Generating Differential Invariants

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Abstract. The equivariant method of moving frames is used to specify systems of generating differential invariants for finite-dimensional Lie group actions.

1. Introduction.

Differential invariants are the fundamental building blocks for constructing invariant differential equations and variational problems, and determining their explicit solutions and conservation laws. The equivalence, symmetry and rigidity properties of submanifolds are all governed by their differential invariants. Additional applications abound in differential geometry and relativity, computer vision, integrable systems, geometric numerical integration, classical invariant theory, and many other fields of both pure and applied mathematics, [**17**, **20**, **24**].

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The paper [**3**] initiated the rapid development of a new and far-reaching generalization of the Cartan method of moving frames, which exploits their (re-)interpretation as equivariant maps back to the transformation group. In particular, the equivariant approach has endowed us with a number of new, powerful tools for producing and classifying the differential invariants for general Lie group actions. See [**20**] for a recent survey of progress and current directions of research. Further applications can be found, for instance, in the work of Marí Beffa, [**14**, **15**, **16**], on the Poisson geometry of curves and surfaces in homogeneous spaces, and Mansfield, [**13**], on symmetric differential equations.

However, it has recently become apparent that one of the key results claimed in [**3**; Theorem 13.3] characterizing the generators of the algebra of differential invariants is not correct as stated. The goal of this note is to formulate and prove a corrected version of the theorem that applies to moving frames of minimal order. In addition, an explicit counterexample to the claimed non-minimal order result, which arises in the familiar Euclidean geometry of space curves, is presented.

We will assume that the reader has some familiarity with the equivariant approach to moving frames, as developed in [**3**, **20**]. General results on group actions, jet spaces, prolongation, and differential invariants can be found, for instance, in [**17**].

2. Moving Frames and Differential Invariants.

Let G be a Lie group that acts (locally) on an m-dimensional manifold M. We are interested in the action of G on p-dimensional submanifolds $N \subset M$ which, in local coordinates, we identify with the graphs of functions. For each positive integer n, let $G^{(n)}$ denote the prolonged group action on the associated n^{th} order submanifold jet space $\mathrm{J}^n = \mathrm{J}^n(M,p)$, whose overall dimension equals

$$\dim \mathrm{J}^n = q^{(n)} = p + q\binom{p+n}{n}. \tag{2.1}$$

A real-valued function† $I\colon \mathrm{J}^n \to \mathbb{R}$ is known as a *differential invariant* if it is unaffected by the prolonged group transformations, so $I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)})$ for all $z^{(n)} \in \mathrm{J}^n$ and all $g \in G$ such that both $z^{(n)}$ and $g^{(n)} \cdot z^{(n)}$ lie in the domain of I. Any finite-dimensional group action admits an infinite number of functionally independent differential invariants of progressively higher and higher order. The *Basis Theorem* for differential invariants first formulated by Lie, [**12**; p. 760], and then extended by Tresse, [**26**], to infinite-dimensional pseudo-group actions, states that all the differential invariants can be generated from a finite number of low order invariants by repeated invariant differentiation. Modern proofs can be found in [**17**, **24**].

Theorem 2.1. *Given a finite-dimensional Lie group G acting on p-dimensional submanifolds $N \subset M$, then, locally, there exist finitely many* generating differential invariants

† Throughout, functions, maps, etc., may only be defined on an open subset of their indicated domain.

$I_1, \ldots, I_\ell$, along with exactly p invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$, with the property that every differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives: $\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_k} I_\kappa$.

The invariant differential operators do not necessarily commute, and so the order of differentiation is important. However, each commutator can be re-expressed as

$$[\, \mathcal{D}_i, \mathcal{D}_j \,] = \sum_{k=1}^{p} J_{ij}^k \mathcal{D}_k, \tag{2.2}$$

where the coefficients J_{ij}^k are certain differential invariants, and hence functions of the $\mathcal{D}_J I_\kappa$. Moreover, the differentiated invariants are *not* necessarily functionally independent, but may be subject to certain functional relations or *differential syzygies* of the form

$$H(\ \ldots \ \mathcal{D}_J I_\kappa \ \ldots \ ) \equiv 0. \tag{2.3}$$

In [**3**], it was proved that there are a finite number of generating differential syzygies; see also [**23**] for extensions to pseudo-group actions. Together, the commutation relations (2.2) and syzygies (2.3) completely prescribe the structure of the algebra† of differential invariants.

A familiar example is $G = \mathrm{SE}(3)$, the (special) Euclidean group consisting of all rigid, orientation-preserving motions of $M = \mathbb{R}^3$, acting on space curves, i.e., one-dimensional submanifolds. The differential invariant algebra is generated by the curvature κ, torsion τ, and their successive derivatives with respect to arc length, [**4**, **7**]. Similarly, the differential invariants for the action of SE(3) on surfaces $S \subset \mathbb{R}^3$ are the Gauss and mean curvatures and their derivatives with respect to two non-commuting invariant differential operators, which are closely related to, but not exactly the same as, the standard covariant derivatives, cf. [**17**]. In this case, there is a single fundamental differential syzygy among the curvature invariants: the Gauss–Codazzi formula, [**11**].

In general, for most‡ Lie group actions on curves, so $p = 1$, the number of generating differential invariants is equal to $q = m - 1$, and there are no syzygies. However, when dealing with higher dimensional submanifolds, where $p > 1$, the number of generating differential invariants can vary with the transformation group. For example, according to recent results in [**22**], the differential invariants of surfaces under the three-dimensional Euclidean group action are generated by just the mean curvature through invariant differentiation; similarly, the equi- (volume-preserving) affine group SA(3) acting on surfaces in $\mathbb{R}^3$ also requires only one generting differential invariant — the third order Pick invariant, cf. [**10**, **25**]. At the other extreme, the following rather trivial abelian group actions

† Technically, because differential invariants may only be locally defined, we should speak of the "sheaf of differential invariants". However, as we work locally on suitable open subsets, this extra level of abstraction is not required; moreover, experts can readily translate our constructions into sheaf-theoretic language.

‡ More precisely, we require that the group action be "ordinary", [**17**], meaning it is transitive on M and does not pseudo-stabilize when prolonged. Non-ordinary actions on curves require one additional generating invariant.

on surfaces demonstrates that, when the submanifolds have dimension $p \geq 2$, there is no universal upper bound on the required number of generating differential invariants.

Example 2.2. Consider the abelian group G_V acting on $M = \mathbb{R}^3$ via

$$(x, y, u) \longmapsto \big(x + a, y + b, u + \varphi(x, y) \big), \tag{2.4}$$

where $a, b \in \mathbb{R}$ and $\varphi(x, y) \in V \subset \mathbb{R}[x, y]$ is an arbitrary element of a finite-dimensional subspace of the space of polynomial functions of (x, y). The infinitesimal generators are

$$\mathbf{w}_1 = \partial_x, \qquad \mathbf{w}_2 = \partial_y, \qquad \mathbf{v}_j = \varphi_j(x, y)\partial_u, \qquad j = 1, \ldots, s = \dim V, \tag{2.5}$$

where $\varphi_1, \ldots, \varphi_s$ form a basis of V. We are interested in the induced action of this $(s+2)$-dimensional transformation group on graphs of functions $u = f(x, y)$, i.e., surfaces.

In the particular case when $V = V_n$ consists of all polynomials of degree $\leq n$, then it is easy to see that the individual derivatives $u_{i,j} = \partial^{i+j}u/\partial x^i \partial y^j$ for $i + j \geq n + 1$ form a complete system of functionally independent differential invariants. Since the action on the independent variables is just translation, the invariant differential operators are the usual total derivatives:

$$\mathcal{D}_1 = D_x, \qquad \mathcal{D}_2 = D_y.$$

The higher order differential invariants are generated by differentiating the $n+1$ differential invariants $u_{i,j}$ of order $n + 1 = i + j$. Moreover, these invariants clearly form a minimal generating set for this particular action. We conclude that there is no universal bound on the number of required generating differential invariants, even for such an elementary class of group actions.

3. Moving Frames.

The equivariant method of moving frames, inspired by Cartan, [**1**, **7**], and initiated in [**3**], provides an effective means of not only constructing the differential invariants and invariant differential operators for general Lie group actions, but also revealing the structure of their induced non-commutative differential algebra. More recent extensions of these methods to infinite-dimensional pseudo-groups can be found in [**21**, **23**], and many of the techniques and results, suitably interpreted, carry over to this context. However, for simplicity and brevity, in this paper we deal only with finite-dimensional Lie group actions.

Assuming that the prolonged action is free[†] on an open subset of J^n, then one can construct a (locally defined) *moving frame*, which, according to [**3**], is an equivariant map $\rho\colon \mathrm{J}^n \to G$. Equivariance can be with respect to either the right or left multiplication

[†] A theorem of Ovsiannikov, [**24**], slightly corrected in [**18**], guarantees local freeness of the prolonged action at sufficiently high order, provided G acts locally effectively on subsets of M. This is only a technical restriction; for example, all analytic actions can be made effective by dividing by the global isotropy subgroup. Although all known examples of prolonged effective group actions are, in fact, free on an open subset of a sufficiently high order jet space, there is, frustratingly, as yet no general proof, nor known counterexample, to this result.

action of G on itself. All classical moving frames, e.g., those appearing in [**1**, **5**, **6**, **7**, **10**], can be regarded as left equivariant maps, but the right equivariant versions may be easier to compute. Of course, any right moving frame can be converted to a left moving frame by composition with the inversion map $g \mapsto g^{-1}$.

In practice, one constructs a moving frame by the process of normalization, relying on the choice of a local *cross-section* $K^n \subset \mathrm{J}^n$ to the prolonged group orbits. The corresponding value of the right moving frame at a jet $z^{(n)} \in \mathrm{J}^n$ is the unique group element $g = \rho^{(n)}(z^{(n)}) \in G$ that maps it to the cross-section:

$$\rho^{(n)}(z^{(n)}) \cdot z^{(n)} = g^{(n)} \cdot z^{(n)} \in K^n. \tag{3.1}$$

The moving frame $\rho^{(n)}$ clearly depends on the choice of cross-section, which is usually designed so as to simplify the required computations as much as possible.

Typically, simplification requires that one choose the moving frame to have as low an order as possible. Such "minimal order" moving frames will be a focus of this paper. Since the existence of a moving frame requires (local) freeness of the prolonged group action, the minimal order of any moving frame is just the order of the jet space at which the group action first becomes locally free. However, for our purposes, this in itself does not suffice, and we will use the term "minimal order" in a stricter sense, requiring that *all* the cross-section normalization equations have as low an order as possible.

Definition 3.1. A cross-section $K^n \subset \mathrm{J}^n$, and, hence its induced moving frame $\rho^{(n)}\colon \mathrm{J}^n \to G$, is said to be of *minimal order* if, for each $0 \leq k \leq n$, its projection $K^k = \pi^n_k(K^n) \subset \mathrm{J}^k$ forms a cross-section to the orbits of $G^{(k)}$ on J^k. Here $\pi^n_k\colon \mathrm{J}^n \to \mathrm{J}^k$ denotes the standard jet space projection map, [**17**].

Remark: From here on, a *cross-section* will be taken to mean a submanifold $K^k \subset \mathrm{J}^k$ of the complementary dimension transverse to the maximal dimension prolonged group orbits. We do not necessarily require that the cross-section intersect an orbit in a unique point, and so the normalization construction will only produce a locally equivariant moving frame and local differential invariants, that may retain certain discrete ambiguities. See Hubert and Kogan, [**9**], for further details on the use of semi-regular cross-sections for invariantization.

As a specific example, consider the familiar action of the Euclidean group SE(2) on plane curves $C \subset M = \mathbb{R}^2$. The first order prolonged action is only locally free, because a 180° rotation around a point on the curve will preserve its tangent line, and hence has trivial first order prolongation. Indeed, the classical moving frame, consisting of the unit tangent and normal†, is only locally equivariant, since the 180° rotation will reverse the direction of the two frame vectors and also reverse the sign of the curvature differential invariant κ. The second order prolonged action of SE(2) is free on the subset $\{\kappa \neq 0\} \subset \mathrm{J}^2(M,1)$, and so one can resolve the sign ambiguity by going to second order. (Classically, the

† To interpret the classical construction as a left equivariant map to SE(2), we regard the point on the curve as the translation component, and the two orthonormal frame vectors as forming the columns of a rotation matrix. See [**3**, **20**] for details. Section 6 below discusses the three-dimensional counterpart.

ambiguity is resolved by assigning an orientation to the parametrized curve.) Incidentally, the full Euclidean group E(2), which also includes the reflections, introduces a second sign ambiguity owing to the action of a reflection through the tangent line, which is only fully resolved at third order. See [**19**] for a complete discussion.

Classical moving frames are inevitably of minimal order. Indeed, the normalization procedure advocated in [**1**, **5**, **6**, **7**, **10**] proceeds inductively by order, and one seeks to normalize as many jet coordinates as possible before proceeding to the next higher order. A key innovation of [**3**] was to point out the possibility of using non-minimal order moving frames to generate differential invariants and thereby resolve equivalence problems even at singularities where the classical minimal order moving frame breaks down, e.g., non-degenerate inflection points of space curves, or nondegenerate umbilics of surfaces in Euclidean geometry.

In general, for each $k \geq 0$, let $1 \leq r_k \leq r$ denote the maximal orbit dimension[†] of the k^{th} order prolonged action of $G^{(k)}$ on J^k. The action is locally free at order n if and only if $r_n = r = \dim G$. A jet $z^{(k)} \in \mathrm{J}^k$ is called *regular* if it lies in an r_k–dimensional orbit of $G^{(k)}$. Let $V^k \subset \mathrm{J}^k$ be the open (and necessarily dense if the action is analytic) subset consisting of the regular jets. A jet $z^{(n)} \in \mathrm{J}^n$ is called *completely regular* if it and its projections $z^{(k)} = \pi^n_k(z^{(n)}) \in V^k$ are regular for all $k = 0, \ldots, n$.

Assuming local freeness of $G^{(n)}$, every cross-section $K^n \subset V^n$ has dimension

$$\dim K^n = \dim \mathrm{J}^n - r = q^{(n)} - r. \tag{3.2}$$

According to Definition 3.1, the moving frame is of minimal order if, in addition,

$$\dim K^k = \dim \pi^n_k(K^n) = \dim \mathrm{J}^k - r_k = q^{(k)} - r_k \qquad \text{for all} \qquad k = 0, \ldots, n. \tag{3.3}$$

In particular, minimality requires that every jet $z^{(n)} \in K^n$ be completely regular. Examples of minimal and non-minimal cross-sections appear below.

To compute, we introduce local coordinates $z = (x, u) = (x^1, \ldots, x^p, u^1, \ldots, u^q)$ on M — considering the first p as independent variables, and the latter $q = m - p$ as dependent variables. We locally identify the submanifolds with graphs of functions $u = f(x)$. (This omits submanifolds that are not transversal to the vertical fibers $x = c$, but these can be handled by using an alternative coordinate chart.) The induced local coordinates on J^n are denoted $z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )$, with u^α_J, for $0 \leq \#J \leq n$ and $1 \leq \alpha \leq q$, representing the partial derivatives of the dependent variables with respect to the independent variables, [**17**]. Each jet space coordinate x^i or u^α_J is indexed by either a single integer $1 \leq i \leq p$, or a multi-index pair $(J; \alpha)$ where $J = (j_1, \ldots, j_k)$ is an unordered multi-index with each $1 \leq j_\nu \leq p$ and $1 \leq \alpha \leq q$. In particular, the dependent variable u^α corresponds to the pair $(0; \alpha)$, where 0 denotes the empty multi-index. We let $\mathcal{T}$ denote the set of all such indices — i or $(J; \alpha)$ — and $\mathcal{T}^{(n)}$ those of order $k = \#J \leq n$. (By convention, the single index i has order 0.) If $\mathcal{S} \subset \mathcal{T}$ is any subset, we set $\mathcal{S}^{(n)} = \mathcal{S} \cap \mathcal{T}^{(n)}$.

[†] If any $r_n = 0$, then all $r_k = 0$, and the action is purely discrete. We are not interested in discrete actions here.

In most cases, one selects a *coordinate cross-section* defined by setting a number of the coordinate functions to specified constant values, so the resulting $K^n \subset \mathrm{J}^n$ is parallel to the coordinate axes. (See [**13**] for examples based on non-coordinate cross-sections; adapting our constructions to this more general context situation is not difficult, but we will stick with coordinate cross-section to avoid technical complications.) Each coordinate cross-section passing through a fixed regular jet $z_0^{(n)} \in V^n \subset \mathrm{J}^n$ corresponds to a subset $\mathcal{P} \subset \mathcal{T}^{(n)}$ of cardinality $|\,\mathcal{P}\,| = r = \dim G$. Assuming transversality, the coordinate cross-section associated with $\mathcal{P}$ is prescribed by the equations

$$x^i = c^i, \qquad u_K^\beta = c_K^\beta, \qquad \text{for all} \qquad i, (K;\beta) \in \mathcal{P}, \tag{3.4}$$

where c^i, c_K^β denote the values of the corresponding coordinates of the jet $z_0^{(n)}$.

According to Definition 3.1, if the cross-section through $z_0^{(n)}$ defined by $\mathcal{P}$ is of minimal order, then the number of normalization equations of each order $0 \le k \le n$, or, equivalently, the cardinality of $\mathcal{P}^{(k)} = \mathcal{P} \cap \mathcal{T}^{(k)}$, must be as large as possible, namely $|\,\mathcal{P}^{(k)}\,| = r_k$, the maximal prolonged orbit dimension on J^k. Note that $\mathcal{P}^{(k)}$ indexes all the normalization equations of order $\le k$. Therefore:

Lemma 3.2. *If the normalization equations* (3.4) *define a minimal order cross-section, then the number of equations of order* $= k$ *is* $r_k - r_{k-1}$.

Keep in mind that, to define a bona fide cross-section, there is also a transversality condition, that will be properly dealt with below.

Once the cross-section has been fixed, the induced moving frame engenders an invariantization process, that effectively maps functions to invariants, differential forms to invariant differential forms, and so on, [**3**, **20**]. Geometrically, the *invariantization* of any object is defined as the unique invariant object that coincides with its progenitor when restricted to the cross-section. In particular, invariantization does not affect invariants, and hence defines a morphism that projects the algebra of differential functions onto the algebra of differential invariants.

Computationally, the invariantization of a differential function is constructed by first writing out how it is transformed by the prolonged group action: $F(z^{(n)}) \mapsto F(g^{(n)} \cdot z^{(n)})$. One then replaces all the group parameters by their *right* moving frame formulae $g = \rho^{(n)}(z^{(n)})$, resulting in the differential invariant

$$\iota\big[\,F(z^{(n)})\,\big] = F\big(\,\rho^{(n)}(z^{(n)}) \cdot z^{(n)}\,\big). \tag{3.5}$$

Differential forms and differential operators are handled in an analogous fashion — see [**3**, **11**] for complete details.

In particular, the *normalized differential invariants* induced by the moving frame are obtained by invariantization of the basic jet coordinates

$$H^i = \iota(x^i), \qquad I_J^\alpha = \iota(u_J^\alpha). \tag{3.6}$$

These naturally split into two classes: Those corresponding to the cross-section coordinates (3.4) are constant, and known as the *phantom differential invariants*. The remainder, known as the *basic differential invariants*, form a complete system of functionally independent differential invariants. Thus, the index set $\mathcal{P}$ used to prescribe the coordinate cross-section (3.4) also serves to index the phantom differential invariants, and so its elements will be called *phantom indices*. The complement $\mathcal{B} = \mathcal{T} \setminus \mathcal{P}$ indexes the basic differential invariants, and hence its elements will be called *basic indices*. Note in particular that *every* index $(J; \alpha)$ of order $\# J > n$ strictly greater than the moving frame is basic. (This property distinguishes finite-dimensional Lie group actions from infinite-dimensional pseudo-groups, [**21**].)

We call $\# J$ the *degree* of the differential invariant I^α_J, with the convention that the H^i also are of degree 0. If the moving frame is of order n, then

$$\operatorname{order} H^i \leq n, \qquad\qquad \operatorname{order} I^\alpha_J \leq \max\{n, \# J\}. \tag{3.7}$$

Of course, the phantom invariants are constant, and hence of order 0. We use $(H, I) = (\ \ldots\ H^i\ \ldots\ I^\alpha\ \ldots\ )$ to denote the degree 0 differential invariants — all of which are constant if the group acts transitively on M and we choose to normalize all of the base coordinates (x, u) — as would be required for a minimal order moving frame — and $(H, I^{(n)}) = (\ldots H^i \ldots I^\alpha_J \ldots)$ with $\# J \leq n$ to denote the complete system of normalized differential invariants of degree $\leq n$.

Once the normalized differential invariants are known, the invariantization process (3.5) is implemented by simply replacing each jet coordinate by the corresponding normalized differential invariant (3.6), so that

$$\iota\big[\, F(x, u^{(n)}) \,\big] = \iota\big[\, F(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ ) \,\big] = F(\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ ) = F(H, I^{(n)}). \tag{3.8}$$

In particular, if we start with a differential invariant, it is not affected by this we recover the remarkable (but trivial) *Replacement Theorem*:

$$I(x, u^{(n)}) = I(H, I^{(n)}) \quad \text{whenever } I \text{ is a differential invariant.} \tag{3.9}$$

This permits one to straightforwardly rewrite any known differential invariant in terms the basic invariants, and thereby establishes their completeness.

4. Infinitesimal Generators and the Lie Matrix.

Suppose the vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \tag{4.1}$$

represents an infinitesimal generator of the action of G on M. Let

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{0 \leq k = \# J \leq n} \varphi^\alpha_J(x, u^{(k)}) \frac{\partial}{\partial u^\alpha_J} \tag{4.2}$$

denote the corresponding prolonged infinitesimal generator of the action of $G^{(n)}$ on J^n. Its coefficient functions φ^α_J are prescribed by the well-known prolongation formula, [**17**],

$$\varphi^\alpha_J = D_J \left(\varphi^\alpha - \sum_{i=1}^{p} \xi^i\, u^\alpha_i \right) + \sum_{i=1}^{p} \xi^i u^\alpha_{J,i}, \tag{4.3}$$

where $D_J = D_{j_1} \cdots D_{j_k}$ is the corresponding iterated total derivative.

From here on, we will fix a basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$ for the Lie algebra $\mathfrak{g}$ of infinitesimal generators of our transformation group.

Definition 4.1. The *Lie matrix* of order n is the $q^{(n)} \times r$ matrix†

$$\mathbf{L}^{(n)}(x, u^{(n)}) = \begin{pmatrix} \xi_1^1 & \ldots & \xi_r^1 \\ \vdots & \ddots & \vdots \\ \xi_1^p & \ldots & \xi_r^p \\ \varphi_1^1 & \ldots & \varphi_r^1 \\ \vdots & \ddots & \vdots \\ \varphi_1^q & \ldots & \varphi_r^q \\ \vdots & \ddots & \vdots \\ \varphi_{J,1}^\alpha & \ldots & \varphi_{J,r}^\alpha \\ \vdots & \ddots & \vdots \end{pmatrix}, \qquad \text{where} \qquad 0 \leq \#J \leq n. \tag{4.4}$$

Its entries $\xi_\kappa^i, \varphi_\kappa^\alpha, \varphi_{J,\kappa}^\alpha$ are the coefficients (4.2) of the n^{th} order prolongations, $\mathbf{v}_1^{(n)}, \ldots, \mathbf{v}_r^{(n)}$, of the chosen basis infinitesimal generators.

At a jet $z^{(n)} = (x, u^{(n)}) \in \mathrm{J}^n$, the rank of the Lie matrix $\mathbf{L}^{(n)}(z^{(n)})$ equals the dimension of the prolonged group orbit passing through $z^{(n)}$. In particular, $z^{(n)}$ is a regular jet if and only if $\operatorname{rank} \mathbf{L}^{(n)}(z^{(n)}) = r_n = r = \dim G$. Moreover, $z^{(n)}$ is completely regular if and only if $\operatorname{rank} \mathbf{L}^{(k)}(z^{(k)}) = r_k$ for all $0 \leq k \leq n$, where $z^{(k)} = \pi_k^n(z^{(n)})$.

The rows of the n^{th} order Lie matrix are indexed by the elements of $\mathcal{T}^{(n)}$, and we indicate them by the corresponding bold face symbol: $\boldsymbol{\xi}^i = (\xi_1^i, \ldots, \xi_r^i)$ or $\boldsymbol{\varphi}_J^\alpha = (\varphi_{J,1}^\alpha \ldots \varphi_{J,r}^\alpha)$. The *order* of a row is that of its associated index, namely $\operatorname{order} \boldsymbol{\xi}^i = 0$, while $\operatorname{order} \boldsymbol{\varphi}_J^\alpha = \#J$. Given any subset $\mathcal{S} \subset \mathcal{T}^{(n)}$ of row indices, we let $\mathbf{L}_{\mathcal{S}}^{(n)} = \mathbf{L}_{\mathcal{S}}^{(n)}(z^{(n)})$ denote the corresponding $|\,\mathcal{S}\,| \times r$ Lie submatrix formed by the rows indexed by $\mathcal{S}$.

Lemma 4.2. *A subset $\mathcal{P} \subset \mathcal{T}^{(n)}$ containing $|\,\mathcal{P}\,| = r$ indices defines a cross-section* (3.4) *through the regular jet $z_0^{(n)}$ if and only if the corresponding $r \times r$* Lie minor *is nonsingular:*

$$\det \mathbf{L}_{\mathcal{P}}^{(n)}(z_0^{(n)}) \neq 0. \tag{4.5}$$

We call a row of the Lie matrix $\mathbf{L}^{(n)}(z^{(n)})$ either *phantom* or *basic* according to whether its index belongs to $\mathcal{P}$ or $\mathcal{B}^{(n)} = \mathcal{T}^{(n)} \setminus \mathcal{P}$. (In linear algebraic terms, the phantom rows would correspond to the free variables and the basic rows to the basic variables following from the appropriate column echelon form that results from (transposed) Gaussian Elimination.) A straightforward translation of Definition 3.1 yields the following characterization of minimal order moving frames.

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† *Warning*: In many texts, e.g., [**18**], the transpose of this matrix is known as the Lie matrix. To avoid unnecessary transpose notation, we will adopt this convention throughout.

Lemma 4.3. *At a completely regular jet $z_0^{(n)} \in \mathrm{J}^n$, the moving frame defined by $\mathcal{P} \subset \mathcal{T}^{(n)}$ is of minimal order if and only if, for each $k = 0, \ldots, n$, the rank of the Lie submatrix consisting of the phantom rows of order $\leq k$ equals*

$$\operatorname{rank} \mathbf{L}^{(k)}_{\mathcal{P}^{(k)}}(z_0^{(k)}) = |\, \mathcal{P}^{(k)} \,| = r_k = \operatorname{rank} \mathbf{L}^{(k)}(z_0^{(k)}).$$

Corollary 4.4. *The moving frame defined by $\mathcal{P} \subset \mathcal{T}^{(n)}$ is of minimal order if and only if each basic row of the Lie matrix can be written as a linear combination of the phantom rows of equal or lower order*:

$$\begin{aligned} \boldsymbol{\xi}^i &= \sum_{l \in \mathcal{P}^{(0)}} h^i_l(x,u)\, \boldsymbol{\xi}^l \;+ \sum_{(0,\beta) \in \mathcal{P}^{(0)}} h^i_\beta(x,u)\, \boldsymbol{\varphi}^\beta, \\ \boldsymbol{\varphi}^\alpha_J &= \sum_{l \in \mathcal{P}^{(0)}} h^\alpha_{J,l}(x,u^{(k)})\, \boldsymbol{\xi}^l \;+ \sum_{(K;\beta) \in \mathcal{P}^{(k)}} h^{\alpha,K}_{J,\beta}(x,u^{(k)})\, \boldsymbol{\varphi}^\beta_K, \qquad \textit{where} \qquad k = \#J. \end{aligned} \tag{4.6}$$

Proof: If a basic row of order k were not a linear combination of phantom rows of that order or less, this would mean that the rank of $\mathbf{L}^{(k)}(z_0^{(k)})$ would be strictly greater than the cardinality of $\mathcal{P}^{(k)}$, which would contradict Lemma 4.3. *Q.E.D.*

5. Recurrence Formulae.

Given a moving frame, the associated invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are obtained by invariantization of the total derivatives:

$$\mathcal{D}_i = \iota(D_i), \qquad i = 1, \ldots, p, \tag{5.1}$$

Equivalently, they can be defined as the dual differential operators arising from the invariant horizontal forms

$$\omega^i = \iota(dx^i), \qquad i = 1, \ldots, p, \tag{5.2}$$

obtained by (horizontal) invariantization of the basic horizontal one-forms $dx^1, \ldots, dx^p$. Details can be found in [**3**, **11**].

Each invariant differential operator maps differential invariants to differential invariants. Moreover, the differentiated invariants $\mathcal{D}_i H^j$ and $\mathcal{D}_i I^\alpha_J$ can be written in terms of the normalized differential invariants. Understanding these so-called *recurrence formulae* is the master key that unlocks the structure of the algebra of differential invariants, the determination of generators, and the classification of syzygies. Remarkably, [**3**, **21**], the recurrence formulae can be explicitly determined without knowing the actual formulas for either the differential invariants, or the invariant differential operators, or even the moving frame! The only required ingredients are the prolongation formulas for the infinitesimal generators, or, equivalently, the Lie matrix, along with the specification of the cross-section normalizations.

To formulate the construction, we introduce the *invariantized Lie matrix*

$$\mathbf{M}^{(n)}(H,I^{(n)}) = \iota\big(\,\mathbf{M}^{(n)}(x,u^{(n)})\,\big) = \begin{pmatrix} \eta_1^1 & \ldots & \eta_r^1 \\ \vdots & \ddots & \vdots \\ \eta_1^p & \ldots & \eta_r^p \\ \psi_1^1 & \ldots & \psi_r^1 \\ \vdots & \ddots & \vdots \\ \psi_1^q & \ldots & \psi_r^q \\ \vdots & \ddots & \vdots \\ \psi_{J,1}^\alpha & \ldots & \psi_{J,r}^\alpha \\ \vdots & \ddots & \vdots \end{pmatrix}, \quad \text{where} \quad 0 \leq \#J \leq n, \tag{5.3}$$

whose entries are obtained by invariantizing the infinitesimal generator coefficients:

$$\eta_\kappa^i(H,I) = \iota\big[\,\xi_\kappa^i(x,u))\,\big], \qquad \psi_\kappa^\alpha(H,I) = \iota\big[\,\varphi_\kappa^\alpha(x,u)\,\big], \qquad \psi_{J,\kappa}^\alpha(H,I^{(k)}) = \iota\big[\,\varphi_{J,\kappa}^\alpha(x,u^{(k)}))\,\big]. \tag{5.4}$$

We also employ the corresponding bold face symbols,

$$\boldsymbol{\eta}^i = \iota(\boldsymbol{\xi}^i), \qquad\qquad \boldsymbol{\psi}_J^\alpha = \iota(\boldsymbol{\varphi}_J^\alpha), \tag{5.5}$$

to indicate the individual rows of the invariantized Lie matrix. Keep in mind that the invariantized and ordinary Lie matrices agree when restricted to the cross-section, and hence have isomorphic algebraic structure.

Theorem 5.1. *The* recurrence formulae *for the differentiated invariants are*

$$\mathcal{D}_i H^j = \delta_i^j + \boldsymbol{\eta}^j(H,I)\,\mathbf{R}_i, \qquad\qquad \mathcal{D}_i I_J^\alpha = I_{J,i}^\alpha + \boldsymbol{\psi}_J^\alpha(H,I^{(k)})\,\mathbf{R}_i. \tag{5.6}$$

In these formulas, δ_i^j is the usual Kronecker symbol, while each $\mathbf{R}_i = (R_i^1,\ldots,R_i^r)^T$, for $i = 1,\ldots,p$, is a column vector whose $r = \dim G$ entries are certain differential invariants.

The entries R_i^κ of the $\mathbf{R}_i$ will be called the *Maurer–Cartan invariants*, because, according to [**3**], they can be identified as the coefficients of the invariant horizontal one-forms ω^i in the moving frame pull-backs

$$\gamma^\kappa = (\rho^{(n)})^*(\mu^\kappa) = \sum_{i=1}^p R_i^\kappa\,\omega^i + \cdots\,, \tag{5.7}$$

where the dots indicate contact forms, while $\mu^1,\ldots,\mu^r$ form the basis of Maurer–Cartan forms dual to the chosen infinitesimal generator basis $\mathbf{v}_1,\ldots,\mathbf{v}_r$. To explain all this in any detail would take several paragraphs. Fortunately, this turns out to be completely unnecessary from an algorithmic viewpoint. The Maurer–Cartan invariants are, in fact, uniquely prescribed by the recurrence formulae, and so, for computational purposes, one can remain blissfully unaware of how they arise from the Maurer–Cartan forms! (However, the proof of the recurrence formulae (5.6) *does* rely essentially on this identification, [**3**].)

Indeed, given a coordinate cross-section prescribed by a set of phantom indices $\mathcal{P} \subset \mathcal{T}$, subject to the transversality constraint (4.5), the full system of recurrence formulae (5.6) naturally splits into two subsystems. Since the phantom differential invariants are constant, the corresponding *phantom recurrence relations* have the form

$$0 = \mathcal{D}_i H^l = \delta^l_i + \boldsymbol{\eta}^l \,\mathbf{R}_i, \qquad 0 = \mathcal{D}_i I^\beta_K = I^\beta_{K,i} + \boldsymbol{\psi}^\beta_K \,\mathbf{R}_i, \quad \text{for all} \quad l,\ (K;\beta) \in \mathcal{P}. \tag{5.8}$$

For each fixed i, (5.8) forms a system of r linear algebraic equations in the r unknown entries of $\mathbf{R}_i$. In fact, its coefficient matrix, whose rows are $\boldsymbol{\eta}^l, \boldsymbol{\psi}^\beta_K$, is nothing but the invariantized Lie matrix minor corresponding to the phantom indices:

$$\mathbf{M}^{(n)}_{\mathcal{P}}(H, I^{(n)}) = \iota\big[\,\mathbf{L}^{(n)}_{\mathcal{P}}(x, u^{(n)})\,\big].$$

Since we are using a bona fide cross-section, Lemma 4.2 implies that the coefficient matrix is invertible. (Here we are using the fact that the invariantized Lie matrix agrees with the oridinary Lie matrix when restricted to the cross-section.) We conclude that the phantom recurrence equations (5.8) can be uniquely solved for the Maurer–Cartan invariants $\mathbf{R}_i$. They are then substituted into the remaining *basic recurrence formulae*

$$\mathcal{D}_i H^j = \delta^j_i + \boldsymbol{\eta}^j \,\mathbf{R}_i, \qquad \mathcal{D}_i I^\alpha_J = I^\alpha_{J,i} + \boldsymbol{\psi}^\alpha_J \,\mathbf{R}_i \qquad \text{for} \qquad j,\ (J;\alpha) \in \mathcal{B}, \tag{5.9}$$

that explicitly relate the normalized and differentiated invariants. The resulting *fundamental recurrence formulae* serve to completely characterize the algebra of differential invariants, and, through their detailed analysis, allow us to pinpoint the generating differential invariants and their syzygies. Examples of this procedure can be found in [**3**, **11**, **20**, **19**] and below.

Remark: It is well known, [**17**], that the coefficients of the prolonged infinitesimal generators of any group action are polynomial functions of the jet coordinates u^α_J for all $\# J \geq 1$. Therefore, the Maurer–Cartan invariants, being solutions to a linear system with polynomially varying coefficients, are rational functions of the generating invariants, except possibly those of index 0, namely H^i, I^α. In particular, if the action is transitive on M, and we normalize all the order zero coordinates — or, more generally, the infinitesimal generators on M depend rationally on the coordinates (x, u) — then we conclude that the Maurer–Cartan invariants, and hence all the higher order normalized differential invariants, are rational functions of the generating differential invariants. The same holds for a large class of pseudo-group actions, [**23**]: the differential invariant algebra is intrinsically *rational*, in the sense that all recurrence formula, commutation relations and syzygies involve rational functions of the basic differential invariants (of order ≥ 1).

Definition 5.2. Given phantom indices $\mathcal{P} \subset \mathcal{T}$, we define the set of *edge indices* $\mathcal{E} \subset \mathcal{B} = \mathcal{T} \setminus \mathcal{P}$ to consist of all zero$^{\text{th}}$ order basic indices $i, (0;\alpha) \in \mathcal{B}^{(0)}$, if any, along with all basic indices of the form $(J, i;\alpha) \in \mathcal{B}$ with $(J;\alpha) \in \mathcal{P}$ a phantom index.

Remark: The edge indices lie on the "edges" of the subset $\mathcal{B} \subset \mathcal{T}$ of all basic indices, meaning that they appear next to a phantom index of lower order. For instance, if $p = 2,\ q = 1$, the edge indices corresponding to $\mathcal{P} = \big\{\,1, 2, (0;1), (1;1), (2,2,2;1)\,\big\}$ are $\mathcal{E} = \big\{(2;1), (1,1;1),\ (1,2;1), (1,2,2,2;1), (2,2,2,2;1)\big\}$.

In this terminology, Theorem 13.3 in [**3**] states that the *edge differential invariants*, meaning those normalized differential invariants indexed by the elements of $\mathcal{E}$, form a generating set. In the following section, we present an explicit counterexample to this claim, which is based on a non-minimal order moving frame for the Euclidean geometry of space curves.

6. An Instructive Example.

Consider the standard action of the $r = 6$ – dimensional Euclidean group SE(3) on space curves $C \subset M = \mathbb{R}^3$. We use coordinates $z = (x, u, v)$ and, to avoid having to deal with the infinite-dimensional reparametrization pseudo-group, restrict our attention to curves given by the graphs of functions $u = u(x),\ v = v(x)$. However, all our results remain valid for general parametrized curves $z(t) = (\, x(t), u(t), v(t)\,)^T$. We use

$$z_t = \begin{pmatrix} x_t \\ u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 1 \\ u_x \\ v_x \end{pmatrix}, \qquad z_{tt} = \begin{pmatrix} x_{tt} \\ u_{tt} \\ v_{tt} \end{pmatrix} = \begin{pmatrix} 0 \\ u_{xx} \\ v_{xx} \end{pmatrix}, \qquad z_{ttt} = \begin{pmatrix} x_{ttt} \\ u_{ttt} \\ v_{ttt} \end{pmatrix} = \begin{pmatrix} 0 \\ u_{xxx} \\ v_{xxx} \end{pmatrix}, \tag{6.1}$$

and so on, to denote the derivative vectors along the curve, where the second expression can be used in the special case of a graph, parametrized by $t = x$.

A basis for the infinitesimal generators is provided by the vector fields

$$\begin{aligned} &\mathbf{v}_1 = \partial_x, && \mathbf{v}_2 = \partial_u, && \mathbf{v}_3 = \partial_v, \\ &\mathbf{v}_4 = v\partial_u - u\partial_v, && \mathbf{v}_5 = -u\partial_x + x\partial_u, && \mathbf{v}_6 = -v\partial_x + x\partial_v. \end{aligned} \tag{6.2}$$

The Lie matrices are easily computed; at order 4, say, $\mathbf{L}^{(4)}$ equals

$$\begin{pmatrix}
1 & 0 & 0 & 0 & -u & -v \\
0 & 1 & 0 & v & x & 0 \\
0 & 0 & 1 & -u & 0 & x \\
0 & 0 & 0 & v_x & 1+u_x^2 & u_x v_x \\
0 & 0 & 0 & -u_x & u_x v_x & 1+v_x^2 \\
0 & 0 & 0 & v_{xx} & 3\,u_x u_{xx} & 2\,u_{xx} v_x + u_x v_{xx} \\
0 & 0 & 0 & -u_{xx} & u_{xx} v_x + 2\,u_x v_{xx} & 3\,v_x v_{xx} \\
0 & 0 & 0 & v_{xxx} & 4\,u_x u_{xxx} + 3\,u_{xx}^2 & 3u_{xxx} v_x + 3\,u_{xx} v_{xx} + u_x v_{xxx} \\
0 & 0 & 0 & -u_{xxx} & u_{xxx} v_x + 3\,u_{xx} v_{xx} + 3\,u_x v_{xxx} & 4v_x v_{xxx} + 3v_{xx}^2 \\
0 & 0 & 0 & v_{xxxx} & 5\,u_x u_{xxxx} + 10\,u_{xx} u_{xxx} & \begin{matrix} 4\,u_{xxxx} v_x + 6\,u_{xxx} v_{xx} + \\ + 4\,u_{xx} v_{xxx} + u_x v_{xxxx} \end{matrix} \\
0 & 0 & 0 & -u_{xxxx} & \begin{matrix} u_{xxxx} v_x + 4\,u_{xxx} v_{xx} + \\ + 6\,u_{xx} v_{xxx} + 4\,u_x v_{xxxx} \end{matrix} & 5\,v_x v_{xxxx} + 10\,v_{xx} v_{xxx}
\end{pmatrix} \tag{6.3}$$

Its rows are indexed by the jet variables $x, u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, \ldots$, while each column represents a prolonged infinitesimal generator. The corresponding index set $\mathcal{T}$ consists of the single index 1, corresponding to x, and the multi-indices† $(k;1),(k;2)$ representing, respectively, the jet coordinates $u_k = D_x^k u$ and $v_k = D_x^k v$.

The classical moving frame, [**7**], relies on the equations

$$x = 0, \qquad u = 0, \qquad v = 0, \qquad u_x = 0, \qquad v_x = 0, \qquad v_{xx} = 0,$$

which serve to define a coordinate cross-section provided $u_{xx} \neq 0$. (Indeed, the classical moving frame is not defined at inflection points of the space curve, [**4**, **7**].) The classical cross-section is of minimal order, because the maximal prolonged orbit dimensions (or, equivalently, Lie matrix ranks) are $r_0 = 3$, $r_1 = 5$, $r_2 = 6$, and, in agreement with Lemma 3.2, we are normalizing all $3 = r_0$ zero$^{\text{th}}$ order variables, $2 = r_1 - r_0$ additional first order variables, and $1 = r_2 - r_1$ second order variable. This particular coordinate cross-section corresponds to the phantom indices

$$\mathcal{P} = \left\{ 1, (0;1), (0;2), (1;1), (1;2), (2;2) \right\},$$

while the complementary set of basic indices

$$\mathcal{B} = \left\{ (k;1), (l;2) \ \text{ for all } \ k \geq 2,\ l \geq 3 \right\}$$

serves to index the complete system of functionally independent basic differential invariants. The edge indices in this case are

$$\mathcal{E} = \left\{ (2;1), (3;2) \right\},$$

and represent the jet coordinates u_{xx}, v_{xxx} as well as their invariantizations.

For this particular cross-section, the left moving frame has the form

$$\rho(x, u, v, u_x \ldots, v_{xx}) = (R, z) \in \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3,$$

where the translational component $z = (x, u, v)$ is the point on the curve, while the columns of the rotational component $R = [\,\mathbf{t}, \mathbf{n}, \mathbf{b}\,] \in \mathrm{SO}(3)$ are the unit tangent, unit normal, and unit binormal frame vectors at z. However, keep in mind that these explicit identifications are not required to generate the recurrence formulae for the differential invariant algebra. The resulting invariantization map ι produces the phantom invariants

$$\begin{aligned} &H = \iota(x) = 0, \qquad && I_0 = \iota(u) = 0, \qquad && J_0 = \iota(v) = 0, \\ &I_1 = \iota(u_x) = 0, \qquad && J_1 = \iota(v_x) = 0, \qquad && J_2 = \iota(v_{xx}) = 0, \end{aligned}$$

along with the independent normalized differential invariants

$$I_2 = \iota(u_{xx}), \qquad I_3 = \iota(u_{xxx}), \qquad J_3 = \iota(v_{xxx}), \qquad I_4 = \iota(u_{xxxx}), \qquad \ldots,$$

† Technically, to be in accord with our general index notation, we should write $(\overbrace{1 \ \cdots \ 1}^{k\ 1'\mathrm{s}};\alpha)$ instead of $(k;\alpha)$, but this is, of course, a less convenient notation in this situation.

and so on. One can identify the edge invariants: $I_2 = \kappa$ is, up to a sign, the curvature†, while $J_3 = \kappa\,\tau$ is the product of curvature and torsion, [**19**]. The non-edge basic invariants,

$$I_3 = \kappa_s, \qquad I_4 = \kappa_{ss} + 3\,\kappa^3 - \kappa\,\tau^2, \qquad J_4 = 2\,\kappa_s\tau + \kappa\,\tau_s,$$

and so on are all obtained by invariant differentiation with respect to arc length, and so will not be required in the generating system; this is well known, and can be readily deduced from the recurrence formulae derived below. Thus, with this choice of minimal order cross-section, the edge invariants do generate the rest. We note the classical formulas

$$\begin{aligned}\kappa &= \frac{\|\, z_t \times z_{tt} \,\|}{\|\, z_t \,\|^3} = \frac{\sqrt{(u_x v_{xx} - u_{xx} v_x)^2 + u_{xx}^2 + v_{xx}^2}}{(1 + u_x^2 + v_x^2)^{3/2}}\,, \\ \tau &= \frac{z_t \times z_{tt} \cdot z_{ttt}}{\|\, z_t \times z_{tt} \,\|^2} = \frac{u_{xx} v_{xxx} - u_{xxx} v_{xx}}{(u_x v_{xx} - u_{xx} v_x)^2 + u_{xx}^2 + v_{xx}^2}\,,\end{aligned} \tag{6.4}$$

which can be obtained by fully implementing the moving frame construction, [**4**]. The first expression is valid for arbitrary parametrized curves, and the second is for graphs.

The invariant differential operator is the usual arc length derivative:

$$\mathcal{D} = \frac{1}{\sqrt{1 + u_x^2 + v_x^2}}\, D_x = \iota(D_x). \tag{6.5}$$

To establish the recurrence formulas for the arc length derivatives of the normalized invariants, we implement the algorithm of Section 5. The invariantized Lie matrix is obtained by replacing each jet coordinate in (6.3) by the corresponding normalized differential invariant, and so

$$\mathbf{M}^{(4)} = \iota\big(\,\mathbf{L}^{(4)}\,\big) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_2 & 0 & 0 \\ 0 & 0 & 0 & J_3 & 3\,I_2^2 & 0 \\ 0 & 0 & 0 & -I_3 & 0 & 0 \\ 0 & 0 & 0 & J_4 & 10\,I_2 I_3 & 4\,I_2 J_3 \\ 0 & 0 & 0 & -I_4 & 6\,I_2 J_3 & 0 \end{pmatrix}. \tag{6.6}$$

† As in the planar version, there is an ambiguous sign resulting from a 180° rotation, and one usually sets $\kappa = |\, I_2 \,|$ to ensure full invariance. To avoid technicalities, we shall ignore this minor complication here, and refer the reader to [**19**] for further details.

Therefore, the recurrence formulae (5.6) are given by

$$0 = \mathcal{D}H = 1 + R_1,$$

$$\begin{aligned} 0 &= \mathcal{D}I_0 = I_1 + R_2 = R_2, & 0 &= \mathcal{D}J_0 = J_1 + R_3 = R_3, \\ 0 &= \mathcal{D}I_1 = I_2 + R_5, & 0 &= \mathcal{D}J_1 = J_2 + R_6 = R_6, \\ &\mathcal{D}I_2 = I_3, & 0 &= \mathcal{D}J_2 = J_3 - I_2 R_4, \\ &\mathcal{D}I_3 = I_4 + J_3 R_4 + 3\,I_2^2 R_5, & &\mathcal{D}J_3 = J_4 - I_3 R_4, \\ &\mathcal{D}I_4 = I_5 + J_4 R_4 + 10\,I_2 I_3 R_5 + 4\,I_2 J_3 R_6, & &\mathcal{D}J_4 = J_5 - I_4 R_4 + 6\,I_2 J_3 R_5, \end{aligned}$$

and so on. Note that we do not require the explicit formulas for either the moving frame or the differential invariants in order to write out these formulas. The six phantom recurrence relations are to be solved for the Maurer–Cartan invariants:

$$R_1 = -1, \qquad R_2 = 0, \qquad R_3 = 0, \qquad R_4 = J_3/I_2, \qquad R_5 = -I_2, \qquad R_6 = 0.$$

Substituting these expressions into the remaining basic recurrence formulas leads to the explicit recurrence relations

$$\begin{aligned} &\mathcal{D}I_2 = I_3, \qquad \mathcal{D}I_3 = I_4 - 3\,I_2^3 + J_3^2/I_2, \qquad \mathcal{D}J_3 = J_4 - I_3 J_3/I_2, \\ &\mathcal{D}I_4 = I_5 - 10\,I_2^2 I_3 + J_3 J_4/I_2, \qquad \mathcal{D}J_4 = J_5 - 6\,I_2^2 J_3 + J_3 I_4/I_2, \end{aligned} \tag{6.7}$$

and, in general,

$$\begin{aligned} \mathcal{D}I_k &= I_{k+1} + \frac{1}{I_2}\,P_k(I_2,\ldots,I_k,J_3,\ldots,J_k), \\ \mathcal{D}J_k &= J_{k+1} + \frac{1}{I_2}\,Q_k(I_2,\ldots,I_k,J_3,\ldots,J_k), \end{aligned} \qquad \text{for all} \qquad k \geq 3, \tag{6.8}$$

where P_k, Q_k are certain polynomials whose precise forms are not difficult to determine, but are not required here. With these in hand, it is easy to see that the two edge invariants I_2 and J_3 do indeed generate all higher differential invariants.

On the other hand, suppose we were to construct the non-traditional moving frame based on the equations

$$x = 0, \qquad u = 0, \qquad v = 0, \qquad v_x = 0, \qquad v_{xx} = 0, \qquad v_{xxx} = 1,$$

which define a coordinate cross-section provided $u_x u_{xx} \neq 0$. In this case, the phantom indices are

$$\mathcal{P} = \big\{\, 1, (0;1), (0;2), (1;2), (2;2), (3;2) \,\big\};$$

the basic indices are

$$\mathcal{B} = \big\{\, (k\,;1), (l\,;2) \ \ \text{for all} \ \ k \geq 1, \ l \geq 4 \,\big\};$$

while the edge indices are

$$\mathcal{E} = \big\{\, (1;1), (4;2) \,\big\}.$$

The resulting moving frame invariantizations produce the phantom invariants[†]

$$\begin{aligned} &\widetilde{H} = \widetilde{\iota}(x) = 0, && \widetilde{I}_0 = \widetilde{\iota}(u) = 0, && \widetilde{J}_0 = \widetilde{\iota}(v) = 0, \\ &\widetilde{J}_1 = \widetilde{\iota}(v_x) = 0, && \widetilde{J}_2 = \widetilde{\iota}(v_{xx}) = 0, && \widetilde{J}_3 = \widetilde{\iota}(v_{xxx}) = 1, \end{aligned} \tag{6.9}$$

along with the independent basic differential invariants

$$\widetilde{I}_1 = \widetilde{\iota}(u_x), \qquad \widetilde{I}_2 = \widetilde{\iota}(u_{xx}), \qquad \widetilde{I}_3 = \widetilde{\iota}(u_{xxx}), \qquad \widetilde{I}_4 = \widetilde{\iota}(u_{xxxx}), \qquad \widetilde{J}_4 = \widetilde{\iota}(v_{xxxx}), \tag{6.10}$$

and so on. Let $\widetilde{\mathcal{D}} = \widetilde{\iota}(D_x)$ denote the associated invariant differential operator.

We will show that, in contradiction to the general claim in [**3**], the edge invariants $\widetilde{I}_1, \widetilde{J}_4$ in this case *do not* generate the complete system of differential invariants through invariant differentiation. To this end, we need to write out the recurrence formulae associated with this choice of cross-section. In view of (6.9–10), the invariantized Lie matrix is

$$\mathbf{M}^{(4)} = \widetilde{\iota}\big(\mathbf{L}^{(4)}\big) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\widetilde{I}_1^2 & 0 \\ 0 & 0 & 0 & -\widetilde{I}_1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3\widetilde{I}_1\widetilde{I}_2 & 0 \\ 0 & 0 & 0 & -\widetilde{I}_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4\widetilde{I}_1\widetilde{I}_3+3\widetilde{I}_2^2 & \widetilde{I}_1 \\ 0 & 0 & 0 & -\widetilde{I}_3 & 3\widetilde{I}_1 & 0 \\ 0 & 0 & 0 & \widetilde{J}_4 & 5\widetilde{I}_1\widetilde{I}_4+10\widetilde{I}_2\widetilde{I}_3 & \widetilde{I}_1\widetilde{J}_4+4\widetilde{I}_2 \\ 0 & 0 & 0 & -\widetilde{I}_4 & 4\widetilde{I}_1\widetilde{J}_4+6\widetilde{I}_2 & 0 \end{pmatrix}. \tag{6.11}$$

Thus, the phantom recurrence formulae are

$$\begin{aligned} 0 &= \widetilde{\mathcal{D}} H = 1 + \widetilde{R}_1, \\ 0 &= \widetilde{\mathcal{D}} \widetilde{I}_0 = \widetilde{I}_1 + \widetilde{R}_2, \\ 0 &= \widetilde{\mathcal{D}} \widetilde{J}_0 = \widetilde{J}_1 + \widetilde{R}_3 = \widetilde{R}_3, \\ 0 &= \widetilde{\mathcal{D}} \widetilde{J}_1 = \widetilde{J}_2 - \widetilde{I}_1\widetilde{R}_4 + \widetilde{R}_6 = -\widetilde{I}_1\widetilde{R}_4 + \widetilde{R}_6, \\ 0 &= \widetilde{\mathcal{D}} \widetilde{J}_2 = \widetilde{J}_3 - \widetilde{I}_2\widetilde{R}_4 = 1 - \widetilde{I}_2\widetilde{R}_4, \\ 0 &= \widetilde{\mathcal{D}} \widetilde{J}_3 = \widetilde{J}_4 - \widetilde{I}_3\widetilde{R}_4 + 3\widetilde{I}_1\widetilde{R}_5. \end{aligned}$$

Since $\widetilde{I}_1\widetilde{I}_2 = \widetilde{\iota}(u_x u_{xx}) \neq 0$ by virtue of our cross-section condition, these equations can be solved for the Maurer–Cartan invariants:

$$\widetilde{R}_1 = -1, \quad \widetilde{R}_2 = -\widetilde{I}_1, \quad \widetilde{R}_3 = 0, \quad \widetilde{R}_4 = \frac{1}{\widetilde{I}_2}, \quad \widetilde{R}_5 = \frac{1}{3\widetilde{I}_1}\left(\frac{\widetilde{I}_3}{\widetilde{I}_2} - \widetilde{J}_4\right), \quad \widetilde{R}_6 = \frac{\widetilde{I}_1}{\widetilde{I}_2}.$$

[†] We use tildes to distinguish these from the classical differential invariants derived above.

Substituting these expressions into the basic recurrence formulae

$$\begin{aligned}
\widetilde{\mathcal{D}}\widetilde{I}_1 &= \widetilde{I}_2 + (1+\widetilde{I}_1^2)\widetilde{R}_5, \\
\widetilde{\mathcal{D}}\widetilde{I}_2 &= \widetilde{I}_3 + 3\widetilde{I}_1\widetilde{I}_2\widetilde{R}_5, \\
\widetilde{\mathcal{D}}\widetilde{I}_3 &= \widetilde{I}_4 + \widetilde{R}_4 + (4\widetilde{I}_1\widetilde{I}_3 + 3\widetilde{I}_2^2)\widetilde{R}_5 + \widetilde{I}_1\widetilde{R}_6, \\
\widetilde{\mathcal{D}}\widetilde{I}_4 &= \widetilde{I}_5 + \widetilde{J}_4\widetilde{R}_4 + (5\widetilde{I}_1\widetilde{I}_4 + 10\widetilde{I}_2\widetilde{I}_3)\widetilde{R}_5 + (\widetilde{I}_1\widetilde{J}_4 + 4\widetilde{I}_2)\widetilde{R}_6, \\
\widetilde{\mathcal{D}}\widetilde{J}_4 &= \widetilde{J}_5 - \widetilde{I}_4\widetilde{R}_4 + (4\widetilde{I}_1\widetilde{J}_4 + 6\widetilde{I}_2)\widetilde{R}_5,
\end{aligned}$$

and so on, leads to the fundamental recurrence formulas

$$\begin{gathered}
\widetilde{\mathcal{D}}\widetilde{I}_1 = \widetilde{I}_2 + \frac{1+\widetilde{I}_1^2}{3\widetilde{I}_1}\left(\frac{\widetilde{I}_3}{\widetilde{I}_2} - \widetilde{J}_4\right), \qquad \widetilde{\mathcal{D}}\widetilde{I}_2 = 2\widetilde{I}_3 - \widetilde{I}_2\widetilde{J}_4, \\
\widetilde{\mathcal{D}}\widetilde{I}_3 = \widetilde{I}_4 + \frac{4\widetilde{I}_1\widetilde{I}_3 + 3\widetilde{I}_2^2}{3\widetilde{I}_1}\left(\frac{\widetilde{I}_3}{\widetilde{I}_2} - \widetilde{J}_4\right) + \frac{1+\widetilde{I}_1^2}{\widetilde{I}_2},
\end{gathered} \tag{6.12}$$

and, in general,

$$\begin{aligned}
\widetilde{\mathcal{D}}\widetilde{I}_k &= \widetilde{I}_{k+1} + \frac{1}{\widetilde{I}_1\widetilde{I}_2}\,P_k(\widetilde{I}_1,\ldots,\widetilde{I}_k,\widetilde{J}_4,\ldots,\widetilde{J}_k), \\
\widetilde{\mathcal{D}}\widetilde{J}_k &= \widetilde{J}_{k+1} + \frac{1}{\widetilde{I}_1\widetilde{I}_2}\,Q_k(\widetilde{I}_1,\ldots,\widetilde{I}_k,\widetilde{J}_4,\ldots,\widetilde{J}_k),
\end{aligned} \tag{6.13}$$

in which P_k, Q_k are certain polynomials whose precise forms are not required here. The higher order formulae (6.13) imply that the normalized invariants $\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3, \widetilde{I}_4, \widetilde{J}_4$ of degree less than 4 serve to generate all the higher order differential invariants, which is in accordance with the general result given in Theorem 7.1 below.

Let us now show that the edge invariants $\widetilde{I}_1$ and $\widetilde{J}_4$ do not generate the complete system of differential invariants. Indeed, while the second and third formulas in (6.12) allow us to express both

$$\widetilde{I}_3 = \tfrac{1}{2}\,\widetilde{\mathcal{D}}\widetilde{I}_2 + \tfrac{1}{2}\,\widetilde{I}_2\widetilde{J}_4, \qquad \widetilde{I}_4 = \widetilde{\mathcal{D}}\widetilde{I}_3 - \frac{4\widetilde{I}_1\widetilde{I}_3 + 3\widetilde{I}_2^2}{3\widetilde{I}_1}\left(\frac{\widetilde{I}_3}{\widetilde{I}_2} - \widetilde{J}_4\right) + \frac{1+\widetilde{I}_1^2}{\widetilde{I}_2}, \tag{6.14}$$

in terms of derivatives of $\widetilde{I}_1, \widetilde{I}_2$ and $\widetilde{J}_4$, the resulting initial recurrence formula

$$\widetilde{\mathcal{D}}\widetilde{I}_1 = \widetilde{I}_2 + \frac{1+\widetilde{I}_1^2}{6\widetilde{I}_1}\left(\frac{\widetilde{\mathcal{D}}\widetilde{I}_2}{\widetilde{I}_2} - \widetilde{J}_4\right) \tag{6.15}$$

is a *differential equation* for $\widetilde{I}_2$, and cannot be used to express $\widetilde{I}_2$ algebraically in terms of $\widetilde{I}_1$ and $\widetilde{J}_4$ and their invariant derivatives. Also, the higher order recurrence formulae (6.13) are of no help, since they always introduce a new, higher order functionally independent differential invariant, namely $\widetilde{I}_{k+1}$ or $\widetilde{J}_{k+1}$, and this precludes any further syzygies among the lower order invariants. (On the other hand, one can solve (6.15) for $\widetilde{J}_4$ in terms of

$\widetilde{I}_1, \widetilde{I}_2$ and their derivatives, and hence the latter pair of differential invariants do form a generating system.)

To reconfirm our conclusion, let us rewrite the lower order normalized differential invariants in terms of the classical curvature and torsion invariants. Applying the invariantization map $\widetilde{\iota}$ defined by our non-traditional moving frame, as specified by (6.9–10), to the classical differential invariants (6.4), and invoking the Replacement Rule (3.9), we find

$$\begin{aligned} \kappa &= \widetilde{\iota}\left(\frac{\sqrt{(u_x v_{xx} - u_{xx} v_x)^2 + u_{xx}^2 + v_{xx}^2}}{(1 + u_x^2 + v_x^2)^{3/2}} \right) = \frac{|\, \widetilde{I}_2 \,|}{(1 + \widetilde{I}_1^2)^{3/2}}\,, \\ \tau &= \widetilde{\iota}\left(\frac{u_{xx} v_{xxx} - u_{xxx} v_{xx}}{(u_x v_{xx} - u_{xx} v_x)^2 + u_{xx}^2 + v_{xx}^2} \right) = \frac{\widetilde{I}_2}{\widetilde{I}_2^2} = \frac{1}{\widetilde{I}_2}\,. \end{aligned} \tag{6.16}$$

Solving, we find†

$$\widetilde{I}_1 = \sqrt{(\kappa\tau)^{-2/3} - 1}\,, \qquad \widetilde{I}_2 = \frac{1}{\tau}\,. \tag{6.17}$$

In particular, we discover that this cross-section and resulting moving frame are only valid for curves with $\kappa\tau > 1$. (Changing the last cross-section equation in (6.9) to $v_{xxx} = c$ will produce a moving frame with a somewhat wider range of validity.) Thus, the invariants $\widetilde{I}_1, \widetilde{I}_2$ are essentially equivalent to the classical curvature and torision, which explains why they serve to generate the full differential invariant algebra.

The corresponding invariant differential operator is expressed by applying invariantization to the arc length derivative (6.5):

$$\mathcal{D} = \widetilde{\iota}(\mathcal{D}) = \widetilde{\iota}\left(\frac{1}{\sqrt{1 + u_x^2 + v_x^2}}\, D_x \right) = \frac{1}{\sqrt{1 + \widetilde{I}_1^2}}\, \widetilde{\mathcal{D}},$$

and hence, using (6.16),

$$\widetilde{\mathcal{D}} = (\kappa\tau)^{-1/3} \mathcal{D} = (\kappa\tau)^{-1/3}\, \frac{d}{ds}\,. \tag{6.18}$$

Substituting (6.17–18) into (6.15) produces

$$\widetilde{J}_4 = \frac{2\kappa_s \tau + \kappa \tau_s}{(\kappa\tau)^{4/3}} \;+\; 6\, \frac{\kappa^{2/3}}{\tau^{1/3}} \sqrt{(\kappa\tau)^{-2/3} - 1}\,. \tag{6.19}$$

Obeserve that $\widetilde{J}_4$ depends on τ_s and κ_s, and hence we cannot generate both κ and τ by differentiating the edge invariants $\widetilde{I}_1, \widetilde{J}_4$, reconfirming our earlier observations. We also note that

$$\widetilde{I}_3 = \frac{\kappa_s}{(\kappa\tau)^{4/3}} \;+\; 3\, \frac{\kappa^{2/3}}{\tau^{4/3}} \sqrt{(\kappa\tau)^{-2/3} - 1}\,, \tag{6.20}$$

† There is a sign ambiguity in the square root throughout.

which results from substituting (6.17–19) into the first recurrence formula in (6.14). An alternative means of deriving these formulae (or for checking the preceding computations) is to differentiate the classical formulae (6.4) with respect to arc length, and then apply the invariantization map $\widetilde{\iota}$.

7. Generating Differential Invariants.

Let us now present corrected, rigorous results on generating differential invariants, for both minimal and non-minimal order moving frames. Theorem 13.3 in [**3**] claims that the *edge differential invariants*, meaning those normalized differential invariants indexed by the elements of $\mathcal{E}$, form a generating set. The justification relied on an induction argument, which was based on the erroneous claim that the Maurer–Cartan invariants, being solutions to the phantom recurrence equations (5.8), only depended on the edge invariants. First, it was correctly noted that the leading terms $I^\alpha_{J,i}$ in (5.8) are all either phantom invariants, and hence constant, or, if non-constant, edge invariants. However, the rows $\boldsymbol{\psi}^\beta_K$ of the coefficient matrix can, in certain scenarios, depend on some of the non-edge basic differential invariants — as we witnessed in the preceding example — thereby precipitating a breakdown of the proposed inductive argument. However, the following less powerful result does follow from the original argument.

Theorem 7.1. *Given a moving frame of order n, the normalized differential invariants corresponding to indices in $\mathcal{B}^{(n)} \cup \mathcal{E}$ form a generating system.*

Proof: Indeed, the linear system (5.8) determining the Maurer–Cartan invariants only involves the basic invariants of order $\leq n$ and the edge invariants of order $n+1$, and hence the Maurer–Cartan invariants can be expressed as functions of the listed generating invariants. Moreover, the higher order basic recurrence formulae

$$I^\alpha_{J,i} = \mathcal{D}_i I^\alpha_J - \boldsymbol{\psi}^\alpha_J \, \mathbf{R}_i \qquad \text{for} \qquad (J;\alpha) \in \mathcal{B}, \qquad k = \#J \geq n,$$

express the invariants of order $k+1$ in terms of generating and lower order differential invariants. A straightforward induction argument completes the proof. *Q.E.D.*

Thus, to find a complete system of generating differential invariants, one may require all basic differential invariants of order $\leq n$ along with any edge invariants that appear at order $n+1$. Theorem 7.1 is a slight improvement on the classical result, [**24**], that requires all differential invariants of order $n+1$. Keep in mind that it is *not* claimed that the differential invariants indexed by $\mathcal{B}^{(n)} \cup \mathcal{E}$ form a minimal generating system. Indeed, in practice, many of these invariants can be generated by lower order invariants, and so are not required in a generating system. However, as the example in Section 6 makes clear, the edge invariants by themselves may not suffice.

But, if a minimal order moving frame is employed, Theorem 13.3 in [**3**] *does* remain valid as originally formulated.

Theorem 7.2. *The edge differential invariants arising from a minimal order moving frame form a generating system of differential invariants.*

Proof: According to Corollary 4.4, we can rewrite every non-phantom row of the Lie matrix as a linear combination of phantom rows. Applying the invariantization process to the resulting linear dependencies (4.6) leads to similar dependencies amongst the rows of the invariantized Lie matrix:

$$\boldsymbol{\eta}^i = \sum_{l\in\mathcal{P}^{(0)}} h^i_l(H,I)\,\boldsymbol{\eta}^l \ + \sum_{(0,\beta)\in\mathcal{P}^{(0)}} h^i_\beta(H,I)\,\boldsymbol{\psi}^\beta, \tag{7.1}$$

$$\boldsymbol{\psi}^\alpha_J = \sum_{l\in\mathcal{P}^{(0)}} h^\alpha_{J,l}(H,I^{(k)})\,\boldsymbol{\eta}^l \ + \sum_{(K;\beta)\in\mathcal{P}^{(k)}} h^{\alpha,K}_{J,\beta}(H,I^{(k)})\,\boldsymbol{\psi}^\beta_K, \qquad \text{where} \qquad k=\#J.$$

Substituting these formulae into (5.9) and then using the phantom recurrence formulae (5.8) leads to

$$\begin{aligned}\mathcal{D}_i I^\alpha_J &= I^\alpha_{J,i} + \boldsymbol{\psi}^\alpha_J\,\mathbf{R}_i \\ &= I^\alpha_{J,i} + \sum_{l\in\mathcal{P}^{(0)}} h^\alpha_{J,l}(H,I^{(k)})\,\boldsymbol{\eta}^l\,\mathbf{R}_i \ + \sum_{(K;\beta)\in\mathcal{P}^{(j)}} h^{\alpha,K}_{J,\beta}(H,I^{(k)})\,\boldsymbol{\psi}^\beta_K\,\mathbf{R}_i \\ &= I^\alpha_{J,i} - \sum_{l\in\mathcal{P}^{(0)}} h^\alpha_{J,l}(H,I^{(k)})\,\delta^l_i \ - \sum_{(K;\beta)\in\mathcal{P}^{(k)}} h^{\alpha,K}_{J,\beta}(H,I^{(k)})\,I^\beta_{K,i}.\end{aligned} \tag{7.2}$$

We conclude that, for any basic index $(J;\alpha)\in\mathcal{B}$,

$$I^\alpha_{J,i} = \mathcal{D}_i I^\alpha_J \ + \sum_{l\in\mathcal{P}^{(0)}} h^\alpha_{J,l}(H,I^{(k)})\,\delta^l_i \ + \sum_{(K;\beta)\in\mathcal{P}^{(k)}} h^{\alpha,K}_{J,\beta}(H,I^{(k)})\,I^\beta_{K,i}, \qquad \#J=k. \tag{7.3}$$

To complete the proof, we use induction on the degree of the differential invariant, noting that all basic degree zero invariants H^i, I^α (which only appear if the group acts intransitively on M) are automatically included in our generating set. The only differential invariants of degree $> k = \#J$ that appear on the right hand side of (7.3) are the $I^\beta_{K,i}$ for phantom indices $(K;\beta)\in\mathcal{P}^{(k)}$. But, in this case, either $(K,i;\beta)\in\mathcal{P}^{(k+1)}$ is another phantom index, in which case $I^\beta_{K,i}$ is constant, or $(K,i;\beta)\in\mathcal{E}$ is an edge index, in which case $I^\beta_{K,i}$ is one of the generating differential invariants. Thus, by our induction hypothesis, any degree $k+1$ non-edge normalized differential invariant $I^\alpha_{J,i}$ can be written in terms of the generating edge invariants of degree $k+1$ and the differential invariants of degree $\leq k$ and their invariant derivatives. This completes the induction step. *Q.E.D.*

Remark: Since, according to [**17**; Theorems 5.37 and 5.49], the order at which the prolongation of a locally effective r-dimensional Lie group action becomes locally free is bounded by r, Theorem 7.2 can be used to bound the number of differential invariants in terms of the dimension of the group. Details will appear elsewhere.

The edge invariants may still not form a minimal generating system. In general, given an edge index of the form $(J,i;\alpha)\in\mathcal{E}$ with $(J;\alpha)\in\mathcal{B}$ basic, then (5.9) relates the edge invariant $I^\alpha_{J,i}$ to the differentiated basic invariant $\mathcal{D}_i I^\alpha_J$. Let us call the edge indices/invariants that are *not* of this form *essential*. Thus, one might expect that only the essential invariants are needed in a generating set. Unfortunately, while often true, this is not always the case as the following example demonstrates.

Remark: As in [**23**], we can identify each index $(J;\alpha) \in \mathcal{T}$ with a monomial $s_J S^\alpha$ in the $\mathbb{R}[s]$ polynomial module

$$\mathcal{S} = \left\{ \sigma(s,S) = \sum_{\alpha=1}^{q} \sigma_\alpha(s)\, S^\alpha \right\} \simeq \mathbb{R}[s] \otimes \mathbb{R}^q,$$

consisting of polynomials in variables $s = (s_1, \ldots, s_p)$, $S = (S^1, \ldots, S^q)$, that are linear in the latter. A cross-section and its associated moving frame are called *algebraic* if the subspace spanned by the *basic monomials* $s_J S^\alpha$ for $(J;\alpha) \in \mathcal{B}$ forms a submodule. In this case, the essential indices correspond to a (Gröbner) basis for the monomial submodule. See [**23**] for further developments in this direction.

Example 7.3. Consider the group action G_V on $\mathbb{R}^3$ discussed in Example 2.2, in the special case when the subspace V is spanned by the polynomials

$$\begin{gathered} \varphi_1(x,y) = 1, \qquad \varphi_2(x,y) = x, \qquad \varphi_3(x,y) = y, \qquad \varphi_4(x,y) = xy - \tfrac{1}{2}x^2, \\ \varphi_5(x,y) = xy - \tfrac{1}{2}y^2, \qquad \varphi_6(x,y) = -\tfrac{1}{6}x^3 + \tfrac{1}{4}x^2 y + \tfrac{1}{4}xy^2 - \tfrac{1}{6}y^3. \end{gathered} \tag{7.4}$$

The cross-section normalizations

$$x = y = u = u_x = u_y = u_{xx} = u_{yy} = u_{xxx} = 0 \tag{7.5}$$

serve to define a minimal order moving frame, since the rank of the second order Lie matrix is $r_2 = 7$. The basic differential invariants are

$$\begin{gathered} I_{xy} = \iota(u_{xy}), \qquad I_{xxy} = \iota(u_{xxy}), \qquad I_{xyy} = \iota(u_{xyy}), \qquad I_{yyy} = \iota(u_{yyy}), \\ I_{jk} = \iota(u_{jk}) \qquad \text{for all} \qquad j+k \geq 5. \end{gathered}$$

The edge invariants are $I_{xy}, I_{xxy}, I_{xyy}, I_{yyy}, I_{xxxx}, I_{xxxy}$, while the essential invariants are $I_{xy}, I_{yyy}, I_{xxxx}$.

The recurrence formulas are readily established:

$$\begin{aligned} \mathcal{D}_1 I_{xy} &= I_{xxy} + I_{xyy}, & \mathcal{D}_2 I_{xy} &= I_{xxy} + I_{xyy} + I_{yyy}, \\ \mathcal{D}_1 I_{xxy} &= \tfrac{1}{2} I_{xxxx} + I_{xxxy}, & \mathcal{D}_2 I_{xxy} &= \tfrac{1}{2} I_{xxxy} + I_{xxyy}, \\ \mathcal{D}_1 I_{xyy} &= \tfrac{1}{2} I_{xxxx} + I_{xxyy}, & \mathcal{D}_2 I_{xyy} &= \tfrac{1}{2} I_{xxxy} + I_{xyyy}, \\ \mathcal{D}_1 I_{yyy} &= -I_{xxxx} + I_{xyyy}, & \mathcal{D}_2 I_{yyy} &= -I_{xxxy} + I_{yyyy}, \end{aligned} \tag{7.6}$$

while

$$\mathcal{D}_1 I_{jk} = I_{j+1,k}, \qquad \mathcal{D}_2 I_{jk} = I_{j,k+1}, \qquad \text{whenever} \qquad j+k \geq 4. \tag{7.7}$$

In accordance with Theorem 7.2, we can generate all higher order differential invariants from $I_{xy}, I_{xxy}, I_{xyy}, I_{yyy}, I_{xxxx}$. However, we cannot generate both I_{xxy}, I_{xyy} from I_{xy}, and so the essential invariants do not form a generating system in this particular case.

The cause of the difficulty in this example appears to be that we are not paying proper attention to the algebraic structure associated with this group action. We are not able to fully develop this remark here, but will make the following preliminary observations. The polynomial subspace spanned by (7.4) is the solution space to the following overdetermined system of partial differential equations

$$\varphi_{xx} + \varphi_{xy} + \varphi_{yy} = 0, \qquad \varphi_{xxy} - \varphi_{xyy} = 0. \tag{7.8}$$

The symbol module associated with this system is generated by the polynomials

$$s^2 + st + t^2, \qquad s^2 t - s t^2.$$

The arguments in [**23**] inspire us to choose an "algebraic" coordinate cross-section prescribed by the set of complementary monomials to the prolonged symbol module relative to some term ordering, [**2**]. For example, using lexicographic ordering based on $s < t$, the complementary monomials are $1, s, t, s^2, st, s^3$, leading to the cross-section equations

$$x = y = u = u_x = u_y = u_{xx} = u_{xy} = u_{xxx} = 0, \tag{7.9}$$

which also define a minimal order moving frame. (In contrast, the monomials $1, s, t, s^2, t^2, s^3$, corresponding to (7.5) do not form a complementary set with respect to any term ordering.) In this case, the basic differential invariants are

$$I_{yy} = \iota(u_{yy}), \qquad I_{xxy} = \iota(u_{xxy}), \qquad I_{xyy} = \iota(u_{xyy}), \qquad I_{yyy} = \iota(u_{yyy}),$$
$$I_{jk} = \iota(u_{jk}) \qquad \text{for all} \qquad j + k \geq 5.$$

But in this case, the essential invariants $I_{yy}, I_{xxy}, I_{xxxx}$ do generate all higher order differential invariants. Indeed, the first two recurrence formulas are

$$\mathcal{D}_1 I_{yy} = I_{xxy} + I_{xyy}, \qquad \mathcal{D}_2 I_{yy} = I_{xxy} + I_{xyy} + I_{yyy}, \tag{7.10}$$

while all the rest are exactly the same as in (7.6–7). Observe that we are now able to write the non-essential edge invariants I_{xyy}, I_{xxxy}, as well as all the non-edge differential invariants, in terms of $I_{yy}, I_{xxy}, I_{xxxx}$.

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