Differential Invariants of Conformal and Projective Surfaces

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November 15, 2009

Abstract

We show that, for both the conformal and projective groups, all the differential invariants of a generic surface in three-dimensional space can be written as combinations of the invariant derivatives of a single differential invariant. The proof is based on the equivariant method of moving frames.

1 Introduction

According to Cartan, the local geometry of submanifolds under transformation groups, including equivalence and symmetry properties, are entirely governed by their differential invariants. Familiar examples are curvature and torsion of a curve in three-dimensional Euclidean space, and the Gauss and mean curvatures of a surface, [11, 30, 37].

In general, given a Lie group G acting on a manifold M, we are interested in studying its induced action on submanifolds $S \subset M$ of a prescribed dimension, say $p < m = \dim M$. To this end, we prolong the group action to the submanifold jet bundles $\mathrm{J}^n = \mathrm{J}^n(M,p)$ of order $n \geq 0$, [30]. A *differential invariant* is a (perhaps locally defined) real-valued function $I \colon \mathrm{J}^n \to \mathbb{R}$ that is invariant under the prolonged group action. Any finite-dimensional Lie group action admits an infinite number of functionally independent differential invariants of progressively higher and higher order. Moreover, there always exist $p = \dim S$ linearly independent invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$. For curves, the invariant differentiation is with respect to the group-invariant arc length parameter; for Euclidean surfaces, with respect to the diagonalizing Frenet frame, [11, 22, 24, 25, 26]. The *Fundamental Basis Theorem*, first formulated by Lie, [23, p. 760], states that all the differential invariants can be generated from a finite number of low order invariants by repeated invariant differentiation. A modern statement and proof of Lie's Theorem can be found, for instance, in [30].

A basic question, then, is to find a minimal set of generating differential invariants. For curves, where $p = 1$, the answer is known: under mild restrictions on the group action (specifically transitivity and no pseudo-stabilization under prolongation), there are exactly $m-1$ generating differential invariants, and any other differential invariant is a function of the generating invariants and their successive derivatives with respect to arc length [30]. Thus, for instance, the differential invariants of a space curve $C \subset \mathbb{R}^3$ under the action of the Euclidean group SE(3), are generated by $m-1=2$ differential invariants, namely its curvature and torsion.

In [34], it was proved, surprisingly that, for generic surfaces in three-dimensional space under the action of either the Euclidean or equi-affine (volume-preserving affine) groups, a minimal system of generating differential invariants consists of a *single* differential invariant. In the Euclidean case, the mean curvature serves as a generator of the Euclidean differential invariants under invariant differentiation. In particular, an explicit, apparently new formula expressing the Gauss curvature as a rational function of derivatives of the mean curvature with respect to the Frenet frame was found. In the equi-affine case, there is a single third order differential invariant, known as the Pick invariant, [36, 37], which was shown to generate all the equi-affine differential invariants through invariant differentiation.

In this paper, we extend this research program to study the differential invariants of surfaces in $\mathbb{R}^3$ under the action of the conformal and the projective groups. Tresse classified the differential invariants in both cases in 1894, [38]. Subsequent developments in conformal geometry can be found in [2, 3, 7, 39], as well as the work of Tom Branson and collaborators surveyed in the papers in this special issue, while [1, 8, 27] present results on the projective geometry of submanifolds.

The goal of this note is to prove that, just as in the Euclidean and equi-affine cases, the differential invariants of both actions are generated by a single differential invariant though invariant differentiation with respect to the induced Frenet frame. However, lest one be tempted to naïvely generalize these results, [33] gives examples of finite-dimensional Lie groups acting on surfaces in $\mathbb{R}^3$ which require an arbitrarily large number of generating differential invariants. Our two main results are:

Theorem 1. *Every differential invariant of a generic surface $S \subset \mathbb{R}^3$ under the action of the conformal group* $\mathrm{SO}(4,1)$ *can be written in terms of a single third order invariant and its invariant derivatives.*

Theorem 2. *Every differential invariant of a generic surface $S \subset \mathbb{R}^3$ under the action of the projective group* $\mathrm{PSL}(4)$ *can be written in terms of a single fourth order invariant and its invariant derivatives.*

The proofs follow the methods developed in [34]. They are based on [6], where moving frames were introduced as equivariant maps from the manifold to the group. A recent survey of the many developments and applications this approach has entailed can be found in [32]. Further extensions are in [18, 19, 16, 17, 33].

A moving frame induces an invariantization process that maps differential functions and differential operators to differential invariants and (non-commuting) invariant differential operators. Normalized differential invariants are the invariantizations of the standard jet coordinates and are shown to generate differential invariants at each order: any differential invariant can be written as a function of the normalized invariants. This rewriting is actually a trivial replacement.

The key to the explicit, finite description of differential invariants of any order lies in the *recurrence formulae* that explicitly relate the differentiated and normalized differential invariants. Those formulae show that any differential invariant can be written in terms of a finite set of normalized differential invariants and their invariant derivatives. Combined with the replacement rule, the formulae make the rewriting process effective. Remarkably, these fundamental relations can be constructed using only the (prolonged) infinitesimal generators of the group action and the moving frame normalization equations. One does *not* need to know the explicit formulas for either the group action, or the moving frame, or even the differential invariants and invariant differential operators, in order to completely characterize generating sets of differential invariants and their syzygies. Moreover the syzygies and recurrence relations are given by rational functions and are thus amenable to algebraic algorithms and symbolic software [13, 14, 15, 19] that we have used for this paper.

2 Moving Frames and Differential Invariants

In this section we review the construction of differential invariants and invariant derivations proposed in [6]; see also [19, 16, 33, 34]. Let G be an r-dimensional Lie group that acts (locally) on an m-dimensional manifold M. We are interested in the action of G on p-dimensional submanifolds $N \subset M$ which, in local coordinates, we identify with the graphs of functions $u = f(x)$. For each positive integer n, let $G^{(n)}$ denote the prolonged group action on the associated n-th order submanifold jet space $\mathrm{J}^n = \mathrm{J}^n(M,p)$, defined as the set of equivalence classes of p-dimensional submanifolds of M under the equivalence relation of n-th order contact. Local coordinates on J^n are denoted $z^{(n)} = (x, u^{(n)}) = (\ \dots\ x^i\ \dots\ u^\alpha_J\ \dots\ )$, with u^α_J representing the partial derivatives of the dependent variables $u = (u^1, \dots, u^q)$ with respect to the independent variables $x = (x^1, \dots, x^p)$, where $p + q = m$, [30].

Assuming that the prolonged action is free[1] on an open subset of J^n, then one can construct a (locally defined) *moving frame*, which, according to [6], is an equivariant map $\rho\colon V^n \to G$ defined on an open subset $V^n \subset \mathrm{J}^n$. Equivariance can be with respect to either the right or left multiplication action of G on itself. All classical moving frames, e.g., those appearing in [5, 9, 10, 11, 21, 20], can be regarded as left equivariant maps, but the right equivariant versions may be easier to compute, and will be the version used here. Of course, any right moving frame can be converted to a left moving frame by composition with the inversion map $g \mapsto g^{-1}$.

In practice, one constructs a moving frame by the process of normalization, relying on the choice of a local *cross-section* $K^n \subset \mathrm{J}^n$ to the prolonged group orbits, meaning a submanifold

[1] A theorem of Ovsiannikov, [35], slightly corrected in [31], guarantees local freeness of the prolonged action at sufficiently high order, provided G acts locally effectively on subsets of M. This is only a technical restriction; for example, all analytic actions can be made effective by dividing by the global isotropy subgroup. Although all known examples of prolonged effective group actions are, in fact, free on an open subset of a sufficiently high order jet space, there is, frustratingly, as yet no general proof, nor known counterexample, to this result.

of the complementary dimension that intersects each orbit transversally. A general cross-section is prescribed implicitly by setting $r = \dim G$ differential functions $Z = (Z_1, \ldots, Z_r)$ to constants:

$$Z_1(x, u^{(n)}) = c_1, \ \ldots \ Z_r(x, u^{(n)}) = c_r. \tag{2.1}$$

Usually — but not always, [28, 34] — the functions are selected from the jet space coordinates x^i, u^α_J, resulting in a *coordinate cross-section*. The corresponding value of the right moving frame at a jet $z^{(n)} \in \mathrm{J}^n$ is the unique group element $g = \rho^{(n)}(z^{(n)}) \in G$ that maps it to the cross-section:

$$\rho^{(n)}(z^{(n)}) \cdot z^{(n)} = g^{(n)} \cdot z^{(n)} \in K^n. \tag{2.2}$$

The moving frame $\rho^{(n)}$ clearly depends on the choice of cross-section, which is usually designed so as to simplify the required computations as much as possible.

Once the cross-section has been fixed, the induced moving frame engenders an invariantization process, that effectively maps functions to invariants, differential forms to invariant differential forms, and so on, [6, 32]. Geometrically, the *invariantization* of any object is defined as the unique invariant object that coincides with its progenitor when restricted to the cross-section. In the special case of functions, invariantization is actually entirely defined by the cross-section, and therefore doesn't require the action to be (locally) free. It is a projection from the ring of differential functions to the ring of differential invariants, the latter being isomorphic to the ring of smooth functions on the cross-section [19].

Pragmatically, the invariantization of a differential function is constructed by first writing out how it is transformed by the prolonged group action: $F(z^{(n)}) \mapsto F(g^{(n)} \cdot z^{(n)})$. One then replaces all the group parameters by their *right* moving frame formulae $g = \rho^{(n)}(z^{(n)})$, resulting in the differential invariant

$$\iota\big[\, F(z^{(n)}) \,\big] = F\big(\, \rho^{(n)}(z^{(n)}) \cdot z^{(n)} \,\big). \tag{2.3}$$

Differential forms and differential operators are handled in an analogous fashion — see [6, 22] for complete details. Alternatively, the algebraic construction for the invariantization of functions in [19] works with the knowledge of the cross-section only, i.e. without the explicit formulae for the moving frame, and applies to non-free actions as well.

In particular, the *normalized differential invariants* induced by the moving frame are obtained by invariantization of the basic jet coordinates:

$$H^i = \iota(x^i), \qquad\qquad I^\alpha_J = \iota(u^\alpha_J), \tag{2.4}$$

which we collectively denote by $(H, I^{(n)}) = (\ldots H^i \ldots I^\alpha_J \ldots)$ for $\#J \leq n$. In the case of a coordinate cross-section, these naturally split into two classes: Those corresponding to the cross-section functions Z_κ are constant, and known as the *phantom differential invariants*. The remainder, known as the *basic differential invariants*, form a complete system of functionally independent differential invariants.

Once the normalized differential invariants are known, the invariantization process (2.3) is implemented by simply replacing each jet coordinate by the corresponding normalized differential invariant (2.4), so that

$$\iota\big[\, F(x, u^{(n)}) \,\big] = \iota\big[\, F(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ ) \,\big] = F(\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ ) = F(H, I^{(n)}). \tag{2.5}$$

In particular, a differential invariant is not affected by invariantization, leading to the very useful *Replacement Theorem*:

$$J(x, u^{(n)}) = J(H, I^{(n)}) \quad \text{whenever } J \text{ is a differential invariant.} \tag{2.6}$$

This permits one to straightforwardly rewrite any known differential invariant in terms the normalized invariants, and thereby establishes their completeness.

A contact-invariant coframe is obtained by taking the horizontal part (i.e., deleting any contact forms) of the invariantization of the basic horizontal one-forms:

$$\omega^i \equiv \iota(dx^i) \qquad \text{modulo contact forms,} \qquad\qquad i = 1, \ldots, p, \tag{2.7}$$

Invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ can then be defined as the associated dual differential operators, defined so that

$$dF \equiv \sum_{i=1}^{p} (\mathcal{D}_i F)\, \omega^i \qquad \text{modulo contact forms,}$$

for any differential function F. Details can be found in [6, 22]. The invariant differential operators do not commute in general, but are subject to the commutation formulae

$$[\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y^i_{jk}\, \mathcal{D}_i, \tag{2.8}$$

where the coefficients $Y^i_{jk} = -Y^i_{kj}$ are certain differential invariants known as the *commutator invariants*.

3 Recurrence and Syzygies.

In general, invariantization and differentiation do not commute. By a *recurrence relation*, we mean an equation expressing an invariantly differentiated invariant in terms of the basic differential invariants. Remarkably, the recurrence relations can be deduced knowing only the (prolonged) infinitesimal generators of the group action and the choice of cross-section.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be a basis for the infinitesimal generators of our transformation group. We prolong each infinitesimal generator to J^n, resulting in the vector fields

$$\mathbf{v}^{(n)}_\kappa = \sum_{i=1}^{p} \xi^i_\kappa(x, u)\, \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{j=\#J=0}^{n} \varphi^\alpha_{J,\kappa}(x, u^{(j)})\, \frac{\partial}{\partial u^\alpha_J}\,, \qquad \kappa = 1, \ldots, r, \tag{3.1}$$

on J^n. The coefficients $\varphi^\alpha_{J,\kappa} = \mathbf{v}^{(n)}_\kappa(u^\alpha_J)$ are given by the prolongation formula, [29, 30]:

$$\varphi^\alpha_{J,\kappa} = D_J\left(\varphi^\alpha_\kappa - \sum_{i=1}^{p} \xi^i\, u^\alpha_i\right) + \sum_{i=1}^{p} \xi^i_\kappa u^\alpha_{J,i}, \tag{3.2}$$

where $D_1, \ldots, D_p$ are the usual (commuting) total derivative operators, and $D_J = D_{j_1} \cdots D_{j_k}$ the corresponding iterated total derivative.

Given a collection $F = (F_1, \ldots, F_k)$ of differential functions, let

$$\mathbf{v}(F) = \left(\mathbf{v}_\kappa^{(n)}(F_j) \right) \tag{3.3}$$

denote the $k \times r$ *generalized Lie matrix* obtained by applying the prolonged infinitesimal generators to the differential functions. In particular, $L^{(n)}(x, u^{(n)}) = \mathbf{v}(x, u^{(n)})$ is the classical Lie matrix of order n whose entries are the infinitesimal generator coefficients $\xi^i_\kappa, \varphi^\alpha_{J,\kappa}$, [30, 33]. The rank of the classical Lie matrix $L^{(n)}(x, u^{(n)})$ equals the dimension of the prolonged group orbit passing through the point $(x, u^{(n)}) \in \mathrm{J}^n$. We set

$$r_n = \max \left\{ \operatorname{rank} L^{(n)}(x, u^{(n)}) \;\middle|\; (x, u^{(n)}) \in \mathrm{J}^n \right\} \tag{3.4}$$

to be the maximal prolonged orbit dimension. Clearly, $r_0 \leq r_1 \leq r_1 \leq \cdots \leq r = \dim G$, and $r_n = r$ if and only if the action is locally free on an open subset of J^n. Assuming G acts locally effectively on subsets, [31], this holds for n sufficiently large, and we define the *stabilization order* s to be the minimal n such that $r_n = r$. Locally, the number of functionally independent differential invariants of order $\leq n$ equals $\dim \mathrm{J}^n - r_n$.

The fundamental moving frame recurrence formulae were first established in [6] and written as follows; see also [33] for additional details.

Theorem 3. *The* recurrence formulae *for the normalized differential invariants have the form*

$$\mathcal{D}_i H^j = \delta_i^j + \sum_{\kappa=1}^{r} R_i^\kappa \, \iota(\xi_\kappa^j), \qquad \mathcal{D}_i I_J^\alpha = I_{Ji}^\alpha + \sum_{\kappa=1}^{r} R_i^\kappa \, \iota(\varphi_{J,\kappa}^\alpha), \tag{3.5}$$

where δ_i^j is the usual Kronecker delta, and R_i^κ are certain differential invariants.

The recurrence formulae (3.5) imply the following commutator syzygies among the normalized differential invariants:

$$\mathcal{D}_i I_{Jj}^\alpha - \mathcal{D}_j I_{Ji}^\alpha = \sum_{\kappa=1}^{r} \left[R_i^\kappa \, \iota(\varphi_{Jj,\kappa}^\alpha) - R_j^\kappa \, \iota(\varphi_{Ji,\kappa}^\alpha) \right], \tag{3.6}$$

for all $1 \leq i, j \leq p$ and all multi-indices J. We can show that a subset of these relationships (3.5), (3.6) form a complete set of syzygies, [16]. By formally manipulating those syzygies, performing differential elimination [4, 12, 13, 14], we are able to obtain expressions of some of the differential invariants in terms of the invariant derivatives of others. This is the strategy for the main results of this paper.

In the case of coordinate cross-section, if we single out the recurrence formulae for the constant *phantom differential invariants* prescribed by the cross-section, the left hand sides are all zero, and hence we obtain a linear algebraic system that can be uniquely solved for the invariants R_i^κ. Substituting the resulting formulae back into the recurrence formulae for the remaining, non-constant basic differential invariants leads to a complete system of relations among the normalized differential invariants [6, 33].

More generally, if we think of the R_i^κ as the entries of a $p \times r$ matrix

$$R = \left(R_i^\kappa \right), \tag{3.7}$$

then they are given explicitly by

$$R = -\,\iota\left[\, D(Z)\, \mathbf{v}(Z)^{-1} \,\right], \tag{3.8}$$

where $Z = (Z_1, \ldots, Z_r)$ are the cross-section functions (2.1), while

$$D(Z) = \left(D_i Z_j \right) \tag{3.9}$$

is the $p \times r$ matrix of their total derivatives. The recurrence formulae are then covered by the matricial equation [16]

$$\mathcal{D}\left(\iota(F) \right) = \iota\left(D(F) \right) + R\, \iota\left(\mathbf{v}(F) \right), \tag{3.10}$$

for any set of differential functions $F = (F_1, \ldots, F_k)$. The left hand side denotes the $p \times k$ matrix

$$\mathcal{D}\left(\iota(F) \right) = \left(\mathcal{D}_i\left(\iota(F_j) \right) \right) \tag{3.11}$$

obtained by invariant differentiation.

The invariants R_i^κ actually arise in the proof of (3.5) as the coefficients of the horizontal parts of the pull-back of the Maurer–Cartan forms via the moving frame, [6]. Explicitly, if $\mu^1, \ldots, \mu^r$ are a basis for the Maurer–Cartan forms on G dual to the Lie algebra basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$, then the horizontal part of their moving frame pull-back can be expressed in terms of the contact-invariant coframe (2.7):

$$\gamma^\kappa = \rho^* \mu^\kappa \equiv \sum_{i=1}^{p} R_i^\kappa\, \omega^i \qquad \text{modulo contact forms.} \tag{3.12}$$

We shall therefore refer to R_i^κ as the *Maurer–Cartan invariants*, while R in (3.7) will be called *the Maurer-Cartan matrix*. In the case of curves, when $G \subset \mathrm{GL}(N)$ is a matrix Lie group, the Maurer–Cartan matrix $R = \mathcal{D}\rho^{(n)}(x, u^{(n)}) \cdot \rho^{(n)}(x, u^{(n)})^{-1}$ can be identified with the Frenet–Serret matrix, [11, 26], with $\mathcal{D}$ the invariant arc-length derivative.

The identification (3.12) of the Maurer–Cartan invariants as the coefficients of the (horizontal parts of) the pulled-back Maurer–Cartan forms can be used to deduce their syzygies, [17]. The Maurer-Cartan forms on G satisfy the usual Lie group structure equations

$$d\mu^c = -\sum_{a<b} C_{ab}^c\, \mu^a \wedge \mu^b, \qquad c = 1, \ldots, r, \tag{3.13}$$

where C_{ab}^c are the structure constants of the Lie algebra relative to the basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$. It follows that their pull-backs (3.12) satisfy the same equations:

$$d\gamma^c = -\sum_{a<b} C_{ab}^c\, \gamma^a \wedge \gamma^b, \qquad c = 1, \ldots, r. \tag{3.14}$$

The purely horizontal components of these identities provide the following syzygies among the Maurer–Cartan invariants, [17]:

Theorem 4. *The Maurer–Cartan invariants satisfy the following identities*:

$$\mathcal{D}_j(R^i_c) - \mathcal{D}_i(R^j_c) + \sum_{1\le a<b\le r} C^c_{ab}\left(R^i_a R^j_b - R^j_a R^i_b \right) + \sum_{k=1}^{p} Y^i_{jk}\, R^k_c = 0, \tag{3.15}$$

for $1 \le c \le r$, $1 \le i < j \le p$, *and where* Y^i_{jk} *are the commutator invariants* (2.8).

Finally, we note the recurrence formulas for the invariant differential forms established in [6] produce the explicit formulas for the commutator invariants:

$$Y^i_{jk} = \sum_{\kappa=1}^{r} \sum_{j=1}^{p} R^\kappa_k\, \iota(D_j \xi^i_\kappa) - R^\kappa_j\, \iota(D_k \xi^i_\kappa)\,. \tag{3.16}$$

4 Generating Differential Invariants

A set of differential invariants $\mathfrak{I} = \{I_1, \ldots, I_k\}$ is called *generating* if, locally, every differential invariant can be expressed as a function of them and their iterated invariant derivatives $\mathcal{D}_J I_\nu$. A key issue is to find a minimal set of generating invariants, which (except for curves) must be done on a case by case basis. Before investigating the minimality question in the conformal and projective examples, let us state general results characterizing (usually non-minimal) generating systems. These results are all consequences of the recurrence formulae (3.5) or, equivalently, (3.10).

Let

$$\mathfrak{I}^n = \{H^1, \ldots, H^p\} \cup \{\, I^\alpha_J \mid \alpha = 1, \ldots, q, \#J \le n \,\} \tag{4.17}$$

denote the complete set of normalized differential invariants of order $\le n$. In particular, assuming we choose a cross-section that projects to a cross-section on M (e.g., a minimal order cross-section) then $\mathfrak{I}^0 = \{H^1, \ldots, H^p, I^1, \ldots I^q\}$ are the ordinary invariants for the action on M. In particular, if, as in the examples treated here, the action is transitive on M, the normalized order 0 invariants are all constant, and hence are superfluous for the following generating systems.

Theorem 5. *If the moving frame has order* n, *then the set of normalized differential invariants* $\mathfrak{I}^{n+1}$ *of order* $n+1$ *forms a generating set. Indeed, the rewriting process is effective using the Replacement Rule* (2.6).

For cross-section of *minimal order* there is an additional important set of invariants that is generating. This was proved for coordinate cross-sections in [33] and then generalized in [16]. For each $k \ge 0$, let r_k denote the maximal orbit dimension of the action of $G^{(k)}$ on J^k.

Theorem 6. *Let* $Z = (Z_1, \ldots, Z_r)$ *define a* minimal order cross-section *in the sense that for each* $k = 0, 1, \ldots, s$, *where* s *is the stabilization order*, $Z_k = (Z_1, \ldots, Z_{r_k})$ *defines a cross-section for the action of* $G^{(k)}$ *on* J^k. *Then* $\mathfrak{I}^0 \cup \mathfrak{Z}$, *where*

$$\mathfrak{Z} = \left\{\, \iota(D_i(Z_j)) \;\middle|\; 1 \le i \le p,\ 1 \le j \le r \,\right\}, \tag{4.18}$$

form a generating set of differential invariants.

Another interesting consequence of Theorem 3 observed in [17] is that the Maurer–Cartan invariants

$$\mathfrak{R} = \{ R^i_a \mid 1 \le i \le p,\ 1 \le a \le r \} \tag{4.19}$$

also form a generating set when the action is transitive on M. More precisely:

Theorem 7. *The differential invariants $\mathfrak{I}^0 \cup \mathfrak{R}$ form a generating set.*

In [34], the following device for generating the commutator invariants was introduced, and then applied to the differential invariants of Euclidean and equi-affine surfaces. We will employ the same trick here.

Theorem 8. *Let $I = (I_1, \ldots, I_p)$ be a set of differential invariants such that $\mathcal{D}(I)$, cf.* (3.11), *forms a nonsingular $p \times p$ matrix of differentiated invariants. Then one can express the commutator invariants as rational functions of the invariant derivatives, of order ≤ 2, of $I_1, \ldots, I_p$.*

Proof: In view of (2.8), we have

$$\mathcal{D}_i \mathcal{D}_j I_l - \mathcal{D}_j \mathcal{D}_i I_l = \sum_{k=1}^{p} Y^i_{jk}\, \mathcal{D}_k I_l. \tag{4.20}$$

We regard (4.20) as a system of p linear equations for the commutator invariants $Y^i_{j1}, \ldots Y^i_{jp}$. Our assumption implies that coefficient matrix is nonsingular. Solving the linear system by, say, Cramer's rule, produces the formulae for the Y^i_{jk}. *Q.E.D.*

In particular, if I is any single differential invariant with sufficiently many nontrivial invariant derivatives, the differential invariants in the proposition can be taken as invariant derivatives of I. Typically we choose I of order at least n, the order of the moving frame, and $p-1$ of its first order invariant derivatives. If I is a basic invariant, nonsingularity of the matrix of differentiated invariants is then a consequence of the recurrence formulae. As a result, one is, in fact, able to generate all of the commutator invariants as combinations of derivatives of a *single differential invariant*!

5 Differential Invariants of Surfaces

Let us specialize the preceding general constructions to the case of two-dimensional surfaces in three-dimensional space. Let G be a r-dimensional Lie group acting transitively and effectively on $M = \mathbb{R}^3$. Let $\mathrm{J}^n = \mathrm{J}^n(\mathbb{R}^3, 2)$ denote the n-th order surface jet bundle, with the usual induced coordinates $z^{(n)} = (x, y, u, u_x, u_y, u_{xx},\ \ldots\ , u_{jk},\ \ldots\ )$ for $j + k \le n$.

Let $n \geq s$, the stabilization order of G. Given a cross-section $K^n \subset \mathrm{J}^n$, let $\rho : V^n \to G$ be the induced right moving frame defined on a suitable open subset $V^n \subset \mathrm{J}^n$ containing K^n. Invariantization of the basic jet coordinates results in the *normalized differential invariants*

$$H_1 = \iota(x), \qquad H_2 = \iota(y), \qquad I_{jk} = \iota(u_{jk}), \qquad j, k \geq 0. \tag{5.21}$$

In view of our transitivity assumption, we will only consider cross-sections that normalize the order 0 variables, $x = y = u = 0$, and so the order 0 normalized invariants are trivial: $H_1 = H_2 = I_{00} = 0$. We use

$$I^{(n)} = (0, I_{10}, I_{01}, I_{20}, I_{11}, \ \ldots \ , I_{0n}) = \iota(u^{(n)}) \tag{5.22}$$

to denote all the normalized differential invariants, both phantom and basic, of order $\leq n$ obtained by invariantizing the dependent variable u and its derivatives.

In addition, the two invariant differential operators are obtained by invariantizing the total derivatives $\mathcal{D}_1 = \iota(D_x)$, $\mathcal{D}_2 = \iota(D_y)$, or, equivalently, are given as the dual differentiations with respect to the contact-invariant coframe

$$\omega_1 = \iota(dx), \qquad \omega_2 = \iota(dy). \tag{5.23}$$

Specializing the general moving frame recurrence formulae in Theorem 3, we have:

Theorem 9. *The* recurrence formulae *for the differentiated invariants are*

$$\begin{aligned} \mathcal{D}_1 I_{jk} &= I_{j+1,k} + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(0,0,I^{(j+k)}) R_1^\kappa, \\ \mathcal{D}_2 I_{jk} &= I_{j,k+1} + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(0,0,I^{(j+k)}) R_2^\kappa, \end{aligned} \qquad j + k \geq 1, \tag{5.24}$$

where R_i^κ are the Maurer–Cartan invariants, which multiply the invariantizations of the coefficients of the prolonged infinitesimal generator

$$\mathbf{v}_\kappa = \xi_\kappa(x,y,u) \frac{\partial}{\partial x} + \eta_\kappa(x,y,u) \frac{\partial}{\partial y} + \sum_{0 \leq j+k \leq n} \varphi_\kappa^{jk}(x,y,u^{(j+k)}) \frac{\partial}{\partial u_{jk}} \ , \tag{5.25}$$

which are given explicitly by the usual prolongation formula (3.2)*:*

$$\varphi_\kappa^{jk} = D_x^j D_y^k \left(\varphi_\kappa - \xi_\kappa \, u_x - \eta_\kappa \, u_y \right) + \xi_\kappa \, u_{j+1,k} + \eta_\kappa \, u_{k,j+1}. \tag{5.26}$$

6 Surfaces in Conformal Geometry

In this section, we focus our attention on the standard action of the conformal group SO(4, 1) on surfaces in $\mathbb{R}^3$, [2]. Note that $\dim \mathrm{SO}(4,1) = 10$. A basis for its infinitesimal generators is

$$\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y},\quad \frac{\partial}{\partial u},\quad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\quad x\frac{\partial}{\partial u} - u\frac{\partial}{\partial x},\quad y\frac{\partial}{\partial u} - u\frac{\partial}{\partial y},\quad x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u},$$
$$(x^2 - y^2 - u^2)\frac{\partial}{\partial x} + 2\,x\,y\,\frac{\partial}{\partial y} + 2\,x\,u\,\frac{\partial}{\partial u},\qquad 2\,x\,y\,\frac{\partial}{\partial x} + (y^2 - x^2 - u^2)\frac{\partial}{\partial y} + 2\,y\,u\,\frac{\partial}{\partial u},$$
$$2\,x\,u\,\frac{\partial}{\partial u} + 2\,y\,u\,\frac{\partial}{\partial y} + (u^2 - x^2 - y^2)\,\frac{\partial}{\partial u}.$$

The maximal prolonged orbit dimensions (3.4) are $r_0 = 3,\ r_1 = 5,\ r_2 = 8$ and $r_3 = 10$. The stabilization order is thus $s = 3$. The action is transitive on an open subset of J^2 and there are two independent differential invariants of order 3. Thus, by Theorem 5, the differential invariants of order 3 and 4 form a generating set. In this section we shall show that, under a certain non-degeneracy condition, all the differential invariants can be written in terms of the derivatives of a single third order differential invariant.

The argument goes in two steps. We first show that all the differential invariants of fourth order can be written in terms of the two third order differential invariants and their monotone derivatives, i.e., those obtained by applying the operators $\mathcal{D}_1^i \mathcal{D}_2^j$. Then, the commutator trick of Theorem 8 allows us to reduce to a single generator.

We give two computational proofs of the first step. First using the properties of normalized invariants, Theorems 3 and 6, and a cross-section that corresponds to a hyperbolic quadratic form, second by using the properties of the Maurer-Cartan invariants, thsmcgenmcsyz, along with a cross-section that corresponds to a degenerate quadratic form. We have used the symbolic computation software AIDA [15] to compute the Maurer-Cartan matrix, the commutation rules and the syzygies, and the software *diffalg* [4, 13] to operate the differential elimination.

6.1 Hyperbolic cross-section

The cross-section implicitly used in [38] is:

$$x = y = u = u_x = u_y = u_{xx} = u_{yy} = u_{xxy} = u_{xyy} = 0, \qquad u_{xy} = 1. \tag{6.27}$$

Thus, there are two basic third order differential invariants:

$$I_{30} = \iota(u_{xxx}), \qquad\qquad I_{03} = \iota(u_{yyy}),$$

and 5 of order 4, given by invariantization of the fourth order jet coordinates: $I_{jk} = \iota(u_{jk})$, $j + k = 4$. Since (6.27) defines a minimal order cross-section, Theorem 6 implies that $\{I_{30}, I_{03}, I_{31}, I_{22}, I_{13}\}$ is a generating set of differential invariants.

To prove Theorem 1, we first show that I_{31}, I_{13} and I_{22} can be written in terms of $\{I_{30}, I_{03}\}$ and their monotone derivatives. Using formula (3.8), the Maurer-Cartan matrix is found to have the form

$$R = -\begin{pmatrix} 1 & 0 & 0 & \phi & 0 & 1 & 0 & \kappa & \sigma & \phi \\ 0 & 1 & 0 & \psi & 1 & 0 & 0 & \sigma & \tau & -\psi \end{pmatrix}, \tag{6.28}$$

where

$$\phi = -\tfrac{1}{4}\, I_{30}, \qquad \psi = \tfrac{1}{4}\, I_{03}, \qquad \tau = 1 - \tfrac{1}{2}\, I_{13} - \tfrac{1}{8}\, {I_{03}}^2,$$
$$\sigma = \tfrac{1}{8}\, I_{30} I_{03} - \tfrac{1}{2}\, I_{22}, \qquad \kappa = 1 - \tfrac{1}{2}\, I_{31} - \tfrac{1}{8}\, {I_{30}}^2.$$

The first two are, in fact, the commutator invariants since, by (3.16), the invariant derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfy the commutation rule:

$$[\mathcal{D}_2, \mathcal{D}_1] = \phi\, \mathcal{D}_1 + \psi\, \mathcal{D}_2. \tag{6.29}$$

Implementing (3.5), (3.6), we deduce the following relationships among $\{I_{30}, I_{03}, I_{31}, I_{13}, I_{22}\}$:

$$\begin{aligned}
E_{301}: &\quad \mathcal{D}_1(I_{30}) - 3\, I_{22} + \tfrac{3}{4}\, I_{30}\, I_{03} - I_{40}, \\
E_{302}: &\quad \mathcal{D}_2(I_{30}) - 3\, I_{13} - \tfrac{3}{4}\, {I_{03}}^2 + 6 - I_{31}, \\
E_{031}: &\quad \mathcal{D}_1(I_{03}) - 3\, I_{31} - \tfrac{3}{4}\, {I_{30}}^2 + 6 - I_{13}, \\
E_{032}: &\quad \mathcal{D}_2(I_{03}) - 3\, I_{22} + \tfrac{3}{4}\, I_{30}\, I_{03} - I_{04}, \\
S_{14}: &\quad \mathcal{D}_2(I_{13}) - \mathcal{D}_1(I_{04}) + \tfrac{3}{4}\, I_{03}\, I_{22} - \tfrac{1}{4}\, I_{03}\, I_{04} + I_{30}\, I_{13}, \\
S_{23}: &\quad \mathcal{D}_2(I_{22}) - \mathcal{D}_1(I_{13}) - \tfrac{3}{2}\, I_{03}\, (I_{31} + I_{13}) - \tfrac{1}{4}\, I_{30}\, (I_{22} + I_{04}) - \tfrac{1}{4}\, I_{03}\, ({I_{30}}^2 + {I_{03}}^2 - 20), \\
S_{32}: &\quad \mathcal{D}_2(I_{31}) - \mathcal{D}_1(I_{22}) + \tfrac{1}{4}\, I_{03}\, (I_{40} + I_{22}) + \tfrac{3}{2}\, I_{30}\, (I_{13} + I_{31}) + \tfrac{1}{4}\, I_{30}\, ({I_{03}}^2 + {I_{30}}^2 - 20), \\
S_{41}: &\quad \mathcal{D}_2(I_{40}) - \mathcal{D}_1(I_{31}) - I_{03}\, I_{31} - \tfrac{3}{4}\, I_{30}\, I_{22} + \tfrac{1}{4}\, I_{30}\, I_{40}.
\end{aligned}$$

Taking the combination $E_{302} - 3\, E_{031}$ and $E_{031} - 3\, E_{302}$ we obtain:

$$I_{31} = \tfrac{3}{2} - \tfrac{1}{8}\, \mathcal{D}_2(I_{30}) + \tfrac{3}{8}\, \mathcal{D}_1(I_{03}) + \tfrac{3}{32}\, (I_{03})^2 - \tfrac{9}{32}\, (I_{30})^2,$$
$$I_{13} = \tfrac{3}{2} - \tfrac{1}{8}\, \mathcal{D}_1(I_{03}) + \tfrac{3}{8}\, \mathcal{D}_2(I_{30}) - \tfrac{9}{32}\, (I_{03})^2 + \tfrac{3}{32}\, (I_{30})^2.$$

Taking the combination

$$\begin{aligned}
&128\, \mathcal{D}_2(S_{32}) - 48\, \mathcal{D}_1(S_{41}) - 16\, \mathcal{D}_1(S_{23}) - 36\, I_{03} S_{41} - 12\, I_{03} S_{23} + 108\, I_{30} S_{32} + 4\, I_{30} S_{14} \\
&\quad - 48\, \mathcal{D}_1 \mathcal{D}_2(E_{301}) - 16\, \mathcal{D}_2^2(E_{302}) + 48\, \mathcal{D}_2^2(E_{031}) + 16\, \mathcal{D}_1^2(E_{031}) \\
&\quad + 36\, I_{03} \mathcal{D}_1(E_{031}) + 88\, I_{30} \mathcal{D}_2(E_{031}) - 12\, I_{30} \mathcal{D}_1(E_{301}) - 4\, I_{03} \mathcal{D}_2(E_{301}) + 36\, I_{30} \mathcal{D}_2(E_{302}) \\
&\quad + \big(18\, {I_{03}}^2 + 40\, {I_{30}}^2 + 48\, \mathcal{D}_2(I_{30}) + 24\, \mathcal{D}_1(I_{03})\big)\, E_{031} \\
&\quad + (18\, I_{30} I_{03} - 12\, \mathcal{D}_1(I_{30}) + 32\, \mathcal{D}_2(I_{03}))\, E_{301} \\
&\quad + \big(42\, {I_{30}}^2 + 48\, \mathcal{D}_2(I_{30})\big)\, E_{302} + (2\, I_{30} I_{03} + 4\, \mathcal{D}_1(I_{30}))\, E_{032}
\end{aligned}$$

leads to:

$$I_{22} = \frac{A_{22}}{64\, B_{22}},$$

where

$$\begin{aligned}
A_{22} = &-48\,\mathcal{D}_1^2\mathcal{D}_2(I_{30}) - 48\,\mathcal{D}_1\mathcal{D}_2^2(I_{03}) - 64\,\mathcal{D}_1^3(I_{03}) + 64\,\mathcal{D}_2^3(I_{30}) + \\
&+ \left(36\,\mathcal{D}_1^2(I_{03}) + 48\,\mathcal{D}_2^2(I_{03}) - 52\,\mathcal{D}_1\mathcal{D}_2(I_{30})\right) I_{03} - \\
&- \left(36\,\mathcal{D}_2^2(I_{30}) + 24\,\mathcal{D}_1^2(I_{30}) - 28\,\mathcal{D}_1\mathcal{D}_2(I_{03})\right) I_{30} + \\
&+ 36\,\mathcal{D}_2(I_{03})^2 - 24\,\mathcal{D}_1(I_{30})^2 + 24\,\mathcal{D}_1(I_{03})^2 - 24\,\mathcal{D}_2(I_{30})^2 - 12\,\mathcal{D}_2(I_{30})\mathcal{D}_1(I_{03}) + \\
&+ (30\,\mathcal{D}_1(I_{03}) - 8\,\mathcal{D}_2(I_{30}))\, {I_{03}}^2 + (52\,\mathcal{D}_2(I_{03}) - 42\,\mathcal{D}_1(I_{30}))\, I_{30}I_{03} - \\
&- (30\,\mathcal{D}_2(I_{30}) + 2\,\mathcal{D}_1(I_{03}))\, {I_{30}}^2 + 64\,\mathcal{D}_1(I_{03}) - 64\,\mathcal{D}_2(I_{30}) + \\
&+ 3\,{I_{03}}^4 - 3\,I_{30}^4 + 3\,{I_{03}}^2 - 3\,I_{30}^2,
\end{aligned}$$

and

$$B_{22} = \mathcal{D}_2(I_{03}) - \mathcal{D}_1(I_{30}).$$

We conclude that the two third order invariants $I_{3,0}$ and $I_{0,3}$ form a generating system. Moreover, since the generating invariants are, up to constant multiple, commutator invariants, we can use the commutator trick of Theorem 8 to generate them both from any single differential invariant. Indeed, when $\mathcal{D}_2\phi \neq 0$, the commutation rule (6.29) implies that

$$\psi = \frac{\mathcal{D}_2\mathcal{D}_1\phi - \mathcal{D}_1\mathcal{D}_2\phi - \phi\mathcal{D}_1\phi}{\mathcal{D}_2\phi}\,. \tag{6.30}$$

Similarly, when $\mathcal{D}_1\psi \neq 0$ we have

$$\phi = \frac{\mathcal{D}_2\mathcal{D}_1\psi - \mathcal{D}_1\mathcal{D}_2\psi - \psi\mathcal{D}_2\psi}{\mathcal{D}_1\psi}\,. \tag{6.31}$$

Therefore, under the assumption that

$$(\mathcal{D}_1\psi)^2 + (\mathcal{D}_2\phi)^2 \neq 0, \tag{6.32}$$

a single differential invariant, of order 3, generates all the differential invariants for surfaces in conformal geometry.

6.2 Degenerate cross-section

In our second approach, we choose the “degenerate” cross-section

$$x = y = u = u_x = u_y = u_{xx} = u_{xy} = u_{yy} = u_{xxy} = u_{xyy} = 0. \tag{6.33}$$

Implementing (3.8), the new Maurer-Cartan matrix is:

$$R = -\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -\psi & \sigma & \kappa & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \phi & \tau & -\sigma & -\frac{1}{2}\,\phi \end{pmatrix}, \tag{6.34}$$

where

$$\phi = I_{03}, \qquad \psi = I_{30}, \qquad \tau = \tfrac{1}{2}\,I_{13}, \qquad \kappa = -\tfrac{1}{2}\,I_{31}, \qquad \sigma = \tfrac{1}{2}\,I_{22}.$$

Again, ϕ, ψ are the commutator invariants since $[\mathcal{D}_2, \mathcal{D}_1] = \phi\, \mathcal{D}_1 + \psi\, \mathcal{D}_2$. Theorem 7 tells us that the Maurer–Cartan invariants $\{\phi, \psi, \kappa, \tau, \sigma\}$ form a generating set. We will show that $\{\kappa, \tau, \sigma\}$ can be written in terms of $\{\phi, \psi\}$ and their derivatives. We write those as $\phi_{i,j}$ to mean $\mathcal{D}_1^i \mathcal{D}_2^2(\phi)$ and similarly for $\psi, \kappa, \tau, \sigma$.

The non-zero syzygies of Theorem 4 are:

$$
\begin{aligned}
\mathit{\Delta}_{7}: &\quad \phi_{10} + \psi_{01} - 2\,\tau + 2\,\kappa = 0,\\
\mathit{\Delta}_{8}: &\quad \sigma_{01} - \tau_{10} - \tfrac{1}{2}\,\phi - 2\,\phi\,\sigma - 2\,\psi\,\tau = 0,\\
\mathit{\Delta}_{9}: &\quad \sigma_{10} + \kappa_{01} - 2\,\phi\,\kappa + 2\,\psi\,\sigma = 0,\\
\mathit{\Delta}_{10}: &\quad \tfrac{1}{2}\,\phi_{10} - \tau + \psi\,\phi = 0.
\end{aligned}
$$

The following combinations of the syzygies allow us to rewrite τ and κ in terms of ϕ, ψ and their derivatives:

$$
\begin{aligned}
\mathit{\Delta}_7 + 2\,\mathit{\Delta}_{10}: &\quad \kappa = -\tfrac{1}{2}\,\psi_{01} + \psi\phi,\\
\Delta_{10}: &\quad \tau = \tfrac{1}{2}\,\phi_{10} + \psi\phi,
\end{aligned}
$$

while the following combination

$$
2\,\mathcal{D}_2(\Delta_9) - 2\,\mathcal{D}_1(\Delta_8) + 4\,\sigma\Delta_7 - 6\,\psi\Delta_8 - 6\,\phi\Delta_9 - 2\,\Delta_{10}
$$

allows to express σ in terms of ϕ, ψ, τ, κ and their derivatives:

$$
\sigma = \frac{\tau_{20} + \kappa_{02} = 5\,\psi\tau_{10} - 5\,\phi\kappa_{01} + 2\,\psi_{10}\tau - 2\,\phi_{01}\kappa + 6\,\phi^2\kappa + (6\,\psi^2 + 1)\,\tau + \tfrac{1}{2}\,\psi\phi}{4(\kappa - \tau)}\,.
$$

Observe that this exhibits a singular behavior at *umbilic points* where $\kappa = \tau$.

Finally, since the generating invariants are, up to constant multiple, commutator invariants, we can generate one from the other by the same formulas (6.30), (6.31), under the assumption that (6.32) holds.

7 Projective Surfaces

The infinitesimal generators of the projective action of PSL(4) on $\mathbb{R}^3$ are

$$
\begin{gathered}
\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial u},\\
x\,\frac{\partial}{\partial x}, \quad y\,\frac{\partial}{\partial x}, \quad u\,\frac{\partial}{\partial x}, \quad x\,\frac{\partial}{\partial y}, \quad y\,\frac{\partial}{\partial y}, \quad u\,\frac{\partial}{\partial y}, \quad x\,\frac{\partial}{\partial u}, \quad y\,\frac{\partial}{\partial u}, \quad u\,\frac{\partial}{\partial u},\\
x^2\,\frac{\partial}{\partial x} + x\,y\,\frac{\partial}{\partial y} + x\,u\,\frac{\partial}{\partial u}, \qquad x\,y\,\frac{\partial}{\partial x} + y^2\,\frac{\partial}{\partial y} + y\,u\,\frac{\partial}{\partial u}, \qquad x\,u\,\frac{\partial}{\partial x} + y\,u\,\frac{\partial}{\partial y} + u^2\,\frac{\partial}{\partial u}.
\end{gathered}
$$

The generic prolonged orbit dimensions are $r_0 = 3,\ r_1 = 5,\ r_2 = 8,\ r_3 = 12$ and $r_4 = 15 = \dim \mathrm{PSL}(4)$, and so the stabilization order is $s = 4$.

We adopt the same strategy as in previous section to show that the all the differential invariants are generated by a single fourth order differential invariants. The computations and formulae are nonetheless more challenging.

The section implicitly used in [38] is:

$$\begin{gathered} x = y = u = u_x = u_y = u_{xx} = u_{yy} = u_{xxy} = u_{xyy} = u_{xxxy} = u_{xxyy} = u_{xyyy} = 0, \\ u_{xy} = u_{xxx} = u_{yyy} = 1. \end{gathered} \tag{7.35}$$

Thus, there are two basic fourth order differential invariants:

$$I_{40} = \iota(u_{xxxx}), \qquad\qquad I_{04} = \iota(u_{yyyy}),$$

and 6 of order 5, given by invariantization of the fifth order jet coordinates. Theorem 6 implies that the invariants $\{ I_{40}, I_{04}, I_{41}, I_{32}, I_{23}, I_{14} \}$ generate the algebra of projective differential invariants.

The Maurer-Cartan matrix (3.8) is

$$R = - \begin{pmatrix} 1 & 0 & 0 & -2\,\psi & 0 & \kappa & -\frac{1}{2} & -\psi & \tau & 0 & 1 & -3\,\psi & -\tau & \frac{1}{4} - \kappa & \frac{1}{2}\,\sigma - \frac{3}{8}\,\psi \\ 0 & 1 & 0 & \phi & -\frac{1}{2} & \sigma & 0 & 2\,\phi & \eta & 1 & 0 & 3\,\phi & \frac{1}{4} - \eta & -\sigma & \frac{3}{8}\,\phi + \frac{1}{2}\,\tau \end{pmatrix}, \tag{7.36}$$

where

$$\begin{gathered} \phi = -\tfrac{1}{3}\, I_{04}, \qquad \psi = \tfrac{1}{3}\, I_{40}, \qquad \eta = -\tfrac{1}{2}\, I_{14} - \tfrac{1}{4}, \\ \tau = -\tfrac{1}{2}\, I_{23} + \tfrac{1}{4}\, I_{04}, \qquad \sigma = -\tfrac{1}{2}\, I_{32} + \tfrac{1}{4}\, I_{40}, \qquad \kappa = -\tfrac{1}{2}\, I_{41} - \tfrac{1}{4}. \end{gathered}$$

The invariant derivations satisfy the commutation rule;

$$[\mathcal{D}_2, \mathcal{D}_1] = \phi\, \mathcal{D}_1 + \psi\, \mathcal{D}_2$$

and so ϕ, ψ are the commutator invariants.

The nonzero syzygies (3.5), (3.6) of the generating set $\{\phi, \psi, \eta, \sigma, \tau, \kappa\}$ are

$$\begin{aligned} \Delta_4 :&\qquad \phi_{10} + 2\,\psi_{01} + 2\,\eta - \phi\,\psi - \tfrac{1}{2} \;=\; 0, \\ \Delta_6 :&\qquad \sigma_{10} - \kappa_{01} - \tfrac{3}{8}\,\phi + 3\,\phi\,\kappa + 2\,\psi\,\sigma \;=\; 0, \\ \Delta_8 :&\qquad 2\,\phi_{10} + \psi_{01} - 2\,\kappa + \phi\,\psi + \tfrac{1}{2} \;=\; 0, \\ \Delta_9 :&\qquad \eta_{10} - \tau_{01} - \tfrac{3}{8}\,\psi + 2\,\phi\,\tau + 3\,\psi\,\eta \;=\; 0, \\ \Delta_{12} :&\qquad \Delta_4 + \Delta_8, \quad \Delta_{13} :\ -\Delta_9, \quad \Delta_{14} :\ -\Delta_6, \\ \Delta_{15} :&\qquad \tfrac{1}{2}\,\tau_{10} - \tfrac{1}{2}\,\sigma_{01} + \tfrac{3}{8}\,\phi_{10} + \tfrac{3}{8}\,\psi_{01} - \tfrac{1}{4}\,\kappa + \tfrac{1}{4}\,\eta + 2\,\phi\,\sigma + 2\,\psi\,\tau \;=\; 0. \end{aligned}$$

From Δ_4 and Δ_8 we immediately obtain:

$$\eta = \tfrac{1}{4} - \tfrac{1}{2}\,\phi_{10} - \psi_{01} + \tfrac{1}{2}\,\phi\psi, \qquad\qquad \kappa = \tfrac{1}{4} + \phi_{10} + \tfrac{1}{2}\,\psi_{01} + \tfrac{1}{2}\,\phi\psi.$$

Let P_1, P_2, P_3 be the differential polynomials obtained from $\Delta_6, \Delta_9, \Delta_{15}$ after substitution of κ and τ:

$$\begin{aligned} P_1 &= -\tfrac{1}{2}\,\tau_{10} + \tfrac{1}{2}\,\sigma_{01} - 2\,\phi\,\sigma - 2\,\tau\,\psi, \\ P_2 &= \tfrac{1}{2}\,\phi_{20} + \psi_{11} - \tfrac{1}{2}\,\phi_{10}\psi - \tfrac{1}{2}\,\phi\psi_{10} + \tau_{01} - \tfrac{3}{8}\,\psi - 2\,\phi\,\tau + \tfrac{3}{2}\,\psi\,\phi_{10} + 3\,\psi\,\psi_{01} - \tfrac{3}{2}\,\phi\,\psi^2, \\ P_3 &= -\sigma_{10} + \phi_{11} + \phi\phi_{10} + \tfrac{3}{2}\,\psi\phi_{01} + \tfrac{1}{2}\,\psi_{02} + \tfrac{1}{2}\,\phi\psi_{01} - \tfrac{3}{8}\,\phi - 3\,\phi\,\phi_{10} - \tfrac{3}{2}\,\phi\,\psi_{01} - \tfrac{3}{2}\,\phi^2\psi - 2\,\psi\,\sigma. \end{aligned}$$

To obtain τ and σ we proceed with a *differential elimination* [12, 4, 13] on $\{P_1, P_2, P_3\}$. We use a ranking where

$$\psi < \phi < \psi_{01} < \phi_{01} < \psi_{10} < \phi_{10} < \psi_{02} < \psi_{11} < \phi_{11} < \phi_{20} < \cdots$$
$$\cdots < \tau < \sigma < \tau_{01} < \sigma_{01} < \tau_{10} < \sigma_{10} < \tau_{02} < \sigma_{02} < \tau_{11} < \sigma_{11} < \tau_{20} < \sigma_{20} < \cdots .$$

For this ranking, the leaders of P_1, P_2, P_3 are, respectively, $\tau_{10}, \tau_{01}, \sigma_{10}$.

We first form the Δ-*polynomial* (cross-derivative) of P_1 and P_2 and *reduce* it with respect to $\{P_1, P_2, P_3\}$. We obtain a polynomial P_4 with leader σ_{02}. We then take the Δ-polynomial of P_3 and P_4 and reduce it with respect to $\{P_1, P_2, P_3, P_4\}$ to obtain a differential polynomial P_5 with leader σ_{01}. On one hand, if we reduce now P_4 by $\{P_1, P_2, P_3, P_5\}$ we obtain a differential polynomial P with leader σ. On the other hand, if we form the Δ-polynomial of P_3 and P_5, reduce it by $\{P_1, P_2, P_3, P_5\}$ we obtain a differential polynomial Q with leader σ. The polynomial P and Q are linear in σ and τ so that we can solve for those two invariants in terms of ϕ, ψ and their derivatives. The explicit formulas are rather long (available from the authors on request), but not particularly enlightening. We conclude that the commutator invariants ϕ, ψ form a generating set. Finally, we can use either (6.30) or (6.31), to generate one commutator invariant from the other, and thereby establish Theorem 2.

Acknowledgements

This research was initiated during the first author's visit to the Institute for Mathematics and its Applications (I.M.A.) at the University of Minnesota during 2007–8 with support from the Fulbright visiting scholar program. The second author is supported in part by NSF Grant DMS 05–05293.

References

[1] M. A. Akivis and V. V. Goldberg. *Projective Differential Geometry of Submanifolds*, volume 49 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1993.

[2] M. A. Akivis and V. V. Goldberg. *Conformal Differential Geometry and its Generalizations*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1996.

[3] T. N. Bailey, M. G. Eastwood and C. R. Graham. Invariant theory for conformal and CR geometry. *Ann. Math.* 139:491–552, 1994.

[4] F. Boulier and E. Hubert. DIFFALG*: description, help pages and examples of use*. Symbolic Computation Group, University of Waterloo, Ontario, Canada, 1998. `http://www.inria.fr/cafe/Evelyne.Hubert/diffalg`.

[5] E. Cartan. *La théorie des groupes finis et continus et la géométrie différentielle traitées par la méthode du repère mobile*. Number 18 in Cahiers scientifiques. Gauthier-Villars, 1937.

[6] M. Fels and P. J. Olver. Moving coframes. II. Regularization and theoretical foundations. *Acta Appl. Math.*, 55:127–208, 1999.

[7] C. Fefferman and C. R. Graham. Conformal invariants. *Élie Cartan et les Mathématiques d'Aujourd'hui*. Astérisque, hors série, Soc. Math. France, Paris, 1985, pp. 95–116.

[8] G. Fubini and E. Čech. *Introduction à la Géométrie Projective Différentielle des Surfaces*. Gauthier-Villars, 1931.

[9] M. L. Green. The moving frame, differential invariants and rigidity theorems for curves in homogeneous spaces. *Duke Math. J.*, 45:735–779, 1978.

[10] P. A. Griffiths. On Cartan's method of Lie groups as applied to uniqueness and existence questions in differential geometry. *Duke Math. J.*, 41:775–814, 1974.

[11] H. W. Guggenheimer. *Differential Geometry.* McGraw-Hill Book Co., Inc., New York, 1963.

[12] E. Hubert. Notes on triangular sets and triangulation-decomposition algorithms II: Differential systems. In F. Winkler and U. Langer, editors, *Symbolic and Numerical Scientific Computing*, number 2630 in Lecture Notes in Computer Science, pages 40–87. Springer Verlag Heidelberg, 2003.

[13] E. Hubert. *diffalg: extension to non commuting derivations.* INRIA, Sophia Antipolis, 2005. `http://www.inria.fr/cafe/Evelyne.Hubert/diffalg`.

[14] E. Hubert. Differential algebra for derivations with nontrivial commutation rules. *J. Pure Applied Algebra*, 200:163–190, 2005.

[15] E. Hubert. *The* MAPLE *package* AIDA *- Algebraic Invariants and their Differential Algebras.* INRIA, 2007. `http://www.inria.fr/cafe/Evelyne.Hubert/aida`.

[16] E. Hubert. Differential invariants of a Lie group action: syzygies on a generating set. *In preparation.*

[17] E. Hubert. Generation properties of Maurer-Cartan invariants. *In preparation.*

[18] E. Hubert and I. A. Kogan. Rational invariants of a group action. construction and rewriting. *J. Symbolic Comput.*, 42:203–217, 2007.

[19] E. Hubert and I. A. Kogan. Smooth and algebraic invariants of a group action. Local and global constructions. *Found. Comput. Math.*, 7, 2007.

[20] T. A. Ivey and J. M. Landsberg. *Cartan for beginners: differential geometry via moving frames and exterior differential systems*, volume 61 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2003.

[21] G. Jensen. *Higher Order Contact of Submanifolds of Homogeneous Spaces*, volume 610 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin-New York, 1977.

[22] I. A. Kogan and P. J. Olver. Invariant Euler-Lagrange equations and the invariant variational bicomplex. *Acta Appl. Math.*, 76:137–193, 2003.

[23] S. Lie. *Vorlesungen über Continuierliche Gruppen mit Geometrischen und anderen Anwendungen.* Chelsea Publishing Co., Bronx, N.Y., 1971.

[24] G. Marí Beffa. The theory of differential invariants and KdV Hamiltonian evolutions. *Bull. Soc. Math. France*, 127:363–391, 1999.

[25] G. Marí Beffa. Relative and absolute differential invariants for conformal curves. *J. Lie Theory*, 13:213–245, 2003.

[26] G. Marí Beffa. Projective-type differential invariants and geometric curve evolutions of KdV-type in flat homogeneous manifolds, *Ann. Institut Fourier*, to appear.

[27] G. Marí Beffa and P. J. Olver. Differential invariants for parametrized projective surfaces. *Commun. Anal. Geom.*, 7:807–839, 1999

[28] E. L. Mansfield. Algorithms for symmetric differential systems. *Found. Comput. Math.*, 1:335–383, 2001.

[29] P. J. Olver. *Applications of Lie Groups to Differential Equations.* Number 107 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1986.

[30] P. J. Olver. *Equivalence, Invariants and Symmetry.* Cambridge University Press, 1995.

[31] P. J. Olver. Moving frames and singularities of prolonged group actions. *Selecta Math. (N.S.)*, 6:41–77, 2000.

[32] P. J. Olver. A survey of moving frames. In *Computer Algebra and Geometric Algebra with Applications*, H. Li, P. J. Olver, and G. Sommer, eds., Lecture Notes in Computer Science, vol. 3519, Springer–Verlag, New York, 2005, pp. 105–138.

[33] P. J. Olver. Generating differential invariants. *J. Math. Anal. Appl.*, 333:450–471, 2007.

[34] P. J. Olver. Differential invariants of surfaces. *Preprint*, University of Minnesota, 2007.

[35] L. V. Ovsiannikov. *Group Analysis of Differential Equations.* Academic Press, New York, 1982.

[36] U. Simon. The Pick invariant in equiaffine differential geometry. *Abh. Math. Sem. Univ. Hamburg*, 53:225–228, 1983.

[37] M. Spivak. *A Comprehensive Introduction to Differential Geometry. Vol. III.* Publish or Perish Inc., Wilmington, Del., second edition, 1979.

[38] A. Tresse. Sur les invariants des groupes continus de transformations. *Acta Math.*, 18:1–88, 1894.

[39] E. Vessiot. Contribution à la géométrie conforme. Théorie des surfaces. I. *Bull. Soc. Math. France*, 54:139–179, 1926.