On formal integrability of evolution equations and local geometry of surfaces

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Abstract

Some relationships between local differential geometry of surfaces and integrability of evolutionary partial differential equations are studied. It is proven that every second order formally integrable equation describes pseudo-spherical surfaces. A classification of integrable equations of Boussinesq type is presented, and it is shown that they can be interpreted geometrically as "equations describing hyperbolic affine surfaces".

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1 Introduction.

One of the most widely accepted definitions of integrability of partial differential equations requires the existence of soliton solutions, i.e. of special kind of traveling wave solutions that interact "elastically", without changing their shapes. The analytical construction of soliton solutions is based on the general inverse scattering method. In the formulation of Zakharov and Shabat [36], all known integrable systems supporting solitons can be realized as the integrability condition of a linear problem of the form

$$\psi_x = U\,\psi, \qquad \psi_t = V\,\psi, \tag{1.1}$$

where the matrices U and V, which depend on the field variables u and their derivatives as well as a "spectral" parameter λ, take their values in some matrix Lie algebra $\mathfrak{g}$. Introducing the $\mathfrak{g}$–valued one-form

$$\Omega = U\,dx + V\,dt \tag{1.2}$$

allows us to combine the linear system (1.1) into a single one-form equation

$$d\psi = \Omega\,\psi. \tag{1.3}$$

The associated integrability conditions for (1.1) or (1.3), which are obtained by cross differentiation, then take the matrix form

$$d\Omega - \Omega \wedge \Omega = 0, \tag{1.4}$$

that is, they imply that the connection one–form Ω is flat. In terms of the matrices U and V, the system of equations at hand is characterized by a *zero curvature condition*

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [\,U, V\,] = 0. \tag{1.5}$$

Any system of differential equations in the field variables u which can be characterized by such a linear problem will be called "kinematically integrable", after Faddeev and Takhtajan, [15]; see also [27, 29] and references therein.

Among the properties that seem to be characteristic of equations which have been called integrable are the Painlevé property (Weiss, Tabor and Carnevale [35]), the existence of a bi-Hamiltonian formulation (Magri [22]), of an infinite number of generalized symmetries (Olver [25]), of an infinite hierarchy of conservation laws [22], and of a formal symmetry of rank ∞, (Mikhailov, Shabat and Sokolov [23], Mikhailov, Shabat and Yamilov [24]). The existence of higher symmetries is a particularly good test of integrability because they can be computed algorithmically. As an illustration, in Section 2 we prove that the Boussinesq equation is the unique formally integrable (in the sense of possessing a formal symmetry of rank ∞) polynomial system of evolution equations of its particular scaling homogeneity.

An interesting and natural problem is to investigate the relationships among the properties listed above. Several theorems have been proved in this context. For example, the existence of a bi-Hamiltonian formulation implies the existence of a recursion operator, of an infinite number of commuting conservation laws and of an infinite hierarchy of generalized symmetries in involution, [22]. More recently, Reyes [27] has shown that all autonomous second order formally integrable evolution equations possess a zero curvature formulation (1.5).

We prove in Section 3 that all second order formally integrable equations possess a zero curvature formulation, thereby generalizing the main result of Reyes [27]. In principle, this result implies that these systems can be solved analytically by means of inverse scattering techniques. The zero curvature formulation is based on $\mathfrak{sl}(2,\mathfrak{R})$–valued linear problems (1.3), and is obtained by exploiting the fact (Theorem 3.7 below) that all second order equations which are formally integrable belong to an interesting class of equations introduced by Chern and Tenenblat [10] — the "equations describing pseudo-spherical surfaces".

We then reconsider the Boussinesq equation. It is well–known, [18, 13], that it can be formulated as a zero curvature condition for an $\mathfrak{sl}(3,\mathfrak{R})$–valued linear system. Coupled with our classification and comparison results, this lends added weight to the general equivalence between formal and kinematic integrability. We use this zero curvature condition in Section 4 to show that the local solutions of the Boussinesq equation determine the structure of an hyperbolic affine surface on the space of independent variables x, t. Thus, we can find geometrical interpretations for a general class of formally integrable equations beyond the second order case.

2 Formal Integrability.

We begin with a brief review of formal symmetries and the symmetry approach to integrability. We will be considering n^{th} order evolution equations

$$u_t = K[u] = K(x,t,u,u_1,\ldots,u_n), \tag{2.1}$$

in which the solution $u = f(x,t)$ depends on a single spatial variable x. The right hand side is a *differential function*, meaning that it depends on x, u, and a finite number of derivatives

$u_k = D_x^k u$, as well as, in the non-autonomous case, t. Here, D_x and D_t denote the total derivatives with respect to x and t respectively.

A second evolution equation $u_t = Q[u]$ is said to be a *symmetry* of (2.1) if, at least on a formal level, their flows commute. The infinitesimal symmetry criterion, [25], is

$$\frac{\partial Q}{\partial t} + \mathbf{D}_K(Q) - \mathbf{D}_Q(K) = 0. \tag{2.2}$$

Here

$$\mathbf{D}_K = \sum_i \frac{\partial K}{\partial u_i} D_x^i \tag{2.3}$$

denotes the Fréchet derivative or formal linearization of a differential function K.

The definition of formal symmetries relies on the calculus of pseudo-differential operators, that is, formal Laurent series

$$\mathcal{D} = \sum_{-\infty < i \leq k} P_i[u]\, D_x^i \tag{2.4}$$

in the total derivative D_x whose coefficients are differential functions. We call k the *order* of $\mathcal{D}$ provided $P_k \neq 0$. See [23, 24] and [25, chapter 5] for details, as well as [1, 32, 20] for the non-autonomous case.

Definition 2.1 Let $u_t = K[u]$ be an n^{th} order evolution equation. A pseudo-differential operator $\mathcal{D}$ of order m is called a *formal symmetry* of *rank* k if

$$\text{order}\,(\, \mathcal{D}_t + [\,\mathcal{D}, \mathbf{D}_K\,]\,) \leq n + m - k \tag{2.5}$$

on solutions.

Here, given a pseudo-differential operator as in (2.4), we define

$$\mathcal{D}_t = [\,D_t, \mathcal{D}\,] = \sum_{-\infty < i \leq k} (D_t P_i[u])\, D_x^i = \sum_{-\infty < i \leq k} \mathbf{D}_{P_i}(K)\, D_x^i, \tag{2.6}$$

the final equality holding on solutions to (2.1).

Linearization of the infinitesimal symmetry criterion (2.2) proves the following:

Proposition 2.2 *If $u_t = Q[u]$ is an m^{th} order symmetry of an evolution equation* (2.1), *then its Fréchet derivative $\mathbf{D}_Q$ is a formal symmetry of rank m.*

The foregoing analysis extends straightforwardly to systems of equations, see [23, 24].

Definition 2.3 A system of evolution equations is called *formally integrable* if it possesses a formal symmetry of infinite rank.

A recursion operator gives a formal symmetry of infinite rank, [25], and therefore every system possessing a recursion operator is formally integrable. The converse is not known in general; see [30] for further discussion. In the case of scalar equations, the existence of one higher order symmetry, or, more or less equivalently, of a formal symmetry of sufficiently high order, appears to be enough to guarantee formal integrability. This has been rigorously proved for homogeneous, autonomous polynomial scalar evolution equations with linear leading terms by Sanders and Wang, [30], using a remarkable synthesis of the symbolic method of classical invariant theory and results from Diophantine approximation theory on the factorizability of certain algebraic polynomials.

It has been also proven that an autonomous second order evolution equation is integrable if and only if it has a formal symmetry of rank 5, and that an autonomous third order evolution equation is integrable if and only if it has a formal symmetry of symmetry of rank 8. However, it is not known what rank of formal symmetry is required for a general n^{th} order evolution equation to guarantee integrability. The following classification of all formally integrable second order evolution equations can be found in Svinolupov and Sokolov, [32].

Theorem 2.4 *Every formally integrable second order evolution equation is equivalent, under a contact transformation of the form*

$$t = \chi(\overline{t}), \qquad x = \phi(\overline{t}, \overline{x}, \overline{u}, \overline{u}_{\overline{x}}), \qquad u = \psi(\overline{t}, \overline{x}, \overline{u}, \overline{u}_{\overline{x}}),$$

to one of the following:

$$u_t = u_{xx} + h(x,t)u, \tag{2.7}$$

$$u_t = u_{xx} + uu_x + g(x,t), \tag{2.8}$$

$$u_t = D_x(u_x u^{-2}), \tag{2.9}$$

$$u_t = D_x(u_x u^{-2} - x), \tag{2.10}$$

$$u_t = D_x(u_x u^{-2} + x^2 u) + xu. \tag{2.11}$$

In the case of systems in q dependent variables, Fokas conjectured [17] that the existence of q higher symmetries will ensure formal integrability. In an attempt to understand the validity of this claim, the 2 component system

$$\begin{cases} u_t = u_{xxxx} + v^2, \\ v_t = v_{xxxx}. \end{cases} \tag{2.12}$$

was conjectured by Bakirov, [3], and then rigorously proved by Beukers, Sanders and Wang, [4], to only have a single higher (sixth) order symmetry. On the other hand, the Bakirov system does possess a formal symmetry of infinite rank, [5], and so is formally integrable. More recently, van der Kamp and Sanders, [34], have proposed an example of a very complicated two-component system possessing only two higher-order symmetries, but it is still not known whether it possesses a formal symmetry of rank ∞. Thus, the precise relationship between formal integrability and the existence of a finite number of higher order or formal symmetries for systems remains rather unclear.

It is worth noting that both the Bakirov system (2.12), and the van der Kamp–Sanders example can be decoupled, in the sense of the following definition. This means that they are, in a sense, not "true" two-component systems.

Definition 2.5 A two-component system of evolution equations is called *decoupled* if one of the equations depends only on a single dependent variable.

A decoupled system can be effectively considered as a pair of scalar equations, where the solution to the equation involving only one of the dependent variables drives the second equation. In this paper, we only consider genuinely non-decoupled systems. Since all known symmetry pathologies occur in decoupled systems, we will continue to use the existence of higher order (formal) symmetries to detect integrability.

A particularly important example to be studied here is the Boussinesq equation

$$u_{tt} = u_{xxxx} + D_x(u u_x). \tag{2.13}$$

This integrable soliton equation was derived by Boussinesq, [6, p. 258], as a model for the uni-directional propagation of long waves in shallow water. Less well known is the fact that in the 1870's Boussinesq also derived the Korteweg-de Vries (KdV) equation, its first three conservation laws, and its one-soliton and periodic traveling wave solutions, in [7, eq. (30), p. 77], [8, eqs. (283, 291)], some 25 years before the paper of Korteweg and deVries!

We rewrite the Boussinesq equation (2.13) as a system of two evolution equations

$$\begin{cases} u_t = v_x \\ v_t = u_{xxx} + u u_x. \end{cases} \tag{2.14}$$

The system (2.14) has an obvious scaling symmetry

$$(x, t, u, v) \longmapsto (\lambda^{-1} x, \lambda^{-2} t, \lambda^2 u, \lambda v). \tag{2.15}$$

The most general autonomous polynomial evolutionary system that admits such a scaling symmetry is

$$\begin{cases} u_t = a_1 u_{xx} + a_2 v_x + a_3 u^2, \\ v_t = b_1 u_{xxx} + b_2 v_{xx} + b_3 u u_x + b_4 u v, \end{cases} \tag{2.16}$$

where the a_i, b_i are arbitrary constants. The following Theorem demonstrates that the Boussinesq system (2.14) is, in a precise sense, the unique integrable system among all non-decouplable systems of the form (2.16).

Theorem 2.6 *A nonlinear, non-decouplable equation of type* (2.16) *is formally integrable if it is equivalent (up to a scaling) to one of the following systems*

$$\begin{cases} u_t = u_{xx} - v_x + \frac{1}{2} u^2 \\ v_t = u_{xxx} - v_{xx} + u u_x \end{cases} \tag{2.17}$$

$$\begin{cases} u_t = u_{xx} + v_x \\ v_t = (\alpha - 1) u_{xxx} - v_{xx} + u u_x \end{cases} \tag{2.18}$$

$$\begin{cases} u_t = v_x \\ v_t = \alpha u_{xxx} + u u_x \end{cases} \tag{2.19}$$

The proof of Theorem 2.6 relies on extensive symbolic computations based on a Mathematica symbolic manipulation package developed by the second author; see also [26]. The computations demonstrate that the three systems (2.17), (2.18), (2.19), form a complete list of non-decouplable systems of type (2.16) that possess a formal symmetry of rank at least 7. In all three systems, we can eliminate v and obtain a single equation for u. When $\alpha \neq 0$, both (2.18) and (2.19) reduce to a rescaled version of the Boussinesq equation

$$u_{tt} = \alpha u_{xxxx} + D_x(u u_x). \tag{2.20}$$

Indeed, (2.18) seems to be a previously unknown way to write the Boussinesq equation as an integrable system, although it can be reduced to (2.19) by an invertible differential substitution $(u, v) \mapsto (u, v + u_x)$. On the other hand, the first system (2.17) reduces to an ordinary differential equation $u_{tt} = u u_t$ and thus can be solved explicitly!

Remark: In (2.19) we can rescale any positive parameter α to 1 and any negative α to -1, and thus reduce to the usual Boussinesq equations $u_{tt} = \pm u_{xxxx} + D_x(u u_x)$. On the other hand, the parameter α in (2.18) is essential and cannot be scaled away.

In the next two sections, we consider the existence of geometrical interpretations for formally integrable equations.

3 Geometric Integrability.

Geometric integrability was introduced by Chern and Tenenblat [10], motivated by Sasaki's observation [31] (See also Kamran and Tenenblat [21]) that the graphs of solutions to the soliton equations integrable by the AKNS inverse scattering scheme are pseudo-spherical surfaces.

Definition 3.1 A two-dimensional manifold S is called a *pseudo-spherical surface* if there exist one-forms $\overline{\omega}^1, \overline{\omega}^2, \overline{\omega}^3$ on S that satisfy the independence condition $\overline{\omega}^1 \wedge \overline{\omega}^2 \neq 0$, and the structure equations

$$d\,\overline{\omega}^1 = \overline{\omega}^3 \wedge \overline{\omega}^2, \qquad d\,\overline{\omega}^2 = \overline{\omega}^1 \wedge \overline{\omega}^3, \qquad d\,\overline{\omega}^3 = \overline{\omega}^1 \wedge \overline{\omega}^2. \tag{3.1}$$

The pseudo-spherical structure equations (3.1) imply that the induced Riemannian metric $\overline{\omega}^1 \otimes \overline{\omega}^1 + \overline{\omega}^2 \otimes \overline{\omega}^2$ has constant Gaussian curvature -1, and that moreover, $\overline{\omega}^3$ is the unique associated connection form.

Definition 3.2 A system of differential equations

$$\Delta(x, t, u, u_1, \ldots, u_n) = 0, \tag{3.2}$$

in two independent variables is said to be of *pseudo-spherical type* if there exist one-forms

$$\omega^\alpha = f_{\alpha 1}(x, t, u, \ldots, u_r)\, dx + f_{\alpha 2}(x, t, u, \ldots, u_s)\, dt, \tag{3.3}$$

whose coefficients $f_{\alpha\beta}$, $\alpha = 1, 2, 3;\ \beta = 1, 2$, are smooth differential functions, which satisfy the pseudo-spherical structure equations (3.1) whenever $u = u(x,t)$ is a solution to the system (3.2).

We exclude the trivial cases when the differential functions $f_{\alpha\beta}$ all depend only on x, t, and when $\omega^1 \wedge \omega^2 \equiv 0$. Note that the graph $\{(x, t, u(x,t))\}$ of any solution to a pseudo-spherical system for which $\omega^1(u(x,t)) \wedge \omega^2(u(x,t)) \neq 0$, has the structure of a pseudo-spherical surface.

The characterization of a system of differential equations as one describing pseudo-spherical surfaces has several advantages: *(a)* it allows us to study conservation laws and Bäcklund transformations from a geometrical point of view (Sasaki [31], Tenenblat [33], Reyes [28, 29]), *(b)* in accordance with the results of Kamran and Tenenblat [21], it allows us to determine "generic" solutions of a scalar equation of pseudo-spherical type from suitably generic solutions of *any other* such equation, and *(c)* it characterizes the system as the integrability condition for an $\mathfrak{sl}(2,\Re)$–valued linear problem.

Items *(a)* and *(b)* will not figure prominently here. As for *(c)*, we use the one-forms ω^α to define the $\mathfrak{sl}(2,\Re)$–valued one-form

$$\Omega = U dx + V dt = \frac{1}{2}\left(\begin{array}{cc} \omega^2 & \omega^1 - \omega^3 \\ \omega^1 + \omega^3 & -\omega^2 \end{array}\right). \tag{3.4}$$

The structure equations (3.1) imply that the zero-curvature condition (1.4) holds on solutions $u(x,t)$. The converse also holds: each $\mathfrak{sl}(2,\Re)$–valued one-form satisfying (1.4) on solutions can be used, as in (3.4), to construct three one-forms ω^α satisfying the pseudo-spherical structure equations (3.1) on solutions to the system. The additional nondegeneracy condition

$$\omega^1 \wedge \omega^2 \not\equiv 0, \tag{3.5}$$

is not immediate, but can be ensured by applying a suitable gauge transformation to the connection determined by (3.4); see [12] and [29], for details.

Of course, solution by inverse scattering requires a linear problem depending on a "spectral" parameter.

Definition 3.3 A differential equation (or system of equations) is *geometrically integrable* if it describes a non–trivial one–parameter family of pseudo-spherical surfaces.

Classifications of scalar geometrically integrable equations, under the hypothesis that the equation at hand is not only sufficient (as in Definition 3.2) but also *necessary* for the pseudo-spherical structure equations (3.1) to hold, have appeared in Chern and Tenenblat [10], Kamran and Tenenblat [21], Reyes [27], and references therein. In order to formalize this hypothesis, we follow [21].

Given a k^{th} order scalar differential equation $u_t = K(x,t,u,...,u_k)$, consider the differential ideal $\mathcal{I}_K$ generated by the two–forms

$$du \wedge dx + K(x,t,u,...,u_k) dx \wedge dt, \qquad du_i \wedge dt - u_{i+1}\, dx \wedge dt, \qquad 1 \le i \le k-1,$$

on the reduced k^{th} order jet space with coordinates $x, t, u, u_1, \ldots, u_k$. Note that the local solutions to the evolution equation correspond to integral submanifolds of the exterior differential system $\{\mathcal{I}_K, dx \wedge dt\}$ determined by the *equation ideal* $\mathcal{I}_K$. We shall use the terminology "strictly pseudo-spherical" to indicate that $\mathcal{I}_K$ is algebraically equivalent to a system of differential forms satisfying the pseudo-spherical structure equations whenever $u(x,t)$ is a solution of the equation $u_t = K$.

Definition 3.4 A scalar differential equation $u_t = K(x,t,u,...,u_k)$ will be called *strictly pseudo-spherical* if there exist one-forms $\omega^\alpha = f_{\alpha 1}\,dx + f_{\alpha 2}\,dt$ whose coefficients are differential functions $f_{\alpha\beta}$ depending at most on derivatives of order k, such that the two–forms

$$\Omega_1 = d\omega^1 - \omega^3 \wedge \omega^2, \qquad \Omega_2 = d\omega^2 - \omega^1 \wedge \omega^3, \qquad \Omega_3 = d\omega^3 - \omega^1 \wedge \omega^2, \tag{3.6}$$

generate the equation ideal $\mathcal{I}_K$.

An interesting open problem is whether there are any equations that are pseudo-spherical, but not strictly pseudo-spherical.

The following characterizations, taken from [27], will be used in our proof of the implication "formal integrability $\Rightarrow$ kinematic integrability" for second order evolution equations:

Lemma 3.5 *A k^{th} order scalar evolution equation $u_t = K(x,t,u,...,u_k)$ is strictly pseudo-spherical, with associated differential functions $f_{\alpha\beta}$ such that $f_{21} = \lambda$ is a constant "spectral" parameter, if and only if*

a) f_{11} and f_{31} only depend on x,t, and u, and are not both *independent of u,*

b) f_{12} and f_{32} only depend on $x,t,u,\ldots,u_{k-1}$,

c) f_{22} only depends on $x,t,u,\ldots,u_{k-2}$, and

d) the following identities hold:

$$\begin{aligned} &D_x f_{12} + \lambda f_{32} - f_{22} f_{31} = D_t f_{11} = \frac{\partial f_{11}}{\partial t} + K\,\frac{\partial f_{11}}{\partial u}\,, \\ &D_x f_{22} + f_{12} f_{31} - f_{11} f_{32} = 0, \\ &D_x f_{32} + \lambda f_{12} - f_{22} f_{11} = D_t f_{31} = \frac{\partial f_{31}}{\partial t} + K\,\frac{\partial f_{31}}{\partial u}\,. \end{aligned}$$

Theorem 3.6 *Let $f_{\alpha\beta}$ be differential functions satisfying the conditions of Lemma 3.5. Suppose $f_{31} = \varepsilon\, f_{11} \neq 0$ with $\varepsilon = \pm 1$. Then the associated scalar evolution equation $u_t = K$ is strictly pseudo-spherical if and only if f_{22} depends only on t, $f_{32} = \varepsilon f_{12}$, and*

$$K = \frac{D_x f_{12} + \varepsilon(\lambda f_{12} - f_{11} f_{22}) - \dfrac{\partial f_{11}}{\partial t}}{\dfrac{\partial f_{11}}{\partial u}}. \tag{3.7}$$

We are now ready to prove the main result of this section.

Theorem 3.7 *Every second order evolution equation $u_t = K(x,t,u,u_x,u_{xx})$ which possesses a formal symmetry of infinite rank is of pseudo-spherical type.*

Proof: We refer back to the list of formally integrable second order equations. Equations (2.8)–(2.10) are simple modifications of the equations considered in Theorem 6 of Reyes [27]. For completeness, however, their associated one–forms are also collected here. We begin with Equation (2.7):

1. Pseudo-spherical one–forms associated with Equation (2.7) are

$$\begin{aligned}\omega^1 &= a(x,t)u\,dx + \left[\,a(x,t)u_x - \left(\varepsilon\,\lambda a(x,t) + \frac{\partial a(x,t)}{\partial x}\right) u\,\right] dt,\\ \omega^2 &= \lambda\,dx - \frac{\lambda^2}{\varepsilon}\,dt,\\ \omega^3 &= \varepsilon\,\omega^1,\end{aligned}$$

in which $\varepsilon = \pm 1$, and the function $a(x,t) \neq 0$ is a solution of the linear equation

$$-2\,\varepsilon\,\lambda\,\frac{\partial a(x,t)}{\partial x} - \frac{\partial^2 a(x,t)}{\partial x^2} - \frac{\partial a(x,t)}{\partial t} = a(x,t)h(x,t).$$

The derivation of these one–forms is similar to the case of (2.11) discussed below, and will, in the interests of brevity, be omitted.

2. For equation (2.8), the associated pseudo-spherical one–forms are

$$\begin{aligned}\omega^1 &= \left(\tfrac{1}{2}u + \alpha(x,t)\right) dx + \left(\tfrac{1}{2}u_x + \tfrac{1}{4}u^2 + \beta(x,t)\right) dt,\\ \omega^2 &= \lambda\,dx + \left(\tfrac{1}{2}\lambda\,u - \lambda\,\alpha(x,t)\right) dt,\\ \omega^3 &= -\lambda\,dx + \left(-\tfrac{1}{2}\lambda\,u + \lambda\,\alpha(x,t)\right) dt,\end{aligned}$$

in which the functions $\alpha(x,t)$ and $\beta(x,t)$ satisfy the equations

$$\alpha_x + \alpha^2 + \beta = 0, \qquad \beta_x - \alpha_t - \tfrac{1}{2}g = 0.$$

3. For equation (2.9), we have

$$\omega^1 = e^{-\lambda\,x}u\,dx + \frac{e^{-\lambda\,x}u_x}{u^2}\,dt, \qquad \omega^2 = \lambda\,dx, \qquad \omega^3 = e^{-\lambda\,x}u\,dx + \frac{e^{-\lambda\,x}u_x}{u^2}\,dt.$$

4. For equation (2.10), we have

$$\begin{aligned}\omega^1 &= -e^{-\varepsilon\,\lambda\,x}u\,dx + \left(-e^{-\varepsilon\,\lambda\,x}u^{-2}u_x + \delta(x)\right)\,dt,\\ \omega^2 &= \lambda\,dx,\\ \omega^3 &= -\varepsilon\,e^{-\varepsilon\,\lambda\,x}u\,dx + \left(-\varepsilon\,e^{-\varepsilon\,\lambda\,x}u^{-2}u_x + \varepsilon\,\delta(x)\right)\,dt,\end{aligned}$$

in which $\varepsilon = \pm 1$ and $\delta(x)$ is a solution of the equation

$$\varepsilon\,\lambda\,\delta + \delta_x = e^{-\varepsilon\,\lambda\,x}.$$

5. Finally, let us consider Equation (2.11). First we apply the change of variables $u = \frac{1}{v}$, to transform it into

$$v_t = K = v^2 v_{xx} + x^2 v_x - 3xv. \tag{3.8}$$

Applying (3.7) with $k = 2$ and v replacing u, we find that

$$K = \frac{v_{xx} f_{12,v_x} + v_x f_{12,v} + \varepsilon\,\lambda f_{12} - \varepsilon f_{11} f_{22} + f_{12,x}}{f_{11,v}}\,. \tag{3.9}$$

Here, and below, we use an abbreviated comma notation to denote partial derivatives. To agree with (3.8), the coefficient of v_{xx} in (3.9) must be v^2, and so

$$f_{12} = v^2 f_{11,v} v_x + \alpha(x,v)$$

for some $\alpha(x,v)$. Substituting this expression into (3.9) yields

$$K = v^2 v_{xx} + \frac{(v^2 f_{11,v})_v\, v_x^2 + (\alpha_v + \varepsilon\,\lambda v^2 f_{11,v} + v^2 f_{11,v,x}) v_x + \varepsilon\,\lambda\alpha - \varepsilon f_{11} f_{22} + \alpha_x}{f_{11,v}}\,. \tag{3.10}$$

Since the coefficient of v_x^2 must be zero, we obtain

$$f_{11} = -c(x) v^{-1} + d(x),$$

for some functions $c(x)$ and $d(x)$ to be determined. Since the coefficient of v_x must be x^2, we obtain the following formula for α:

$$\alpha(x,v) = -x^2 c(x) v^{-1} - \lambda\varepsilon c(x) v - c_x(x) v + \beta(x).$$

Substituting into (3.10) and considering the coefficients of v and v^2, we find that $\varepsilon = 1$, $f_{21} = \lambda$, $f_{22} = \beta = d = 0$, and

$$c(x) = x e^{-\lambda\,x}.$$

Summarizing, Equation (3.8) describes one–parameter families of pseudo-spherical surfaces with associated one–forms

$$\omega^1 = -x e^{-\lambda\,x} v^{-1} dx + \left(x e^{-\lambda\,x} v_x - x^3 e^{-\lambda\,x} v^{-1} - e^{-\lambda\,x} v \right) dt \qquad \omega^2 = \lambda\,dx \qquad \omega^3 = \omega^1.$$

This finishes the proof.

Remark: Of course, the one–forms appearing in the proof of the preceding theorem are not unique. For instance, the one–forms

$$\widehat{\omega}^1 = \frac{e^{-\lambda\,x}}{u}\,dx + \frac{e^{-\lambda\,x} u_x}{u^4}\,dt, \qquad \widehat{\omega}^2 = \frac{(u\lambda + 2\,u_x)}{u}\,dx - 2\,\frac{\left(2\,u_x^2 - u_{xx} u\right)}{u^4}\,dt,$$

$$\widehat{\omega}^3 = \frac{e^{-\lambda\,x}}{u}\,dx + \frac{e^{-\lambda\,x} u_x}{u^4}\,dt.$$

satisfy the structure equations of a pseudo-spherical surface whenever $u(x,t)$ is a solution of Equation (2.9). These one–forms do not contradict Lemma 3.5 because the corresponding two-forms (3.6) do not generate the equation ideal. Indeed, if we write $\Delta = u_t + 2u_x^2/u^3 - u_{xx}/u^2$, then

$$\widehat{\Omega}_1 = d\,\widehat{\omega}^1 - \widehat{\omega}^3 \wedge \widehat{\omega}^2 = \frac{e^{-\lambda x}}{u^2}\,\Delta, \qquad \widehat{\Omega}_2 = d\,\widehat{\omega}^2 - \widehat{\omega}^1 \wedge \widehat{\omega}^3 = -\frac{2}{u}\left(D_x\Delta - \frac{u_x}{u}\Delta\right),$$
$$\widehat{\Omega}_3 = d\,\widehat{\omega}^3 - \widehat{\omega}^1 \wedge \widehat{\omega}^2 = \frac{e^{-\lambda x}}{u^2}\,\Delta,$$

involve derivatives of the equation and therefore do not satisfy the algebraic requirements for the equation to be strictly pseudo-spherical.

Remark: A straightforward generalization of the last part of the proof above yields the following new family of evolution equations of pseudo-spherical type:

$$v_t = \left(\frac{F - v_x}{x} + D_x F\right) v^2 - 3\,xv + x^2 v_x + c\,v,$$

in which $F(x, v, v_x)$ is arbitrary, and c is a constant. Indeed, it describes pseudo-spherical surfaces with associated one–forms

$$\omega^1 = -\frac{xe^{-\lambda\,x}}{v}\,dx - \frac{e^{-\lambda\,x}\left(-xF(x,v,v_x)v + x^3 + v^2\right)}{v}\,dt,$$
$$\omega^2 = \lambda\,dx + c\,dt, \qquad \omega^3 = \omega^1.$$

Remark: Ding and Tenenblat [14] have recently developed a theory of differential systems describing surfaces of constant curvature, generalizing the notion of an equation of pseudo-spherical type discussed here. It would be very interesting to check whether one can use this point of view to extend our Theorem 3.7 to formally integrable systems of equations [5, 19, 23, 24, 30].

4 On equations describing affine surfaces.

We now investigate the geometry underlying integrable equations which, like the Boussinesq equation, arise as the integrability or zero curvature conditions for an $\mathfrak{sl}(3,\mathfrak{R})$–valued linear system. We begin by summarizing the (equi-)affine geometry of surfaces in terms of moving frames following Chern and Terng [11] and Flanders [16].

Let E^3 be the three–dimensional affine space equipped with coordinates $x = (x^1, x^2, x^3)$ and volume form $dV = dx^1 \wedge dx^2 \wedge dx^3$. The Lie group G which preserves dV is the equi-affine group $SA(3) = SL(3,\mathbf{R}) \ltimes \mathbf{R}^3$.

Consider a surface $M \subset E^3$. Let $\{e_1(x), e_2(x), e_3(x)\}$ be an affine moving frame on M such that $e_1(x)$ and $e_2(x)$ are tangent to M at x, and

$$\det(e_1(x), e_2(x), e_3(x)) = 1. \tag{4.1}$$

We write

$$dx = \sum_{\alpha=1}^{3} \omega^\alpha \, e_\alpha, \qquad de_\alpha = \sum_{\beta=1}^{3} \omega_\alpha^\beta \, e_\beta, \tag{4.2}$$

so that ω^α and ω_α^β can be identified with the Maurer–Cartan forms of G. Equations (4.1)–(4.2) imply the unimodular constraint

$$\sum_{\beta=1}^{3} \omega_\beta^\beta = 0 \tag{4.3}$$

and the $\mathfrak{sa}(3)$ structure equations

$$d\omega^\alpha = \sum_{\beta=1}^{3} \omega^\beta \wedge \omega_\beta^\alpha, \qquad d\omega_\alpha^\beta = \sum_{\gamma=1}^{3} \omega_\alpha^\gamma \wedge \omega_\gamma^\beta, \qquad \alpha, \beta = 1, 2, 3. \tag{4.4}$$

Let ϖ^α and ϖ_α^β denote the restrictions (pull-backs) of the one-forms ω^α and ω_α^β to the surface M. We deduce the structure equations

$$\sum_{\alpha=1}^{3} \varpi_\alpha^\alpha = 0, \qquad d\varpi_\alpha^\beta = \sum_{\gamma=1}^{3} \varpi_\alpha^\gamma \wedge \varpi_\gamma^\beta, \qquad \alpha, \beta = 1, 2, 3, \tag{4.5}$$

corresponding to the unimodular subgroup $SL(3, \mathbf{R})$, along with the additional structure equations

$$\begin{aligned} d\varpi^1 &= \varpi^1 \wedge \varpi_1^1 + \varpi^2 \wedge \varpi_2^1, & \varpi^3 &= 0, \\ d\varpi^2 &= \varpi^1 \wedge \varpi_1^2 + \varpi^2 \wedge \varpi_2^2, & 0 &= \varpi^1 \wedge \varpi_1^3 + \varpi^2 \wedge \varpi_2^3, \end{aligned} \tag{4.6}$$

arising from the translation components. The "fundamental theorem of the theory of surfaces" says that conversely, given a set of one–forms satisfying (4.5) and (4.6), there exists an affine surface M described locally by a moving frame satisfying (4.1). See Flanders [16] for a proof of this result.

Suppose we are given a system of differential equations $\Delta = 0$ that forms the integrability conditions for a one–parameter family of $\mathfrak{sl}(3, \mathfrak{R})$–valued linear problems of the form (1.3), where

$$\Omega = (\,\omega_\alpha^\beta\,) = U(x, t, u, \ldots, u_r)\, dx + V(x, t, u, \ldots, u_s)\, dt \tag{4.7}$$

is an $\mathfrak{sl}(3, \mathfrak{R})$–valued one-form whose coefficients are differential functions. By construction, the entries ω_α^β of Ω satisfy the unimodular structure equations (4.5) when restricted to solutions to the system. Thus, in order to identify the surface described by solutions of the system $\Delta = 0$, it is enough to find one–forms ω^1, ω^2 such that equations (4.6) are satisfied. Our aim is to show that this can indeed be done for the integrable equations of Boussinesq type classified in Section 2.

Zero curvature representations of the Boussinesq system (2.14) can be found in [18, 13]. We consider the following version of (2.14)

$$v_t = 2w_x, \qquad w_t = -\frac{1}{6}(v_{xxx} + 4vv_x). \tag{4.8}$$

It possesses the standard Lax pair

$$L = \partial_x^3 + v\partial_x + \tfrac{1}{2}v_x + w, \qquad P = \partial_x^2 + \tfrac{2}{3}v. \tag{4.9}$$

We convert the system $L\Psi = \lambda\Psi$, $\Psi_t = P\Psi$ into an equivalent first order system

$$\psi_x = U\,\psi, \qquad \psi_t = V\,\psi, \tag{4.10}$$

with $\mathfrak{sl}(3,\mathfrak{R})$–valued coefficient matrices

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2}v_x - w + \lambda & -v & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} \frac{2}{3}v & 0 & 1 \\ \frac{1}{6}v_x - w + \lambda & -\frac{1}{3}v & 0 \\ \frac{1}{6}v_{xx} - w_x & -\frac{1}{6}v_x - w + \lambda & -\frac{1}{3}v \end{pmatrix}.$$

One can easily check that Equations (4.8) are indeed the integrability condition of (4.10).

The corresponding $\mathfrak{sl}(3,\mathfrak{R})$–valued one-form (4.7) produces the one-forms

$$\begin{aligned} &\omega^1_1 = \tfrac{2}{3}\,v\,dt, && \omega^1_2 = \left(\tfrac{1}{6}\,v_x - w + \lambda\right)\,dt, && \omega^1_3 = \left(-\tfrac{1}{2}\,v_x - w + \lambda\right)\,dx + \left(\tfrac{1}{6}\,v_{xx} - w_x\right)\,dt, \\ &\omega^2_1 = dx, && \omega^2_2 = -\tfrac{1}{3}\,vdt, && \omega^2_3 = -vdx + \left(-\tfrac{1}{6}\,v_x - w + \lambda\right)\,dt, \\ &\omega^3_1 = dt, && \omega^3_2 = dx, && \omega^3_3 = -\tfrac{1}{3}\,v\,dt, \end{aligned} \tag{4.11}$$

that satisfy the structure equations (4.5). As pointed out above, the one remaining task is to find one–forms ω^1, ω^2 so that the structure equations (4.6) are satisfied.

An important simplification occurs if instead of finding simply ω^1 and ω^2, we look for one–forms ω^1, ω^2, ω^3 satisfying (4.4) on solutions of the Boussinesq system (4.8). Clearly, the one-forms

$$\omega^1 = \omega^1_2, \qquad \omega^2 = \omega^2_2, \qquad \omega^3 = \omega^3_2, \tag{4.12}$$

will satisfy these conditions. However, since $\omega^3 \neq 0$, the one-forms (4.11), (4.12) are not *adapted* to the surfaces described by solutions of the Boussinesq system (4.8).

This can be arranged by applying a suitable gauge transformation to the linear system (4.10). A short computation shows that the unimodular matrix

$$S = \begin{pmatrix} -\frac{2}{3}v & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{4.13}$$

leads to the gauge transformation

$$\begin{aligned} \widehat{U} = SUS^{-1} + \partial_x S\, S^{-1} &= \begin{pmatrix} 0 & \frac{1}{3}v & -\frac{1}{6}v_x + w - \lambda \\ -1 & 0 & -\frac{2}{3}v \\ 0 & 1 & 0 \end{pmatrix}, \\ \widehat{V} = SVS^{-1} + \partial_t S\, S^{-1} &= \begin{pmatrix} \frac{1}{3}v & \frac{1}{6}v_x + w - \lambda & -\frac{2}{9}v^2 - \frac{1}{6}v_{xx} - \frac{1}{3}w_x \\ 0 & -\frac{1}{3}v & \frac{1}{6}v_x - w + \lambda \\ -1 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{4.14}$$

on the coefficient matrices of the one–form Ω. The resulting $\mathfrak{sl}(3,\mathfrak{R})$–valued one-form $\widehat{\Omega} = \widehat{U}\,dx + \widehat{V}\,dt$ has entries

$$\begin{aligned} \widehat{\omega}_1^1 &= \tfrac{1}{3}\,vdt, & \widehat{\omega}_2^3 &= -\tfrac{2}{3}\,vdx + \left(\tfrac{1}{6}\,v_x - w + \lambda\right) dt, \\ \widehat{\omega}_1^2 &= \tfrac{1}{3}\,vdx + \left(\tfrac{1}{6}\,v_x + w - \lambda\right)\, dt, & \widehat{\omega}_3^1 &= -dt, \\ \widehat{\omega}_1^3 &= \left(-\tfrac{1}{6}\,v_x + w - \lambda\right) dx + \left(-\tfrac{2}{9}\,v^2 - \tfrac{1}{6}\,v_{xx} - \tfrac{1}{3}\,w_x\right)\, dt, & \widehat{\omega}_3^2 &= dx, \\ \widehat{\omega}_2^1 &= -dx, & \widehat{\omega}_3^3 &= 0. \\ \widehat{\omega}_2^2 &= -\tfrac{1}{3}\,vdt, \end{aligned} \tag{4.15}$$

We then set

$$\widehat{\omega}^1 = dt, \qquad \widehat{\omega}^2 = -dx, \qquad \widehat{\omega}^3 = 0. \tag{4.16}$$

We can easily check that these adapted one–forms (4.15), (4.16) satisfy all the structure equations of an affine surface immersed in E^3 whenever $(v(x,t), w(x,t))$ is a solution of the Boussinesq system (4.8).

Combined with our classification result (Theorem 2.6), and the fact that we can easily transform (2.18) into (2.19) and vice-versa by using simple differential substitutions, the foregoing discussion allows us to conclude the following.

Theorem 4.1 *Every nonlinear, non–decouplable equation of Boussinesq type* (2.16) *describes affine surfaces.*

What kind of affine surfaces have we obtained? Let us go back to Chern and Terng [11] and Flanders [16]. We denote by M any affine surface determined by a solution $(v(x,t), w(x,t))$ of the Boussinesq system (4.8), and we let $\{e_i, N\}$ $(i = 1, 2)$ be a moving frame on M with e_i $(i = 1, 2)$ tangent to M. Note that

$$\widehat{\omega}_1^3 = \left(-\tfrac{2}{9}\, v^2 - \tfrac{1}{6}\, v_{xx} - \tfrac{1}{3}\, w_x\right) \widehat{\omega}^1 + \left(\tfrac{1}{6}\, v_x - w + \lambda\right) \widehat{\omega}^2,$$
$$\widehat{\omega}_2^3 = \left(\tfrac{1}{6}\, v_x - w + \lambda\right) \widehat{\omega}^1 + \tfrac{2}{3} v\, \widehat{\omega}^2,$$

and so we have the linear dependencies

$$\widehat{\omega}_i^3 = \sum_{k=1,2} h_{ik} \widehat{\omega}^k, \qquad i = 1, 2.$$

The sign of $|H|$, the determinant of the matrix (h_{ij}), is invariant under unimodular transformations of the form

$$e_i' = a_i^1 e_1 + a_i^2 e_2, \qquad N' = cN + a_3^1 e_1 + a_3^2 e_2. \tag{4.17}$$

We will assume that $|H| \neq 0$. Note that the non–generic case $|H| = 0$ also occurs, for instance, for some constant solutions of (4.8).

The *affine metric* on M is defined to be the quadratic form

$$II = |H|^{-1/4} \sum_{i,j=1,2} h_{ij}\, \widehat{\omega}^i\, \widehat{\omega}^j.$$

The geometrical properties of interest are those invariant under changes of frame (4.17) keeping the *affine normal vector* fixed. This vector is defined (Flanders [16] p. 364, Chern and Terng [11], p. 113) thus: one normalizes the frame $\{e_i, N\}$ in such a way that Equations (4.2) become

$$d\,x = \omega^1\, e_1 + \omega^2\, e_2, \qquad d\,e_i = \omega_i^j\, e_j + \epsilon\, \omega^i\, N, \qquad d\,N = \omega_3^1\, e_1 + \omega_3^2\, e_2, \tag{4.18}$$

in which $\epsilon = \pm 1$, the precise sign depending on the signature of the quadratic form II, and

$$\omega_1^1 + \omega_2^2 \;=\; \omega_3^3 \;=\; 0.$$

The affine normal vector is then $\nu = N$. This normalization implies

$$\widehat{\omega}_3^i = \sum_{k=1,2} l_i^k \widehat{\omega}^k, \qquad i = 1, 2,$$

and also that the quadratic form

$$III = \sum_{i=1,2} \widehat{\omega}_3^i\, \widehat{\omega}_i^3$$

is invariant under changes of frame fixing ν. The *affine curvatures* of M are the invariants of III relative to II. In particular, the *affine mean curvature* of M is

$$L = \tfrac{1}{2}\,|H|^{1/4} \sum_{i=1,2} l_i^i.$$

Our one–forms (4.15), (4.16) are already normalized. In the present situation,

$$\widehat{\omega}_3^1 = -\widehat{\omega}^1, \qquad \widehat{\omega}_3^2 = -\widehat{\omega}^2,$$

and so

$$l_1^1 = l_2^2 = -1, \qquad l_2^1 = l_1^2 = 0.$$

We conclude that the solutions to the Boussinesq system define affine surfaces with affine mean curvature

$$L = -|H|^{-1/4},$$

and hence are always hyperbolic affine surfaces.

An important remark is that we could have chosen $\widehat{\omega}^i = \widehat{\omega}_3^i$ instead of (4.16). We would have then obtained affine surfaces of *positive* affine mean curvature. In fact, the right choice is related to the signature of the quadratic form II, as pointed out after (4.18). It would be interesting to check whether this signature depends on the solutions of the Boussinesq system (4.8). If it does, can one classify the affine surfaces arising from different solutions of (4.8)?

Finally, we would like to speculate on the possibility of extending the notion of an equation of pseudo-spherical type considered in Section 3 to affine geometry. Chern and Terng [11] proved that *minimal* affine surfaces admit Bäcklund–like transformations. Buyske [9] then showed that, unlike the classical Bäcklund transformation, these transformations are periodic of period two and in fact, essentially trivial, being a combination of an involution and a translation of the affine conormal. However, they can still be used to obtain new solutions of the system of equations underlying the geometry of minimal affine surfaces from old ones, [2, 9, 11]. Thus, it is worth asking how one can generalize this picture. What is the appropriate class of systems describing minimal affine surfaces?

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