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Internal, External, and Generalized Symmetries

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Bäcklund's theorem, which characterizes contact transformations, is generalized to give an analogous characterization of "internal symmetries" of systems of differential equations. For a wide class of systems of differential equations, we prove that every internal symmetry comes from a first order generalized symmetry and, conversely, every first order generalized symmetry satisfying certain explicit contact conditions determines an internal symmetry. We analyze the contact conditions in detail, deducing powerful necessary conditions for a system of differential equations that admit "genuine" internal symmetries, i.e., ones which do not come from classical "external" symmetries. Applications include a direct proof that both the internal symmetry group and the first order generalized symmetries of a remarkable differential equation due to Hilbert and Cartan are the noncompact real form of the exceptional simple Lie group G_2. © 1993 Academic Press, Inc.

1. INTRODUCTION

Classically, the symmetry group of a system of differential equations is a local group of point transformations, meaning local diffeomorphisms on the space of independent and dependent variables, which map solutions of the system to solutions. Connected symmetry groups are effectively calculated using Lie's infinitesimal method and have numerous applications, including integration of ordinary differential equations, group-

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invariant solutions of partial differential equations, conservation laws, bifurcation theory, etc.; cf. [8, 15, 17]. Over the years, a number of different generalizations of the concept of a symmetry group of a system of differential equations have been proposed. One of the principal purposes of this paper is to interconnect several of these generalizations.

Lie himself [13] regarded symmetries of differential equations as groups of contact transformations, which are local diffeomorphisms of the nth order jet bundle J^n which preserve the contact ideal. In general, so as to distinguish Lie's type of symmetries from others, we call them *external symmetries*, since they are also defined external to the system of differential equations. Unfortunately, contact transformations do not significantly extend point transformations except in the special case of one dependent variable, since , according to Bäcklund's theorem [1, 8], any contact transformation on J^n is the prolongation of a first order contact transformation, or, if the fiber dimension (number of dependent variables) is greater than one, of an ordinary point transformation.

A second significant generalization of classical symmetries, which includes Lie's contact symmetries, is the generalized symmetries first introduced by E. Noether [14] in her famous theorem relating variational symmetries and conservation laws; these have received renewed attention due to the role they play in completely integrable (soliton) nonlinear partial differential equations; cf. [15]. Here, as with contact transformations, the infinitesimal generators are allowed to depend on derivatives of the dependent variables, but one relaxes the restriction that the vector field generates a one-parameter transformation group on any finite order jet space. In the case of one dependent variable, every first order generalized symmetry determines a contact symmetry and conversely, whereas higher order generalized symmetries, or first order generalized symmetries in the case of several dependent variables, provide examples of non-geometrical symmetries.

An alternative, more geometrical, generalization of the symmetry group concept is the internal symmetry groups which appear in the works of Elie Cartan (cf. [2–4]); they are also known in the literature as "dynamical symmetries" (cf. [17]). Recall that a (reasonable) system of nth order differential equations will determine a submanifold of the nth order jet bundle: $\mathscr{R} \subset J^n$. Since one is usually only interested in the action of a symmetry group on the solutions of the system of differential equations, one really need only consider its (prolonged) action restricted to the submanifold $\mathscr{R}$, and so the conditions that the transformation preserve the contact ideal need only be imposed on $\mathscr{R}$. Thus, an internal symmetry of the system is defined as a transformation which maps the equation submanifold to itself and also preserves the contact ideal restricted to the submanifold. The restrictions of Bäcklund's theorem no longer apply to internal symmetries, and there are examples of internal symmetry transformations which are not prolongations of first order transformations; see Example 8 below. Note that every external symmetry of a system of differential equations gives rise to an internal symmetry by restricting to the equation manifold. In many cases, all internal symmetries arise this way; see Cartan [2] for the case of a single parabolic partial differential equation, Gardner and Kamran [5] for the hyperbolic and elliptic cases, and Krasil'shchik, Lychagin, and Vinogradov [12] and the final section of this paper for general normal systems of partial differential equations.

The principal objective of this paper is to interrelate the above symmetry concepts. Specifically we are interested in the precise relationship between internal symmetries and generalized symmetries so as to generalize the connection between contact transformations and first order generalized symmetries. This program was motivated by a highly intriguing underdetermined ordinary differential equation studied by Hilbert [6] and Cartan [3, 4]. The Hilbert–Cartan equation, which is just $v' = (u'')^2$, was shown by Cartan, as a consequence of his work on Pfaffian systems in five variables [2], to have as its internal symmetry group the exceptional simple Lie group G_2. In answer to a question posed by Robert Bryant, the first order generalized symmetry group of the Hilbert–Cartan equation was calculated and was found to be the same group G_2. This paper arose out of an attempt to understand why these two computations gave the same answer.

Our results answer this question in general and can be summarized as follows. First, and obvious, is the fact that every external symmetry restricts to an internal symmetry. In many cases, all internal symmetries arise in this way, although the Hilbert–Cartan equation is a significant exception; in the final section we present some preliminary results in this direction. Second, under a certain condition on the system, which we name the "descent property," we prove that every internal symmetry comes from a *first order* generalized symmetry, a result that significantly ameliorates the computation of these symmetries. The systems covered by this result include all second order systems of differential equations, all normal systems of partial differential equations, and a wide class of higher order underdetermined ordinary differential equations; the principal exceptional cases are the normal systems of ordinary differential equations of order three or more. Finally, we prove that every first order generalized symmetry which satisfies additional contact conditions is equivalent to an internal symmetry. In certain cases, such as the "codimension 1" ordinary differential equations, of which the Hilbert–Cartan equation is a particular example, there are no contact restrictions, hence there is a one-to-one correspondence between internal symmetries and first order generalized symmetries. This explains the aforementioned calculations for the Hilbert–Cartan equation. More generally, in the case of systems of ordinary differential equations, the contact conditions naturally split into "tangential" and "normal" components. First order generalized symmetries which satisfy the tangential contact conditions give rise to internal symmetries. In the case of systems of partial differential equations, the contact conditions are much more restrictive and, in many cases, preclude the existence of any "genuine" internal symmetries, meaning ones that do not come from restriction of an external symmetry. In particular, we prove that every internal symmetry of a normal system of partial differential equations (meaning a system that can be placed into Cauchy–Kovalevskaya form) of order at least two extends to an external symmetry, hence only for first order normal systems of partial differential equations can interesting new internal symmetries arise. Further results based on analysis of the characteristic variety of the system for the existence of non-extendable internal symmetries are discussed, including a few examples. However, the complete analysis of the contact conditions remains a significant open problem.

Our principal results can be regarded as a generalization of Bäcklund's theorem to systems of differential equations, in that contact transformations can be viewed as "internal symmetries" of the entire jet space. Internal symmetries form an intermediate and interesting class of symmetries between the classical external symmetry groups and completely general generalized symmetries and are the most general local geometrical transformation groups which map the space of solutions to a system of differential equations to itself. Applications of internal symmetries toward the integration of differential equations and the determination of explicit solutions remain to be investigated in depth.

There are two possible expository styles available for the presentation of our results, the first being a concrete approach using local coordinates and explicit calculations, and the second a more abstract, invariant formulation. In the present paper, results are proved in local coordinates, allowing for concrete calculations and leading to immediate applications; however, this approach is slightly restrictive in that the theorems are not as general as can be proved using a more powerful coordinate-free machinery. We believe that the two approaches are complementary, the first having the advantage of being immediately applicable to most practical examples, whereas the second leads to more synthetic, general formulations of the key results. However, to make the results understandable by as wide an audience as possible, we have chosen to adopt the less abstract mode.

2. Point Symmetries

We begin with a brief review of the classical local theory of symmetry groups of differential equations and refer the reader to [15, 17] for more detailed treatments. Consider a system of differential equations in p independent variables $x=(x^1, ..., x^p)$, which form local coordinates on the base space X, and q dependent variables $u=(u^1, ..., u^q)$, which are fiber coordinates on the space U, the total space being the trivial bundle $E=X\times U$ over X. The derivatives of the u's are denoted by $u^\alpha_J=\partial^J u^\alpha/\partial x^J$, where $J=(j_1, ..., j_n)$, $1\leqslant j_\nu\leqslant p$, is a symmetric multi-index of order $n=\#J$. We let $u^{(n)}$ denote all derivatives of orders $\leqslant n$, which provide (fiber) coordinates on the jet space $J^n=J^nE$ over X.

A system of nth order differential equations

$$\Delta_\kappa(x, u^{(n)})=0, \qquad \kappa=1, ..., r, \tag{2.1}$$

is *nondegenerate* if it is of *maximal rank* and is *locally solvable* [15, Sect. 2.6]. The maximal rank condition is that the full Jacobian matrix of the system have rank r,

$$\operatorname{rank}\left(\frac{\partial\Delta_\kappa}{\partial x^i}, \frac{\partial\Delta_\kappa}{\partial u^\alpha_J}\right)=r, \tag{2.2}$$

at each $(x, u^{(n)})$ satisfying (2.1). In this case, (2.1) defines a submanifold $\mathscr{R}\subset J^n$, and a solution $u=f(x)$ of (2.1) can then be identified with a smooth section of the bundle E the graph of whose n-jet (nth prolongation) $u^{(n)}=j_nf(x)$ is contained in $\mathscr{R}$. The system is locally solvable if, for every point $(x_0, u_0^{(n)})\in\mathscr{R}$, there is a solution $u=f(x)$ defined in a neighborhood of x_0 such that $u_0^{(n)}=j_nf(x_0)$. Local solvability in particular implies that the system is "involutive," i.e., has no integrability conditions. We also assume that the system contains no equations involving solely the independent variables, meaning that the projection $\mathscr{R}\to X$ is onto. We define the kth prolongation of the system (2.1) to be the system of partial differential equations

$$D_K\Delta_\kappa(x, u^{(n)})=0, \qquad \kappa=1, ..., r, \qquad 0\leqslant\#K\leqslant k, \tag{2.3}$$

obtained by differentiating the equations up to order k. Here

$$D_i=\frac{\partial}{\partial x^i}+\sum_{\alpha=1}^{q}\sum_{\#J\geqslant 0}u^\alpha_{J,i}\frac{\partial}{\partial u^\alpha_J} \tag{2.4}$$

denotes the total derivative with respect to x^i, and $D_K=D_{k_1}D_{k_2}\dots D_{k_m}$, the corresponding mth order total derivative . We assume that each prolongation of $\mathscr{R}$ is also nondegenerate, so, for each $k\geqslant 0$, (2.3) defines the prolonged submanifold $\mathrm{pr}^{(k)}\mathscr{R}\subset J^{n+k}$.

A "classical" symmetry group of the system (2.1) is a (local) group G of point transformations $\Phi: E\to E$ which map solutions of the system to solutions. Assuming local solvability, this is equivalent to the requirement that the prolonged transformation $\mathrm{pr}^{(n)}\Phi\colon J^n \to J^n$ preserve the equation manifold $\mathscr{R}$, i.e.,

$$\mathrm{pr}^{(n)}\Phi\colon \mathscr{R} \to \mathscr{R}.$$

Assuming connectivity of the group, we can check this condition using Lie's infinitesimal criterion for invariance. Let Φ_ε be a one-parameter susbgroup of G and let

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u)\frac{\partial}{\partial u^\alpha} \tag{2.5}$$

be the infinitesimal generator of Φ_ε. The prolonged vector field

$$\mathrm{pr}^{(n)}\mathbf{v} = \sum_{i=1}^{p} \xi^i\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\# J \leqslant n} \varphi_J^\alpha\frac{\partial}{\partial u_J^\alpha} \tag{2.6}$$

is the infinitesimal generator of the corresponding prolonged one-parameter subgroup $\mathrm{pr}^{(n)}\Phi_\varepsilon$. Here the coefficients φ_J^α are determined recursively via the standard prolongation formula [15, (2.44)]

$$\varphi_{J,i}^\alpha = D_i\varphi_J^\alpha - \sum_{k=1}^{p} D_i\xi^k u_{J,k}^\alpha. \tag{2.7}$$

THEOREM 1. *Suppose the system of partial differential equations* (2.1) *is nondegenerate. Then a connected group of point transformations G is a symmetry group of the system if and only if the "determining equations"*

$$\mathrm{pr}^{(n)}\mathbf{v}(\Delta_\kappa) = 0, \qquad \kappa = 1, \ldots, r, \tag{2.8}$$

vanish whenever $u = f(x)$ is a solution to (2.1) *for every infinitesimal generator $\mathbf{v}$ of G.*

The effective computation of symmetry groups using this result is well known [8, 15, 17] and has been applied to many examples of interest. Algorithms for computing symmetry groups have been successfully implemented in a number of computer algebra systems, including MACSYMA, REDUCE, and SCRATCHPAD; see [9, 16].

3. CONTACT TRANSFORMATIONS AND EXTERNAL SYMMETRIES

The nth order *contact ideal* $I^{(n)}$ is the differential ideal on J^n annihilated by all n-jets of sections $u = f(x)$ of E. In local coordinates, $I^{(n)}$ is generated by the *contact forms*

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^{p} u_{J,i}^\alpha\, dx^i, \qquad \alpha = 1, \ldots, q, \quad 0 \leqslant \# J < n. \tag{3.1}$$

A contact transformation is a (locally defined) map

$$\Psi\colon J^n \to J^n,$$

which preserves the contact ideal

$$\Psi^* I^{(n)} \subset I^{(n)}. \tag{3.2}$$

Any contact transformation on J^n has a natural prolongation to any higher order jet space J^{n+k}, $k \geqslant 0$. In particular, any prolonged point transformation will determine a contact transformation. Bäcklund's theorem (cf. [1]) imposes significant restrictions on the possible further types of contact transformations.

THEOREM 2. *Let $\Psi\colon J^n \to J^n$ be a contact transformation, where $J^n = J^n E$ is the n-jet space of the bundle E which has q-dimensional fibers. If $q = 1$, then Ψ is the $(n-1)$st prolongation of a first order contact transformation $J^1 E$. If $q > 1$, then Ψ is the nth prolongation of a point transformation on E.*

The infinitesimal generator of a one-parameter group of contact transformations is a vector field

$$X = \sum_{i=1}^{p} \xi^i(x, u^{(n)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J \leqslant n} \varphi_J^\alpha(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha} \tag{3.3}$$

on J^n. The infinitesimal version of the contact condition (3.2) is that the Lie derivative of any contact form with respect to **X** is contained in the contact ideal

$$\mathbf{X}(I^{(n)}) \subset I^{(n)}. \tag{3.4}$$

The standard infinitesimal proof of Bäcklund theorem (cf. [8, 17]) proceeds in outline as follows. (See the proof of Theorem 15 below for details.) Applying **X** to the contact form (3.1) implies that the coefficients φ_J^α of **X** are related by the prolongation formula (2.7). Close inspection of these conditions coupled with the fact that these coefficients can depend on at most nth order derivatives of the u's leads to the fact that **X** is the prolongation of the infinitesimal generator of a first order contact transformation

$$\mathbf{Y} = \sum_{i=1}^{p} \xi^i(x, u^{(1)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u^{(1)}) \frac{\partial}{\partial u^\alpha} + \sum_{\alpha=1}^{q} \sum_{i=1}^{p} \varphi_i^\alpha(x, u^{(1)}) \frac{\partial}{\partial u_i^\alpha}. \tag{3.5}$$

In order that $\mathbf{Y}$ preserve the contact ideal $I^{(1)}$, the coefficients φ_i^α must be given by

$$\varphi_i^\alpha = \frac{\partial \varphi^\alpha}{\partial x^i} + \sum_{\beta=1}^{q} \frac{\partial \varphi^\alpha}{\partial u^\beta} u_i^\beta - \sum_{j=1}^{p} \frac{\partial \xi^j}{\partial x^i} u_j^\alpha - \sum_{j=1}^{p} \sum_{\beta=1}^{q} \frac{\partial \xi^j}{\partial u^\beta} u_j^\alpha u_i^\beta, \tag{3.6}$$

and, moreover, the coefficients ξ^i, φ^α must satisfy the *contact conditions*

$$\frac{\partial \varphi^\alpha}{\partial u_j^\beta} = \sum_{i=1}^{p} u_i^\alpha \frac{\partial \xi^i}{\partial u_j^\beta}. \tag{3.7}$$

Note that (3.6), (3.7) are equivalent to the usual prolongation formula (2.7), i.e.,

$$\varphi_i^\alpha = D_i \varphi^\alpha - \sum_{j=1}^{p} D_i \xi^j u_j^\alpha, \tag{3.8}$$

and the requirement that the right-hand side of this formula only depend on first order derivatives. If $q > 1$, the integrability conditions for the system of partial differential equations (3.7) will require that ξ^i, φ^α depend only on x, u, and so every contact transformation reduces to a point transformation.

The condition that a contact transformation or vector field define a symmetry of a system of differential equations is the same as that discussed above, and the infinitesimal symmetry criterion of Theorem 1 holds as before. We call a group of contact transformations which preserves a given system of differential equations an *external symmetry group* as the transformations are (locally) defined on (open subsets of) the jet space J^n, so as to contrast them with internal symmetry groups to be considered later.

4. Generalized Symmetries

To further generalize the symmetry group concept, we allow the coefficients ξ^i and φ^α of the vector field $\mathbf{v}$ given by (2.5) to depend on higher derivatives $u^{(n)}$, but we relax the requirement that its prolongation generate a geometrical transformation group on the jet space J^n. Thus a kth order *generalized vector field* is a first order partial differential operator of the form

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u^{(k)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}. \tag{4.1}$$

The prolongation of $\mathbf{v}$ has the same form as that above; cf. (2.6), (2.7). There is, however, a more convenient representation for this prolongation formula; cf. [15]. Define the *characteristic* of $\mathbf{v}$ to be the q-tuple of functions $Q = (Q^1, ..., Q^q)$ with

$$Q^\alpha(x, u^{(k)}) = \varphi^\alpha(x, u^{(k)}) - \sum_{i=1}^{p} \xi^i(x, u^{(k)}) \frac{\partial u^\alpha}{\partial x^i}, \qquad \alpha = 1, ..., q. \tag{4.2}$$

By definition, the *evolutionary vector field*

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q^\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha} \tag{4.3}$$

is called the "evolutionary form" of $\mathbf{v}$. Note that $\mathbf{v}_Q$ has elementary prolongation

$$\mathrm{pr}^{(n)} \mathbf{v}_Q = \sum_{\alpha=1}^{q} \sum_{\# J \leqslant n} (D_J Q_\alpha) \frac{\partial}{\partial u_J^\alpha}. \tag{4.4}$$

Then $\mathbf{v}$ itself has prolongation

$$\mathrm{pr}^{(n)} \mathbf{v} = \mathrm{pr}^{(n)} \mathbf{v}_Q + \sum_{i=1}^{p} \xi^i(x, u) \tilde{D}_i, \tag{4.5}$$

where $\tilde{D}_i$ denotes the nth order truncation of the total derivative, i.e., we sum (2.4) only for $\# J \leqslant n$.

The condition that $\mathbf{v}$ be a generalized symmetry of the system of differential equations (2.1) is the same as that given in Theorem 1. Note that, when verifying the symmetry condition (2.8), one must take into account not only the system (2.1), but also its kth prolongation, (2.3). Once one fixes the order of derivatives upon which the coefficient functions ξ^i and φ^α in (4.1) can depend, the determining equations (2.8) can, in most cases, be solved, although the computations are even more tedious than those in the case of point or contact symmetries. There is, however, one simplification which can be effected in this generalized context. Since any linear combination of the total derivatives $\sum \xi^i D_i$ is trivially a generalized symmetry of any system of partial differential equations, we deduce using (4.5) that $\mathbf{v}$ is a generalized symmetry of a system of partial differential equations if and only if its evolutionary form $\mathbf{v}_Q$ is.

An evolutionary vector field $\mathbf{v}_Q$ is a *trivial symmetry* of (2.1) if the characteristic $Q(x, u^{(n)})$ vanishes on all solutions to (2.1). Two generalized symmetries $\mathbf{v}$ and $\mathbf{w}$ are *equivalent* if their respective evolutionary forms differ by a trivial evolutionary symmetry.

A generalized vector field is not usually a well-defined vector field on any jet bundle J^n since its nth prolongation will involve derivatives of orders up to $k+n$, which is greater than n. Beyond prolonged points transformations, the only exceptions to this are the infinitesimal contact transformations, which correspond to first order generalized symmetries in the case $q=1$.

THEOREM 3. *Let* $\mathbf{v}_Q$ *be an evolutionary vector field. Then* $\mathbf{v}_Q$ *is the evolutionary form of an infinitesimal contact transformation if and only if its characteristic* $Q(x, u^{(1)})$ *depends on at most first order derivatives, and there exist functions* $\xi^i(x, u^{(1)})$, $i = 1, ..., p$, *which satisfy the* contact conditions

$$\frac{\partial Q^\alpha}{\partial u_i^\beta} + \delta_\beta^\alpha \xi^i = 0. \tag{4.6}$$

Proof. Indeed. if (4.6) holds, we can define

$$\varphi^\alpha = Q^\alpha + \sum_{i=1}^{p} \xi^i u_i^\alpha \tag{4.7}$$

and see that (4.6) is equivalent to the contact conditions (3.7). In the case of one dependent variable, $q=1$, the contact conditions (4.6) serve to define the coefficients ξ^i. Thus, any first order generalized symmetry will give rise to a contact transformation. Indeed, the characteristic $Q(x, u^{(1)})$ can be identified with the negative of Lie's characteristic function [13] (hence the name). For more than one dependent variable, $q>1$, the integrability conditions for (4.6) will imply that the ξ^i's are independent of the derivatives u_i^α, and hence the symmetry is just the evolutionary form of a point transformation.

5. INTERNAL SYMMETRIES

Any external symmetry group G of a nondegenerate system of differential equations is characterized by two conditions: (1) the prolonged group transformations map the equation manifold $\mathcal{R}$ to itself, and (2) they preserve the contact ideal on J^n. Bäcklund's theorem demonstrates that the second condition is very restrictive. However, since we are usually only interested in what a transformation in G does to solutions of the system of differential equations, and hence its restriction to the equation submanifold $\mathcal{R}$, it makes sense to relax the second condition and only require that the transformation preserve the contact ideal on $\mathcal{R}$, rather than all of J^n. This leads to the definition of an internal symmetry.

DEFINITION 4. Let $\mathcal{R} \subset J^n$ be a system of differential equations. An *internal symmetry* of the system is an invertible transformation $\Psi: \mathcal{R} \to \mathcal{R}$ which maps $\mathcal{R}$ to itself and which preserves the restriction (pull-back) of the contact ideal to $\mathcal{R}$:

$$\Psi^*(I^{(n)} \mid \mathcal{R}) \subset I^{(n)} \mid \mathcal{R}. \tag{5.1}$$

Here, and below, we use the notation $|\mathcal{R}$ to denote the pull-back of differential forms to the submanifold $\mathcal{R}$.

Note that internal symmetries also map solutions of the system to solutions. Clearly any external symmetry restricts to an internal symmetry, but it is not necessarily true that an internal symmetry can be extended off the solution manifold to a contact transformation. Indeed, Bäcklund's theorem in its original form no longer holds for internal symmetries, and, as we shall see, there are nth order internal symmetries which are not the prolongation of any lower order contact map.

In the case of connected local Lie groups of internal symmetries, we can again work infinitesimally. Let $\mathbf{X}$ be a vector field which is tangent to the equation submanifold $\mathcal{R}$. In local coordinates, $\mathbf{X}$ takes the form

$$\mathbf{X} = \sum_{i=1}^{p} \xi^i(x, u^{(n)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\# J \leqslant n} \varphi^\alpha_J(x, u^{(n)}) \frac{\partial}{\partial u^\alpha_J}, \tag{5.2}$$

where the coefficients ξ^i and φ^α_J are just defined on $\mathcal{R}$. Moreover, they must satisfy the tangency (symmetry) condition

$$\mathbf{X}(\Delta_\kappa) = 0 \qquad \text{on} \quad (2.1), \qquad \kappa = 1, ..., r. \tag{5.3}$$

In addition, in analogy with (3.4), $\mathbf{X}$ must preserve the restriction of the contact ideal to the submanifold $\mathcal{R}$,

$$\mathbf{X}(I^{(n)} \mid \mathcal{R}) \subset I^{(n)} \mid \mathcal{R}, \tag{5.4}$$

where the left-hand side refers to the Lie derivative with respect to $\mathbf{X}$. Note that the projection of $\mathbf{X}$ to the bundle E determines an nth order generalized vector field

$$\mathbf{v} = \pi(\mathbf{X}) = \sum_{i=1}^{p} \xi^i(x, u^{(n)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u^{(n)}) \frac{\partial}{\partial u^\alpha}. \tag{5.5}$$

It is not difficult to see that $\mathbf{v}$ is a generalized symmetry of the system whose prolongation agrees with $\mathbf{X}$ when restricted to $\mathcal{R}$. Moreover, since the coefficients ξ^i, φ^α are only defined on $\mathcal{R}$, the generalized vector field $\mathbf{v}$ is only defined up to a trivial generalized symmetry.

THEOREM 5. *Let* $\mathbf{X}$ *be an internal symmetry for the* n*th order system* *of differential equations* $\mathscr{R} \subset J^n$. *Let* $\mathbf{v} = \pi(\mathbf{X})$. *Then* $\mathbf{v}$ *is a generalized symmetry of the system and, moreover,* $\mathbf{X} = \mathrm{pr}^{(n)}\,\mathbf{v}$ *on* $\mathrm{pr}^{(n)}\,\mathscr{R}$.

Proof. Let $\mathbf{w} = \mathbf{X} - \mathrm{pr}^{(n)}\,\mathbf{v}$, so that $\mathbf{w}$ has the form

$$\mathbf{w} = \sum_{\alpha=1}^{q} \sum_{1 \leqslant \# J \leqslant n} \psi_J^\alpha(x, u^{(2n)}) \frac{\partial}{\partial u_J^\alpha}.$$

Since $\mathbf{w}$ has no $\partial/\partial x^i$ or $\partial/\partial u^\alpha$ components, the Lie derivative of the zeroth order contact form θ^α with respect to $\mathbf{w}$ is given by

$$\mathbf{w}(\theta^\alpha) = \mathbf{w}\left(du^\alpha - \sum_j u_j^\alpha\, dx^j\right) = -\sum_j \psi_j^\alpha\, dx^j.$$

This is required to vanish on $\mathrm{pr}^{(n)}\,\mathscr{R}$, and since the dx^i are independent on $\mathscr{R}$, this implies $\psi_j^\alpha = 0$ on $\mathrm{pr}^{(n)}\,\mathscr{R}$. Continuing to the first order contact forms, we find

$$\mathbf{w}(\theta_i^\alpha) = d\psi_i^\alpha - \sum_j \psi_{i,j}^\alpha\, dx^j. \tag{5.6}$$

Note that since $\psi_i^\alpha = 0$ on $\mathrm{pr}^{(n)}\,\mathscr{R}$, the differentials $d\psi_i^\alpha = 0$ vanish on $\mathrm{pr}^{(n)}\,\mathscr{R}$, hence for (5.6) to vanish on $\mathrm{pr}^{(n)}\,\mathscr{R}$, we must have $\psi_{i,j}^\alpha = 0$ on $\mathrm{pr}^{(n)}\,\mathscr{R}$ also. The induction step is now clear, and the proof is easily completed.

6. Normal Systems of Ordinary Differential Equations

Rather than study internal symmetries in general straightaway, it is perhaps easier to work our a few specific examples in detail first. We begin with the simplest case, which is that of a normal system of ordinary differential equations (as opposed to the underdetermined systems to be treated in the next section). By definition, a *normal* system of ordinary differential equations is one of the form

$$u_n^\alpha = F^\alpha(x, u^{(n-1)}), \qquad \alpha = 1, ..., q, \tag{6.1}$$

in which the number of equations is the same number of unknowns, and we have solved for the top order derivatives. (Here $u_n^\alpha = D_x^n u^\alpha$.) Note that by differentiating the system, we can re-express the nth and higher order derivatives of the u's in terms of $(x, u^{(n-1)})$. For example, $u_{n+1}^\alpha = F_1^\alpha(x, u^{(n-1)})$, where

$$F_1^\alpha(x, u^{(n-1)}) = \frac{\partial F^\alpha}{\partial x} + \sum_{\beta=1}^{q} \sum_{j=1}^{n-2} u_{j+1}^\beta \frac{\partial F^\alpha}{\partial u_j^\beta} + \sum_{\alpha=1}^{q} F^\beta(x, u^{(n-1)}) \frac{\partial F^\alpha}{\partial u_{n-1}^\beta}. \tag{6.2}$$

Therefore, when restricted to the system, the prolongation of any generalized vector field is equivalent to a vector field on J^n which still preserves the contact ideal restricted to the system and so is an internal symmetry. Thus the following elementary converse to Theorem 5 is easily established:

THEOREM 6. *Any generalized symmetry of a normal system of ordinary differential equations restricts to an internal symmetry. Conversely, any internal symmetry is equivalent to a generalized symmetry of order at most* $n-1$.

Note that while the nth order truncated total derivative

$$\tilde{D}_x = \frac{\partial}{\partial x} + \sum_{\alpha=1}^{q} \sum_{j=1}^{n} u_{j+1}^{\alpha} \frac{\partial}{\partial u_j^{\alpha}} \tag{6.3}$$

is not a vector field on J^n (it also involves the $(n+1)$st order derivatives), it nevertheless restricts to a vector field

$$\mathbf{d}_x = \frac{\partial}{\partial x} + \sum_{\alpha=1}^{q} \sum_{j=1}^{n-2} u_{j+1}^{\alpha} \frac{\partial}{\partial u_j^{\alpha}} + \sum_{\alpha=1}^{q} F^{\alpha}(x, u^{(n-1)}) \frac{\partial}{\partial u_{n-1}^{\alpha}} + \sum_{\alpha=1}^{q} F_1^{\alpha}(x, u^{(n-1)}) \frac{\partial}{\partial u_n^{\alpha}} \tag{6.4}$$

on the system (6.1). The vector field (6.4) is trivially an internal symmetry. (Note that (6.2) is the same as $F_1^{\alpha} = \mathbf{d}_x F_{\alpha}$, and, indeed, $u_{n+k}^{\alpha} = F_k^{\alpha} = \mathbf{d}_x^k F_{\alpha}$ on solutions.) Moreover, any multiple of the truncated total derivative, $\xi(x, u^{(n-1)}) \tilde{D}_x$, also restricts to an internal symmetry $\xi(x, u^{(n-1)}) \mathbf{d}_x$ of the system. These should be thought of as "trivial" internal symmetries, with two internal symmetries equivalent if they differ by a trivial one. (We remark that this notion of trivial internal symmetries does not extend to underdetermined systems of ordinary differential equations or to systems of partial differential equations.) Geometrically, the trivial internal symmetries have the following interpretation. Since we are dealing with a system of ordinary differential equations, the submanifold $\mathscr{R}$ will be "foliated" by the prolongations (n-jets) of the solution curves $u^{(n)} = \mathrm{pr}^{(n)} f(x)$. The vector field $\mathbf{d}_x$ is then just the infinitesimal generator of the translation group along these solution curves: the group element $\Psi_{\varepsilon} = \exp(\varepsilon \mathbf{d}_x)$ takes the point $(x, u^{(n)}) = (x, \mathrm{pr}^{(n)} f(x)) \in \mathscr{R}$ to the point $\Psi_{\varepsilon}(x, u^{(n)}) = (x+\varepsilon, \mathrm{pr}^{(n)} f(x+\varepsilon))$ on the same curve. A more general trivial internal symmetry $\xi(x, u^{(n-1)}) \mathbf{d}_x$ will accordingly determine a reparametrized translation along the same solution curves of the system.

Thus, the correspondence between internal symmetries and generalized symmetries is not one-to-one in the case of normal systems of ordinary differential equations. (See also Stephani [17, (12.2.1), p. 114].) Moreover, not every internal symmetry comes from a first order generalized symmetry; a simple counterexample is provided by the third order equation $u''' = 0$, which has the second order generalized symmetry $\mathbf{v} = u'' \partial_u$. Note that, in view of the equation, $\mathrm{pr}^{(3)} \mathbf{v} = \mathbf{v}$, so the symmetry is internal, as guaranteed by Theorem 6, but is clearly not equivalent to any first order generalized symmetry. More generally, suppose we rewrite the nth order system (6.1) as an equivalent first order system of ordinary differential equations in the standard manner by introducing new variables representing the derivatives of the u^α's up to order $n-1$. Then any symmetry of the first order system will correspond to a generalized symmetry of order $\leqslant n-1$ of the original system (6.1). But, at least away from singular points, any first order system of ordinary differential equations has infinitely many symmetries, [15], so we conclude that, locally, any normal system of ordinary differential equations has an infinite number of generalized symmetries of order $\leqslant n-1$.

7. Underdetermined Systems of Ordinary Differential Equations

Theorem 6 completes the general determination of internal symmetries of a normal system of ordinary differential equations. We now shift our attention to the case of underdetermined systems of ordinary differential equations. Our general results are easier to understand if we begin by considering the following instructive example, which includes the remarkable Hilbert–Cartan equation as a special case.

Theorem 7. *For a single underdetermined second order equation*

$$u'' = F(x, u, u', v, v', v''), \tag{7.1}$$

in two unknowns, there is a one-to-one correspondence between first order generalized symmetries and internal symmetries.

It should be emphasized that the correspondence between first order generalized symmetries and internal symmetries is genuinely one-to-one; there are no trivial symmetries of either type because (a) the equation is of second order, and (b) the total derivative does not truncate to form an internal symmetry as in the normal case discussed above.

Proof. First suppose

$$\mathbf{v} = Q(x, u, v, u', v') \partial_u + R(x, u, v, u', v') \partial_v \tag{7.2}$$

is a first order generalized symmetry, which we assume, without loss of generality, to be in evolutionary form. Its prolongation to J^2 is

$$\mathrm{pr}^{(2)}\mathbf{v} = Q\,\partial_u + R\,\partial_v + D_x Q\,\partial_{u'} + D_x R\,\partial_{v'} + D_x^2 Q\,\partial_{u''} + D_x^2 R\,\partial_{v''},$$

which may depend on third order derivatives. The goal is to find an equivalent generalized symmetry whose prolongation, when restricted to the equation submanifold $\mathscr{R}$ determined by (7.1), depends only on second order derivatives. Note that we have the freedom of (a) using the equation and its derivatives to replace second and higher order derivatives of u, and (b) using the equivalence condition between generalized vector fields to add on any multiple $\xi \tilde{D}_x$ of the second order truncated total derivative to $\mathrm{pr}^{(2)}\mathbf{v}$.

First of all, differentiating the equation, we find that

$$u''' = D_x F = F_{v''} v''' + O(2),$$

where $O(k)$ indicates terms that depend only on kth and lower order derivatives of u and v. Hence u''' can be rewritten in terms of the variables x, u, v, u', v', u'', v'', v'''. Therefore, to obtain a genuine vector field on the equation manifold, we need only eliminate the v''' dependency in $\mathrm{pr}^{(2)}\mathbf{v}$. Second, since

$$D_x^2 R = R_{u'} u''' + R_{v'} v''' + O(2) = (R_{u'} F_{v''} + R_{v'})\, v''' + O(2),$$

we see that, when restricted to the equation (7.1),

$$\mathrm{pr}^{(2)}\mathbf{v} = (R_{u'} F_{v''} + R_{v'})\, v'''\,\partial_{v''} + (Q_{u'} F_{v''} + Q_{v'})\, v'''\,\partial_{u''} + \hat{\mathbf{X}},$$

where the coefficients of $\hat{\mathbf{X}}$ only depend on x, u, v, u', v', v'' and so $\hat{\mathbf{X}}$ is a genuine vector field on J^2. The first term can be absorbed by a suitable multiple of the truncated total derivative, so, on the equation (7.1),

$$\begin{aligned}\mathrm{pr}^{(2)}\mathbf{v} = {} & (R_{u'} F_{v''} + R_{v'})\, \tilde{D}_x \\ & + [(Q_{u'} F_{v''} + Q_{v'}) - (R_{u'} F_{v''} + R_{v'})\, F_{v''}]\, v'''\,\partial_{u''} + \mathbf{X},\end{aligned}$$

where $\mathbf{X}$ is also a genuine vector field on J^2. The first term on the right-hand side is a trivial generalized symmetry. Therefore, if we can prove that the second term vanishes, i.e., show that

$$Q_{u'} F_{v''} + Q_{v'} = (R_{u'} F_{v''} + R_{v'})\, F_{v''}, \tag{7.3}$$

then we can deduce that $\mathrm{pr}^{(2)}\mathbf{v}$ is equivalent to the internal symmetry $\mathbf{X}$.

(Note that $\mathbf{X}$ preserves the contact ideal on $\mathscr{R}$, because both $\mathrm{pr}^{(2)}\,\mathbf{v}$ and $\tilde{D}_x$ do.) In fact, we can then replace $\mathbf{v}$ by the equivalent generalized symmetry

$$\tilde{\mathbf{v}} = -(R_{u'}F_{v''} + R_{v'})\,\partial_x + [Q - u'(R_{u'}F_{v''} + R_{v'})]\,\partial_u + [R - v'(R_{u'}F_{v''} + R_{v'})]\,\partial_v, \tag{7.4}$$

so that $\mathbf{v}$ is the evolutionary form of $\tilde{\mathbf{v}}$. The prolongation formula (4.5) shows that

$$\mathrm{pr}^{(2)}\,\tilde{\mathbf{v}} = \mathbf{X} \qquad \text{on } \mathscr{R}, \tag{7.5}$$

so $\mathbf{X}$ is an internal symmetry which is equal to the prolongation of the generalized symmetry $\tilde{\mathbf{v}}$.

To prove (7.3), we use the symmetry condition (2.8). Applying $\mathrm{pr}^{(2)}\,\mathbf{v}$ to the equation, we deduce that

$$D_x^2 Q = \mathrm{pr}^{(2)}\,\mathbf{v}(F) = D_x^2 R \cdot F_{v''} + O(2)$$

on (7.1). The coefficient of v''' in this equation is just

$$Q_{u'}F_{v''} + Q_{v'} = (R_{u'}F_{v''} + R_{v'})\,F_{v''},$$

which is exactly the condition (7.3), as required.

Conversely, given an internal symmetry, its projection onto E will be a generalized vector field. The problem now is to show that there is an equivalent first order generalized symmetry; cf. Theorem 5. In fact, we will prove that, on the equation, the characteristic of the internal symmetry is necessarily a function depending on at most first order derivatives. Let

$$\mathbf{X} = \xi\,\partial_x + \varphi\,\partial_u + \psi\,\partial_v + \varphi^1\,\partial_{u'} + \psi^1\,\partial_{v'} + \varphi^2\,\partial_{u''} + \psi^2\,\partial_{v''} \tag{7.6}$$

be a vector field on $\mathscr{R}$. Since (x, u, v, u', v', v'') provide local coordinates for the points of $\mathscr{R}$, we can assume that all the coefficients depend on these variables. Moreover, according to the tangency condition (5.3), the coefficient

$$\varphi^2 = \mathbf{X}[F] \tag{7.7}$$

is automatically determined from the other coefficients of $\mathbf{X}$. The characteristic of the projection

$$\mathbf{v} = \pi(\mathbf{X}) = \xi\,\partial_x + \varphi\,\partial_u + \psi\,\partial_v$$

is the pair of functions

$$Q = \varphi - u'\xi, \qquad R = \psi - v'\xi.$$

The goal is to prove that these are defined on J^1, i.e., they do not depend on v''.

The contact ideal $I^{(2)}$ is generated by the one-forms

$$du - u'\,dx, \qquad dv - v'\,dx, \qquad du' - u''\,dx, \qquad dv' - v''\,dx.$$

Requiring that $\mathbf{X}$ preserve the contact ideal on $\mathscr{R}$ says, for instance,

$$\begin{aligned}\mathbf{X}[du - u'\,dx]& \\ &= d\varphi - \varphi^1\,dx - u'\,d\xi \\ &= \{\mathbf{d}_x\varphi - u'\,\mathbf{d}_x\xi - \varphi^1\}\,dx + \{\varphi_{v''} - u'\xi_{v''}\}\,dv'' \qquad \bmod I^{(2)}|\mathscr{R},\end{aligned}$$

where

$$\mathbf{d}_x = \partial_x + u'\,\partial_u + v'\,\partial_v + F(x, u, v, u', v', v'')\,\partial_{u'} + v''\,\partial_{v'}$$

is the restriction of the total derivative to $\mathscr{R}$ (cf. (6.4)). This will lie in $I^{(2)}|\mathscr{R}$ if and only if the two conditions

$$\varphi^1 = \mathbf{d}_x\varphi - u'\,\mathbf{d}_x\xi$$

and

$$\varphi_{v''} - u'\xi_{v''} = \partial_{v''}(\varphi - u'\xi) = Q_{v''} = 0$$

hold. Similarly, using the contact form $dv - v'\,dx$, we deduce that

$$\psi^1 = \mathbf{d}_x\psi - v'\,\mathbf{d}_x\xi$$

and

$$\psi_{v''} - v'\xi_{v''} = \partial_{v''}(\psi - v'\xi) = R_{v''} = 0.$$

Thus Q and R only depend on x, u, v, u', v'. Finally, the conditions that $\mathbf{X}(du' - u''\,dx)$ and $\mathbf{X}(dv' - v''\,dx)$ also lie in $I^{(2)}|\mathscr{R}$ imply, respectively, the tangency condition (7.7), the prolongation formula for the coefficient ψ^2, and, upon eliminating ξ, the top order symmetry conditions (7.3). Together, these all imply that the generalized vector field

$$\mathbf{X} - \xi\,\mathbf{d}_x = Q\,\partial_u + R\,\partial_v + \dots$$

coincides with the prolongation of the first order generalized symmetry

$$\mathbf{v} = Q\,\partial_u + R\,\partial_v.$$

Therefore, every internal symmetry of (7.1) comes from a first order generalized symmetry, and the proof is complete.

EXAMPLE 8 (The Hilbert–Cartan equation). The underdetermined ordinary differential equation

$$v' = (u'')^2 \tag{7.8}$$

was introduced by Hilbert [6] as an example of an equation whose general solution cannot be expressed in terms of an arbitrary function and a finite number of its derivatives. Subsequently, Cartan [3, 4], as an example of his theory of Pfaffian systems in five variables [2], proved that this equation has, as an internal symmetry group, the real noncompact form of the 14 dimensional exceptional Lie group G_2. We verify this result directly using Theorem 7. We begin with the calculation of the first order generalized symmetries.

THEOREM 9. *Every first order evolutionary generalized symmetry of the Hilbert–Cartan equation is a linear constant coefficient combination of the symmetries*

$$\begin{aligned}
\mathbf{v}_1 &= (\tfrac{1}{2}uv - \tfrac{2}{9}u'^3)\,\partial_u + (\tfrac{1}{2}v^2 - \tfrac{2}{3}u'^2u''^2 + \tfrac{2}{3}uu''^3)\,\partial_v \\
\mathbf{v}_2 &= (\tfrac{1}{6}x^3v - \tfrac{2}{3}x^2u'^2 + 2xuu' - 2u^2)\,\partial_u \\
&\quad + (2xu'v - 2uv + \tfrac{2}{9}x^3u''^3 - \tfrac{4}{3}x^2u'u''^2 + 2xuu''^2 - \tfrac{8}{9}u'^3)\,\partial_v. \\
\mathbf{v}_3 &= (\tfrac{1}{2}x^2v - \tfrac{4}{3}xu'^2 + 2uu')\,\partial_u \\
&\quad + (2u'v + 2uu''^2 + \tfrac{2}{3}x^2u''^3 - \tfrac{8}{3}xu'u''^2)\,\partial_v \\
\mathbf{v}_4 &= (xv - \tfrac{4}{3}u'^2)\,\partial_u + (\tfrac{4}{3}xu''^3 - \tfrac{8}{3}u'u''^2)\,\partial_v \\
\mathbf{v}_5 &= v\,\partial_u + \tfrac{4}{3}u''^3\,\partial_v \\
\mathbf{v}_6 &= \tfrac{1}{2}u\,\partial_u + v\,\partial_v \\
\mathbf{v}_7 &= (\tfrac{1}{2}x^2u' - \tfrac{3}{2}xu)\,\partial_u + (\tfrac{1}{2}x^2u''^2 - 2u'^2)\,\partial_v \\
\mathbf{v}_8 &= (xu' - \tfrac{3}{2}u)\,\partial_u + xu''^2\,\partial_v \\
\mathbf{v}_9 &= u'\,\partial_u + u''^2\,\partial_v \\
\mathbf{v}_{10} &= \tfrac{1}{6}x^3\,\partial_u + 2(xu' - u)\,\partial_v \\
\mathbf{v}_{11} &= \tfrac{1}{2}x^2\,\partial_u + 2u'\,\partial_v \\
\mathbf{v}_{12} &= x\,\partial_u \\
\mathbf{v}_{13} &= \partial_u \\
\mathbf{v}_{14} &= \partial_v.
\end{aligned} \tag{7.9}$$

Proof. We implement the standard algorithm [15, Chap. 5]. Let

$$\mathbf{v} = Q\,\partial_u + R\,\partial_v$$

be a first order evolutionary symmetry, so Q, R are functions of x, u, v, u', v'. Since we need only work modulo trivial symmetries, though, we can replace v' by $(u'')^2$, and it is slightly simpler to assume that Q, R are functions of x, u, v, u', u'' instead. The infinitesimal symmetry condition is

$$D_x R = 2u'' D_x^2 Q, \tag{7.10}$$

which is required to hold whenever

$$v' = (u'')^2 \qquad \text{and} \qquad v'' = 2u''u'''. \tag{7.11}$$

Note first that each of the 14 vector fields (7.9) satisfies this condition and so defines a symmetry. We now expand the total derivatives in (7.10), using (7.11) to eliminate v' and v'', and equate the various coefficients of the derivatives of u and v to zero. The coefficient of u''' shows that Q is independent of u''. The coefficients of u''^5 and u''^4 imply that

$$Q = A(x, u)v + B(x, u, u').$$

Now since ∂_v commutes with the total derivative D_x and the substitutions given by (7.11), we can differentiate the symmetry conditions with respect to v to deduce that

$$D_x R_v = 2u'' D_x^2 Q_v, \qquad D_x R_{vv} = 0$$

(subscripts on Q and R indicating derivatives). The second of these implies that R_{vv} is a constant, say c_1. Substituting the consequential form of R and the previous form of Q into the first equation leads, after some fairly routine calculations, to the fact that Q and R have the forms

$$Q = [\tfrac{1}{2}c_1 u + \tfrac{1}{6}c_2 x^3 + \tfrac{1}{2}c_3 x^2 + c_4 x + c_5]v + S(x, u, u'),$$

$$R = \tfrac{1}{2}c_1 v^2 + [2c_2(xu' - u) + 2c_3 u' + c_6]v + T(x, u, u', u''),$$

where $c_1, \ldots, c_6$ are constants. However, from the table of symmetries we see that

$$\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 + c_4 \mathbf{v}_4 + c_5 \mathbf{v}_5 + c_6 \mathbf{v}_6 + \tilde{\mathbf{v}},$$

where

$$\tilde{\mathbf{v}} = \tilde{Q}(x, u, u')\, \partial_u + \tilde{R}(x, u, u', u'')\, \partial_v \tag{7.12}$$

is a first order generalized symmetry which does not depend explicitly on v. Substituting (7.12) into the symmetry conditions (7.10), the coefficients of u''' and u''^2 now imply that

$$\tilde{Q} = A(x, u)\, u' + B(x, u), \qquad \tilde{R} = A(x, u)\, u''^2 + C(x, u, u').$$

The coefficient of u''^2 in the symmetry condition implies that

$$A_u = 2A_x + B_x = B_{uu} = 0,$$

and the symmetry condition has reduced to

$$C_x + u'C_u + u''C_{u'} = 2u''[u'A_{xx} + B_{xx} + 2u'B_{xy}]. \tag{7.13}$$

It is now straightforward to solve (7.13) explicitly; the general solution is

$$A = c_7 x^2 + c_8 x + c_9, \quad B = -\tfrac{3}{2} c_7 x u - \tfrac{3}{2} c_8 u + \tfrac{1}{6} c_{10} x^3 + \tfrac{1}{2} c_{11} x^2 + c_{12} x + c_{13},$$
$$C = -2c_7 u^{12} 2c_{10}(x^2 u' - u) + 2c_{11} u' + c_{14},$$

which yields the remaining eight symmetries, completing the proof. (This computation was subsequently reverified by P. H. M. Kersten using his REDUCE symmetry package [9].)

Since each of the vector fields in Theorem 9 corresponds to a unique internal symmetry, we deduce that these vector fields close under the Lie bracket operation to form a Lie algebra when restricted to the equation; however, *on the entire jet space they may not close.* For example,

$$[\mathbf{v}_4, \mathbf{v}_5] = \tfrac{8}{3} u'(v' - u''^2)\, \partial_u + \tfrac{16}{3} u''[u'(v'' - 2u''u''') - 2u''(v' - u''^2)]\, \partial_v,$$

which is not in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_{14}$ but which vanishes on the equation and so forms a trivial generalized symmetry.

The corresponding internal symmetries are, according to (7.4), the prolongations of

$$
\begin{aligned}
\mathbf{X}_1 &= (\tfrac{2}{3}u'^2 - uu'')\,\partial_x + (\tfrac{1}{2}uv + \tfrac{4}{9}u'^3 - uu'u'')\,\partial_u + (\tfrac{1}{2}v^2 - \tfrac{1}{3}uu''^3)\,\partial_v \\
\mathbf{X}_2 &= (\tfrac{4}{3}x^2u' - 2xu - \tfrac{1}{3}x^3u'')\,\partial_x + (\tfrac{1}{6}x^3v + \tfrac{2}{3}x^2u'^2 - 2u^2 - \tfrac{1}{3}x^3u'u'')\,\partial_u \\
&\quad + (2xu'v - 2uv - \tfrac{1}{9}x^3u''^3 - \tfrac{8}{9}u'^3)\,\partial_v \\
\mathbf{X}_3 &= (\tfrac{8}{3}xu' - 2u - x^2u'')\,\partial_x + (\tfrac{1}{2}x^2v + \tfrac{4}{3}xu'^2 - x^2u'u'')\,\partial_u \\
&\quad + (2vu' - \tfrac{1}{3}x^2u''^3)\,\partial_v \\
\mathbf{X}_4 &= (\tfrac{8}{3}u' - 2xu'')\,\partial_x + (xv + \tfrac{4}{3}u'^2 - 2xu'u'')\,\partial_u - \tfrac{2}{3}xu''^3\,\partial_v \\
\mathbf{X}_5 &= -2u''\,\partial_x + (v - 2u'u'')\,\partial_u - \tfrac{2}{3}u''^3\,\partial_v \\
\mathbf{X}_6 &= \tfrac{1}{2}u\,\partial_u + v\,\partial_v \\
\mathbf{X}_7 &= -\tfrac{1}{2}x^2\,\partial_x - \tfrac{3}{2}xu\,\partial_u - 2u'^2\,\partial_v \\
\mathbf{X}_8 &= -x\,\partial_x - \tfrac{3}{2}u\,\partial_u \\
\mathbf{X}_9 &= -\partial_x \\
\mathbf{X}_{10} &= \tfrac{1}{6}x^3\,\partial_u + 2(xu' - u)\,\partial_v \\
\mathbf{X}_{11} &= \tfrac{1}{2}x^2\,\partial_u + 2u'\,\partial_v \\
\mathbf{X}_{12} &= x\,\partial_u \\
\mathbf{X}_{13} &= \partial_u \\
\mathbf{X}_{14} &= \partial_v.
\end{aligned}
\tag{7.14}
$$

Note that only the vector fields $\mathbf{X}_6$, $\mathbf{X}_8$, $\mathbf{X}_9$, $\mathbf{X}_{12}$, $\mathbf{X}_{13}$, $\mathbf{X}_{14}$ are external symmetries; the remaining eight vector fields provide genuine internal symmetries.

We now determine the structure of the 14 dimensional Lie algebra $\mathfrak{g}$ spanned by these vector fields. Note that we must use the "generalized Lie bracket" $[\cdot,\cdot]^*$ defined by

$$[\mathrm{pr}^{(2)}\mathbf{X}_i, \mathrm{pr}^{(2)}\mathbf{X}_j] = \mathrm{pr}^{(2)}([\mathbf{X}_i, \mathbf{X}_j]^*),$$

in order to recover the usual Lie bracket of the associated internal symmetries. The commutators are shown in Table I.

TABLE I

	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{X}_6$	$\mathbf{X}_7$	$\mathbf{X}_8$	$\mathbf{X}_9$	$\mathbf{X}_{10}$	$\mathbf{X}_{11}$	$\mathbf{X}_{12}$	$\mathbf{X}_{13}$	$\mathbf{X}_{14}$
$\mathbf{X}_1$	0	0	0	0	0	$-\mathbf{X}_1$	0	0	0	$-\frac{1}{2}\mathbf{X}_2$	$-\frac{1}{2}\mathbf{X}_3$	$-\frac{1}{2}\mathbf{X}_4$	$-\frac{1}{2}\mathbf{X}_5$	$-\mathbf{X}_6$
$\mathbf{X}_2$		0	0	0	$4\mathbf{X}_1$	$-\frac{1}{2}\mathbf{X}_2$	0	$\frac{3}{2}\mathbf{X}_2$	$-\mathbf{X}_3$	0	0	$-\frac{4}{3}\mathbf{X}_7$	$2\mathbf{X}_6-2\mathbf{X}_8$	$-\mathbf{X}_{10}$
$\mathbf{X}_3$			0	$-4\mathbf{X}_1$	0	$-\frac{1}{2}\mathbf{X}_3$	$-\frac{3}{2}\mathbf{X}_2$	$\frac{1}{2}\mathbf{X}_3$	$-\mathbf{X}_4$	0	$\frac{4}{3}\mathbf{X}_7$	$\frac{2}{3}\mathbf{X}_8-2\mathbf{X}_6$	$2\mathbf{X}_9$	$-\mathbf{X}_{11}$
$\mathbf{X}_4$				0	0	$-\frac{1}{2}\mathbf{X}_4$	$-2\mathbf{X}_3$	$-\frac{1}{2}\mathbf{X}_4$	$-\mathbf{X}_5$	$-\frac{4}{3}\mathbf{X}_7$	$\frac{2}{3}\mathbf{X}_8+2\mathbf{X}_6$	$-\frac{8}{3}\mathbf{X}_9$	0	$-\mathbf{X}_{12}$
$\mathbf{X}_5$					0	$-\frac{1}{2}\mathbf{X}_5$	$-\frac{3}{2}\mathbf{X}_4$	$-\frac{3}{2}\mathbf{X}_5$	0	$-2\mathbf{X}_8-2\mathbf{X}_6$	$2\mathbf{X}_9$	0	0	$-\mathbf{X}_{13}$
$\mathbf{X}_6$						0	0	0	0	$-\frac{1}{2}\mathbf{X}_{10}$	$-\frac{1}{2}\mathbf{X}_{11}$	$-\frac{1}{2}\mathbf{X}_{12}$	$-\frac{1}{2}\mathbf{X}_{13}$	$-\mathbf{X}_{14}$
$\mathbf{X}_7$							0	$\mathbf{X}_7$	$-\mathbf{X}_8$	0	$\frac{3}{2}\mathbf{X}_{10}$	$2\mathbf{X}_{11}$	$\frac{3}{2}\mathbf{X}_{12}$	0
$\mathbf{X}_8$								0	$\mathbf{X}_9$	$-\frac{3}{2}\mathbf{X}_{10}$	$-\frac{1}{2}\mathbf{X}_{11}$	$\frac{1}{2}\mathbf{X}_{12}$	$\frac{3}{2}\mathbf{X}_{13}$	0
$\mathbf{X}_9$									0	$\mathbf{X}_{11}$	$\mathbf{X}_{12}$	$\mathbf{X}_{13}$	0	0
$\mathbf{X}_{10}$										0	0	0	$2\mathbf{X}_{14}$	0
$\mathbf{X}_{11}$											0	$-2\mathbf{X}_{14}$	0	0
$\mathbf{X}_{12}$												0	0	0
$\mathbf{X}_{13}$													0	0
$\mathbf{X}_{14}$														0

The Killing form for this Lie algebra is

$$K=\left|\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}\right|,$$

which is patently nondegenerate, and hence $\mathfrak{g}$ is a semi-simple Lie algebra. Moreover, K is indefinite, so $\mathfrak{g}$ is noncompact real form of the associated complex semi-simple Lie algebra. We now investigate the structure of this Lie algebra using standard methods; cf. [7].

LEMMA 10. *The two dimensional subalgebra* $\mathfrak{g}_0$ *spanned by* $\{\mathbf{X}_6, \mathbf{X}_8\}$ *is a Cartan subalgebra of* $\mathfrak{g}$.

Proof. The multiplication table shows that $\mathfrak{g}_0$ is an abelian subalgebra. Thus we need to check that it is a maximal abelian subalgebra, by proving that if

$$\mathbf{X} = \sum_{i=1}^{14} a_i \mathbf{X}_i \in \mathfrak{g}$$

satisfies

$$[\mathbf{X}_6, \mathbf{X}] = 0 \qquad \text{and} \qquad [\mathbf{X}_8, \mathbf{X}] = 0, \tag{7.15}$$

then $\mathbf{X} \in \mathfrak{g}_0$. However, since

$$[\mathbf{X}_6, \mathbf{X}_i] = \lambda_i \mathbf{X}_i, \qquad \text{where} \quad \lambda_i \neq 0 \text{ for } i \neq 6, 7, 8, 9, \tag{7.16}$$

and

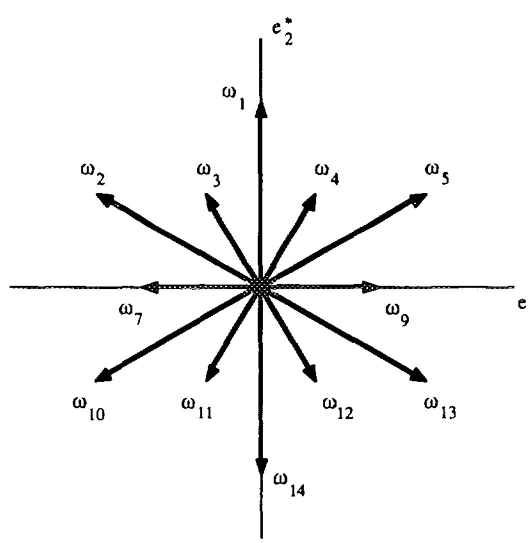
$$[\mathbf{X}_8, \mathbf{X}_i] = \mu_i \mathbf{X}_i, \qquad \text{where} \quad \mu_i \neq 0 \text{ for } i \neq 1, 6, 8, 14, \tag{7.17}$$

Eqs. (7.15) hold if and only if $a_i = 0$ for $i \neq 6, 8$. Thus $\mathbf{X} \in \mathfrak{g}_0$. Moreover, Eqs. (7.16) and (7.17) also show that for every element $\mathbf{X} \in \mathfrak{g}_0$, its adjoint representation ad $\mathbf{X}$ is diagonal with respect to the given basis.

LEMMA 11. *The roots of the Lie algebra* $\mathfrak{g}$ *are*

$$\begin{aligned}
&\omega_1 = \left(0, \frac{1}{2}\right), && \omega_2 = \left(-\frac{\sqrt{3}}{4}, \frac{1}{4}\right), && \omega_3 = \left(-\frac{\sqrt{3}}{12}, \frac{1}{4}\right),\\
&\omega_4 = \left(\frac{\sqrt{3}}{12}, \frac{1}{4}\right), && \omega_5 = \left(\frac{\sqrt{3}}{4}, \frac{1}{4}\right), && \omega_6 = 0,\\
&\omega_7 = \left(-\frac{\sqrt{3}}{6}, 0\right), && \omega_8 = 0, && \omega_9 = \left(\frac{\sqrt{3}}{6}, 0\right),\\
&\omega_{10} = \left(-\frac{\sqrt{3}}{4}, -\frac{1}{4}\right), && \omega_{11} = \left(-\frac{\sqrt{3}}{12}, -\frac{1}{4}\right), && \omega_{12} = \left(\frac{\sqrt{3}}{12}, -\frac{1}{4}\right),\\
&\omega_{13} = \left(\frac{\sqrt{3}}{4}, -\frac{1}{4}\right), && \omega_{14} = \left(0, -\frac{1}{2}\right).
\end{aligned} \tag{7.18}$$

The fundamental roots are ω_9, ω_2. *The root diagram for* $\mathfrak{g}$ *is*



Proof. For the Cartan subalgebra $\mathfrak{g}_0$ from the previous lemma, the restriction of the Killing form K to $\mathfrak{g}_0$ is

$$K_0 = \begin{pmatrix} 4 & 0 \\ 0 & 12 \end{pmatrix}.$$

We normalize our basis for $\mathfrak{g}_0$ by letting

$$\mathbf{e}_1 = \frac{1}{2}\mathbf{X}_6, \qquad \mathbf{e}_2 = \frac{\sqrt{3}}{6}\mathbf{X}_8.$$

Then with respect to this basis K_0 is the identity.

Each basis element $\mathbf{X}_i$ for $\mathfrak{g}$ defines a linear functional ω_i on $\mathfrak{g}_0$ by

$$[\mathbf{X}, \mathbf{X}_i] = \omega_i(\mathbf{X})\,\mathbf{X}_i, \qquad \mathbf{X} \in \mathfrak{g}_0.$$

For example, if $\mathbf{X} = a\mathbf{e}_1 + b\mathbf{e}_2$, then

$$[\mathbf{X}, \mathbf{X}_1] = a[\mathbf{e}_1, \mathbf{X}_1] + b[\mathbf{e}_2, \mathbf{X}_1] = \tfrac{1}{2}b\mathbf{X}_1,$$

and so $\omega_1 = (0, \frac{1}{2})$. From the commutator table, we verify the roots given by (7.18). This completes the proof of the lemma.

Since our root diagram coincides with the root diagram for G_2 (cf. [7]), we conclude that the internal symmetry group of the Hilbert–Cartan equation is the noncompact real form of the semi-simple Lie algebra G_2. This completes our discussion of the internal symmetries and first order generalized symmetries of the Hilbert–Cartan equation.

It is of interest to classify the higher order generalized symmetries of the Hilbert–Cartan equation. A calculation similar to that of Theorem 9 proves that there are no second order generalized symmetries of (7.8) beyond the first order ones already found. However, there *are* new generalized symmetries of arbitrarily high order. Indeed, note that for $k \geqslant 0$, the function $2u_2 u_{2k+3}$ is an x-derivative, so therefore we can find a ψ_k such that $D_x \psi_k = 2u_2 u_{2k+3}$. (The explicit formula for ψ_k is easy to find, but not required.) Then the generalized vector field

$$\mathbf{v} = u_{2k+1}\,\partial_u + \psi_k\,\partial_v$$

is a symmetry. For example, we have the third order symmetry

$$\mathbf{v} = u'''\,\partial_u + (2u''u'''' - u'''^2)\,\partial_v$$

coming from the case $k = 1$. (We call $\mathbf{v}$ "third order" since we can express u'''' in terms of third order derivatives using the second prolongation of the equation.) In a recent paper Kersten [10] provides a complete characterization of all the generalized symmetries to the Hilbert–Cartan equation and, in [11], a method for determining all generalized symmetries to a general underdetermined equation of the form $u_n = F(x, u^{(n-1)}, v^{(k)})$.

We now present two examples illustrating the difficulty of extending Theorem 7 to more general systems of ordinary differential equations.

EXAMPLE 12. The equation

$$v'u'' = w \tag{7.19}$$

has

$$\mathbf{v} = x^2\,\partial_u + 2v'\,\partial_w \tag{7.20}$$

as a first order generalized symmetry, but there is no internal counterpart. Indeed,

$$\mathrm{pr}^{(2)}\,\mathbf{v} = x^2\,\partial_u + 2v'\,\partial_w + 2x\,\partial_{u'} + 2v''\,\partial_{w'} + 2\,\partial_{u''} + 2v'''\,\partial_{w''},$$

but there is no way using the equation and its prolongations, or adding in any multiple of the total derivative, to eliminate all the third order derivatives in this vector field, so it never restricts to a genuine geometrical vector field on $\mathscr{R} \subset J^2$. Moreover, this problem persists even if we replace (7.19) by any (finite order) prolongation.

Let us see what goes wrong if we try to mimic the proof of Theorem 7 for a codimension 2 equation of the form

$$u'' = F(x, u, v, w, u', v', w'). \tag{7.21}$$

(For simplicity, we assume that F does not depend on v'' or w'', but the argument carries through more generally.) Consider a first order generalized vector field

$$\mathbf{v} = Q\,\partial_u + R\,\partial_v + S\,\partial_w,$$

where Q, R, S depend on x, u, v, w, u', v', w'. Its second prolongation $\mathrm{pr}^{(2)}\,\mathbf{v}$ can depend on third order derivatives, and the goal is to produce an equivalent vector field which is defined on J^2. We replace second and higher order derivatives of u using the equation. The remaining terms in $\mathrm{pr}^{(2)}\,\mathbf{v}$ that depends on v''', w''' are

$$(R_{v'}v''' + R_{w'}w''')\,\partial_{v''} + (S_{v'}v''' + S_{w'}w''')\,\partial_{w''}.$$

Now the only remaining freedom is to add in a multiple of the total derivative. This will eliminate all the offending terms if and only if

$$R_{v'} = S_{w'}, \qquad R_{w'} = S_{v'} = 0. \tag{7.22}$$

However, as the above example demonstrates, these conditions are not guaranteed by the symmetry conditions. Therefore, for higher codimension underdetermined systems of ordinary differential equations, there can exist first order generalized symmetries which are not equivalent to internal symmetries.

EXAMPLE 13. Consider a third order equation of the form

$$u''' = F(x, u, v, u', v', u'', v'', v'''). \tag{7.23}$$

The argument that any first order generalized symmetry gives rise to an internal symmetry works as before. However, the converse is not necessarily true. Let

$$\mathbf{X} = \xi\,\partial_x + \varphi\,\partial_u + \psi\,\partial_v + \varphi^1\,\partial_{u'} + \psi^1\,\partial_{v'} + \varphi^2\,\partial_{u''} + \psi^2\,\partial_{v''} + \varphi^3\,\partial_{u'''} + \psi^3\,\partial_{v'''}$$

be an internal symmetry, so that the tangency condition $\varphi^3 = \mathbf{X}[F]$ holds; cf. (7.7). As above, the characteristic of $\mathbf{v} = \pi(\mathbf{X})$ is the pair of functions

$$Q = \varphi - u'\xi, \qquad R = \psi - v'\xi,$$

and we must ascertain whether Q and R are defined on J^1, i.e., subject to (7.23), they do not depend on u'', v'', or v'''.

Applying $\mathbf{X}$ to the lowest order contact forms yields

$$\begin{aligned}\mathbf{X}[du - u'\,dx] \\ &= d\varphi - \varphi^1\,dx - u'\,d\xi \\ &= \{\mathbf{d}_x\varphi - u'\,\mathbf{d}_x\xi - \varphi^1\}\,dx + \{\varphi_{v'''} - u'\xi_{v'''}\}\,dv''' \qquad \mathrm{mod}\ I^{(3)}\,|\,\mathcal{R},\end{aligned}$$

where

$$\mathbf{d}_x = \partial_x + u'\,\partial_u + v'\,\partial_v + u''\,\partial_{u'} + v''\,\partial_{v'} + F\,\partial_{u''} + v'''\,\partial_{v''}$$

is the restriction of the total derivative to $\mathcal{R}$. This will lie in $I^{(3)}\,|\,\mathcal{R}$ if and only if $\varphi^1 = \mathbf{d}_x\,\varphi - u'\,\mathbf{d}_x\xi$, and

$$\varphi_{v'''} - u'\xi_{v'''} = \partial_{v'''}(\varphi - u'\xi) = Q_{v'''} = 0. \tag{7.24}$$

Similarly, applying $\mathbf{X}$ to the contact form $dv - v'\,dx$ implies $\psi^1 = \mathbf{d}_x\psi - v'\,\mathbf{d}_x\xi$ and $R_{v'''} = 0$, and our internal symmetry is equivalent to a second order generalized symmetry. However, at the next order we run into difficulties. Indeed,

$$\begin{aligned}\mathbf{X}[du' - u''\,dx] &= \{\mathbf{d}_x\varphi^1 - u''\,\mathbf{d}_x\xi - \varphi^2\}\,dx \\ &\quad + \{\varphi^1_{v'''} - u''\xi_{v'''}\}\,dv''' \qquad \mathrm{mod}\ I^{(3)}\,|\,\mathrm{R}.\end{aligned}$$

Thus we recover the next part of the prolongation formula, namely $\varphi^2 = \mathbf{d}_x\varphi^1 - u''\,\mathbf{d}_x\xi$, and, using (7.24),

$$0 = \varphi^1_{v'''} - u''\xi_{v'''} = \partial_{v'''}(\mathbf{d}_x Q) = Q_{v''} + F_{v'''}Q_{u''}.$$

Moreover, applying $\partial_{v'''}$ to the latter condition and using (7.24) again, we find

$$F_{v'''v'''} Q_{u''} = 0.$$

Consequently, *if*

$$\frac{\partial^2 F}{(\partial v''')^2} \neq 0, \tag{7.25}$$

then $Q_{u''} = Q_{v''} = 0$, and Q only depends on first order derivatives. Similar arguments involving the contact form $dv' - v''\, dx$ imply that, under the same condition (7.25), R also only depends on first order derivatives, and hence the original symmetry is equivalent to a first order generalized symmetry. The remaining contact forms give the remaining prolongation formulae and top order symmetry conditions. In conclusion, the condition (7.25) is sufficient to guarantee that every internal symmetry of (7.23) is equivalent to unique first order generalized symmetry. Without (7.25), the result is not evident, and, as we saw, in the normal case, when F does not depend on v and its derivatives, not true.

8. Contact Conditions for Ordinary Differential Equations

We now investigate the structure of internal symmetries for a general system of ordinary differential equations. We derive necessary and sufficient conditions under which a first order generalized symmetry of a general system of ordinary differential equations will be equivalent to an internal symmetry. (The more complicated case of systems of partial differential equations is similarly analyzed in the next section.) Consider a system of nth order ordinary differential equations

$$\Delta_\kappa(x, u^{(n)}) = 0, \qquad \kappa = 1, ..., r. \tag{8.1}$$

We assume a slightly strengthened version of the maximal rank condition (2.2), namely that the $q \times r$ Jacobian matrix

$$K = \left(\frac{\partial \Delta_\kappa}{\partial u_n^\alpha}\right) \tag{8.2}$$

with respect to the *top order derivatives* of u has rank r. (In particular, we assume that the system is not overdetermined, i.e., $r \leqslant q$.) This assures us that we can locally solve for r of the top order derivatives, say $u_n^1, ..., u_n^r$, leading to a system of ordinary differential equations of the form

$$u_n^\kappa = F^\kappa(x, u^{(n-1)}, u_n^{r+1}, ..., u_n^q), \qquad \kappa = 1, ..., r. \tag{8.3}$$

With this choice, we refer to the variables $u^1, ..., u^r$ as *normal* directions, and the variables $u^{r+1}, ..., u^q$ as *tangential* directions. (This is in analogy with the case of an implicit submanifold of Euclidean space $\mathbb{R}^p$, where, solving for $x^i = f^i(x)$, $i = 1, ..., r$, splits the variables into tangential, $x^1, ..., x^r$, and normal, $x^{r+1}, ..., x^p$, directions, which can be associated with the tangent and normal spaces to the submanifold.) Every nonsingular $r \times r$ submatrix of the Jacobian matrix K provides a local splitting of the variables into normal and tangential directions.

LEMMA 14. *Let $\mathscr{R}$ be a top order maximal rank system of ordinary differntial equations. If a first order generalized symmetry*

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q^\alpha(x, u^{(1)}) \frac{\partial}{\partial u^\alpha}$$

is equivalent to an internal summetry, then the coefficients Q^α must satisfy the contact conditions

$$\frac{\partial Q^\alpha}{\partial u^\beta_x} + \xi\, \delta^\alpha_\beta = \sum_{\kappa=1}^{r} \lambda^\alpha_\kappa \frac{\partial \Delta_\kappa}{\partial u^\beta_n} \qquad \text{on } \mathscr{R}, \quad \alpha, \beta = 1, ..., q. \tag{8.4}$$

Here ξ, λ^α_κ, $\alpha = 1, ..., q$, $\kappa = 1, ..., r$, are unspecified functions defined on the equation manifold $\mathscr{R}$.

The condition (8.4) may seem a little strange at first glance. The right-hand side can depend on nth order derivatives, whereas the term $\partial Q^\alpha / \partial u^\beta_x$ only depends on first order derivatives. The reason that the conditions do not degenerate for $n \geqslant 2$ is that the functions ξ, λ^α_κ can themselves depend on nth order derivatives and thereby cancel out all the higher order dependence. For example, in the case of the symmetry $\mathbf{v}_5$ of the Hilbert–Cartan equation, the equivalent first order characteristic has $Q^1 = v$, $Q^2 = \frac{4}{3} v'^{3/2}$. We verify (8.4) with $\xi = -2u''$, $\lambda^1 = -1$, $\lambda^2 = 0$. (As there is just one equation, we omit the index κ.)

Proof. Let

$$\mathbf{X} = \mathrm{pr}^{(n)}\, \mathbf{v}_Q + \xi \tilde{D}_x \tag{8.5}$$

be an equivalent generalized vector field on J^n, where $\tilde{D}_x$ denotes the nth order truncation of the total derivative; cf. (4.5). The problem is: when does $\mathbf{X}$ restrict to an ordinary vector field on the submanifold $\mathscr{R}$? This means that (i) after using the system and its prolongations, there can be no $(n+1)$st order derivatives remaining in the formula for $\mathbf{X}$, and (ii) $\mathbf{X}$ is tangent to $\mathscr{R}$. For the proof of the lemma, we just need to analyze the first of these conditions.

The top order terms in the vector field (8.5) take the form

$$\mathbf{X} = \sum_{\alpha=1}^{q} \left(\sum_{\beta=1}^{q} u_{n+1}^{\beta} \frac{\partial Q^{\alpha}}{\partial u_{x}^{\beta}} + \xi u_{n+1}^{\alpha} + O(n) \right) \frac{\partial}{\partial u_{n}^{\alpha}} + \cdots. \tag{8.6}$$

Since

$$D_x \Delta_\nu = \sum_{\alpha=1}^{q} u_{n+1}^{\alpha} \frac{\partial \Delta_\nu}{\partial u_n^{\alpha}} + O(n), \tag{8.7}$$

a simple linear algebra lemma shows that we can use the equations $D_x \Delta_\nu = 0$ to eliminate the derivatives of order $n+1$ in (8.6) if and only if Eqs. (8.4) hold.

If the contact conditions (8.4) hold, then $\mathbf{X}$ can be identified with a genuine vector field on J^n. The symmetry conditions

$$\mathrm{pr}^{(n)}\, \mathbf{v}(\Delta_\kappa) = 0, \qquad \text{on} \quad \Delta_\nu = 0,\, D_x \Delta_\nu = 0, \quad \kappa, \nu = 1, \ldots, r, \tag{8.8}$$

now imply (ii), i.e., that $\mathbf{X}(\Delta_k) = 0$, on (8.1).

THEOREM 15. *Let $\mathscr{R}$ be a top order maximal rank system of ordinary differential equations of order $n \geqslant 2$. Then every internal symmetry is equivalent to a generalized symmetry of order $n-1$.*

Proof. Although the proof is considerably simplified by using the solved form (8.3) for the system, it is better to work directly with the general expressions (8.1) for the system as the proof then more readily generalizes to the case of partial differential equations. Also, in anticipation of later developments, we conduct a more thorough investigations of the contact conditions than would be required just to prove the theorem as stated. Let

$$\mathbf{X} = \xi \frac{\partial}{\partial x} + \sum_{k=0}^{n} \sum_{\alpha=1}^{q} \varphi_k^{\alpha} \frac{\partial}{\partial u_k^{\alpha}} \tag{8.9}$$

be any internal symmetry. We begin by extending $\mathbf{X}$ to a vector field

$$\mathbf{Y} = \hat{\xi} \frac{\partial}{\partial x} + \sum_{k=0}^{n} \sum_{\alpha=1}^{q} \hat{\varphi}_k^{\alpha} \frac{\partial}{\partial u_k^{\alpha}} \tag{8.10}$$

defined (locally) in a neighborhood of the submanifold $\mathscr{R}$. This requires that the coefficients $\hat{\xi}$, $\hat{\varphi}_k^{\alpha}$ of the extension $\mathbf{Y}$ agree with the corresponding coefficients ξ, φ_k^{α} of $\mathbf{X}$ when restricted to $\mathscr{R}$. Apart from this, there are, to begin with, no restrictions on the coefficients of $\mathbf{Y}$, and we make use of this flexibility later in the proof. The associated characteristic

$$\hat{Q}^\alpha = \hat{\varphi}^\alpha - \hat{\xi} u_1^\alpha \tag{8.11}$$

also agrees with the characteristic Q of $\mathbf{X}$ on $\mathscr{R}$.

The contact ideal $I^{(n)}$ is generated by the one-forms

$$\theta_k^\alpha = du_k^\alpha - u_{k+1}^\alpha\, dx, \qquad k = 0, 1, \ldots, n-1, \quad \alpha = 1, \ldots, q.$$

The condition (5.4) that $\mathbf{X}$ be an internal symmetry becomes

$$\mathbf{X}(I^{(n)} \mid \mathscr{R}) = \mathbf{Y}(I^{(n)}) \mid \mathscr{R} \subset I^{(n)} \mid \mathscr{R}, \tag{8.12}$$

which requires that the Lie derivative of any contact form θ_k^α with respect to the vector field (8.10) equal a linear combination of contact forms when restricted (pulled back) to the submanifold $\mathscr{R}$.

Now, in general, given a one-form on $J^{(n)}$,

$$\omega = \sigma\, dx + \sum_{k=0}^{n} \sum_{\alpha=1}^{q} \beta_k^\alpha\, du_k^\alpha,$$

when does it pull back to 0 on $\mathscr{R}$, i.e., when is $\omega \mid \mathscr{R} = 0$? There are two ways in which this could happen: either the coefficients σ, β_k^α vanish on $\mathscr{R}$, or ω is a linear combination of the differentials $d\Delta_\kappa$ of the defining equations (8.1). Thus we have

$$\omega \mid \mathscr{R} = 0 \qquad \text{if and only if} \quad \omega = \sum_{\kappa=1}^{r} \lambda^\kappa\, d\Delta_\kappa \text{ on } \mathscr{R}, \tag{8.13}$$

where by the phrase "on $\mathscr{R}$" in (8.13) we mean that the individual coefficients of the basis one-forms dx, du_k^α of T^*J^n in the equation must agree when restricted to the submanifold $\mathscr{R}$. (This is different from saying the pull-backs agree on $\mathscr{R}$ since we are maintaining the linear independence of all the basis one-forms; the fact that they are no longer linearly independent when pulled back to $\mathscr{R}$ has been already taken care of by the introduction of the coefficients λ^κ in the equation.)

Thus, combining (8.12) and (8.13), we see that the internal contact conditions are equivalent to the conditions

$$\mathbf{Y}[\theta_k^\alpha] = \sum_{\beta=1}^{q} \sum_{j=0}^{n-1} \mu_{k,\beta}^{\alpha,j} \theta_j^\beta + \sum_{\kappa=1}^{r} \lambda_k^{\alpha,\kappa}\, d\Delta_\kappa \qquad \text{on } \mathscr{R},$$
$$k = 0, 1, \ldots, n-1, \quad \alpha = 1, \ldots, q, \tag{8.14}$$

where the $\mu_{k,\beta}^{\alpha,j}$ and $\lambda_k^{\alpha,\kappa}$ are unspecified functions. The left-hand side of (8.14) takes the form

$$\begin{aligned}\mathbf{Y}[\theta_k^\alpha] &= d\hat{\varphi}_k^\alpha - \hat{\varphi}_{k+1}^\alpha\, dx - u_{k+1}^\alpha\, d\hat{\xi} \\ &= [\hat{D}_x \hat{\varphi}_k^\alpha - u_{k+1}^\alpha \hat{D}_x \hat{\xi} - \hat{\varphi}_{k+1}^\alpha]\, dx \\ &\quad + \sum_{\beta=1}^{q} \left(\frac{\partial \hat{\varphi}_k^\alpha}{\partial u_n^\beta} - u_{k+1}^\alpha \frac{\partial \hat{\xi}}{\partial u_n^\beta} \right) du_n^\beta \qquad \operatorname{mod} I^{(n)} | \mathscr{R}, \end{aligned} \tag{8.15}$$

where

$$\hat{D}_x = \frac{\partial}{\partial x} + \sum_{k=0}^{n-1} \sum_{\alpha=1}^{q} u_{k+1}^\alpha \frac{\partial}{\partial u_k^\alpha} \tag{8.16}$$

denotes the $(n-1)$st order truncation of the total derivative. On the other hand

$$d\Delta_\kappa = \hat{D}_x \Delta_\kappa\, dx + \sum_{\beta=1}^{q} \frac{\partial \Delta_\kappa}{\partial u_n^\beta} du_n^\beta \qquad \operatorname{mod} I^{(n)} | \mathscr{R}. \tag{8.17}$$

Substituting (8.16), (8.17) into (8.14) we conclude that, for each $k = 1, ..., n-1$, $\alpha = 1, ..., q$,

$$\hat{\varphi}_{k+1}^\alpha = \hat{D}_x \hat{\varphi}_k^\alpha - u_{k+1}^\alpha \hat{D}_x \hat{\xi} - \sum_{\kappa=1}^{r} \lambda_k^{\alpha,\kappa} \hat{D}_x \Delta_\kappa \qquad \text{on } \mathscr{R}, \tag{8.18}$$

and

$$\frac{\partial \hat{\varphi}_k^\alpha}{\partial u_n^\beta} - u_{k+1}^\alpha \frac{\partial \hat{\xi}}{\partial u_n^\beta} = \sum_{\kappa=1}^{r} \lambda_k^{\alpha,\kappa} \frac{\partial \Delta_k}{\partial u_n^\beta} \qquad \text{on } \mathscr{R}. \tag{8.19}$$

Multiplying (8.19) by u_{n+1}^β and summing over β, and then subtracting the result from (8.18), allow us to conclude that

$$\hat{\varphi}_{k+1}^\alpha = D_x \hat{\varphi}_k^\alpha - u_{k+1}^\alpha D_x \hat{\xi} - \sum_{\kappa=1}^{r} \lambda_k^{\alpha,\kappa} D_x \Delta_x \qquad \text{on } \mathscr{R},$$

or, equivalently, that the coefficients of $\mathbf{Y}$ are connected by usual prolongation formula (2.7) when restricted to the prolonged equation

$$\hat{\varphi}_{k+1}^\alpha = D_x \hat{\varphi}_k^\alpha - u_{k+1}^\alpha D_x \hat{\xi} \qquad \text{on } \operatorname{pr}^{(1)} \mathscr{R}.$$

Using (8.11) and a simple induction, this implies that

$$\hat{\varphi}_k^\alpha = D_x^k \hat{Q}^\alpha + u_{k+1}^\alpha \hat{\xi} \qquad \text{on } \operatorname{pr}^{(k)} \mathscr{R}, \tag{8.20}$$

which agrees with the restriction of the prolongation formula (4.5).

Given a function $f(x, u^{(n)})$ on $J^{(n)}$, let

$$d_n f = \sum_{\beta=1}^{q} \frac{\partial f}{\partial u_n^\beta} du_n^\beta$$

denote its exterior derivative with respect to the top order derivative variables. Then, for $k < n-1$, (8.19) implies that the one-form $d_n[\hat{\varphi}_k^\alpha - u_{k+1}^\alpha \hat{\xi}]$ must vanish when pulled back to $\mathscr{R}$:

$$d_n[\hat{\varphi}_k^\alpha - u_{k+1}^\alpha \hat{\xi}] \,|\, \mathscr{R} = 0. \tag{8.21}$$

Condition (8.21) implies that the expression $\hat{\varphi}_k^\alpha - u_{k+1}^\alpha \hat{\xi}$ is (locally) independent of the top order derivatives u_n^β when restricted to $\mathscr{R}$. Thus, by possibly rechoosing the coefficient $\hat{\varphi}_k^\alpha$ of our extension $\mathbf{Y}$, we can assume that the $\hat{\varphi}_k^\alpha$ satisfy

$$\frac{\partial}{\partial u_n^\beta}[\hat{\varphi}_k^\alpha - u_{k+1}^\alpha \hat{\xi}] = 0 \qquad \text{on } \mathscr{R}. \tag{8.22}$$

Substituting (8.20) into (8.22) yields

$$\frac{\partial}{\partial u_n^\beta}\{D_x^k \hat{Q}^\alpha\} = 0 \qquad \text{on } \mathrm{pr}^{(k)} \mathscr{R}, \qquad 0 \leqslant k < n-1. \tag{8.23}$$

In particular, the case $k = 0$ of (8.23) implies that, on $\mathscr{R}$, the characteristic $\hat{Q}$ does not depend on nth order derivatives of the u's, which proves Theorem 14. (Note that this conclusion only requires that $\mathbf{Y}$ preserve the first order contact forms θ^α on $\mathscr{R}$.)

In the case of Bäcklund's theorem, there is no restriction to an equation manifold, and the conditions (8.23) immediately imply that the characteristic $\hat{Q}$ depends on at most first order derivatives. (This is a simple consequence of the simple result that the total derivative $D_x Q$ of a function Q has order m if and only if Q has order $m-1$.) However, the fact that we are restricting to a submanifold $\mathscr{R}$ makes this result no longer valid. Therefore, to conclude that every internal symmetry of a system is equivalent to a *first* order generalized symmetry necessitates that we impose some additional conditions on the system. This serves to motivate the following definition.

DEFINITION 16. An nth order system of differential equations $\mathscr{R}$ is said to have the *descent property* if the only (smooth) functions $Q(x, u^{(n-1)})$ of order $n-1$ whose total derivative, when restricted to the system, i.e., $D_x Q \,|\, \mathscr{R}$, also has order $n-1$ are the functions $Q(x, u^{(n-2)})$ of order $n-2$.

In other words, if the system $\mathscr{R}$ has the descent property, and $Q(x, u^{(n-1)})$ is such that $D_x Q \mid \mathscr{R}$ does not depend on derivatives of order n, then $Q = Q(x, u^{(n-2)})$ cannot depend on derivatives of order $n-1$. As we remarked above, any open subset of J^n has the descent property. On the other hand, a normal system of ordinary differential equations does *not* have the descent property. Indeed, since we can replace *all* nth order derivatives of the u's by the system, the total derivative of *any* function $Q(x, u^{(n-1)})$, when restricted to the system, still has order at most $n-1$. While not every system of ordinary differential equations has the descent property, there is, however, a significant class of underdetermined systems which have the descent property.

LEMMA 17. *Consider an underdetermined system of ordinary differential equations of the form*

$$u_n^\kappa = F^\kappa(x, u^{(n-1)}, u_n^{r+1}, ..., u_n^q), \qquad \kappa = 1, ..., r. \tag{8.24}$$

If the $\frac{1}{2}(q-r)(q-r+1) \times r$ tangential Hessian *matrix with entries*

$$\frac{\partial^2 F^\kappa}{\partial u_n^\lambda \, \partial u_n^\mu} \qquad \textit{for} \quad r+1 \leqslant \lambda \leqslant \mu \leqslant q \qquad \textit{and} \qquad 1 \leqslant \kappa \leqslant r \tag{8.25}$$

has (maximal) rank r, then the system has the descent property. (In (8.25), the rows of the matrix are indexed by the pair (λ, μ), and the columns by κ.)

Proof. Let $Q = Q(x, u^{(n-1)})$. Then, on the system (8.24),

$$\begin{aligned} D_x Q &= \sum_{\alpha=1}^{q} u_n^\alpha \frac{\partial Q}{\partial u_{n-1}^\alpha} + O(n-1) \\ &= \sum_{\kappa=1}^{r} F^\kappa \frac{\partial Q}{\partial u_{n-1}^\kappa} + \sum_{\lambda=r+1}^{q} u_n^\lambda \frac{\partial Q}{\partial u_{n-1}^\lambda} + O(n-1). \end{aligned}$$

Now, if $D_x Q \mid \mathscr{R}$ is independent of nth order derivatives of u, then

$$0 = \frac{\partial(D_x Q)}{\partial u_n^\lambda} = \sum_{\kappa=1}^{r} \frac{\partial F^\kappa}{\partial u_n^\lambda} \frac{\partial Q}{\partial u_{n-1}^\kappa} + \frac{\partial Q}{\partial u_{n-1}^\lambda}, \qquad \lambda = r+1, ..., q. \tag{8.26}$$

Moreover,

$$0 = \frac{\partial^2(D_x Q)}{\partial u_n^\lambda \, \partial u_n^\mu} = \sum_{\kappa=1}^{r} \frac{\partial^2 F^\kappa}{\partial u_n^\lambda \, \partial u_n^\mu} \frac{\partial Q}{\partial u_{n-1}^\kappa}, \qquad \lambda, \mu = r+1, ..., q. \tag{8.27}$$

Now, the rank assumption on the tangential Hessian implies that $\partial Q / \partial u_{n-1}^\kappa = 0$ for $\kappa = 1, ..., r$, and hence, in view of (8.26), $\partial Q / \partial u_{n-1}^\lambda = 0$ for

$\lambda = r+1, \ldots, q$, also, which proves that Q has order $n-2$, completing the proof.

Note that even if the Hessian matrix does not have the required rank, the system may still have the descent property owing to further integrability conditions among the (overdetermined) system of first order partial differential equations (8.26), (8.27), which force Q to be independent of all derivatives of order $n-1$. However, the general necessary and sufficient conditions for a system to have the descent property appear to be rather more complicated to determine, and we will not pursue them any further here.

LEMMA 18. *Let $\mathscr{R}$ be an nth order system of ordinary differential equations having the descent property. Then, given $0 \leqslant k < n-1$, any $(n-1)$st order function $Q(x, u^{(n-1)})$ which satisfies*

$$D_x^i Q \mid \mathscr{R} \text{ has order } n-1 \qquad \text{for} \quad i = 0, \ldots, k$$

necessarily has order $n-k-1$, i.e., $Q = Q(x, u^{(n-k-1)})$.

Proof. The case $k=1$ is just the definition of the descent property. For $k>1$, we work by induction, and it suffices to note that if Q has order $m < n-1$, then

$$D_x Q = \sum_{\alpha=1}^{q} u_{m+1}^{\alpha} \frac{\partial Q}{\partial u_m^{\alpha}} + O(m).$$

Moreover, since the system is of top order maximal rank, there are no relations among the derivatives of order $<n$, and hence $D_x Q \mid \mathscr{R}$ has order m if and only if Q has order $m-1$. The details are left to the reader.

DEFINITION 19. A first or second order system of differential equations is said to be *fully top order* if it has top order maximal rank. An nth order system of differential equations for $n \geqslant 3$ is said to be *fully top order* if it has top order maximal rank and also has the descent property.

The second order case is singled out in view of Theorem 15, which already implies that any internal symmetry of a top order maximal rank second order system is equivalent to a first order generalized symmetry. (The first order case is, of course, trivial.) The higher order case follows from applying Lemma 18 to the conditions (8.23). We therefore have proven:

THEOREM 20. *Let $\mathscr{R}$ be a fully top order system of ordinary differential equations. Then every internal symmetry is equivalent to a first order* *generalized symmetry which satisfies the contact conditions* (8.4) *on the equation manifold* $\mathscr{R}$.

Next, in order to generalize Theorem 7, we analyze the contact conditions (8.4) for a first order generalized symmetry in more detail. Given a decomposition of the dependent variables u into tangential and normal components, the contact conditions (8.4) correspondingly split into two subsystems. If $u^1, ..., u^r$ are the normal directions, and $u^{r+1}, ..., u^q$ the tangential directions, then the subsystem of (8.4) corresponding to the range of indices $\alpha = 1, ..., r$, $\beta = 1, ..., q$ (i.e., the equations for the normal components of the characteristic Q) will be referred to as the *normal contact conditions*, while the remaining subsystem, corresponding to $\alpha = r+1, ..., q$, $\beta = 1, ..., q$, is the *tangential contact conditions*. It turns out that, given the tangential contact conditions, the normal contact conditions are automatic consequences of the symmetry conditions. Therefore, only the tangential contact conditions impose restrictions on the first order generalized symmetry in order that it determine an internal symmetry.

THEOREM 21. *Let* $\mathscr{R}$ *be a top order maximal rank system of ordinary differential equations. Then every first order generalized symmetry which satisfies the tangential contact conditions, i.e.,* (8.4) *for* $\alpha = r+1, ..., q$, $\beta = 1, ..., q$, *is equivalent to an internal symmetry.*

In other words, there is a one-to-one correspondence between internal symmetries and first order generalized symmetries which satisfy the contact conditions in the tangential components Q^α of the characteristic. There are two extreme cases. First, if there are no equations, i.e., the equation submanifold $\mathscr{R}$ is an open subset of J^n, then every direction is tangent, and the tangential contact conditions (8.4) reduce to the usual contact conditions (4.6) for a contact transformation. In this case, an "internal symetry of J^n" is just an ordinary contact transformation, and Theorem 20 reduces to Theorem 3. In this sense, we are justified in viewing this result as a generalization of Bäcklund's theorem to systems of (ordinary) differential equations.

At the other extreme, consider a normal system of ordinary differential equations, so $r = q$ and the Jacobian matrix (8.2) has rank q. In this case, there are no tangential directions, and so every first order generalized symmetry determines an internal symmetry. In this case, the contact conditions (8.4) form a system of q^2 equations, with $q^2 + 1$ undetermined functions ξ, λ_κ^α, $\alpha, \kappa = 1, ..., q$. Because K has maximal rank, for each value of the function ξ we can prescribe q^2 additional functions λ_κ^α, $\alpha, \kappa = 1, ..., q$, so as to satisfy Eqs. (8.4). Therefore, we recover the result (Theorem 6) that for a determined system of ordinary differential equations, every first order generalized symmetry corresponds to an internal symmetry, and moreover any two such internal symmetries differ by a trivial internal symmetry $\xi \mathbf{d}_x$.

In the codimension 1 case discussed in Section 7, we have $r = q - 1$ and K has rank $q - 1$. There is just one tangential direction, say u^q, and so the tangential contact conditions (8.4) for $\alpha = q$ form a system of q equations with precisely q undetermined functions ξ, λ_κ^q, $\kappa = 1, ..., q - 1$. Therefore, for each first order generalized symmetry, we can uniquely determine the functions ξ, λ_κ^q, $\kappa = 1, ..., q - 1$, so as to satisfy the tangential contact conditions; the remaining normal contact conditions will then follow automatically from the symmetry conditions. We therefore recover our earlier result (Theorem 7) that there is a one-to-one correspondence between first order generalized symmetries and internal symmetries for codimension one systems. For systems of higher codimension, the tangential contact conditions impose additional constraints the first order generalized symmetry must satisfy in order that it correspond to an internal symmetry. In general, given a system of rank r (equivalently, codimension $q - r$), the tangential contact conditions (8.4) form a system of $q(q - r)$ equations containing $r(q - r) + 1$ undetermined functions ξ, λ_κ^α, $\alpha = 1, ..., r$, $\kappa = 1, ..., q - r$. Therefore there will be $q(q - r) - r(q - r) - 1 = (q - r)^2 - 1$ additional equations a first order generalized symmetry must satisfy in order that it correspond to an internal symmetry. For instance, any symmetry of a codimension 2 system (i.e., one of rank $q - 2$) must satisfy three additional constraints for it to be an internal symmetry. For example, in the case of an equation of the form (7.21) it is easy to check that the constraints imposed by (8.4) are precisely (7.22).

Proof. Suppose we have a generalized symmetry with first order characteristic Q. According to Lemma 14, if the characteristic Q satisfies *all* of the contact conditions (8.4), then $\mathbf{v}_Q$ is equivalent to an internal symmetry. We have to prove that it is enough for the Q^α, $\alpha = r + 1, ..., q$, to satisfy the tangential contact conditions, and to this end we prove that the top order symmetry conditions and the tangential contact conditions imply the normal contact conditions, i.e., those for Q^α, $\alpha = 1, ..., r$.

Let $\mathbf{v}_Q$ be a first order generalized symmetry of the system (8.1), written in evolutionary form. The symmetry condition (2.8) is equivalent to the equations

$$\mathrm{pr}^{(n)} \mathbf{v}_Q[\Delta_\kappa] = \sum_{\nu=1}^{r} \sigma_\kappa^\nu D_x \Delta_\nu + \sum_{\nu=1}^{r} \rho_\kappa^\nu \Delta_\nu, \qquad \kappa = 1, ..., r. \tag{8.28}$$

(Note that the σ's and ρ's can be taken to depend on at most nth order derivatives since the left-hand side is linear in the u_{n+1}^α.) Using the prolongation formula (4.4), the terms of order $n+1$ on the left-hand side of (8.28) are found to be

$$\begin{aligned}\mathrm{pr}^{(n)}\mathbf{v}_Q[\Delta_\kappa] &= \sum_{k=0}^{n}\sum_{\alpha=1}^{q} D_x^k Q^\alpha \frac{\partial \Delta_\kappa}{\partial u_k^\alpha} = \sum_{\alpha=1}^{q} D_x^n Q^\alpha \frac{\partial \Delta_k}{\partial u_n^\alpha} + O(n)\\ &= \sum_{\alpha,\beta=1}^{q} u_{n+1}^\beta \frac{\partial Q^\alpha}{\partial u_x^\beta}\frac{\partial \Delta_\kappa}{\partial u_n^\alpha} + O(n). \end{aligned} \tag{8.29}$$

Thus, the top order symmetry conditions are

$$\sum_{\alpha=1}^{q} \frac{\partial \Delta_\kappa}{\partial u_n^\alpha}\frac{\partial Q^\alpha}{\partial u_x^\beta} = \sum_{\nu=1}^{r} \sigma_\kappa^\nu \frac{\partial \Delta_\nu}{\partial u_n^\beta}, \qquad \beta = 1, ..., q, \quad \kappa = 1, ..., r. \tag{8.30}$$

To continue, we introduce some simplifying matrix notation. Along with the $r\times q$ Jacobian matrix K given in (8.2), we introduce the $q\times q$ matrix

$$R = \left(\frac{\partial Q^\alpha}{\partial u_x^\beta}\right), \tag{8.31}$$

and the $q\times r$ and $r\times r$ matrices

$$L = (\lambda_\kappa^\alpha), \qquad S = (\sigma_\kappa^\nu). \tag{8.32}$$

Then the contact conditions (8.4) have the matrix form

$$R + \xi I_q = LK, \tag{8.33}$$

where I_q denotes the $q\times q$ identity matrix. The top order symmetry conditions (8.30) have the matrix form

$$KR = SK. \tag{8.34}$$

Note that L, S, and the scalar function ξ are undetermined. Now assume that we have ordered the variables so that $u^1, ..., u^r$ are normal directions, and $u^{r+1}, ..., u^q$ are the tangential directions. We split the matrices accordingly,

$$K = (K_1, K_2), \qquad R = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}, \qquad L = \begin{pmatrix} L_1 \\ L_3 \end{pmatrix},$$

where K_1, R_1, L_1, S are $r\times r$; K_2, R_2 are $r\times(q-r)$; R_3, L_3 are $(q-r)\times r$; and R_4 is $(q-r)\times(q-r)$. Also, K_1 is invertible. The symmetry conditions (8.34) are

$$K_1R_1 + K_2R_3 = SK_1, \qquad K_1R_2 + K_2R_4 = SK_2. \tag{8.35}$$

Eliminating S we obtain the conditions

$$K_1 R_2 + K_2 R_4 = (K_1 R_1 + K_2 R_3) K_1^{-1} K_2. \tag{8.36}$$

The normal contact conditions from (8.33) are

$$R_1 + \xi I_r = L_1 K_1, \qquad R_2 = L_1 K_2. \tag{8.37}$$

Solving for R_2 by eliminating L_1 gives

$$R_2 = \xi K_1^{-1} K_2 + R_1 K_1^{-1} K_2. \tag{8.38}$$

The tangential contact conditions from (8.33) are

$$R_3 = L_3 K_1, \qquad R_4 + \xi I_{q-r} = L_3 K_2. \tag{8.39}$$

Solving for R_4 by eliminating L_3 gives

$$R_4 = -\xi I_{q-r} + R_3 K_1^{-1} K_2. \tag{8.40}$$

Now if we multiply the normal contact conditions (8.38) by K_1 and add in the tangential contact conditions (8.40) multiplied by K_2, we get exactly the symmetry conditions (8.36). Since K_1 is invertible, this implies that the symmetry conditions plus the tangential contact conditions are enough to give the normal contact conditions. (Conversely, if we have both the normal and tangential contact conditions, these imply that the top order symmetry conditions are satisfied.) This completes the proof of Theorem 20.

9. Contact Conditions for Partial Differential Equations

For systems of partial differential equations, similar considerations apply, although the development is complicated by the presence of several independent variables. Moreover, the conditions for a true internal symmetry (i.e., one which does not come from an external symmetry) are considerably more restrictive than those for ordinary differential equations.

THEOREM 22. *Let $\mathscr{R}$ be an nth order system of differential equations of top order maximal rank. Then every internal symmetry is equivalent to an $(n-1)$st order generalized symmetry.*

The proof is very similar in outline to that of Theorem 15. Let $\mathbf{X}$ (cf. (5.2)) be the infinitesimal generator of an internal symmetry group. Extend $\mathbf{X}$ to a vector field $\mathbf{Y}$ off the submanifold $\mathscr{R}$, and let $\hat{Q}$ be its characteristic. Analysis of the contact conditions (5.4) as before will prove that the coefficients $\hat{\xi}^i$, $\hat{\varphi}^\alpha_J$ of $\mathbf{Y}$ satisfy the usual prolongation formula, modulo the system and its prolongations,

$$\hat{\varphi}^\alpha_{J,i} = D_i\hat{\varphi}^\alpha_J - \sum_{k=1}^{p} D_i\hat{\xi}^k u^\alpha_{J,k}, \qquad \#J \leqslant n-1, \qquad \text{on } \mathrm{pr}^{(1)}\mathcal{R}, \tag{9.1}$$

and the additional constraints

$$\frac{\partial\hat{\varphi}^\alpha_J}{\partial u^\beta_K} - \sum_{i=1}^{p} u^\alpha_{K,i}\frac{\partial\hat{\xi}^i}{\partial u^\beta_K} = \sum_{\kappa=1}^{r} \lambda^{\alpha,\kappa}_J \frac{\partial\Delta_\kappa}{\partial u^\beta_K}, \qquad \#K = n, \qquad \text{on } \mathcal{R}. \tag{9.2}$$

By induction, (9.1) implies that

$$\hat{\varphi}^\alpha_J = D_J\hat{Q}^\alpha + \sum_{i=1}^{p} \hat{\xi}^i u^\alpha_{J,i}, \qquad k = \#J \leqslant n, \quad \text{on } \mathrm{pr}^{(k)}\mathcal{R}.$$

Further, (9.2) then implies

$$\frac{\partial}{\partial u^\alpha_J}(D_K\hat{Q}^\alpha) = 0 \qquad \text{on} \quad \mathrm{pr}^{(k)}\mathcal{R},$$

$$\alpha = 1, ..., q, \ \#J = N, \ 0 \leqslant k = \#K \leqslant n-2. \tag{9.3}$$

In particular, setting $k = 0$ suffices to prove Theorem 22.

DEFINITION 23. An nth order system of partial differential equations $\mathcal{R}$ is said to have *the descent property* if the only functions $Q(x, u^{(n-1)})$ of order $n-1$, *all* of whose total derivatives $D_iQ \mid \mathcal{R}$, $i = 1, ..., p$, restricted to the system, have order $n-1$, are functions $Q(x, u^{(n-2)})$ of order $n-2$.

A system of partial differential equations is called *normal* if can be placed in *Kovalevskaya form*

$$\frac{\partial^n u^\alpha}{\partial t^n} = F^\alpha(y, t, \widetilde{u^{(n)}}), \qquad \alpha = 1, ..., q, \tag{9.4}$$

by introducing appropriate local coordinates $(t, y^1, ..., y^{p-1})$ on the base. In (9.4), the right-hand sides may depend on all derivatives of orders $\leqslant n$ except those explicitly appearing on the left-hand sides. In analytic case, the Kovalevskaya form (9.4) is required for the classical Cauchy–Kovalevskaya Existence Theorem to hold; see [15; Sect. 2.6].

LEMMA 24. *A normal system of partial differential equations in $p > 1$ independent variables of order $n \geqslant 2$ has the descent property.*

Proof. Let $Q = Q(x, u^{(n-1)})$. Note that the total derivatives $D_{y^i}Q$ in the y-directions do not depend on the derivatives $\partial^n u^\alpha/\partial t^n$ appearing on the left-hand side of (9.4). Therefore, given any multi-index K of order $\# K = n-1$, we have

$$0 = \frac{\partial}{\partial u^\alpha_{K,i}} D_{y^i} Q = \frac{\partial Q}{\partial u^\alpha_K} \qquad \text{on } \mathscr{R}, \qquad \alpha = 1, ..., q, \quad \# K = n-1.$$

Therefore $Q = Q(x, u^{(n-2)})$ has order $n-2$, and the proof is complete.

Inspection of the proof of Lemma 23 shows that, unlike systems of ordinary differential equations, only rather overdetermined system of partial differential equations will fail to have the descent property. Definition 19 of *fully top order* holds word for word in the case of systems of partial differential equations. Applying the direct analogue of Lemma 17 to conditions (9.3), we immediately deduce that every internal symmetry of a fully top order system of partial differential equations is equivalent to a first order generalized symmetry. As for the converse, we need to determine the appropriate contact conditions.

Let $\mathbf{v}_Q$ be a first order evolutionary vector field. Then, for $\mathbf{v}_Q$ to be equivalent to an internal symmetry, there must exist functions $\xi^1, ..., \xi^p$ such that the generalized vector field $\operatorname{pr}^{(n)} \mathbf{v}_Q + \sum \xi^j D_j$, when restricted to the equation manifold $\mathscr{R}$, can only depend on at most nth order derivatives of u and hence is an ordinary vector field on $\mathscr{R}$. The only way that higher order derivatives could appear is in the coefficient of $\partial/\partial u^\alpha_J$, with $\# J = n$, which is

$$D_J Q^\alpha + \sum_j u^\alpha_{J,j} \xi^j = \sum_{j,\beta} M^{\alpha,j}_\beta u^\beta_{J,j} + O(n), \qquad \# J = n, \tag{9.5}$$

where we define

$$M^{\alpha,j}_\beta = \frac{\partial Q^\alpha}{\partial u^\beta_j} + \xi^j \delta^\alpha_\beta = \frac{\partial \varphi^\alpha}{\partial u^\beta_j} - \sum_i u^\alpha_i \frac{\partial \xi^i}{\partial u^\beta_j}. \tag{9.6}$$

On the other hand, the total derivatives of the system (2.1) have the form

$$0 = D_j \Delta_\kappa = \sum_K \partial^K_\beta \Delta_\kappa u^\beta_{K,j} + O(n), \tag{9.7}$$

where

$$\partial^K_\beta \Delta_\kappa = \frac{\partial \Delta_\kappa}{\partial u^\beta_K}. \tag{9.8}$$

In order that (9.5) only depend on nth order derivatives when (9.7) holds, we must have the contact conditions

$$M_{\beta}^{\alpha,(j}\delta_J^{K)} = \sum_{\kappa} \lambda_J^{\alpha\kappa,(j}\,\partial_{\beta}^{K)}\,\Delta_{\kappa}, \tag{9.9}$$

holding on the equation manifold $\mathscr{R}$, for all $\alpha, \beta = 1, ..., q$, and all symmetric multi-indices J, K of order n. Here the parentheses denote symmetrization on the $(n+1)$st order multi-index (j, K).

To express (9.9) in a more transparent form, it is convenient to introduce some auxiliary variables $\zeta = (\zeta_1, ..., \zeta_p)$. Define the matrices

$$\mathbf{K}(\zeta) = \Big(\sum_{\# J = n} \frac{\partial \Delta_{\kappa}}{\partial u_J^{\beta}} \zeta_J \Big), \qquad \mathbf{M}(\zeta) = \Big(\sum_{j=1}^{p} M_{\beta}^{\alpha, j} \zeta_j \Big), \tag{9.10}$$

where, for $J = (j_1, ..., j_n)$, we set $\zeta_J = \zeta_{j_1}\zeta_{j_2}\cdots\zeta_{j_n}$. The matrix $\mathbf{K}(\zeta)$ is an $r \times q$ matrix of n^{th} degree homogeneous polynomials in the ζ's. It appears in the definition of the classical characteristic directions of the system of partial differential equations (2.1), where the ζ's are interpreted as cotangent bundle coordinates on the base X. The matrix $\mathbf{M}(\zeta)$ is a $q \times q$ matrix of linear functions of the ζ's. According to Theorem 3 and (9.6), an internal symmetry will extend to an external symmetry if and only if the corresponding matrix $\mathbf{M}(\zeta)$ is identically zero. Internal symmetries which do not extend to external symmetries, i.e., ones for which $\mathbf{M}(\zeta) \not\equiv 0$, will be called *non-extendable*, and these are, in a sense, the only "true" internal symmetries. For each nth order multi-index we also set

$$\mathbf{L}_J(\zeta) = \Big(\sum_{j=1}^{p} \lambda_J^{\alpha\kappa, j} \zeta_j \Big), \qquad \# J = n, \tag{9.11}$$

so $\mathbf{L}_J$ is a $q \times r$ matrix of linear polynomials in the ζ's. The contact conditions (9.9) can then be written the simple matrix form

$$\zeta_J \mathbf{M}(\zeta) = \mathbf{L}_J(\zeta) \cdot \mathbf{K}(\zeta).$$

In other words, for each multi-index J we must find a matrix $\mathbf{L}_J(\zeta)$ of linear functions of the ζ's such that $\mathbf{L}_J(\zeta)\,\mathbf{K}(\zeta)$ is the product of the scalar monomial ζ_J and the matrix $\mathbf{M}(\zeta)$. We have thus proven the following characterization of first order generalized symmetries which are equivalent to internal symmetries, analogous to Lemma 14.

LEMMA 25. *Let* (2.1) *be a nondegenerate system of differential equations. Let* $\mathbf{v}_Q$ *be a first order generalized symmetry. Define matrices* $\mathbf{K}(\zeta)$, $\mathbf{M}(\zeta)$ *as in* (9.10). *If* $\mathbf{v}_Q$ *is equivalent to an internal symmetry of the system, then for* any *homogeneous scalar polynomial* $P(\zeta)$ *of degree* n, *there exists a matrix of linear polynomials* $\mathbf{L}_P(\zeta)$ *such that*

$$P(\zeta)\,\mathbf{M}(\zeta) = \mathbf{L}_P(\zeta)\cdot\mathbf{K}(\zeta). \tag{9.12}$$

When $p = 1$, the single variable $\zeta = \zeta_1$ factors out, and (9.12) reduces to our previous contact conditions (8.4) for ordinary differential equations. For $p > 1$, we show that (9.12) is a very restrictive condition, and, in most cases, we immediately deduce that $\mathbf{M}(\zeta) \equiv 0$, and hence every internal symmetry must be extendable to an external symmetry.

THEOREM 26. *Let* $\mathscr{R}$ *be a fully top order system of differential equations. Then every internal symmetry is equivalent to a first order generalized symmetry which satisfies the contact conditions* (9.12) *on the equation manifold* $\mathscr{R}$.

The last question to address is to find useful necessary conditions that allow a system of partial differential equations to possess a non-extendable internal symmetry. Here we explicitly assume that we are not in the ordinary differential equation case, i.e., $p > 1$. The matrix $\mathbf{K}(\zeta)$ is related to the characteristic directions for the system of partial differential equations. In particular, if $r = q$, so that we have the same number of equations as unknowns, then a complex direction ζ (which should be thought of a defining coordinates in the complexified cotangent bundle $T^*_{\mathbb{C}}X = T^*X \otimes \mathbb{C}$) determines a characteristic direction if and only if $\det \mathbf{K}(\zeta) = 0$. Such a system is called *normal* if not every direction is characteristic, i.e., $\det \mathbf{K}(\zeta) \not\equiv 0$, and, by introducing appropriate local coordinates, can be placed in Kovalevskaya form (9.4); cf. [15, Theorem 2.79]. We now easily prove that a normal system of partial differential equations of order at least 2 cannot have any internal symmetries. (Compare Stephani [17, p. 225].)

THEOREM 27. *If* $\mathscr{R}$ *is a normal system of partial differential equations in* $p > 1$ *independent variables of order* $n \geqslant 2$, *then every internal symmetry extends to an external symmetry.*

Proof. First, according to Lemma 24, any normal system is fully top order. Next, by Theorem 26, if $\mathbf{X}$ is an internal symmetry, then $\mathbf{X}$ agrees with the prolongation of a first order generalized symmetry $\mathbf{v}_Q$ on $\mathscr{R}$. By Lemma 25, Q satisfies (9.12). Using the cordinates (t, y), let $\zeta = (\tau, \eta)$ be the corresponding cotangent bundle coordinates. Since the system is in Kovalevskaya form (9.4), we have

$$\mathbf{K}(\tau, \eta) = \tau^n I + \tilde{\mathbf{K}}(\tau, \eta),$$

where $\tilde{\mathbf{K}}$ has degree at most $n-1$ in the variable τ, and I is the $q \times q$

identity matrix. Choose the particular polynomial $P(\tau, \eta) = (\eta_1)^n$ in the contact conditions (9.12), which then take the form

$$(\eta_1)^n (M_0\tau + M_1\eta_1 + \cdots + M_{n-1}\eta_{n-1}) \\ = (L_0\tau + L_1\eta_1 + \cdots + L_{n-1}\eta_{n-1})(\tau^n I + \tilde{\mathbf{K}}(\tau, \eta)).$$

The only term in this equality involving τ^{n+1} is $L_0\tau^{n+1}$, hence we must have $L_0 = 0$. Then, since $n \geq 2$, the only term involving $\eta_j \tau^n$ is $L_j \eta_j \tau^n$, which must also vanish. Therefore $\mathbf{L}(\zeta) \equiv 0$, which implies that $\mathbf{M}(\zeta) \equiv 0$, and hence the symmetry $\mathbf{v}_Q$ must be external. But this implies $\mathbf{X}$ is an external symmetry, and we are done.

An extension of this argument implies that any higher order system of partial differential equations must be "considerably" overdetermined to admit any internal symmetries. To investigate what this means, we restrict attention to the simplest case of just one unknown, $q = 1$. Consider an overdetermined system of partial differential equations

$$\Delta_\kappa(x, u^{(n)}) = 0, \qquad \kappa = 1, ..., r, \tag{9.13}$$

in $p > 1$ independent variables and $q = 1$ dependent variable. Assume, for simplicity, that the system is fully top order, so we need only consider internal symmetries which are equivalent to first order generalized symmetries. Define the characteristic ideal $\mathscr{C}_z$ at a fixed point $z \in \mathscr{R}$ to be the homogeneous polynomial ideal generated by the m complex-valued polynomials determining the entries of the $1 \times r$ characteristic matrix $\mathbf{K}(\zeta)$:

$$\mathscr{C}_z = \mathscr{C} = \langle \chi_1(\zeta), ..., \chi_r(\zeta) \rangle, \qquad \chi_\kappa(\zeta) = \sum_K \frac{\partial \Delta_\kappa}{\partial u_K} \zeta_K.$$

The characteristic variety of the system at z is, by definition, the complex algebraic variety determined by the characteristic ideal, which we can regard as a subvariety of the projectivized complex cotangent bundle

$$\mathscr{V}_z = \mathscr{V} = \{\zeta \in \mathbb{CP}^{p-1} \mid \chi(\zeta) = 0 \text{ for all } \chi \in \mathscr{C}\} \subset \pi_n^* \mathbb{P} T_{\mathbb{C}}^* X,$$

where π_n^* denotes the pull-back of the cotangent bundle of X to the cotangent space of J^n. (This reflects the fact that generally, for nonlinear systems, the characteristics depend on which point on the equation manifold $\mathscr{R}$ is being considered.)

In the case of one dependent variable, the matrix $\mathbf{M}(\zeta)$ just consists of a single linear polynomial $\mu(\zeta)$, and (9.12) becomes

$$P(\zeta)\,\mu(\zeta) = \sum_{\kappa=1}^{r} \lambda_\kappa(\zeta)\,\chi_\kappa(\zeta), \tag{9.14}$$

for some collection of linear polynomials λ_κ, which may depend on $P(\zeta)$. If the symmetry is not extendable, then $\mu(\zeta)$ is not zero and so vanishes on a hyperplane. Equation (9.14), which holds for all polynomials $P(\zeta)$, immediately implies that $\mathscr{V}$ must be contained in the hyperplane. Thus an immediate necessary condition for a system to admit a non-extendable internal symmetry is that its characteristic variety be contained in a hyperplane. This condition already considerable restricts the types of system which admit non-extendable internal symmetries. Moreover, it can be easily strengthened.

Let $\partial/\partial n$ denote the normal derivative to the given hyperplane. Explicitly, if $\mu(\zeta) = \sum \mu_i \zeta_i$, then, by definition,

$$\frac{\partial}{\partial n} = \sum_{i=1}^{p} \bar{\mu}_i \frac{\partial}{\partial \zeta_i}.$$

In particular,

$$\frac{\partial \mu}{\partial n} = \sum_{i=1}^{p} |\mu_i|^2,$$

which is a nonzero constant. Define the *normal derivative ideal*

$$\mathscr{C}_n = \left\{ \frac{\partial \chi}{\partial n} \,\middle|\, \chi \in \mathscr{C} \right\},$$

and let $\mathscr{V}_n = \{\zeta \mid \rho(\zeta) = 0 \text{ for all } \rho \in \mathscr{C}_n\}$ be its associated variety. Note that since $\mathscr{C}_n \supset \mathscr{C}$ we must have $\mathscr{V}_n \subset \mathscr{V}$. We show that, in order for a non-extendable internal symmetry to exist, this inclusion must be strict.

THEOREM 28. *If an overdetermined system of partial differential equations $\mathscr{R}$ in a single unknown admits a non-extendable internal symmetry, then for each point $z \in \mathscr{R}$,*

1. *the characteristic variety is contained in a hyperplane: $\mathscr{V} \subset \mathscr{H}$; and*
2. *the normal derivative with respect to $\mathscr{H}$ is strictly smaller: $\mathscr{V}_n \neq \mathscr{V}$.*

Proof. By the above remarks, the first condition is necessary. Applying the normal derivative to the contact condition (9.14), we find

$$P \frac{\partial \mu}{\partial n} + \frac{\partial P}{\partial n} \mu = \sum_{\kappa=1}^{r} \left(\frac{\partial \lambda_\kappa}{\partial n} \chi_\kappa + \lambda_\kappa \frac{\partial \chi_\kappa}{\partial n} \right).$$

Now on $\mathscr{V}$, $\chi_\kappa = \mu = 0$, while $\partial \mu/\partial n$ is a nonzero constant. Since this must hold for all polynomials P, we conclude that not all the normal derivatives of the χ_κ vanish on $\mathscr{V}$, as otherwise the last equation would lead to a contradiction. This completes the proof of the theorem.

For example, consider a single partial differential equation of order $n \geqslant 2$ for the function u, which is automatically fully top order. At each point, its characteristic variety $\mathscr{V}$ is specified by a single polynomial $\chi(\zeta)$. If $\mathscr{V}$ is contained in a hyperplane, then $\chi(\zeta) = (a \cdot \zeta)^n$ must be the nth power of a linear polynomial. Thus, condition 1 of Theorem 28 alone does not reproduce our earlier result on normal systems of partial differential equations in this context. However, in this case $\mathscr{V}_n = \mathscr{V}$, and hence there are still no internal symmetries, reverifying Theorem 27 in the one dependent variable case.

A similar argument proves that the nonlinear Monge–Ampère type equation

$$u_{xx}u_{yy} - u_{xy}^2 = F(x, y, u, u_x, u_y) \tag{9.15}$$

has no non-extendable internal symmetries. As before, the characteristic variety $\mathscr{V}$ is specified by the single polynomial

$$\chi(\xi, \eta) = u_{yy}\xi^2 - 2u_{xy}\xi\eta + u_{xx}\eta^2 = 0.$$

This will be contained in a hyperplane if and only if χ is a perfect square, which requires that its discriminant vanish:

$$4u_{xy}^2 - 4u_{xx}u_{yy} = 0.$$

Thus, unless F vanishes in an open set, we immediately conclude that (9.13) has no nonextendable internal symmetries. In the particular case of the equation $u_{xx}u_{yy} - u_{xy}^2 = 0$, the first condition of Theorem 27 will not eliminate the possibility of genuine internal symmetries, since

$$\chi(\xi, \eta) = (a\xi - b\eta)^2, \qquad \text{where} \quad a^2 = u_{yy},\ ab = u_{xy},\ b^2 = u_{xx},$$

and $\mathscr{V}$ is the hyperplane $a\xi = b\eta$. However, the normal derivative $\mathscr{V}_n$ is the same hyperplane, and condition 2 of Theorem 28 eliminates the possibility of non-extendable internal symmetries.

As another example, consider the overdetermined system

$$u_{xx} - \lambda u_{zz} = 0, \qquad u_{yy} - \mu u_{zz} = 0. \tag{9.16}$$

Its characteristic variety is a collection of lines given by the intersection of the two degenerate quadrics

$$\xi^2 - \lambda\zeta^2 = 0, \qquad \eta^2 - \mu\zeta^2 = 0,$$

where we view $[\xi, \eta, \zeta]$ as homogeneous coordinates on $\mathbb{CP}^2$. For $\lambda\mu \neq 0$, the characteristic variety is not contained in a hyperplane, so there are no internal symmetries. If, however, $\lambda = 0$, $\mu \neq 0$, the characteristic variety is a pair of points contained in the projective line $\xi = 0$. However, in this case $\partial/\partial n = \partial/\partial\xi$, and $\mathscr{C}_n$ is generated by the polynomials

$$\xi, \qquad \eta^2 - \mu\zeta^2.$$

Therefore $\mathscr{V}_n = \mathscr{V}$, and hence there are still no internal symmetries.

As an example of a system which does admit internal symmetries, consider the rather trivial system

$$u_{xx} + u_{xy} = 0, \qquad u_{xy} + u_{yy} = 0. \tag{9.17}$$

It is easy to see that any vector field of the form

$$\mathbf{v} = f(u_x + u_y)\frac{\partial}{\partial u},$$

where f is any scalar function, prolongs to a true internal symmetry. It is easy to check that the characteristic variety of this system, which is the line $\xi + \eta = 0$, satisfies the conditions of Theorem 28.

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