omega)$, for any choice of $\boldsymbol{u} \in C^1(\bar{\Omega})$.

In the *homogeneous* case, corresponding to $L(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) = \Phi(\nabla \boldsymbol{u})$, it is known that a necessary and sufficient condition for L to be a null Lagrangian is that $\Phi(\nabla \boldsymbol{u})$ is an affine combination of subdeterminants of $\nabla \boldsymbol{u}$ of all orders. In this paper we show that all *inhomogeneous* null Lagrangians may be constructed from these homogeneous ones by introducing appropriate potentials.

In this paper we consider null Lagrangians $L(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u})$, where $\boldsymbol{x} \in \Omega \subset \mathbb{R}^m$, $\boldsymbol{u}: \bar{\Omega} \to \mathbb{R}^n$ and $\nabla \boldsymbol{u} = (\partial u^\alpha / \partial x^i)$. These are integrands for which the corresponding integral

$$\mathscr{E}(\boldsymbol{u}) = \int_\Omega L(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}x \tag{1}$$

has the property that $\mathscr{E}(\boldsymbol{u}+\boldsymbol{\phi}) = \mathscr{E}(\boldsymbol{u}) \quad \forall\, \boldsymbol{\phi} \in C_0^\infty(\Omega)$ for any choice of $\boldsymbol{u} \in C^1(\bar{\Omega})$. It then follows by an approximation argument that $\mathscr{E}(\boldsymbol{u}_1) = \mathscr{E}(\boldsymbol{u}_2)$ whenever

$$\boldsymbol{u}_1, \boldsymbol{u}_2 \in C^1(\bar{\Omega}) \quad \text{and} \quad \boldsymbol{u}_1 = \boldsymbol{u}_2 \quad \text{on} \quad \partial\Omega. \tag{2}$$

In the case when $L(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) = \Phi(\nabla \boldsymbol{u})$ then an explicit representation for L can be given, namely that $\Phi(\nabla \boldsymbol{u})$ is an affine combination of subdeterminants of $\nabla \boldsymbol{u}$ of all orders (see e.g., Ball *et al* 1981, Edelen 1962, Ericksen 1962, Landers 1942, Rund 1966).

In this paper we show how all null Lagrangians $L(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u})$ may be explicitly constructed from those that depend solely on $\nabla \boldsymbol{u}$. This result has an obvious generalisation to the case when $L(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}, \nabla^2 \boldsymbol{u}, \ldots, \nabla^k \boldsymbol{u})$, $k > 1$ (i.e. when L depends on derivatives of $\boldsymbol{u}$ up to kth order), to show that all null Lagrangians of this form may be explicitly constructed from those that depend solely on $\nabla^k \boldsymbol{u}$, but for ease of presentation we only consider the case $k = 1$. (The recent paper of Edelen and Lagoudas (1986) gives a representation for L, obtained by computer algebra through the use of differential forms, for the case $k = 1$, $m = n = 3$.)

Our result is based on the following observation.

Suppose that $S:\bar{\Omega}\times\mathbb{R}^n\to\mathbb{R}^n$ is an arbitrary C^1 function and that $\Phi(\nabla u)$ is a null Lagrangian, then

$$L(x, u, \nabla u)\stackrel{\text{def}}{=}\Phi(\nabla S(x, u))=\Phi\left(\left(\frac{\partial S^\alpha}{\partial x^i}+\frac{\partial S^\alpha}{\partial u^\beta}\frac{\partial u^\beta}{\partial x^i}\right)\right) \tag{3}$$

is also a null Lagrangian. (This follows from (2) since $S(x, u_1(x))=S(x, u_2(x))$ $\forall x\in\partial\Omega$ whenever $u_1(x)=u_2(x)$ $\forall x\in\partial\Omega$, i.e. the functions $S(x, u(x))$ agree on $\partial\Omega$ whenever the functions $u(x)$ do.). The question then arises as to whether *all* null Lagrangians are expressible in this form. The answer to this is no, as shown by example 2, but our result is that all null Lagrangians are expressible as the *finite sum* of terms of the form (3). (Consisting of at most N terms where N is given by (13).)

One of the interests in null Lagrangians is their use in the field theory of the calculus of variations. Basically the problem in this theory is to show that some given map u_0 minimises a given integral functional $I(u)$ on some set of admissible maps $\mathscr{A}$. As noted in Sivaloganathan (1988), a necessary and sufficient condition for this is that there exists a functional $\mathscr{F}:\mathscr{A}\to\mathbb{R}$ satisfying

(i) $I(u)\geqslant\mathscr{F}(u)\quad\forall u\in\mathscr{A}$,
(ii) $\mathscr{F}(u)\geqslant\mathscr{F}(u_0)\quad\forall u\in\mathscr{A}$ and
(iii) $\mathscr{F}(u_0)=I(u_0)$.

(The proof that u_0 minimises $I(u)$ if there exists such a functional $\mathscr{F}$ is trivial and the converse follows on choosing $\mathscr{F}\equiv I$).

In the field theory, conditions (ii) and (iii) are satisfied by choosing $\mathscr{F}$ to be the integral of a null Lagrangian so that

$$\mathscr{F}(u)=\mathscr{F}(u_0)=I(u_0)\quad\forall u\in\mathscr{A}.$$

To our knowledge there are two classical field theories for multiple integral problems due to Carathéodory (1929) and Weyl (1935) (Weyl's theory being a generalisation of the one-dimensional field theory of Hilbert and Weierstrass). In our notation, Weyl's theory corresponds to choosing a null Lagrangian of the form (3) with $\Phi(\nabla u)=\mathrm{Tr}(\nabla u)$ and Carathéodory's corresponds to choosing $\Phi(\nabla u)=\det(\nabla u)$. (The problem is then to find an appropriate function S.) More recently, Armensen (1975) has constructed a field theory, again using a null Lagrangian of the form (3), in which he takes $\Phi(\nabla u)$ as the sum of all $r\times r$ minors of ∇u for some fixed r. As shown by theorem 7, the null Lagrangians used in these field theories are far from being the most general ones and this indicates that there are a large number that may still be exploited. For further details of field theories for multiple integral problems we refer to Morrey (1966), Rund (1966) and the references therein.

Applications of null Lagrangians to nonlinear elasticity are contained in the works by Ball (1977), Ball *et al* (1981), Davini and Parry (1988), Edelen and Lagoudos (1986), Ericksen (1962) and Sivaloganathan (1986, 1988).

We next outline our strategy of the proof.

We first introduce the notion of a null Lagrangian and in proposition 2 we give a necessary condition for a function $L(x, u, \nabla u)$ to be a null Lagrangian in terms of an expansion using homogeneous null Lagrangians (i.e. null Lagrangians that depend only on ∇u).

We next introduce the related notion of a null divergence which is a vector-valued function $\bar{P}(x, u, \nabla u)$ with the property that its total divergence,

$\operatorname{Div}\bar{\boldsymbol{P}}(\boldsymbol{x}, \boldsymbol{u}, \nabla\boldsymbol{u})$, is zero. In theorem 3 we give a characterisation of the homogeneous null divergences (which is the analogue, for null divergences, of theorem 1).

In theorem 7 we establish the link between null Lagrangians and null divergences, namely that $L(\boldsymbol{x}, \boldsymbol{u}, \nabla\boldsymbol{u})$ is a null Lagrangian if and only if it is expressible as $\operatorname{Div}\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{u}, \nabla\boldsymbol{u})$ for a suitable $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{u}, \nabla\boldsymbol{u})$. Proposition 4 then shows that, for fixed $\boldsymbol{x}_0$, $\boldsymbol{u}_0$, $\boldsymbol{P}(\boldsymbol{x}_0, \boldsymbol{u}_0, \nabla\boldsymbol{u})$ is a null divergence and gives a necessary form for $\boldsymbol{P}$ in terms of an expansion using homogeneous null divergences. A basic problem is to show that the coefficients in this expansion may be chosen to be smooth. This is dealt with in lemmas 5 and 6 by recasting the problem using differential forms in a way that allows us to obtain the required smoothness of $\boldsymbol{P}$ by use of the Poincaré lemma.

This characterisation of null Lagrangians given by theorem 7 when combined with proposition 4 yields our main result in theorem 8.

We will assume throughout this paper that $\Omega \subset\subset \mathbb{R}^m$, $m \geq 1$, is open and that $L: V \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R}$ is C^1, where $n \geq 1$ and $V \subset \mathbb{R}^m$ is open with $\bar{\Omega} \subset V$. The symbol F will be used to denote a typical element of $\mathbb{R}^{mn}$.

Definition 1. We say that $L(\boldsymbol{x}, \boldsymbol{u}, F)$ is a null Lagrangian if and only if

$$\mathscr{E}(\boldsymbol{u}) = \int_\Omega L(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla\boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}x \tag{4}$$

satisfies

$$\mathscr{E}(\boldsymbol{u} + \boldsymbol{\phi}) = \mathscr{E}(\boldsymbol{u}) \quad \forall\, \boldsymbol{u} \in C^1(\bar{\Omega}) \quad \forall\, \boldsymbol{\phi} \in C_0^\infty(\Omega).$$

We introduce the following notation for Jacobian determinants

$$J_\kappa^\alpha = \frac{\partial(u^{\alpha_1}, u^{\alpha_2}, \ldots, u^{\alpha_r})}{\partial(x^{\kappa_1}, x^{\kappa_2}, \ldots, x^{\kappa_r})} \tag{5}$$

where

$$\begin{aligned} &\alpha, \kappa \in \mathbb{R}^r \\ &\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \\ &\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_r) \end{aligned} \tag{6}$$

have integer entries and $r \leq \min(m, n)$.

The next theorem is taken from Olver (1983) and is a classification of all null Lagrangians that depend only on $\nabla\boldsymbol{u}$; these are known as the homogeneous null Lagrangians.

Theorem 1. $L(F)$ is a null Lagrangian if and only if

$$L(\nabla\boldsymbol{u}) = C_0 + \sum_{\alpha,\kappa} C_\kappa^\alpha J_\kappa^\alpha \quad \forall\, \boldsymbol{u} \in C^1(\Omega) \tag{7}$$

for some constants C_0, $C_\kappa^\alpha \in \mathbb{R}$, where the sum is taken over all α, κ and all $r \leq \min(m, n)$.

The next proposition gives a representation of inhomogeneous null Lagrangians.

Proposition 2. Suppose that $L(\boldsymbol{x}, \boldsymbol{u}, F)$ is a null Lagrangian, then L considered as a function of $\nabla \boldsymbol{u}$ is a homogeneous null Lagrangian, i.e. $L(\boldsymbol{x}_0, \boldsymbol{u}_0, F)$ is a homogeneous null Lagrangian for fixed $\boldsymbol{x}_0 \in \Omega$, $\boldsymbol{u}_0 \in \mathbb{R}^n$. Moreover

$$L(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) = C_0(\boldsymbol{x}, \boldsymbol{u}) + \sum_{\alpha, \kappa} C_\kappa^\alpha(\boldsymbol{x}, \boldsymbol{u}) J_\kappa^\alpha \quad \forall \boldsymbol{u} \in C^1(\Omega) \tag{8}$$

for some C^1 functions $C_0(.,.)$, $C_\kappa^\alpha(.,.)$, where J_κ^α are given by (5) and the sum is taken over all α, κ and all $r \leqslant \min(m, n)$.

The proof of this proposition follows from Olver (1983).

Closely related to the concept of a null Lagrangian is that of a null divergence.

Definition 2. A null divergence is an m-tuple of functions denoted by $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}, \ldots, \nabla^\kappa \boldsymbol{u})$ satisfying $\operatorname{Div} \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{u}, \ldots, \nabla^\kappa \boldsymbol{u}) = 0$ in the sense of distributions for all $\boldsymbol{u}$ which are κ times continuously differentiable and lie in the domain of definition $\boldsymbol{P}$.

We introduce the following notation for null divergences based on that of Olver (1983).

Given α as in (6) and $M \in \mathbb{R}^{r+1}$, where $M = (m_1, m_2, \ldots, m_{r+1})$ has integer entries that satisfy $1 \leqslant m_1 < m_2 \ldots < m_{r+1} \leqslant m$, we define $N_M^\alpha \in \mathbb{R}^m$ by

$$(N_M^\alpha)_i = \begin{cases} 0 & \text{if } i \neq m_s \text{ for some } s \\ (-1)^{s-1} J_{M_{\hat{s}}}^\alpha & \text{if } i = m_s \end{cases} \tag{9}$$

where $M_{\hat{s}} \in \mathbb{R}^r$

$$M_{\hat{s}} = (m_1, m_2, \ldots, m_{s-1}, m_{s+1}, \ldots, m_{r+1}).$$

It is easily verified that N_M^α is a null divergence. This is equivalent to the statement that the Euler Lagrange equations for the integral of any subdeterminant of $\nabla \boldsymbol{u}$ are identically satisfied (since any subdeterminant of $\nabla \boldsymbol{u}$ is a null Lagrangian by theorem 1).

The next theorem, which is taken from Olver (1983), is the analogue of theorem 1 for null divergences that depend purely on $\nabla \boldsymbol{u}$, i.e. homogeneous null divergences.

Theorem 3. $\boldsymbol{P}(F)$ is a null divergence if and only if

$$\boldsymbol{P}(\nabla \boldsymbol{u}) = \boldsymbol{P}_0 + \sum_{\alpha, M} P_M^\alpha N_M^\alpha \quad \forall\, \boldsymbol{u} \in C^1(\Omega) \tag{10}$$

for some $\boldsymbol{P}_0 \in \mathbb{R}^m$, $P_M^\alpha \in \mathbb{R}$, where the sum is taken over all α, M and all $r < \min(m, n+1)$.

Proposition 4. Suppose that $\boldsymbol{P}(.,.,.) : \bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}^m$ is C^1 and has the property that $\operatorname{Div} \boldsymbol{P}$ is a function of $\boldsymbol{x}$, $\boldsymbol{u}$ and $\nabla \boldsymbol{u}$ only (for any C^1 function $\boldsymbol{u}$), then $\boldsymbol{P}$ considered only as a function of $\nabla \boldsymbol{u}$ is a null divergence, i.e. $\boldsymbol{P}(\boldsymbol{x}_0, \boldsymbol{u}_0, \nabla \boldsymbol{u})$ is a null divergence for fixed $\boldsymbol{x}_0 \in \Omega$, $\boldsymbol{u}_0 \in \mathbb{R}^n$. Moreover

$$\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) = \boldsymbol{P}_0(\boldsymbol{x}, \boldsymbol{u}) + \sum_{\alpha, M} P_M^\alpha(\boldsymbol{x}, \boldsymbol{u}) N_M^\alpha \quad \forall\, \boldsymbol{u} \in C^1(\Omega) \tag{11}$$

for some C^1 functions $\boldsymbol{P}_0(.,.)$, $P^{\alpha}_M(.,.)$, where the N^{α}_M are given by (9) and the sum is taken over all α, M and all $r < \min(m, n+1)$.

The proof of this proposition follows from Olver (1983 proposition 4.2).

The next section, culminating in theorem 7, establishes the link between inhomogeneous null Lagrangians and null divergences.

The next two lemmas are mainly concerned, in the process of proving theorem 7, with showing that the coefficients of the vector function $\boldsymbol{P}$ of the theorem (given in the expression (11)) may be chosen to be smooth.

By proposition 2 any null Lagrangian $L(\boldsymbol{x}, \boldsymbol{u}, F)$ is expressible as

$$L(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) = C_0(\boldsymbol{x}, \boldsymbol{u}) + \sum_{\alpha,\kappa} C^{\alpha}_{\kappa}(\boldsymbol{x}, \boldsymbol{u}) J^{\alpha}_{\kappa} \quad \forall\, \boldsymbol{u} \in C^1(\Omega).$$

Given a multi-index $\kappa = (\kappa_1, \ldots, \kappa_r)$, set $\kappa' = (\kappa'_1, \ldots, \kappa'_{m-r})$ to be the complementary multi-index, consisting of all the integers from 1 to m which do not appear in κ. Define the sign of κ, written $\operatorname{sgn} \kappa$, to be the sign of the permutation that rearranges $(\kappa, \kappa') = (\kappa_1, \ldots, \kappa_r, \kappa'_1, \ldots, \kappa'_{m-r})$ in increasing order, i.e. $(1, 2, \ldots, m)$ for example, if $m = 5$ and $\kappa = (1, 2, 4)$ then $\kappa' = (3, 5)$ and $\operatorname{sgn} \kappa = -1$. Finally define the differential m-form:

$$\omega = C_0(\boldsymbol{x}, \boldsymbol{u})\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m + \sum_{\alpha,\kappa} (\operatorname{sgn} \kappa) C^{\alpha}_{\kappa}(\boldsymbol{x}, \boldsymbol{u})\, \mathrm{d}u^{\alpha_1} \wedge \ldots \wedge \mathrm{d}u^{\alpha_r} \wedge \mathrm{d}x^{\kappa'_1} \wedge \ldots \wedge \mathrm{d}x^{\kappa'_{m-r}}.$$

This is a differential form on the whole space $\mathbb{R}^m \times \mathbb{R}^n$, i.e. we are not viewing the u as functions of the x (see example 1). The motivation for defining the differential form ω is contained in the next lemma.

Lemma 5. $L(\boldsymbol{x}, \boldsymbol{u}, F)$ is a null Lagrangian if and only if $\mathrm{d}\omega = 0$ (i.e. ω is a closed form on $\mathbb{R}^m \times \mathbb{R}^n$).

The proof of this lemma consists of a straightforward verification that the conditions on the coefficient functions C^{α}_{κ} are the same in both instances.

Similarly, by proposition 4 any m-tuple of functions $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{u}, F)$ whose divergence depends on only first-order derivatives must be of the form

$$\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{u}, \nabla u) = \boldsymbol{P}_0(\boldsymbol{x}, \boldsymbol{u}) + \sum_{\alpha,\kappa} P^{\alpha}_{\kappa}(\boldsymbol{x}, \boldsymbol{u}) N^{\alpha}_{\kappa} \quad \forall\, \boldsymbol{u} \in C^1(\Omega).$$

Define the differential $(m-1)$ form:

$$\zeta = \sum_i (-1)^{i-1} P_{0i}(\boldsymbol{x}, \boldsymbol{u})\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \ldots \wedge \mathrm{d}x^m + \sum_{\alpha,\kappa} (\operatorname{sgn} \kappa) P^{\alpha}_{\kappa}(\boldsymbol{x}, \boldsymbol{u})\, \mathrm{d}u^{\alpha_1} \wedge \ldots \wedge \mathrm{d}u^{\alpha_r} \wedge \mathrm{d}x^{\kappa'_1} \wedge \ldots \wedge \mathrm{d}x^{\kappa'_{m-r-1}} \quad \forall\, \boldsymbol{u} \in C^1(\Omega)$$

where the P_{0i} are the components of the m-tuple $\boldsymbol{P}_0$ (all other P^{α}_{κ} are scalars). The connection between ω and ζ is given by the next lemma.

Lemma 6. Let $L(\boldsymbol{x}, \boldsymbol{u}, F)$, $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{u}, F)$ be given as above. Then

$$\operatorname{Div} P(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) = L(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) \quad \forall\, \boldsymbol{u} \in C^1(\Omega) \quad \text{if and only if } \omega = \mathrm{d}\zeta.$$

The proof of this lemma follows from proposition 4 and is a straightforward calculation.

Example 1. Let $m = n = 2$, $x = (x, y)$, $u = (u, v)$. Then

$$L = A(u_x v_y - u_y v_x) + Bu_x + Cu_y + Dv_x + Ev_y + F$$

is a null Lagrangian, A, B, C, D, E, F being functions of x, y, u, v if and only if the first-order partial differential equations

$$B_v = D_u + A_y \qquad E_u = C_v + A_x$$
$$F_u = B_x + C_y \qquad F_v = D_x + E_y$$

are satisfied. These are the same as the conditions that the differential 2-form

$$\omega = A\,\mathrm{d}u \wedge \mathrm{d}v + B\,\mathrm{d}u \wedge \mathrm{d}y - C\,\mathrm{d}u \wedge \mathrm{d}x + D\,\mathrm{d}v \wedge \mathrm{d}y - E\,\mathrm{d}v \wedge \mathrm{d}x + F\,\mathrm{d}x \wedge \mathrm{d}y$$

be closed, i.e. $\mathrm{d}\omega = 0$.

Similarly, $L = D_x P + D_y Q$ if and only if

$$P = Ru_y + Sv_y + T \qquad Q = -Ru_x - Sv_x + U$$

$R, S, T, U,$ again being functions of $x, y, u, v,$ satisfying

$$A = R_v - S_u \qquad B = T_u - R_y \qquad C = U_u + R_x$$
$$D = T_v - S_y \qquad E = U_v + S_x \qquad F = T_x + U_y$$

which are just the conditions that ω be the differential of the 1-form

$$\zeta = R\,\mathrm{d}u + S\,\mathrm{d}v - U\,\mathrm{d}x + T\,\mathrm{d}y$$

i.e. $\omega = \mathrm{d}\zeta$.

Theorem 7. Suppose that $\Omega \subset \mathbb{R}^m$ is star shaped. Then $L(\boldsymbol{x}, \boldsymbol{u}, F)$ is a null Lagrangian if and only if there exists an m-tuple of C^1 functions $\boldsymbol{P}: \bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}^m$ such that

$$L(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) = \operatorname{Div} \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) \quad \forall\, \boldsymbol{u} \in C^1(\Omega).$$

Proof. It follows from lemmas 5 and 6 that the relation of the coefficients of $\boldsymbol{P}$ and L is exactly the same as the coefficients of ω to those of ζ. We can thus invoke the Poincaré lemma on (star-shaped) subdomains of the Euclidean space $\mathbb{R}^m \times \mathbb{R}^n$. The standard homotopy formula (see e.g. Olver 1986, p 65) immediately shows that if the coefficients of ω are C^k then those of ζ can also be taken as C^k since they are obtained by integration in the radial direction.

Theorem 8. Suppose that $\Omega \subset \mathbb{R}^m$ is star shaped. Then $L(\boldsymbol{x}, \boldsymbol{u}, F)$ is a null lagrangian iff there exist N C^1 vector functions

$$\left\{\overset{(1)}{\boldsymbol{S}}, \overset{(2)}{\boldsymbol{S}} \ldots \overset{(N)}{\boldsymbol{S}}\right\} \qquad \overset{(i)}{\boldsymbol{S}}: \bar{\Omega} \times \mathbb{R}^n \to \mathbb{R}^n \qquad i = 1, 2, \ldots, N$$

and N corresponding *homogeneous* null Lagrangians $\left\{\overset{(1)}{\Phi}, \overset{(2)}{\Phi}, \ldots, \overset{(N)}{\Phi}\right\}$ such that

$$L(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) = \sum_{i=1}^{N} \overset{(i)}{\Phi}(\nabla \overset{(i)}{\boldsymbol{S}}(\boldsymbol{x}, \boldsymbol{u})) \quad \forall\, \boldsymbol{u} \in C^1(\Omega) \tag{12}$$

where

$$N = 1 + \frac{nm(m-1)}{2!} + \frac{n(n-1)}{2!}\frac{m(m-1)(m-2)}{3!} + \dots \tag{13}$$

Proof. It follows from the observations made in the introduction that given

$$\left\{\overset{(1)}{\boldsymbol{S}}, \overset{(2)}{\boldsymbol{S}} \dots \overset{(N)}{\boldsymbol{S}}\right\} \qquad \text{and} \qquad \left\{\overset{(1)}{\Phi}, \overset{(2)}{\Phi} \dots \overset{(N)}{\Phi}\right\}$$

the L defined by (12) is then a null Lagrangian.

Conversely, if $L(\boldsymbol{x}, \boldsymbol{u}, F)$ is a null Lagrangian then by proposition 4 and theorem 7

$$L(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) = \mathrm{Div}\left[\boldsymbol{P}_0(\boldsymbol{x}, \boldsymbol{u}) + \sum_{\alpha, M} P_M^\alpha(\boldsymbol{x}, \boldsymbol{u}) N_M^\alpha\right] \quad \forall\, \boldsymbol{u} \in C^1(\Omega) \tag{14}$$

for some functions $\boldsymbol{P}_0(\boldsymbol{x}, \boldsymbol{u})$, $P_M^\alpha(\boldsymbol{x}, \boldsymbol{u})$, where N_M^α is given by (9) and the sum is taken over all α, M and all $r < \min(m, n+1)$.

The theorem follows on observing that, apart from the first term, (14) is the sum of terms of the form

$$\mathrm{Div}[P_M^\alpha(\boldsymbol{x}, \boldsymbol{u}) N_M^\alpha] = \frac{\partial(P_M^\alpha, u^{\alpha_1}, u^{\alpha_2}, \dots, u^{\alpha_r})}{\partial(x^{m_1}, x^{m_2}, \dots, x^{m_{r+1}})} \tag{15}$$

where $\alpha \in \mathbb{R}^r$, $M \subset \mathbb{R}^{r+1}$ and $r < \min(m, n+1)$.

This is expressible in the form

$$\Phi(\nabla \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{u})) \tag{16}$$

with $\Phi(F)$ a homogeneous null Lagrangian and $\boldsymbol{S} : \bar{\Omega} \times \mathbb{R}^n \to \mathbb{R}^n$ by choosing

$$\Phi(\nabla \boldsymbol{u}) = J_M^{\bar{\alpha}}$$

where $\bar{\alpha} \in \mathbb{R}^{r+1}$ is given by

$$\bar{\alpha} = (1, 2, 3, \dots, r+1)$$

and $M = (m_1, m_2, \dots, m_{r+1})$ and on setting

$$(\boldsymbol{S}(\boldsymbol{x}, \boldsymbol{u}))_i = \begin{cases} P_M^\alpha(\boldsymbol{x}, \boldsymbol{u}) & \text{if } i = 1 \\ u^{\alpha_{i-1}} & \text{if } i = 2, 3, \dots, r+1 \\ 0 & \text{if } i = r+2, \dots, n. \end{cases}$$

Thus each term in (14), other than the first, is expressible in the form (16). The first term in (14) may be expressed in this form by choosing $\Phi(F) = \mathrm{Tr}\, F$ and setting $\boldsymbol{S}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{P}_0(\boldsymbol{x}, \boldsymbol{u})$. The theorem now follows on counting the number of vector terms in the sum

$$\boldsymbol{P}_0(\boldsymbol{x}, \boldsymbol{u}) + \sum_{\alpha, M} P_M^\alpha(\boldsymbol{x}, \boldsymbol{u}) N_M^\alpha.$$

Remark. If Ω is not star shaped then the representation (12) is locally valid on star shaped subdomains of Ω.

It is clear from the proof of theorem 8 that the number of arbitrary scalar potentials in the sum (12) is given by $(m-1) + N$ where N is given by (13).

The following simple example shows that we will in general require more than one term in the sum (12).

Example 2. Let $n=1$, $m=2$, $\boldsymbol{x}=(x, y)$, $\boldsymbol{u}=u$. Then by theorem 7, $L(\boldsymbol{x}, u, F)$ given by

$$L(\boldsymbol{x}, u, \nabla u) = \operatorname{Div} \boldsymbol{P}_0(\boldsymbol{x}, u) = \operatorname{Div}\begin{pmatrix} P_{01}(\boldsymbol{x}, u) \\ P_{02}(\boldsymbol{x}, u) \end{pmatrix}$$

$$= \frac{\partial P_{01}}{\partial x} + \frac{\partial P_{02}}{\partial y} + \frac{\partial P_{01}}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial P_{02}}{\partial u}\frac{\partial u}{\partial y} \qquad \forall\, u \in C^1(\Omega)$$

is a null Lagrangian. By proposition 2 the most general homogeneous null Lagrangian $\Phi(F)$ is given by

$$\Phi(\nabla u) = A\partial u/\partial x + B\partial u/\partial y + C \quad \forall\, u \in C^1(\Omega) \qquad \text{where } A, B, C \in \mathbb{R}.$$

Now let $S : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be C^1. In order to express $L(\boldsymbol{x}, u, \nabla u)$ as $\Phi(\nabla S(\boldsymbol{x}, u))$, where

$$\Phi(\nabla S(\boldsymbol{x}, u)) = A\left(\frac{\partial S}{\partial x}(\boldsymbol{x}, u) + \frac{\partial S}{\partial u}(\boldsymbol{x}, u)\frac{\partial u}{\partial x}\right) + B\left(\frac{\partial S}{\partial y}(\boldsymbol{x}, u) + \frac{\partial S}{\partial u}(\boldsymbol{x}, u)\frac{\partial u}{\partial y}\right) + C$$
$$\forall\, u \in C^1(\Omega)$$

it is necessary to solve the following system of equations for S and the constants A, B, and C:

(i) $A\partial S/\partial x + B\partial S/\partial y + C = \partial P_{01}/\partial x + \partial P_{02}/\partial y$
(ii) $A\partial S/\partial u = \partial P_{01}/\partial u$
(iii) $B\partial S/\partial u = \partial P_{02}/\partial u$.

Clearly by choosing $\partial P_{01}/\partial u$ and $\partial P_{02}/\partial u$ to be linearly independent functions we can ensure that the system (i)–(iii) has no solution.

Our next example shows that, in general, the expression (12) will be non-unique in a non-trivial way.

Example 3. Let $m=n=2$. Let $\mathbf{A}=(a_i^\alpha)$, $\mathbf{B}=(b_i^\alpha)$ be 2×2 matrices. We will make use of the following identity for 2×2 determinants

$$\det(\mathbf{A}+\mathbf{B}) = \begin{vmatrix} a_1^1+b_1^1 & a_2^1+b_2^1 \\ a_1^2+b_1^2 & a_2^2+b_2^2 \end{vmatrix} = \begin{vmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{vmatrix} + \begin{vmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{vmatrix} + \begin{vmatrix} a_1^1 & a_2^1 \\ b_1^2 & b_2^2 \end{vmatrix} + \begin{vmatrix} b_1^1 & b_2^1 \\ a_1^2 & a_2^2 \end{vmatrix}.$$

Now let

$$\overset{(i)}{\boldsymbol{S}}(\boldsymbol{x}, \boldsymbol{u}) = \begin{pmatrix} \overset{(i)}{S}{}^1(\boldsymbol{x}, \boldsymbol{u}) \\ \overset{(i)}{S}{}^2(\boldsymbol{x}, \boldsymbol{u}) \end{pmatrix} \qquad i = 1, 2$$

be two arbitrary vector valued functions and let

$$\mathbf{A} = \nabla\overset{(1)}{\boldsymbol{S}}(\boldsymbol{x}, \boldsymbol{u}) \qquad \mathbf{B} = \nabla\overset{(2)}{\boldsymbol{S}}(\boldsymbol{x}, \boldsymbol{u})$$

then

$$\det\left(\nabla\left(\overset{(1)}{\boldsymbol{S}}+\overset{(2)}{\boldsymbol{S}}\right)\right)=\sum_{i=1}^{4}\det\left(\nabla\overset{(i)}{\boldsymbol{S}}(\boldsymbol{x},\boldsymbol{u})\right)$$

where

$$\overset{(3)}{\boldsymbol{S}}(\boldsymbol{x},\boldsymbol{u})=\begin{pmatrix}\overset{(1)}{S}{}^1(\boldsymbol{x},\boldsymbol{u})\\ \overset{(2)}{S}{}^2(\boldsymbol{x},\boldsymbol{u})\end{pmatrix}\qquad\text{and}\qquad\overset{(4)}{\boldsymbol{S}}(\boldsymbol{x},\boldsymbol{u})=\begin{pmatrix}\overset{(2)}{S}{}^1(\boldsymbol{x},\boldsymbol{u})\\ \overset{(1)}{S}{}^2(\boldsymbol{x},\boldsymbol{u})\end{pmatrix}.$$

Example 4.

(i) Let $m=n=2$, $\boldsymbol{x}=(x,y)$, $\boldsymbol{u}=(u,v)$. Then the most general null Lagrangian $L(\boldsymbol{x},\boldsymbol{u},F)$ is given by

$$L(\boldsymbol{x},\boldsymbol{u},\nabla\boldsymbol{u})=\operatorname{Div}\boldsymbol{P}_0(\boldsymbol{x},\boldsymbol{u})+\frac{\partial(P^1(\boldsymbol{x},u),u)}{\partial(x,y)}+\frac{\partial(P^2(\boldsymbol{x},\boldsymbol{u}),v)}{\partial(x,y)}$$
$$\forall\,\boldsymbol{u}\in C^1(\Omega)$$

where $\boldsymbol{P}_0(\boldsymbol{x},\boldsymbol{u})=(P_{01}(\boldsymbol{x},\boldsymbol{u}),P_{02}(\boldsymbol{x},\boldsymbol{u}))$, $P^1(\boldsymbol{x},\boldsymbol{u})$, $P^2(\boldsymbol{x},\boldsymbol{u})$ are abitrary functions.

(ii) Let $m=n=3$, $\boldsymbol{x}=(x,y,z)$, $\boldsymbol{u}=(u^1,u^2,u^3)$. Then the most general null Lagrangian $L(\boldsymbol{x},\boldsymbol{u},F)$ is given by

$$\begin{aligned}L(\boldsymbol{x},\boldsymbol{u},\nabla\boldsymbol{u})=\operatorname{Div}\boldsymbol{P}_0(\boldsymbol{x},\boldsymbol{u})&+\sum_{i=1}^{3}\frac{\partial(P_i^1,u^i)}{\partial(x,y)}\\&+\frac{\partial(P_i^2,u^i)}{\partial(y,z)}+\frac{\partial(P_i^3,u^i)}{\partial(x,y)}+\frac{\partial(\bar P^1,u^1,u^2)}{\partial(x,y,z)}\\&+\frac{\partial(\bar P^2,u^2,u^3)}{\partial(x,y,z)}+\frac{\partial(\bar P^3,u^1,u^3)}{\partial(x,y,z)}\\&\forall\,u\in C^1(\Omega)\end{aligned}$$

where $\boldsymbol{P}_0(\boldsymbol{x},\boldsymbol{u})=(P_{01}(\boldsymbol{x},\boldsymbol{u}),P_{02}(\boldsymbol{x},\boldsymbol{u}),P_{03}(\boldsymbol{x},\boldsymbol{u}))$, $P_j^i(\boldsymbol{x},\boldsymbol{u})$, $\bar P^i(\boldsymbol{x},\boldsymbol{u})$ $i,j=1,2,3$ are arbitrary functions.

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