Moving Frames — in Geometry, Algebra, Computer Vision, and Numerical Analysis

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Abstract. This paper surveys the new, algorithmic theory of moving frames developed by the author and M. Fels. Applications in geometry, computer vision, classical invariant theory, and numerical analysis are indicated.

1. Introduction.

The method of moving frames ("repères mobiles") was forged by Élie Cartan, [**13**, **14**], into a powerful and algorithmic tool for studying the geometric properties of submanifolds and their invariants under the action of a transformation group. However, Cartan's methods remained incompletely understood and the applications were exclusively concentrated

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in classical differential geometry; see [**22**, **23**, **26**]. Three years ago, [**20**, **21**], Mark Fels and I formulated a new approach to the moving frame theory that can be systematically applied to general transformation groups. The key idea is to formulate a moving frame as an equivariant map to the transformation group. All classical moving frames can be reinterpreted in this manner, but the new approach applies in far wider generality. Cartan's construction of the moving frame through the normalization process is interpreted with the choice of a cross-section to the group orbits. Building on these two simple ideas, one may algorithmically construct moving frames and complete systems of invariants for completely general group actions. The existence of a moving frame requires freeness of the underlying group action. Classically, non-free actions are made free by prolonging to jet space, leading to differential invariants and the solution to equivalence and symmetry problems via the differential invariant signature. More recently, the moving frame method was also applied to Cartesian product "prolongations" of group actions, leading to classification of joint invariants and joint differential invariants, [**42**]. The combination of jet and Cartesian product actions known as multi-space was proposed in [**43**] as a framework for the geometric analysis of numerical approximations, and, via the application of the moving frame method, to the systematic construction of invariant numerical algorithms.

New and significant applications of these results have been developed in a wide variety of directions. In [**40**, **1**, **29**], the theory was applied to produce new algorithms for solving the basic symmetry and equivalence problems of polynomials that form the foundation of classical invariant theory. In [**32**], the differential invariants of projective surfaces were classified and applied to generate integrable Poisson flows arising in soliton theory. In [**20**], the moving frame algorithm was extended to include infinite-dimensional pseudo-group actions. Faugeras, [**19**], initiated the applications of moving frames in computer vision, and In [**12**], the characterization of submanifolds via their differential invariant signatures was applied to the problem of object recognition and symmetry detection, [**4**, **5**, **7**, **45**]. The moving frame method provides a direct route to the classification of joint invariants and joint differential invariants, [**21**, **42**], establishing a geometric counterpart of what Weyl, [**51**], in the algebraic framework, calls the first main theorem for the transformation group. In computer vision, joint differential invariants have been proposed as noise-resistant alternatives to the standard differential invariant signatures, [**6**, **10**, **16**, **36**, **49**, **50**]. The approximation of higher order differential invariants by joint differential invariants and, generally, ordinary joint invariants leads to fully invariant finite difference numerical schemes, first proposed in [**11**, **12**, **3**, **43**]. Finally, a complete solution to the calculus of variations problem of directly constructing differential invariant Euler-Lagrange equations from their differential invariant Lagrangians in the has been recently effected, [**30**].

2. Moving Frames.

We begin by outlining the basic moving frame construction in [**21**]. Let G be an r-dimensional Lie group acting smoothly on an m-dimensional manifold M. Let $G_S = \{ g \in G \mid g \cdot S = S \}$ denote the *isotropy subgroup* of a subset $S \subset M$, and $G_S^* = \bigcap_{z \in S} G_z$ its *global isotropy subgroup*, which consists of those group elements which fix all points in S. We always assume, without any significant loss of generality, that G acts *effectively on* *subsets*, and so $G_U^* = \{e\}$ for any open $U \subset M$, i.e., there are no group elements other than the identity which act completely trivially on an open subset of M.

The crucial idea is to decouple the moving frame theory from reliance on any form of frame bundle. In other words,

$$\text{Moving frames} \neq \text{Frames!}$$

A careful study of Cartan's analysis of the case of projective curves, [**13**], reveals that Cartan was well aware of this fact. However, this important and instructive example did not receive the attention it deserves.

Definition 2.1. A *moving frame* is a smooth, G-equivariant map $\rho : M \to G$.

The group G acts on itself by left or right multiplication. If $\rho(z)$ is any right-equivariant moving frame then $\widetilde{\rho}(z) = \rho(z)^{-1}$ is left-equivariant and conversely. All classical moving frames are left equivariant, but, in many cases, the right versions are easier to compute. In many geometrical situations, one can identify our left moving frames with the usual frame-based versions, but these identifications break down for more general transformation groups.

Theorem 2.2. *A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.*

Recall that G acts *freely* if the isotropy subgroup of each point is trivial, $G_z = \{e\}$ for all $z \in M$. This implies that the orbits all have the same dimension as G itself. *Regularity* requires that, in addition, each point $x \in M$ has a system of arbitrarily small neighborhoods whose intersection with each orbit is connected, cf. [**38**].

The practical construction of a moving frame is based on Cartan's method of *normalization*, [**28, 13**], which requires the choice of a (local) *cross-section* to the group orbits.

Theorem 2.3. *Let G act freely and regularly on M, and let $K \subset M$ be a cross-section. Given $z \in M$, let $g = \rho(z)$ be the unique group element that maps z to the cross-section: $g \cdot z = \rho(z) \cdot z \in K$. Then $\rho : M \to G$ is a right moving frame for the group action.*

Given local coordinates $z = (z_1, \ldots, z_m)$ on M, let $w(g, z) = g \cdot z$ be the explicit formulae for the group transformations. The right† moving frame $g = \rho(z)$ associated with a *coordinate cross-section* $K = \{ z_1 = c_1, \ldots, z_r = c_r \}$ is obtained by solving the *normalization equations*

$$w_1(g, z) = c_1, \qquad \ldots \qquad w_r(g, z) = c_r, \tag{2.1}$$

for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the coordinates $z = (z_1, \ldots, z_m)$. Substituting the moving frame formulae into the remaining transformation rules leads to a complete system of invariants for the group action.

† The left version can be obtained directly by replacing g by g^{-1} throughout the construction.

Theorem 2.4. *If $g = \rho(z)$ is the moving frame solution to the normalization equations* (2.1), *then the functions*

$$I_1(z) = w_{r+1}(\rho(z), z), \qquad \ldots \qquad I_{m-r}(z) = w_m(\rho(z), z), \tag{2.2}$$

form a complete system of functionally independent invariants.

Definition 2.5. The *invariantization* of a scalar function $F: M \to \mathbb{R}$ with respect to a right moving frame ρ is the the invariant function $I = \iota(F)$ defined by $I(z) = F(\rho(z) \cdot z)$.

Invariantization amounts to restricting F to the cross-section, $I \mid K = F \mid K$, and then requiring that I be constant along the orbits. In particular, if $I(z)$ is an invariant, then $\iota(I) = I$, so invariantization defines a projection, depending on the moving frame, from functions to invariants. Thus, a moving frame provides a canonical method of associating an invariant with an arbitrary function.

Of course, most interesting group actions are *not* free, and therefore do not admit moving frames in the sense of Definition 2.1. There are two common methods for converting a non-free (but effective) action into a free action. In the traditional moving frame theory, [**13, 23, 26**], this is accomplished by prolonging the action to a jet space J^n of suitably high order; the consequential invariants are the classical differential invariants for the group, [**21, 38**]. Alternatively, one may consider the product action of G on a sufficiently large Cartesian product $M^{\times(n+1)}$; here, the invariants are joint invariants, [**42**], of particular interest in classical algebra, [**40, 51**]. In neither case is there a general theorem guaranteeing the freeness and regularity of the prolonged or product actions, (indeed, there are counterexamples in the product case), but such pathologies never occur in practical examples. In our approach to invariant numerical approximations, we will amalgamate the two methods by prolonging to an appropriate multi-space, as defined below.

3. Prolongation and Differential Invariants.

Traditional moving frames are obtained by prolonging the group action to the n^{th} order (extended) jet bundle $\mathrm{J}^n = \mathrm{J}^n(M, p)$ consisting of equivalence classes of p-dimensional submanifolds $S \subset M$ modulo n^{th} order contact at a single point; see [**38**; Chapter 3] for details. Since G preserves the contact equivalence relation, it induces an action on the jet space J^n, known as its n^{th} order *prolongation* and denoted by $G^{(n)}$.

An n^{th} *order moving frame* $\rho^{(n)}: \mathrm{J}^n \to G$ is an equivariant map defined on an open subset of the jet space. In practical examples, for n sufficiently large, the prolonged action $G^{(n)}$ becomes regular and free on a dense open subset $\mathcal{V}^n \subset \mathrm{J}^n$, the set of *regular jets*. It has been rigorously proved that, for $n \gg 0$ sufficiently large, if G acts effectively on subsets, then $G^{(n)}$ acts locally freely on an open subset $\mathcal{V}^n \subset \mathrm{J}^n$, [**41**].

Theorem 3.1. *An n^{th} order moving frame exists in a neighborhood of a point $z^{(n)} \in \mathrm{J}^n$ if and only if $z^{(n)} \in \mathcal{V}^n$ is a regular jet.*

Our normalization construction will produce a moving frame and a complete system of differential invariants in the neighborhood of any regular jet. Local coordinates $z = (x, u)$ on M — considering the first p components $x = (x^1, \ldots, x^p)$ as independent variables, and the latter $q = m - p$ components $u = (u^1, \ldots, u^q)$ as dependent variables — induce local coordinates $z^{(n)} = (x, u^{(n)})$ on J^n with components u^α_J representing the partial derivatives of the dependent variables with respect to the independent variables, [**38**, **39**]. We compute the prolonged transformation formulae

$$w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}, \qquad \text{or} \qquad (y, v^{(n)}) = g^{(n)} \cdot (x, u^{(n)}),$$

by implicit differentiation of the v's with respect to the y's. For simplicity, we restrict to a coordinate cross-section by choosing $r = \dim G$ components of $w^{(n)}$ to normalize to constants:

$$w_1(g, z^{(n)}) = c_1, \qquad \ldots \qquad w_r(g, z^{(n)}) = c_r. \tag{3.1}$$

Solving the normalization equations (3.1) for the group transformations leads to the explicit formulae $g = \rho^{(n)}(z^{(n)})$ for the right moving frame. As in Theorem 2.4, substituting the moving frame formulae into the unnormalized components of $w^{(n)}$ leads to the *fundamental* n^{th} *order differential invariants*

$$I^{(n)}(z^{(n)}) = w^{(n)}(\rho^{(n)}(z^{(n)}), z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot z^{(n)}. \tag{3.2}$$

Once the moving frame is established, the *invariantization* process will map general differential functions $F(x, u^{(n)})$ to differential invariants $I = \iota(F) = F \circ I^{(n)}$. As before, invariantization defines a projection, depending on the moving frame, from the space of differential functions to the space of differential invariants. The fundamental differential invariants $I^{(n)}$ are obtained by invariantization of the coordinate functions

$$\begin{aligned} H^i(x, u^{(n)}) &= \iota(x^i) = y^i(\rho^{(n)}(x, u^{(n)}), x, u), \\ I^\alpha_K(x, u^{(k)}) &= \iota(u^\alpha_J) = v^\alpha_K(\rho^{(n)}(x, u^{(n)}), x, u^{(k)}). \end{aligned} \tag{3.3}$$

In particular, those corresponding to the normalization components (3.1) of $w^{(n)}$ will be constant, and are known as the *phantom differential invariants*.

Theorem 3.2. *Let $\rho^{(n)}\colon \mathrm{J}^n \to G$ be a moving frame of order $\le n$. Every n^{th} order differential invariant can be locally written as a function $J = \Phi(I^{(n)})$ of the fundamental n^{th} order differential invariants* (3.3)*. The function Φ is unique provided it does not depend on the phantom invariants.*

Example 3.3. Let us illustrate the theory with a very simple, well-known example: curves in the Euclidean plane. The orientation-preserving Euclidean group SE(2) acts on $M = \mathbb{R}^2$, mapping a point $z = (x, u)$ to

$$y = x\cos\theta - u\sin\theta + a, \qquad v = x\sin\theta + u\cos\theta + b. \tag{3.4}$$

For a general parametrized† curve $z(t) = (x(t), u(t))$, the prolonged group transformations

$$v_y = \frac{dv}{dy} = \frac{\dot x\sin\theta + \dot u\cos\theta}{\dot x\cos\theta - \dot u\sin\theta}\,, \qquad v_{yy} = \frac{d^2 v}{dy^2} = \frac{\dot x\ddot u - \ddot x\dot u}{(\dot x\cos\theta - \dot u\sin\theta)^3}\,, \tag{3.5}$$

† While the local coordinates $(x, u, u_x, u_{xx}, \ldots)$ on the jet space assume that the curve is given as the graph of a function $u = f(x)$, the moving frame computations also apply, as indicated in this example, to general parametrized curves. Two parametrized curves are equivalent if and only if one can be mapped to the other under a suitable reparametrization.

and so on, are found by successively applying the implicit differentiation operator

$$\frac{d}{dy} = \frac{1}{\dot{x}\cos\theta - \dot{u}\sin\theta}\,\frac{d}{dt} \tag{3.6}$$

to v. The classical Euclidean moving frame for planar curves, [**23**], follows from the cross-section normalizations

$$y = 0, \qquad\qquad v = 0, \qquad\qquad v_y = 0. \tag{3.7}$$

Solving for the group parameters $g = (\theta, a, b)$ leads to the right-equivariant moving frame

$$\theta = -\tan^{-1}\frac{\dot{u}}{\dot{x}}, \qquad a = -\frac{x\dot{x} + u\dot{u}}{\sqrt{\dot{x}^2 + \dot{u}^2}} = \frac{z\cdot\dot{z}}{\|\,\dot{z}\,\|}, \qquad b = \frac{x\dot{u} - u\dot{x}}{\sqrt{\dot{x}^2 + \dot{u}^2}} = \frac{z\wedge\dot{z}}{\|\,\dot{z}\,\|}. \tag{3.8}$$

The inverse group transformation $g^{-1} = (\widetilde{\theta}, \widetilde{a}, \widetilde{b})$ is the classical left moving frame, [**13, 23**]: one identifies the translation component $(\widetilde{a}, \widetilde{b}) = (x, u) = z$ as the point on the curve, while the columns of the rotation matrix $\widetilde{R}(\widetilde{\theta}) = (\mathbf{t}, \mathbf{n})$ are the unit tangent and unit normal vectors. Substituting the moving frame normalizations (3.8) into the prolonged transformation formulae (3.5), results in the fundamental differential invariants

$$v_{yy} \longmapsto \kappa = \frac{\dot{x}\ddot{u} - \ddot{x}\dot{u}}{(\dot{x}^2 + \dot{u}^2)^{3/2}} = \frac{\dot{z}\wedge\ddot{z}}{\|\,\dot{z}\,\|^3}, \qquad v_{yyy} \longmapsto \frac{d\kappa}{ds}, \qquad v_{yyyy} \longmapsto \frac{d^2\kappa}{ds^2} + 3\kappa^3, \tag{3.9}$$

where $d/ds = \|\,\dot{z}\,\|^{-1}\, d/dt$ is the arc length derivative — which is itself found by substituting the moving frame formulae (3.8) into the implicit differentiation operator (3.6). A complete system of differential invariants for the planar Euclidean group is provided by the curvature and its successive derivatives with respect to arc length: $\kappa, \kappa_s, \kappa_{ss}, \ldots$.

The one caveat is that the first prolongation of SE(2) is only locally free on J^1 since a 180° rotation has trivial first prolongation. The even derivatives of κ with respect to s change sign under a 180° rotation, and so only their absolute values are fully invariant. The ambiguity can be removed by including the second order constraint $v_{yy} > 0$ in the derivation of the moving frame. Extending the analysis to the full Euclidean group E(2) adds in a second sign ambiguity which can only be resolved at third order. See [**42**] for complete details.

Example 3.4. Let $n \neq 0, 1$. In classical invariant theory, the planar actions

$$y = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad\qquad \overline{u} = (\gamma x + \delta)^{-n} u, \tag{3.10}$$

of $G = \mathrm{GL}(2)$ play a key role in the equivalence and symmetry properties of binary forms, when $u = q(x)$ is a polynomial of degree $\leq n$, [**24, 40, 1**]. We identify the graph of the function $u = q(x)$ as a plane curve. The prolonged action on such graphs is found by implicit differentiation:

$$v_y = \frac{\sigma u_x - n\gamma u}{\Delta\sigma^{n-1}}, \qquad v_{yy} = \frac{\sigma^2 u_{xx} - 2(n-1)\gamma\sigma u_x + n(n-1)\gamma^2 u}{\Delta^2\sigma^{n-2}},$$
$$v_{yyy} = \frac{\sigma^3 u_{xxx} - 3(n-2)\gamma\sigma^2 u_{xx} + 3(n-1)(n-2)\gamma^2\sigma u_x - n(n-1)(n-2)\gamma^3 u}{\Delta^3\sigma^{n-3}},$$

and so on, where $\sigma = \gamma p + \delta$, $\Delta = \alpha\delta - \beta\gamma \neq 0$. On the regular subdomain

$$\mathcal{V}^2 = \{ uH \neq 0 \} \subset \mathrm{J}^2, \qquad \text{where} \qquad H = u u_{xx} - \frac{n-1}{n}\, u_x^2$$

is the classical Hessian covariant of u, we can choose the cross-section defined by the normalizations

$$y = 0, \qquad v = 1, \qquad v_y = 0, \qquad v_{yy} = 1.$$

Solving for the group parameters gives the right moving frame formulae†

$$\begin{aligned} &\alpha = u^{(1-n)/n}\sqrt{H}, && \beta = -x\, u^{(1-n)/n}\sqrt{H}, \\ &\gamma = \tfrac{1}{n}\, u^{(1-n)/n} u_x, && \delta = u^{1/n} - \tfrac{1}{n}\, x\, u^{(1-n)/n} u_x. \end{aligned} \tag{3.11}$$

Substituting the normalizations (3.11) into the higher order transformation rules gives us the differential invariants, the first two of which are

$$v_{yyy} \longmapsto J = \frac{T}{H^{3/2}}, \qquad v_{yyyy} \longmapsto K = \frac{V}{H^2}, \tag{3.12}$$

where

$$\begin{aligned} T &= u^2 u_{xxx} - 3\,\frac{n-2}{n}\, u u_x u_{xx} + 2\,\frac{(n-1)(n-2)}{n^2}\, u_x^3, \\ V &= u^3 u_{xxxx} - 4\,\frac{n-3}{n}\, u^2 u_x u_{xx} + 6\,\frac{(n-2)(n-3)}{n^2}\, u {u_x}^2 u_{xx} - 3\,\frac{(n-1)(n-2)(n-3)}{n^3}\, u_x^4, \end{aligned}$$

and can be identified with classical covariants, which may be constructed using the basic transvectant process of classical invariant theory, cf. [**24**, **40**]. Using $J^2 = T^2/H^3$ as the fundamental differential invariant will remove the ambiguity caused by the square root. As in the Euclidean case, higher order differential invariants are found by successive application of the normalized implicit differentiation operator $\mathcal{D}_s = uH^{-1/2} D_x$ to the fundamental invariant J.

4. Equivalence and Signatures.

The moving frame method was developed by Cartan expressly for the solution to problems of equivalence and symmetry of submanifolds under group actions. Two submanifolds $S, \overline{S} \subset M$ are said to be *equivalent* if $\overline{S} = g \cdot S$ for some $g \in G$. A *symmetry* of a submanifold is a group transformation that maps S to itself, and so is an element $g \in G_S$. As emphasized by Cartan, [**13**], the solution to the equivalence and symmetry problems for submanifolds is based on the functional interrelationships among the fundamental differential invariants restricted to the submanifold.

Suppose we have constructed an n^{th} order moving frame $\rho^{(n)} \colon \mathrm{J}^n \to G$ defined on an open subset of jet space. A submanifold S is called *regular* if its n-jet $\mathrm{j}_n S$ lies in the

† See [**1**] for a detailed discussion of how to resolve the square root ambiguities.

domain of definition of the moving frame. For any $k \geq n$, we use $J^{(k)} = I^{(k)} \mid S = I^{(k)} \circ \mathrm{j}_k S$ to denote the k^{th} order *restricted differential invariants*. The k^{th} order *signature* $\mathcal{S}^{(k)} = \mathcal{S}^{(k)}(S)$ is the set parametrized by the restricted differential invariants; S is called *fully regular* if $J^{(k)}$ has constant rank $0 \leq t_k \leq p = \dim S$ for all $k \geq n$. In this case, $\mathcal{S}^{(k)}$ forms a submanifold of dimension t_k — perhaps with self-intersections. In the fully regular case,

$$t_n < t_{n+1} < t_{n+2} < \cdots < t_s = t_{s+1} = \cdots = t \leq p,$$

where t is the *differential invariant rank* and s the *differential invariant order* of S.

Theorem 4.1. *Two fully regular p-dimensional submanifolds $S, \overline{S} \subset M$ are (locally) equivalent, $\overline{S} = g \cdot S$, if and only if they have the same differential invariant order s and their signature manifolds of order $s+1$ are identical: $\mathcal{S}^{(s+1)}(\overline{S}) = \mathcal{S}^{(s+1)}(S)$.*

Since symmetries are the same as self-equivalences, the signature also determines the symmetry group of the submanifold.

Theorem 4.2. *If $S \subset M$ is a fully regular p-dimensional submanifold of differential invariant rank t, then its symmetry group G_S is an $(r-t)$–dimensional subgroup of G that acts locally freely on S.*

A submanifold with maximal differential invariant rank $t = p$, and hence only a discrete symmetry group, is called *nonsingular*. The number of symmetries is determined by the *index* of the submanifold, defined as the number of points in S map to a single generic point of its signature:

$$\operatorname{ind} S = \min \left\{ \# \, (J^{(s+1)})^{-1}\{\zeta\} \;\middle|\; \zeta \in \mathcal{S}^{(s+1)} \right\}.$$

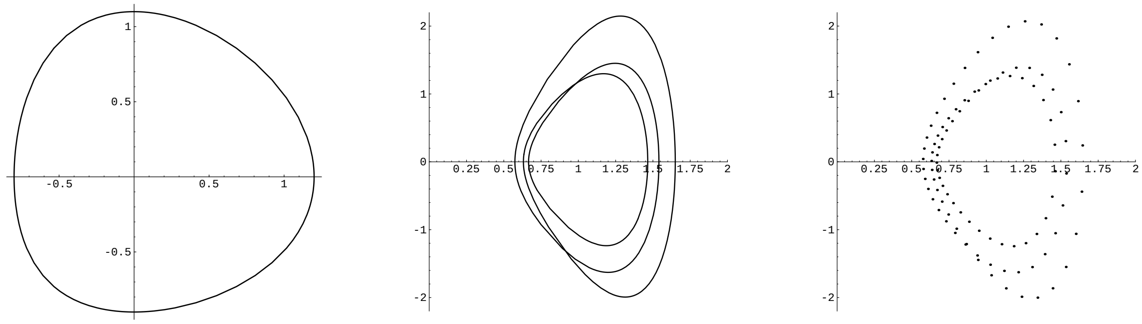
Theorem 4.3. *If S is a nonsingular submanifold, then its symmetry group is a discrete subgroup of cardinality $\# G_S = \operatorname{ind} S$.*

At the other extreme, a rank 0 or *maximally symmetric* submanifold has all constant differential invariants, and so its signature degenerates to a single point.

Theorem 4.4. *A regular p-dimensional submanifold S has differential invariant rank 0 if and only if its symmetry group is a p-dimensional subgroup $H = G_S \subset G$ and an H–orbit: $S = H \cdot z_0$.*

Remark: "Totally singular" submanifolds may have even larger, non-free symmetry groups, but these are not covered by the preceding results. See [**41**] for details and precise characterization of such submanifolds.

Example 4.5. The *Euclidean signature* for a curve in the Euclidean plane is the planar curve $\mathcal{S}(C) = \{(\kappa, \kappa_s)\}$ parametrized by the curvature invariant κ and its first derivative with respect to arc length. Two planar curves are equivalent under oriented rigid motions if and only if they have the same signature curves. The maximally symmetric curves have constant Euclidean curvature, and so their signature curve degenerates to a single point. These are the circles and straight lines, and, in accordance with Theorem 4.4,



The Original Curve Euclidean Signature Curve Discrete Signature

Figure 1. The Curve $x = \cos t + \frac{1}{5}\cos^2 t$, $y = \sin t + \frac{1}{10}\sin^2 t$.

each is the orbit of its one-parameter symmetry subgroup of SE(2). The number of Euclidean symmetries of a curve is equal to its index — the number of times the Euclidean signature is retraced as we go around the curve.

An example of a Euclidean signature curve is displayed in Figure 1. The first figure shows the curve, and the second its Euclidean signature; the axes are κ and κ_s in the signature plot. Note in particular the approximate three-fold symmetry of the curve is reflected in the fact that its signature has winding number three. If the symmetries were exact, the signature would be exactly retraced three times on top of itself. The final figure gives a discrete approximation to the signature which is based on the invariant numerical algorithms to be discussed below.

In Figure 2 we display some signature curves computed from an actual medical image — a 70×70, 8-bit gray-scale image of a cross section of a canine heart, obtained from an MRI scan. We then display an enlargement of the left ventricle. The boundary of the ventricle has been automatically segmented through use of the conformally Riemannian moving contour or snake flow that was proposed in [**27**] and sucessfully applied to a wide variety of 2D and 3D medical imagery, including MRI, ultrasound and CT data, [**52**]. Underneath these images, we display the ventricle boundary curve along with two successive smoothed versions obtained application of the standard Euclidean-invariant curve shortening procedure. Below each curve is the associated spline-interpolated discrete signature curves for the smoothed boundary, as computed using the invariant numerical approximations to κ and κ_s discussed below. As the evolving curves approach circularity the signature curves exhibit less variation in curvature and appear to be winding more and more tightly around a single point, which is the signature of a circle of area equal to the area inside the evolving curve. Despite the rather extensive smoothing involved, except for an overall shrinking as the contour approaches circularity, the basic qualitative features of the different signature curves, and particularly their winding behavior, appear to be remarkably robust.

Thus, the signature curve method has the potential to be of practical use in the general problem of object recognition and symmetry classification. It offer several advantages over more traditional approaches. First, it is purely local, and therefore immediately applicable to occluded objects. Second, it provides a mechanism for recognizing symmetries and approximate symmetries of the object. The design of a suitably robust "signature metric"

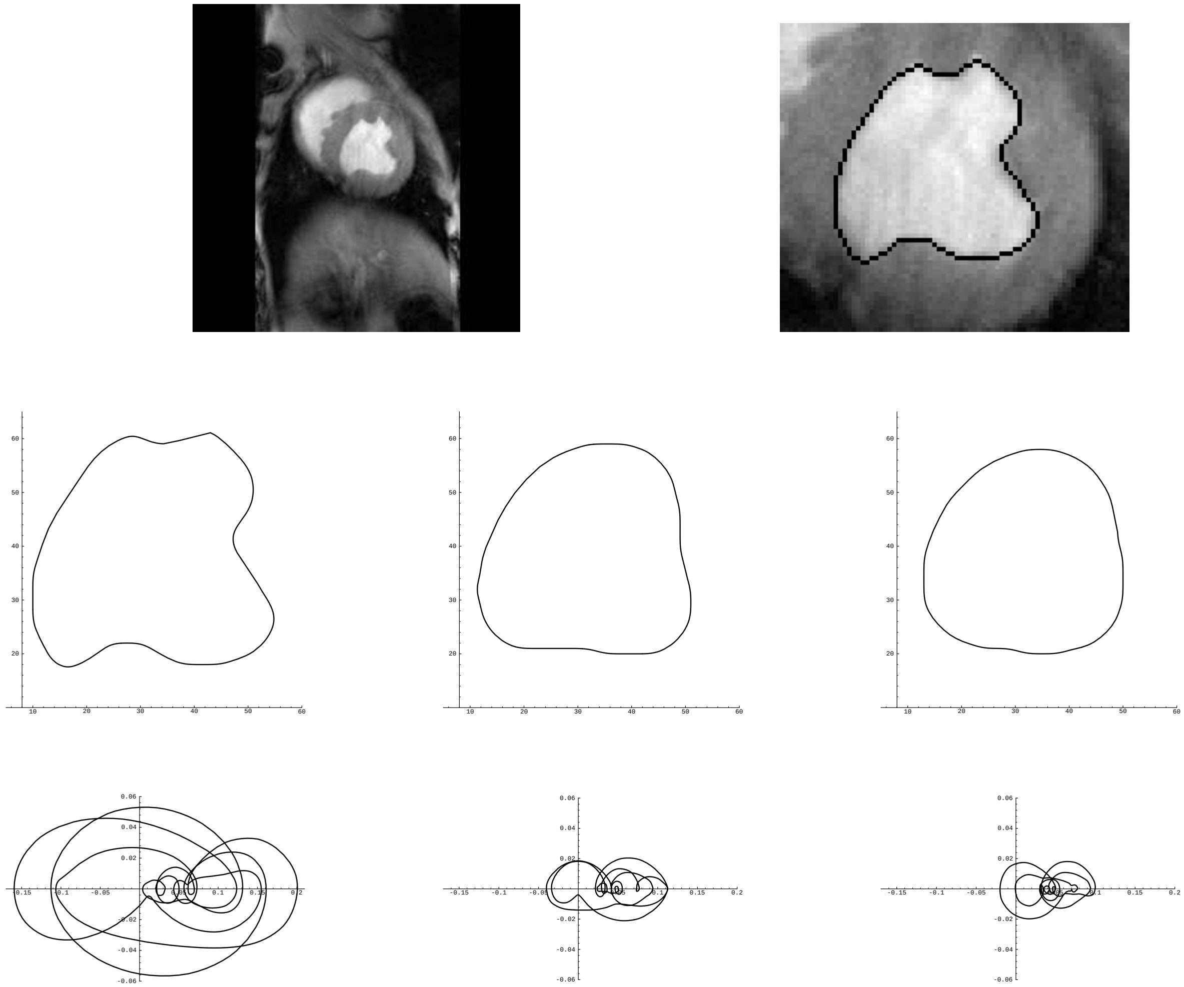


Figure 2. Signature of a Canine Heart Image.

for practical comparison of signatures is the subject of ongoing research.

Example 4.6. Let us next consider the equivalence and symmetry problems for binary forms. According to the general moving frame construction in Example 3.4, the signature curve $\mathcal{S} = \mathcal{S}(q)$ of a function (polynomial) $u = q(x)$ is parametrized by the covariants J^2 and K, as given in (3.12). The following solution to the equivalence problem for complex-valued binary forms, [**1, 37, 40**], is an immediate consequence of the general equivalence Theorem 4.1.

Theorem 4.7. *Two nondegenerate complex-valued forms* $q(x)$ *and* $\overline{q}(x)$ *are equivalent if and only if their signature curves are identical*: $\mathcal{S}(q) = \mathcal{S}(\overline{q})$.

All equivalence maps $\overline{x} = \varphi(x)$ solve the two rational equations

$$J(x)^2 = \overline{J}(\overline{x})^2, \qquad K(x) = \overline{K}(\overline{x}). \tag{4.1}$$

In particular, the theory guarantees φ is necessarily a linear fractional transformation!

Theorem 4.8. *A nondegenerate binary form $q(x)$ is maximally symmetric if and only if it satisfies the following equivalent conditions:*

(a) *q is complex-equivalent to a monomial x^k, with $k \neq 0, n$.*

(b) *The covariant T^2 is a constant multiple of $H^3 \not\equiv 0$.*

(c) *The signature is just a single point.*

(d) *q admits a one-parameter symmetry group.*

(e) *The graph of q coincides with the orbit of a one-parameter subgroup of* GL(2).

A binary form $q(x)$ is nonsingular if and only if it is not complex-equivalent to a monomial if and only if it has a finite symmetry group.

The symmetries of a nonsingular form can be explicitly determined by solving the rational equations (4.1) with $\overline{J} = J$, $\overline{K} = K$. See [**1**] for a MAPLE implementation of this method for computing discrete symmetries and classification of univariate polynomials. In particular, we obtain the following useful bounds on the number of symmetries.

Theorem 4.9. *If $q(x)$ is a binary form of degree n which is not complex-equivalent to a monomial, then its projective symmetry group has cardinality*

$$k \leq \begin{cases} 6n - 12 & \text{if } V = cH^2 \text{ for some constant } c, \text{ or} \\ 4n - 8 & \text{in all other cases.} \end{cases}$$

In her thesis, Kogan, [**29**] extends these results to forms in several variables. In particular, a complete signature for ternary forms leads to a practical algorithm for computing discrete symmetries of, among other cases, elliptic curves.

5. Joint Invariants and Joint Differential Invariants.

One practical difficulty with the differential invariant signature is its dependence upon high order derivatives, which makes it very sensitive to data noise. For this reason, a new signature paradigm, based on joint invariants, was proposed in [**42**]. We consider now the joint action

$$g \cdot (z_0, \ldots, z_n) = (g \cdot z_0, \ldots, g \cdot z_n), \qquad g \in G, \quad z_0, \ldots, z_n \in M. \tag{5.1}$$

of the group G on the $(n+1)$-fold Cartesian product $M^{\times (n+1)} = M \times \cdots \times M$. An invariant $I(z_0, \ldots, z_n)$ of (5.1) is an $(n+1)$*-point joint invariant* of the original transformation group. In most cases of interest, although not in general, if G acts effectively on M, then, for $n \gg 0$ sufficiently large, the product action is free and regular on an open subset of $M^{\times (n+1)}$. Consequently, the moving frame method outlined in Section 2 can be applied to such joint actions, and thereby establish complete classifications of joint invariants and, via prolongation to Cartesian products of jet spaces, joint differential invariants. We will discuss two particular examples — planar curves in Euclidean geometry and projective geometry, referring to [**42**] for details.

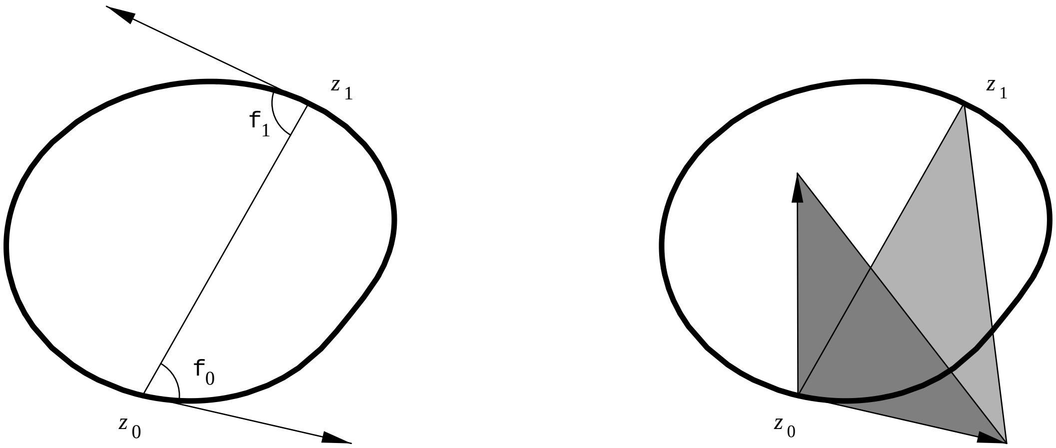


Figure 3. First and Second Order Joint Euclidean Differential Invariants.

Example 5.1. *Euclidean joint differential invariants.* Consider the proper Euclidean group SE(2) acting on oriented curves in the plane $M = \mathbb{R}^2$. We begin with the Cartesian product action on $M^{\times 2} \simeq \mathbb{R}^4$. Taking the simplest cross-section $x_0 = u_0 = x_1 = 0, u_1 > 0$ leads to the normalization equations

$$\begin{gathered} y_0 = x_0 \cos\theta - u_0 \sin\theta + a = 0, \qquad\qquad v_0 = x_0 \sin\theta + u_0 \cos\theta + b = 0, \\ y_1 = x_1 \cos\theta - u_1 \sin\theta + a = 0. \end{gathered} \tag{5.2}$$

Solving, we obtain a right moving frame

$$\theta = \tan^{-1}\left(\frac{x_1 - x_0}{u_1 - u_0}\right), \qquad a = -x_0\cos\theta + u_0\sin\theta, \qquad b = -x_0 \sin\theta - u_0\cos\theta, \tag{5.3}$$

along with the fundamental interpoint distance invariant

$$v_1 = x_1 \sin\theta + u_1\cos\theta + b \quad\longmapsto\quad I = \|\, z_1 - z_0 \,\|. \tag{5.4}$$

Substituting (5.3) into the prolongation formulae (3.5) leads to the the normalized first and second order joint differential invariants

$$\frac{dv_k}{dy} \quad\longmapsto\quad J_k = -\,\frac{(z_1 - z_0)\cdot \dot z_k}{(z_1 - z_0)\wedge \dot z_k}\,, \qquad \frac{d^2 v_k}{dy^2} \quad\longmapsto\quad K_k = -\,\frac{\|\, z_1 - z_0 \,\|^3\,(\dot z_k \wedge \ddot z_k)}{\big[\,(z_1 - z_0)\wedge \dot z_0\,\big]^3}\,, \tag{5.5}$$

for $k = 0, 1$. Note that

$$J_0 = -\cot\phi_0, \qquad J_1 = +\cot\phi_1, \tag{5.6}$$

where $\phi_k = \sphericalangle(z_1 - z_0, \dot z_k)$ denotes the angle between the chord connecting z_0, z_1 and the tangent vector at z_k, as illustrated in Figure 3. The modified second order joint differential invariant

$$\widehat K_0 = -\|\, z_1 - z_0 \,\|^{-3}\, K_0 = \frac{\dot z_0 \wedge \ddot z_0}{\big[\,(z_1 - z_0)\wedge \dot z_0\,\big]^3} \tag{5.7}$$

equals the ratio of the area of triangle whose sides are the first and second derivative vectors $\dot{z}_0$, $\ddot{z}_0$ at the point z_0 over the *cube* of the area of triangle whose sides are the chord from z_0 to z_1 and the tangent vector at z_0; see Figure 3.

On the other hand, we can construct the joint differential invariants by invariant differentiation of the basic distance invariant (5.4). The normalized invariant differential operators are

$$D_{y_k} \quad \longmapsto \quad \mathcal{D}_k = -\frac{\| z_1 - z_0 \|}{(z_1 - z_0) \wedge \dot{z}_k} D_{t_k} \,. \tag{5.8}$$

Proposition 5.2. *Every two-point Euclidean joint differential invariant is a function of the interpoint distance* $I = \| z_1 - z_0 \|$ *and its invariant derivatives with respect to* (5.8).

A generic product curve $\mathbf{C} = C_0 \times C_1 \subset M^{\times 2}$ has joint differential invariant rank $2 = \dim \mathbf{C}$, and its joint signature $\mathcal{S}^{(2)}(\mathbf{C})$ will be a two-dimensional submanifold parametrized by the joint differential invariants I, J_0, J_1, K_0, K_1 of order ≤ 2. There will exist a (local) syzygy $\Phi(I, J_0, J_1) = 0$ among the three first order joint differential invariants.

Theorem 5.3. *A curve* C *or, more generally, a pair of curves* $C_0, C_1 \subset \mathbb{R}^2$, *is uniquely determined up to a Euclidean transformation by its reduced joint signature, which is parametrized by the first order joint differential invariants* I, J_0, J_1. *The curve(s) have a one-dimensional symmetry group if and only if their signature is a one-dimensional curve if and only if they are orbits of a common one-parameter subgroup (i.e., concentric circles or parallel straight lines); otherwise the signature is a two-dimensional surface, and the curve(s) have only discrete symmetries.*

For $n > 2$ points, we can use the two-point moving frame (5.3) to construct the additional joint invariants

$$y_k \longmapsto H_k = \| z_k - z_0 \| \cos \psi_k, \qquad v_k \longmapsto I_k = \| z_k - z_0 \| \sin \psi_k,$$

where $\psi_k = \sphericalangle(z_k - z_0, z_1 - z_0)$. Therefore, a complete system of joint invariants for SE(2) consists of the angles ψ_k, $k \geq 2$, and distances $\| z_k - z_0 \|$, $k \geq 1$. The other interpoint distances can all be recovered from these angles; vice versa, given the distances, and the sign of one angle, one can recover all other angles. In this manner, we establish a "First Main Theorem" for joint Euclidean differential invariants.

Theorem 5.4. *If* $n \geq 2$, *then every* n*-point joint* E(2) *differential invariant is a function of the interpoint distances* $\| z_i - z_j \|$ *and their invariant derivatives with respect to* (5.8). *For the proper Euclidean group* SE(2), *one must also include the sign of one of the angles, say* $\psi_2 = \sphericalangle(z_2 - z_0, z_1 - z_0)$.

Generic three-pointed Euclidean curves still require first order signature invariants. To create a Euclidean signature based entirely on joint invariants, we take four points z_0, z_1, z_2, z_3 on our curve $C \subset \mathbb{R}^2$. As illustrated in Figure 4, there are six different interpoint distance invariants

$$\begin{aligned} a &= \| z_1 - z_0 \|, & b &= \| z_2 - z_0 \|, & c &= \| z_3 - z_0 \|, \\ d &= \| z_2 - z_1 \|, & e &= \| z_3 - z_1 \|, & f &= \| z_3 - z_2 \|, \end{aligned} \tag{5.9}$$

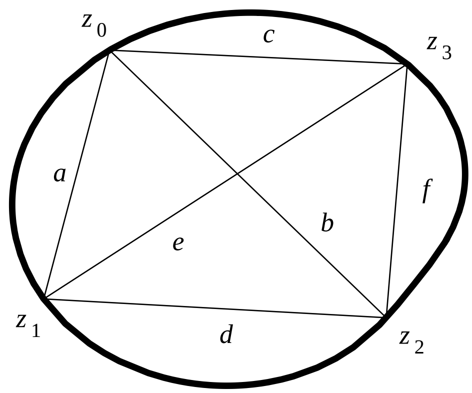


Figure 4. Four-Point Euclidean Curve Invariants.

which parametrize the joint signature $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}(C)$ that uniquely characterizes the curve C up to Euclidean motion. This signature has the advantage of requiring no differentiation, and so is not sensitive to noisy image data. There are two local syzygies

$$\Phi_1(a,b,c,d,e,f) = 0, \qquad\qquad \Phi_2(a,b,c,d,e,f) = 0, \tag{5.10}$$

among the the six interpoint distances. One of these is the universal *Cayley–Menger syzygy*

$$\det \begin{vmatrix} 2a^2 & a^2+b^2-d^2 & a^2+c^2-e^2 \\ a^2+b^2-d^2 & 2b^2 & b^2+c^2-f^2 \\ a^2+c^2-e^2 & b^2+c^2-f^2 & 2c^2 \end{vmatrix} = 0, \tag{5.11}$$

which is valid for all possible configurations of the four points, and is a consequence of their coplanarity, cf. [**2**, **34**]. The second syzygy in (5.10) is curve-dependent and serves to effectively characterize the joint invariant signature. Euclidean symmetries of the curve, both continuous and discrete, are characterized by this joint signature. For example, the number of discrete symmetries equals the signature index — the number of points in the original curve that map to a single, generic point in $\mathcal{S}$.

A wide variety of additional cases, including curves and surfaces in two and three-dimensional space under the Euclidean, equi-affine, affine and projective groups, are investigated in detail in [**42**].

6. Multi-Space for Curves.

In modern numerical analysis, the development of numerical schemes that incorporate additional structure enjoyed by the problem being approximated have become quite popular in recent years. The first instances of such schemes are the symplectic integrators arising in Hamiltonian mechanics, and the related energy conserving methods, [**15**, **31**, **48**].

The design of symmetry-based numerical approximation schemes for differential equations has been studied by various authors, including Shokin, [**47**], Dorodnitsyn, [**17**, **18**], Axford and Jaegers, [**25**], and Budd and Collins, [**8**]. These methods are closely related to the active area of geometric integration of ordinary differential equations on Lie groups, [**9**, **33**]. In practical applications of invariant theory to computer vision, group-invariant numerical schemes to approximate differential invariants have been applied to the problem of symmetry-based object recognition, [**3**, **12**, **11**].

In this section, we outline the basic construction of multi-space that forms the foundation for the study of the geometric properties of discrete approximations to derivatives and numerical solutions to differential equations. We will only discuss the case of curves, which correspond to functions of a single independent variable, and hence satisfy ordinary differential equations. The more difficult case of higher dimensional submanifolds, corresponding to functions of several variables that satisfy partial differential equations, relies on a new approach to multi-dimensional interpolation theory, and hence will be the subject of a subsequent paper, [**44**].

Numerical finite difference approximations to the derivatives of a function $u = f(x)$ rely on its values $u_0 = f(x_0), \ldots, u_n = f(x_n)$ at several distinct points $z_i = (x_i, u_i) = (x_i, f(x_i))$ on the curve. Thus, discrete approximations to jet coordinates on J^n are functions $F(z_0, \ldots, z_n)$ defined on the $(n+1)$-fold Cartesian product space $M^{\times(n+1)} = M \times \cdots \times M$. In order to seamlessly connect the jet coordinates with their discrete approximations, then, we need to relate the jet space for curves, $\mathrm{J}^n = \mathrm{J}^n(M,1)$, to the Cartesian product space $M^{\times(n+1)}$. Now, as the points $z_0, \ldots, z_n$ coalesce, the approximation $F(z_0, \ldots, z_n)$ will not be well-defined unless we specify the "direction" of convergence. Thus, strictly speaking, F is not defined on all of $M^{\times(n+1)}$, but, rather, on the "off-diagonal" part, by which we mean the subset

$$M^{\diamond(n+1)} = \{\ (z_0, \ldots, z_n)\ \big|\ \ z_i \neq z_j \text{ for all } i \neq j\ \} \subset M^{\times(n+1)}$$

consisting of all *distinct* $(n+1)$-tuples of points. As two or more points come together, the limiting value of $F(z_0, \ldots, z_n)$ will be governed by the derivatives (or jet) of the appropriate order governing the direction of convergence. This observation serves to motivate our construction of the n^{th} *order multi-space* $M^{(n)}$, which shall contain both the jet space J^n and the off-diagonal Cartesian product space $M^{\diamond(n+1)}$ in a consistent manner.

Definition 6.1. An $(n+1)$-*pointed curve* $\mathbf{C} = (z_0, \ldots, z_n; C)$ consists of a smooth curve C and $n+1$ not necessarily distinct points $z_0, \ldots, z_n \in C$ thereon. Given $\mathbf{C}$, we let $\#i = \#\{\, j \mid z_j = z_i \,\}$ Two $(n+1)$-pointed curves $\mathbf{C} = (z_0, \ldots, z_n; C)$, $\widetilde{\mathbf{C}} = (\widetilde{z}_0, \ldots, \widetilde{z}_n; \widetilde{C})$, have n^{th} *order multi-contact* if and only if

$$z_i = \widetilde{z}_i, \qquad \text{and} \qquad \mathrm{j}_{\#i-1} C|_{z_i} = \mathrm{j}_{\#i-1} \widetilde{C}|_{z_i}, \qquad \text{for each} \qquad i = 0, \ldots, n.$$

Definition 6.2. The n^{th} order *multi-space*, denoted $M^{(n)}$ is the set of equivalence classes of $(n+1)$-pointed curves in M under the equivalence relation of n^{th} order multi-contact. The equivalence class of an $(n+1)$-pointed curves $\mathbf{C}$ is called its n^{th} order *multi-jet*, and denoted $\mathbf{j}_n \mathbf{C} \in M^{(n)}$.

In particular, if the points on $\mathbf{C} = (z_0, \ldots, z_n; C)$ are all distinct, then $\mathbf{j}_n\mathbf{C} = \mathbf{j}_n\widetilde{\mathbf{C}}$ if and only if $z_i = \widetilde{z}_i$ for all i, which means that $\mathbf{C}$ and $\widetilde{\mathbf{C}}$ have all $n+1$ points in common. Therefore, we can identify the subset of multi-jets of multi-pointed curves having distinct points with the off-diagonal Cartesian product space $M^{\diamond(n+1)} \subset \mathrm{J}^n$. On the other hand, if all $n+1$ points coincide, $z_0 = \cdots = z_n$, then $\mathbf{j}_n\mathbf{C} = \mathbf{j}_n\widetilde{\mathbf{C}}$ if and only if $\mathbf{C}$ and $\widetilde{\mathbf{C}}$ have n^{th} order contact at their common point $z_0 = \widetilde{z}_0$. Therefore, the multi-space equivalence relation reduces to the ordinary jet space equivalence relation on the set of coincident multi-pointed curves, and in this way $\mathrm{J}^n \subset M^{(n)}$. These two extremes do not exhaust the possibilities, since one can have some but not all points coincide. Intermediate cases correspond to “off-diagonal” Cartesian products of jet spaces

$$\mathrm{J}^{k_1} \diamond \cdots \diamond \mathrm{J}^{k_i} \equiv \left\{ (z_0^{(k_1)}, \ldots, z_i^{(k_i)}) \in \mathrm{J}^{k_1} \times \cdots \times \mathrm{J}^{k_i} \;\middle|\; \pi(z_\nu^{(k_\nu)}) \text{ are distinct} \right\}, \tag{6.1}$$

where $\sum k_\nu = n$ and $\pi\colon \mathrm{J}^k \to M$ is the usual jet space projection. These *multi-jet spaces* appear in the work of Dhooghe, [**16**], on the theory of “semi-differential invariants” in computer vision.

Theorem 6.3. *If M is a smooth m-dimensional manifold, then its n^{th} order multi-space $M^{(n)}$ is a smooth manifold of dimension $(n+1)m$, which contains the off-diagonal part $M^{\diamond(n+1)}$ of the Cartesian product space as an open, dense submanifold, and the n^{th} order jet space J^n as a smooth submanifold.*

The proof of Theorem 6.3 requires the introduction of coordinate charts on $M^{(n)}$. Just as the local coordinates on J^n are provided by the coefficients of Taylor polynomials, the local coordinates on $M^{(n)}$ are provided by the coefficients of interpolating polynomials, which are the classical divided differences of numerical interpolation theory, [**35, 46**].

Definition 6.4. Given an $(n+1)$-pointed graph $\mathbf{C} = (z_0, \ldots, z_n; C)$, its divided differences are defined by $[\, z_j \,]_C = f(x_j)$, and

$$[\, z_0 z_1 \ldots z_{k-1} z_k \,]_C = \lim_{z \to z_k} \frac{[\, z_0 z_1 z_2 \ldots z_{k-2} z \,]_C - [\, z_0 z_1 z_2 \ldots z_{k-2} z_{k-1} \,]_C}{x - x_{k-1}}. \tag{6.2}$$

When taking the limit, the point $z = (x, f(x))$ must lie on the curve C, and take limiting values $x \to x_k$ and $f(x) \to f(x_k)$.

In the non-confluent case $z_k \neq z_{k-1}$ we can replace z by z_k directly in the difference quotient (6.2) and so ignore the limit. On the other hand, when all $k+1$ points coincide, the k^{th} order confluent divided difference converges to

$$[\, z_0 \ldots z_0 \,]_C = \frac{f^{(k)}(x_0)}{k!}\,. \tag{6.3}$$

Remark: Classically, one employs the simpler notation $[\, u_0 u_1 \ldots u_k \,]$ for the divided difference $[\, z_0 z_1 \ldots z_k \,]_C$. However, the classical notation is ambiguous since it assumes that the mesh $x_0, \ldots, x_n$ is fixed throughout. Because we are regarding the independent and dependent variables on the same footing — and, indeed, are allowing changes of variables that scramble the two — it is important to adopt an unambiguous divided difference notation here.

Theorem 6.5. *Two $(n+1)$-pointed graphs $\mathbf{C}, \widetilde{\mathbf{C}}$ have n^{th} order multi-contact if and only if they have the same divided differences:*

$$[\, z_0 z_1 \ldots z_k \,]_C = [\, z_0 z_1 \ldots z_k \,]_{\widetilde{C}}, \qquad k = 0, \ldots, n.$$

The reauired local coordinates on multi-space $M^{(n)}$ consist of the independent variables along with all the divided differences

$$x_0, \ldots, x_n, \qquad \begin{array}{ll} u^{(0)} = u_0 = [\, z_0 \,]_C, & u^{(1)} = [\, z_0 z_1 \,]_C, \\ u^{(2)} = 2\, [\, z_0 z_1 z_2 \,]_C \quad \ldots & u^{(n)} = n!\, [\, z_0 z_1 \ldots z_n \,]_C, \end{array} \tag{6.4}$$

prescribed by $(n+1)$-pointed graphs $\mathbf{C} = (z_0, \ldots, z_n; C)$. The $n!$ factor is included so that $u^{(n)}$ agrees with the usual derivative coordinate when restricted to J^n, cf. (6.3).

7. Invariant Numerical Methods.

To implement a numerical solution to a system of differential equations

$$\Delta_1(x, u^{(n)}) = \cdots = \Delta_k(x, u^{(n)}) = 0. \tag{7.1}$$

by finite difference methods, one relies on suitable discrete approximations to each of its defining differential functions Δ_ν, and this requires extending the differential functions from the jet space to the associated multi-space, in accordance with the following definition.

Definition 7.1. An *$(n+1)$-point numerical approximation of order k* to a differential function $\Delta\colon \mathrm{J}^n \to \mathbb{R}$ is an function $F\colon M^{(n)} \to \mathbb{R}$ that, when restricted to the jet space, agrees with Δ to order k.

The simplest illustration of Definition 7.1 is provided by the divided difference coordinates (6.4). Each divided difference $u^{(n)}$ forms an $(n+1)$-point numerical approximation to the n^{th} order derivative coordinate on J^n. According to the usual Taylor expansion, the order of the approximation is $k = 1$. More generally, any differential function $\Delta(x, u, u^{(1)}, \ldots, u^{(n)})$ can immediately be assigned an $(n+1)$-point numerical approximation $F = \Delta(x_0, u^{(0)}, u^{(1)}, \ldots, u^{(n)})$ by replacing each derivative by its divided difference coordinate approximation. However, these are by no means the only numerical approximations possible.

Now let us consider an r-dimensional Lie group G which acts smoothly on M. Since G evidently maps multi-pointed curves to multi-pointed curves while preserving the multi-contact equivalence relation, it induces an action on the multi-space $M^{(n)}$ that will be called the n^{th} *multi-prolongation* of G and denoted by $G^{(n)}$. On the jet subset $\mathrm{J}^n \subset M^{(n)}$ the multi-prolonged action reduced to the usual jet space prolongation. On the other hand, on the off-diagonal part $M^{\diamond(n+1)} \subset M^{(n)}$ the action coincides with the $(n+1)$-fold Cartesian product action of G on $M^{\times(n+1)}$.

We define a *multi-invariant* to be a function $K\colon M^{(n)} \to \mathbb{R}$ on multi-space which is invariant under the multi-prolonged action of $G^{(n)}$. The restriction of a multi-invariant K to jet space will be a differential invariant, $I = K \mid \mathrm{J}^n$, while restriction to $M^{\diamond(n+1)}$ will define a joint invariant $J = K \mid M^{\diamond(n+1)}$. Smoothness of K will imply that the joint invariant J is an *invariant n^{th} order numerical approximation to the differential invariant I.*

Moreover, every invariant finite difference numerical approximation arises in this manner. Thus, the theory of multi-invariants *is* the theory of invariant numerical approximations!

Furthermore, the restriction of a multi-invariant to an intermediate multi-jet subspace, as in (6.1), will define a joint differential invariant, [**42**] — also known as a semi-differential invariant in the computer vision literature, [**16**, **36**]. The approximation of differential invariants by joint differential invariants is, therefore, based on the extension of the differential invariant from the jet space to a suitable multi-jet subspace (6.1). The invariant numerical approximations to joint differential invariants are, in turn, obtained by extending them from the multi-jet subspace to the entire multi-space. Thus, multi-invariants also include invariant semi-differential approximations to differential invariants as well as joint invariant numerical approximations to differential invariants and semi-differential invariants — all in one seamless geometric framework.

Effectiveness of the group action on M implies, typically, freeness and regularity of the multi-prolonged action on an open subset of $M^{(n)}$. Thus, we can apply the basic moving frame construction. The resulting *multi-frame* $\rho^{(n)}: M^{(n)} \to G$ will lead us immediately to the required multi-invariants and hence a general, systematic construction for invariant numerical approximations to differential invariants. Any multi-frame will evidently restrict to a classical moving frame $\rho^{(n)}: \mathrm{J}^n \to G$ on the jet space along with a suitably compatible product frame $\rho^{\diamond(n+1)}: M^{\diamond(n+1)} \to G$.

In local coordinates, we use $w_k = (y_k, v_k) = g \cdot z_k$ to denote the transformation formulae for the individual points on a multi-pointed curve. The multi-prolonged action on the divided difference coordinates gives

$$y_0, \ldots, y_n, \qquad \begin{array}{lll} v^{(0)} = v_0 = [\, w_0 \,], & v^{(1)} = [\, w_0 w_1 \,], & \\ v^{(2)} = [\, w_0 w_1 w_2 \,], & \ldots & v^{(n)} = n!\, [\, w_0, \ldots, w_n \,], \end{array} \tag{7.2}$$

where the formulae are most easily computed via the difference quotients

$$[\, w_0 w_1 \ldots w_{k-1} w_k \,] = \frac{[\, w_0 w_1 w_2 \ldots w_{k-2} w_k \,] - [\, w_0 w_1 w_2 \ldots w_{k-2} w_{k-1} \,]}{y_k - y_{k-1}}, \qquad [\, w_j \,] = v_j, \tag{7.3}$$

and then taking appropriate limits to cover the case of coalescing points. Inspired by the constructions in [**21**], we will refer to (7.2) as the *lifted divided difference invariants*.

To construct a multi-frame, we need to normalize by choosing a cross-section to the group orbits in $M^{(n)}$, which amounts to setting $r = \dim G$ of the lifted divided difference invariants (7.2) equal to suitably chosen constants. An important observation is that in order to obtain the limiting differential invariants, we must require our local cross-section to pass through the jet space, and define, by intersection, a cross-section for the prolonged action on J^n. This compatibility constraint implies that we are only allowed to normalize the first lifted independent variable $y_0 = c_0$.

With the aid of the multi-frame, the most direct construction of the requisite multi-invariants and associated invariant numerical differentiation formulae is through the invariantization of the original finite difference quotients (6.2). Substituting the multi-frame formulae for the group parameters into the lifted coordinates (7.2) provides a complete system of multi-invariants on $M^{(n)}$; this follows immediately from Theorem 2.4. We denote the fundamental multi-invariants by

$$y_i \longmapsto H_i = \iota(x_i), \qquad v^{(n)} \longmapsto K^{(n)} = \iota(u^{(n)}), \tag{7.4}$$

where ι denotes the invariantization map associated with the multi-frame. The fundamental differential invariants for the prolonged action of G on J^n can all be obtained by restriction, so that $I^{(n)} = K^{(n)} \,|\, \mathrm{J}^n$. On the jet space, the points are coincident, and so the multi-invariants H_i will all restrict to the *same* differential invariant $c_0 = H = H_i \,|\, \mathrm{J}^n$ — the normalization value of y_0. On the other hand, the fundamental joint invariants on $M^{\diamond(n+1)}$ are obtained by restricting the multi-invariants $H_i = \iota(x_i)$ and $K_i = \iota(u_i)$. The multi-invariants can computed by using a multi-invariant divided difference recursion

$$[\,I_j\,] = K_j = \iota(u_j) \qquad [\,I_0 \ldots I_k\,] = \iota([\,z_0 z_1 \ldots z_k\,]) = \frac{[\,I_0 \ldots I_{k-2} I_k\,] - [\,I_0 \ldots I_{k-2} I_{k-1}\,]}{H_k - H_{k-1}}, \tag{7.5}$$

and then relying on continuity to extend the formulae to coincident points. The multi-invariants

$$K^{(n)} = n!\,[\,I_0 \ldots I_n\,] = \iota(\,u^{(n)}\,) \tag{7.6}$$

define the fundamental first order invariant numerical approximations to the differential invariants $I^{(n)}$. Higher order invariant approximations can be obtained by invariantization of the higher order divided difference approximations. The moving frame construction has a significant advantage over the infinitesimal approach used by Dorodnitsyn, [**17**, **18**], in that it does not require the solution of partial differential equations in order to construct the multi-invariants.

Given a regular G-invariant differential equation

$$\Delta(x, u^{(n)}) = 0, \tag{7.7}$$

we can invariantize the left hand side to rewrite the differential equation in terms of the fundamental differential invariants:

$$\iota(\Delta(x, u^{(n)})) = \Delta(H, I^{(0)}, \ldots, I^{(n)}) = 0.$$

The invariant finite difference approximation to the differential equation is then obtained by replacing the differential invariants $I^{(k)}$ by their multi-invariant counterparts $K^{(k)}$:

$$\Delta(c_0, K^{(0)}, \ldots, K^{(n)}) = 0. \tag{7.8}$$

Example 7.2. Consider the elementary action

$$(x, u) \longmapsto (\lambda^{-1} x + a, \lambda u + b)$$

of the three-parameter similarity group $G = \mathbb{R}^2 \ltimes \mathbb{R}$ on $M = \mathbb{R}^2$. To obtain the multi-prolonged action, we compute the divided differences (7.2) of the basic lifted invariants

$$y_k = \lambda^{-1} x_k + a, \qquad v_k = \lambda\, u_k + b.$$

We find

$$v^{(1)} = [\,w_0 w_1\,] = \frac{v_1 - v_0}{y_1 - y_0} = \lambda^2\, \frac{u_1 - u_0}{x_1 - x_0} = \lambda^2\,[\,z_0 z_1\,] = \lambda^2\, u^{(1)}.$$

More generally,

$$v^{(n)} = \lambda^{n+1}\, u^{(n)}, \qquad\qquad n \geq 1. \tag{7.9}$$

Note that we may compute the multi-space transformation formulae assuming initially that the points are distinct, and then extending to coincident cases by continuity. (In fact, this gives an alternative method for computing the standard jet space prolongations of group actions!) In particular, when all the points coincide, each $u^{(n)}$ reduces to the n^{th} order derivative coordinate, and (7.9) reduces to the prolonged action of G on J^n. We choose the normalization cross-section defined by

$$y_0 = 0, \qquad\qquad v_0 = 0, \qquad\qquad v^{(1)} = 1,$$

which, upon solving for the group parameters, leads to the basic moving frame

$$a = -\sqrt{u^{(1)}}\; x_0, \qquad\qquad b = -\,\frac{u_0}{\sqrt{u^{(1)}}}\,, \qquad\qquad \lambda = \frac{1}{\sqrt{u^{(1)}}}\,, \tag{7.10}$$

where, for simplicity, we restrict to the subset where $u^{(1)} = [\, z_0 z_1 \,] > 0$. The fundamental joint similarity invariants are obtained by substituting these formulae into

$$y_k \longmapsto H_k = (x_k - x_0)\sqrt{u^{(1)}} = (x_k - x_0)\,\sqrt{\frac{u_1 - u_0}{x_1 - x_0}}\,,$$

$$v_k \longmapsto K_k = \frac{u_k - u_0}{\sqrt{u^{(1)}}} = (u_k - u_0)\,\sqrt{\frac{x_1 - x_0}{u_1 - u_0}}\,,$$

both of which reduce to the trivial zero differential invariant on J^n. Higher order multi-invariants are obtained by substituting (7.10) into the lifted invariants (7.9), leading to

$$K^{(n)} = \frac{u^{(n)}}{(u^{(1)})^{(n+1)/2}} = \frac{n!\,[\, z_0 z_1 \ldots z_n \,]}{[\, z_0 z_1 z_2 \,]^{(n+1)/2}}\,.$$

In the limit, these reduce to the differential invariants $I^{(n)} = (u^{(1)})^{-(n+1)/2}\, u^{(n)}$, and so $K^{(n)}$ give the desired similarity-invariant, first order numerical approximations. To construct an invariant numerical scheme for any similarity-invariant ordinary differential equation

$$\Delta(x, u, u^{(1)}, u^{(2)}, \ldots u^{(n)}) = 0,$$

we merely invariantize the defining differential function, leading to the general similarity–invariant numerical approximation

$$\Delta(0, 0, 1, K^{(2)}, \ldots, K^{(n)}) = 0.$$

Example 7.3. For the action (3.4) of the proper Euclidean group of SE(2) on $M = \mathbb{R}^2$, the multi-prolonged action is free on $M^{(n)}$ for $n \geq 1$. We can thereby determine a first order multi-frame and use it to completely classify Euclidean multi-invariants. The first order transformation formulae are

$$\begin{aligned} y_0 &= x_0 \cos\theta - u_0 \sin\theta + a, & v_0 &= x_0 \sin\theta + u_0 \cos\theta + b, \\ y_1 &= x_1 \cos\theta - u_1 \sin\theta + a, & v^{(1)} &= \frac{\sin\theta + u^{(1)}\cos\theta}{\cos\theta - u^{(1)}\sin\theta}\,, \end{aligned} \tag{7.11}$$

where $u^{(1)} = [\,z_0 z_1\,]$. Normalization based on the cross-section $y_0 = v_0 = v^{(1)} = 0$ results in the right moving frame

$$\begin{aligned} a &= -x_0 \cos\theta + u_0 \sin\theta = -\frac{x_0 + u^{(1)}\,u_0}{\sqrt{1 + (u^{(1)})^2}}\,, \\ b &= -x_0 \sin\theta - u_0 \cos\theta = \frac{x_0\,u^{(1)} - u_0}{\sqrt{1 + (u^{(1)})^2}}\,, \end{aligned} \qquad \tan\theta = -u^{(1)}\,. \tag{7.12}$$

Substituting the moving frame formulae (7.12) into the lifted divided differences results in a complete system of (oriented) Euclidean multi-invariants. These are easily computed by beginning with the fundamental joint invariants $I_k = (H_k, K_k) = \iota(x_k, u_k)$, where

$$\begin{aligned} y_k \longmapsto H_k &= \frac{(x_k - x_0) + u^{(1)}\,(u_k - u_0)}{\sqrt{1 + (u^{(1)})^2}} = (x_k - x_0)\,\frac{1 + [\,z_0 z_1\,]\,[\,z_0 z_k\,]}{\sqrt{1 + [\,z_0 z_1\,]^2}}\,, \\ v_k \longmapsto K_k &= \frac{(u_k - u_0) - u^{(1)}\,(x_k - x_0)}{\sqrt{1 + (u^{(1)})^2}} = (x_k - x_0)\,\frac{[\,z_0 z_k\,] - [\,z_0 z_1\,]}{\sqrt{1 + [\,z_0 z_1\,]^2}}\,. \end{aligned}$$

The multi-invariants are obtained by forming divided difference quotients

$$[\,I_0 I_k\,] = \frac{K_k - K_0}{H_k - H_0} = \frac{K_k}{H_k} = \frac{(x_k - x_1)[\,z_0 z_1 z_k\,]}{1 + [\,z_0 z_k\,]\,[\,z_0 z_1\,]}\,,$$

where, in particular, $I^{(1)} = [\,I_0 I_1\,] = 0$. The second order multi-invariant

$$\begin{aligned} I^{(2)} = 2\,[\,I_0 I_1 I_2\,] = 2\,\frac{[\,I_0 I_2\,] - [\,I_0 I_1\,]}{H_2 - H_1} &= \frac{2\,[\,z_0 z_1 z_2\,]\sqrt{1 + [\,z_0 z_1\,]^2}}{\big(1 + [\,z_0 z_1\,]\,[\,z_1 z_2\,]\big)\big(1 + [\,z_0 z_1\,]\,[\,z_0 z_2\,]\big)} \\ &= \frac{u^{(2)}\sqrt{1 + (u^{(1)})^2}}{\big[\,1 + (u^{(1)})^2 + \tfrac12 u^{(1)} u^{(2)} (x_2 - x_0)\,\big]\,\big[\,1 + (u^{(1)})^2 + \tfrac12 u^{(1)} u^{(2)} (x_2 - x_1)\,\big]} \end{aligned}$$

provides a Euclidean–invariant numerical approximation to the Euclidean curvature:

$$\lim_{z_1, z_2 \to z_0} I^{(2)} = \kappa = \frac{u^{(2)}}{(1 + (u^{(1)})^2)^{3/2}}\,.$$

Similarly, the third order multi-invariant

$$I^{(3)} = 6\,[\,I_0 I_1 I_2 I_3\,] = 6\,\frac{[\,I_0 I_1 I_3\,] - [\,I_0 I_1 I_2\,]}{H_3 - H_2}$$

will form a Euclidean–invariant approximation for the normalized differential invariant $\kappa_s = \iota(u_{xxx})$, the derivative of curvature with respect to arc length, [**12**, **21**].

To compare these with the invariant numerical approximations proposed in [**11**, **12**, **3**], we reformulate the divided difference formulae in terms of the geometrical configurations of the four distinct points z_0, z_1, z_2, z_3 on our curve. We find

$$\begin{aligned} H_k &= \frac{(z_1 - z_0)\cdot(z_k - z_0)}{\|\,z_1 - z_0\,\|} = r_k \cos\phi_k, \\ K_k &= \frac{(z_1 - z_0)\wedge(z_k - z_0)}{\|\,z_1 - z_0\,\|} = r_k \sin\phi_k, \end{aligned} \qquad [\,I_0 I_k\,] = \tan\phi_k,$$

where

$$r_k = \| z_k - z_0 \|, \qquad \phi_k = \sphericalangle(z_k - z_0, z_1 - z_0),$$

denotes the distance and the angle between the indicated vectors. Therefore,

$$\begin{aligned} I^{(2)} &= 2\,\frac{\tan\phi_2}{r_2\cos\phi_2 - r_1}\,, \\ I^{(3)} &= 6\,\frac{(r_2\cos\phi_2 - r_1)\tan\phi_3 - (r_3\cos\phi_3 - r_1)\tan\phi_2}{(r_2\cos\phi_2 - r_1)(r_3\cos\phi_3 - r_1)(r_3\cos\phi_3 - r_2\cos\phi_2)}\,. \end{aligned} \tag{7.13}$$

Interestingly, $I^{(2)}$ is *not* the same Euclidean approximation to the curvature that was used in [**12**, **11**]. The latter was based on the Heron formula for the radius of a circle through three points:

$$I^\star = \frac{4\Delta}{abc} = \frac{2\,\sin\phi_2}{\| z_1 - z_2 \|}\,. \tag{7.14}$$

Here Δ denotes the area of the triangle connecting z_0, z_1, z_2 and

$$a = r_1 = \| z_1 - z_0 \|, \qquad b = r_2 = \| z_2 - z_0 \|, \qquad c = \| z_2 - z_1 \|,$$

are its side lengths. The ratio tends to a limit $I^\star / I^{(2)} \to 1$ as the points coalesce. The geometrical approximation (7.14) has the advantage that it is symmetric under permutations of the points; one can achieve the same thing by symmetrizing the divided difference version $I^{(2)}$. Furthermore, $I^{(3)}$ is an invariant approximation for the differential invariant κ_s, that, like the approximations constructed by Boutin, [**3**], converges properly for arbitrary spacings of the points on the curve.

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