Invariant Submanifold Flows

Peter J. Olver[†]
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
olver@umn.edu
http://www.math.umn.edu/∼olver

Abstract. Given a Lie group acting on a manifold, our aim is to analyze the evolution of differential invariants under invariant submanifold flows. The constructions are based on the equivariant method of moving frames and the induced invariant variational bicomplex. Applications to integrable soliton dynamics, and to the evolution of differential invariant signatures, used in equivalence problems and object recognition and symmetry detection in images, are discussed.

1. Introduction.

Let G be a transformation group acting smoothly on an m-dimensional manifold M. By an *invariant submanifold flow*, we mean a G-invariant evolutionary partial differential equation

$$\frac{\partial S}{\partial t} = \Phi[S]$$

governing the motion of p-dimensional submanifolds $S \subset M$. Invariance requires that G is a symmetry group of the partial differential equation, [**46**]: if $S(t)$ is any solution and $g \in G$ any fixed group transformation, then $\widetilde{S}(t) = g \cdot S(t)$ is another solution. General classification results for invariant evolution equations can be found in [**47**, **52**].

Invariant curve flows, where $p = 1$, and surface flows, where $p = 2$, arise in an impressive range of applications, including geometric optics, [**6**], elastodynamics, [**33**], computer vision, [**52**, **53**, **58**, **60**, **62**], visual tracking and control, [**43**], vortex dynamics, [**25**, **32**],

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interface motions, [**62**], thermal grooving, [**7**], and elsewhere. A celebrated example is the Euclidean invariant curve shortening flow, [**20**, **21**], in which a plane curve moves in its normal direction in proportion to its curvature. In computer vision, Euclidean curve shortening and its equi-affine counterpart have been successfully applied to image denoising and segmentation, [**52**, **59**, **60**]. In three dimensional space, Euclidean-invariant curve flows include the integrable vortex filament flow, [**25**, **32**], while mean curvature and Willmore flows of surfaces have been the subject of extensive analysis and applications, [**5**, **13**].

Given an invariant submanifold flow, a key issue is to track the induced evolution of its basic geometric invariants — curvature, torsion and the like. While a number of particular examples have been worked out by direct computation, e.g., in [**20**, **39**, **40**, **41**], many cases of interest have yet to appear in the literature, owing in part to the complexity of the required calculations. Therefore, it is worth developing general, practical computational tools to facilitate this often tedious task. Mansfield and van der Kamp, [**34**], have already proposed applying the equivariant moving frame methods developed by the author and many collaborators, cf. [**15**, **49**], to this issue. Their approach focusses on the differential invariant syzygies. The present paper takes a direct approach, applying computational tools developed in [**29**] for handling the G–invariant variational bicomplex. Certain invariant differential operators used in the analysis of invariant variational problems also play a key role here.

Any submanifold flow — invariant or otherwise — naturally splits into tangential and normal components. As far as the extrinsic properties of the submanifold are concerned, the tangential component is irrelevant, in that it only induces a reparametrization. On the other hand, tangential flows do affect the evolution of differential invariants as the points move around within the submanifold. Our computational techniques are designed to handle any desired combination of tangential and normal evolution. In practice, there are two principal variants: Flows with no tangential components will be called *normal flows*, keeping in mind that the "normal direction" is specified not by an underlying metric (indeed, G need not act isometrically or conformally), but rather by the (or, more correctly, a) moving frame induced by the transformation group. Normal flows play the predominant role in engineering, computer vision, and most geometric applications.

Alternatively, one can require that the flow be *intrinsic*, meaning that it preserves the group-adapted (co)frame as the submanifold evolves. In the case of curves, a flow is intrinsic if and only if it preserves arc length. Remarkably, in many classical geometries, certain basic intrinsic curve flows induce integrable, soliton evolutions for the differential invariants. The prototypical example is the Euclidean–invariant vortex filament flow studied by Hasimoto, [**25**, **31**, **32**]. The curvature and torsion invariants of the evolving filament satisfy an integrable dynamical system, which can be mapped to the completely integrable nonlinear Schrödinger equation, [**1**]. This led Lamb, [**30**], to draw attention to the surprisingly common, but still poorly understood connection between invariant curve flows and integrable soliton dynamics; since then, many other examples have been found, [**4**, **10**, **12**, **14**, **22**, **24**, **28**, **35**, **36**, **37**, **42**, **54**, **56**]. By "integrable" we shall mean that the evolution equation possesses a recursion operator, [**44**], inducing an infinite hierarchy of higher order symmetries. As we will see, the invariant variational bicomplex provides a natural candidate that turns out to be the recursion operator in many examples.

However, not all induced differential invariant evolutions are integrable, and, at present, we do not understand the general conditions on the group action and invariant curve flow needed to guarantee integrability. Extensions to surface evolutions can be found in [**9**, **11**, **16**, **17**, **38**, **54**].

As a consequence of the Cartan solution to the equivalence problem for submanifolds under group actions, [**47**], the differential invariant signature associated with a submanifold, [**8**], was proposed as a general, mathematically rigorous method for object recognition in the presence of symmetry groups. For example, the signature of a plane curve under Euclidean transformations is the curve parametrized by its curvature invariant and the derivative of curvature with respect to arc length. Several numerical experiments involving the effect of the curve shortening flow on the differential invariant signature were conducted, with encouraging results. However, to date there has been no systematic effort to investigate the behavior of the induced signature flow, and our first task is to show how the induced signature flow follows from the differential invariant evolution, in preparation for subsequent analysis and applications.

2. The Invariant Variational Bicomplex.

We begin by quickly reviewing the basics of prolonged group actions on submanifold jets, moving frames and the induced invariant variational bicomplex. Basic references include [**46**, **47**] for jets, contact forms, and prolonged Lie group actions, [**2**, **63**] for the variational bicomplex, [**15**, **49**, **50**] for the equivariant approach to moving frames, and [**29**] for the moving frame construction of the invariant variational bicomplex. For simplicity, we will only deal with finite-dimensional Lie group actions in this paper, although the general ideas can be straightforwardly adapted to infinite-dimensional pseudo-group actions using more recent extensions of the moving frame technology, [**51**].

Let G be an r-dimensional Lie group, acting smoothly on a m-dimensional manifold M. We will study the induced action on p-dimensional submanifolds $S \subset M$. For $0 \leq n \leq \infty$, let $\mathrm{J}^n = \mathrm{J}^n(M,p)$ denote the n^{th} order (extended) jet bundle for such submanifolds, [**47**]. The action of G on M naturally prolongs to an action on J^n. Since the prolonged group actions are all mutually compatible under projection $\mathrm{J}^n \to \mathrm{J}^k$, we will avoid explicit reference to the order of prolongation, and just use $g \cdot z^{(n)}$ for the action of $g \in G$ on the jet $z^{(n)} \in \mathrm{J}^n$, rather than the more traditional notation $g^{(n)} \cdot z^{(n)}$.

Let† $\rho : \mathrm{J}^n \to G$ be a right-equivariant‡ moving frame, meaning that $\rho(g \cdot z^{(n)}) = \rho(z^{(n)}) \cdot g^{-1}$ for all $g \in G$ and all $z^{(n)} \in \mathrm{J}^n$. Moving frames require freeness and regularity of the prolonged group action, and are explicitly constructed by a normalization process based on the choice of a compatible cross-section $K^n \subset \mathrm{J}^n$ to the group orbits. Specifically, given $z^{(n)} \in \mathrm{J}^n$, we set $g = \rho(z^{(n)})$ to be the unique group element such that $g \cdot z^{(n)} \in K^n$,

† All maps, differential forms, differential functions, etc., need only be locally defined; thus, the domain of ρ is typically a suitable open subset of J^n.

‡ All classical moving frames, [**23**], are left-equivariant, and can be obtained by composing ρ with the group inversion map $g \longmapsto g^{-1}$. We choose to concentrate on the right-equivariant version to (slightly) simplify some of the required calculations.

where defined. Compatibility of moving frames under the jet space projections allows us to also suppress the order in the notation of ρ.

We use ι to denote the *invariantization* process induced by the moving frame. The invariantization of a differential form Ω is the unique differential form $\iota(\Omega)$ that agrees with Ω when restricted to the cross-section. Invariantization defines an (exterior) algebra morphism that projects differential functions and forms on J^n to invariant differential functions and forms.

Calculations take place in local coordinates. Let $(x,u) = (x^1,\ldots,x^p,u^1,\ldots,u^q)$ be local coordinates on M. Viewing the x's as independent variables and the u's as dependent variables, we let $u^\alpha_J = \partial^{\#J} u/\partial x^J$ be the usual induced local coordinates on J^n. Invariantization of the jet coordinate functions produces the *fundamental differential invariants*:

$$H^i = \iota(x^i), \qquad I^\alpha_J = \iota(u^\alpha_J), \qquad \alpha = 1,\ldots,q, \qquad \#J \geq 0. \tag{2.1}$$

These naturally split into two classes: The $r = \dim G$ combinations of fundamental differential invariants appearing in the cross-section equations are constant, and known as the *phantom differential invariants*. The remainder, called the *basic differential invariants*, form a complete system of functionally independent differential invariants.

Separating the local coordinates (x,u) on M into independent and dependent variables naturally splits the differential one-forms on J^∞ into *horizontal forms*, spanned by $dx^1,\ldots,dx^p$, and *vertical forms*, spanned by the *basic contact one-forms*

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^p u^\alpha_{J,i}\,dx^i, \qquad \alpha = 1,\ldots,q, \qquad \#J \geq 0. \tag{2.2}$$

Let π_H and π_V denote the projections mapping one-forms on J^∞ to their horizontal and vertical (contact) components, respectively. The induced splitting $d = d_H + d_V$ of the differential into horizontal and vertical components results in the *variational bicomplex*. In particular, if $F(x,u^{(n)})$ is any differential function, its horizontal and vertical differentials are

$$d_H F = \sum_{i=1}^p (D_i F)\,dx^i, \qquad d_V F = \mathbf{D}_F(\theta) = \sum_{\alpha,J} \frac{\partial F}{\partial u^\alpha_J}\, D_J\theta^\alpha = \sum_{\alpha,J} \frac{\partial F}{\partial u^\alpha_J}\,\theta^\alpha_J, \tag{2.3}$$

in which $D_i = D_{x^i}$ denote the total derivative operators with respect to the independent variables, $D_J = D_{j_1} \cdots D_{j_k}$ are the higher order total derivatives, $\theta = (\theta^1,\ldots,\theta^q)^T$ is the column vector containing the order zero contact forms, while $\mathbf{D}_F = (\mathbf{D}_{F,1},\ldots,\mathbf{D}_{F,q})$ is the *Fréchet derivative* or *formal linearization* of the differential function F.

We will employ our moving frame to invariantize the variational bicomplex as follows. First, let

$$\varpi^i = \omega^i + \eta^i = \iota(dx^i), \quad \text{where} \quad \omega^i = \pi_H(\varpi^i), \quad \eta^i = \pi_V(\varpi^i), \qquad i = 1,\ldots,p, \tag{2.4}$$

denote the *invariantized horizontal one-forms*. Their horizontal components $\omega^1,\ldots,\omega^p$ form, in the language of [**47**], a contact-invariant coframe for the prolonged group action, while $\eta^1,\ldots,\eta^p$ supply "contact corrections" that make the one-forms $\varpi^1,\ldots,\varpi^p$ fully

G-invariant. The corresponding dual invariant total differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are defined so that

$$d_{\mathcal{H}} F = \sum_{i=1}^{p} (\mathcal{D}_i F)\, \varpi^i, \qquad\qquad d_{\mathcal{H}} \Omega = \sum_{i=1}^{p} \varpi^i \wedge \mathcal{D}_i \Omega, \tag{2.5}$$

for any differential function F and, more generally, differential form Ω, on which the $\mathcal{D}_i$ act via Lie differentiation. Finally, let

$$\vartheta^\alpha_J = \iota(\theta^\alpha_J), \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0, \tag{2.6}$$

be the *invariantized basis contact forms*.

As in the usual, non-invariant bicomplex construction, the decomposition of invariant one-forms on J^∞ into invariant horizontal and invariant contact components induces a decomposition of the differential. However, now $d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$ splits into three constituents, where $d_{\mathcal{H}}$ adds an invariant horizontal form, $d_{\mathcal{V}}$ adds a invariant contact form, while $d_{\mathcal{W}}$ replaces an invariant horizontal one-form with a combination of wedge products of two invariant contact forms. In other words, if we let $\widetilde{\mathbf{\Omega}}^{r,s}$ denote the space of differential forms of degree $r+s$ spanned by wedge products of r invariant horizontal one-forms (2.4) and s invariant contact one-forms (2.6), then

$$d_{\mathcal{H}} : \widetilde{\mathbf{\Omega}}^{r,s} \longrightarrow \widetilde{\mathbf{\Omega}}^{r+1,s}, \qquad d_{\mathcal{V}} : \widetilde{\mathbf{\Omega}}^{r,s} \longrightarrow \widetilde{\mathbf{\Omega}}^{r,s+1}, \qquad d_{\mathcal{W}} : \widetilde{\mathbf{\Omega}}^{r,s} \longrightarrow \widetilde{\mathbf{\Omega}}^{r-1,s+2}. \tag{2.7}$$

The resulting *invariant variational quasi-tricomplex* is characterized by the formulae

$$\begin{aligned} d_{\mathcal{H}}^2 &= 0, & d_{\mathcal{H}}\, d_{\mathcal{V}} + d_{\mathcal{V}}\, d_{\mathcal{H}} &= 0, \\ d_{\mathcal{W}}^2 &= 0, & d_{\mathcal{V}}\, d_{\mathcal{W}} + d_{\mathcal{W}}\, d_{\mathcal{V}} &= 0, \end{aligned} \qquad d_{\mathcal{V}}^2 + d_{\mathcal{H}}\, d_{\mathcal{W}} + d_{\mathcal{W}}\, d_{\mathcal{H}} = 0. \tag{2.8}$$

Fortunately, the third, anomalous component $d_{\mathcal{W}}$ plays no role in the applications considered here; in particular, $d_{\mathcal{W}} F = 0$ for any differential function F.

Example 2.1. *Euclidean geometry of plane curves*: Consider the usual action

$$y = x \cos\phi - u \sin\phi + a, \qquad\qquad v = x \cos\phi + u \sin\phi + b, \tag{2.9}$$

of the $r = 3$ – dimensional planar *Euclidean group* $\mathrm{SE}(2) \simeq \mathrm{SO}(2) \ltimes \mathbb{R}^2$ acting on plane curves $C \subset M = \mathbb{R}^2$. To expedite the computations, we will assume the curves are, at least locally, given as the graphs of functions $u = f(x)$. Extending the ensuing analysis to arbitrarily parametrized curves is straightforward; indeed, while the resulting invariants have more complicated formulae, their algebraic and differential interrelationships are exactly the same.

The prolonged group transformations

$$v_y = \frac{\sin\phi + u_x \cos\phi}{\cos\phi - u_x \sin\phi}\,, \qquad v_{yy} = \frac{u_{xx}}{(\cos\phi - u_x \sin\phi)^3}\,, \qquad \text{etc.}, \tag{2.10}$$

are found by implicit differentiation. The classical Euclidean moving frame, [**23**], relies on the coordinate cross-section $K^1 = \{\, x = u = u_x = 0 \,\} \subset \mathrm{J}^1$, resulting in the normalization equations $y = 0$, $v = 0$, $v_y = 0$. Solving these for the group parameters $g = (\phi, a, b)$ yields the right-equivariant[†] moving frame

$$\phi = -\tan^{-1} u_x\,, \qquad a = -\frac{x + u u_x}{\sqrt{1+u_x^2}}\,, \qquad b = \frac{x u_x - u}{\sqrt{1+u_x^2}}\,. \tag{2.11}$$

The fundamental differential invariants (2.11) are obtained by substituting the moving frame formulas (2.11) into the transformed coordinates (2.10), leading to

$$\begin{aligned} &H = \iota(x) = 0, && I_0 = \iota(u) = 0, && I_1 = \iota(u_x) = 0, \\ &I_2 = \iota(u_{xx}) = \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}\,, && I_3 = \iota(u_{xxx}) = \kappa_s, && I_4 = \iota(u_{xxxx}) = \kappa_{ss} + 3\kappa^3, \end{aligned} \tag{2.12}$$

and so on. In particular, H, I_0, I_1 are the phantom invariants, while $I_2 = \kappa$ is the Euclidean curvature. To obtain the invariant differential forms, we substitute the moving frame formulae (2.11) into

$$dy = (\cos\phi)\,dx - (\sin\phi)\,du = (\cos\phi - u_x \sin\phi)\,dx - (\sin\phi)\,\theta,$$

where $\theta = du - u_x\,dx$ is the order 0 contact one-form. This results in the invariantized horizontal one-form

$$\varpi = \iota(dx) = \omega + \eta = \sqrt{1+u_x^2}\;dx + \frac{u_x}{\sqrt{1+u_x^2}}\,\theta, \tag{2.13}$$

which is a combination of the contact-invariant arc length form $\omega = ds$ and the contact correction η. The dual invariant differential operator

$$\mathcal{D} = D_s = (1+u_x^2)^{-1/2}\,D_x \tag{2.14}$$

is the usual arc length derivative, and can be employed to generate the higher order differential invariants. In a similar fashion, we construct the invariantized contact forms

$$\vartheta = \frac{\theta}{\sqrt{1+u_x^2}}\,, \qquad \vartheta_1 = \frac{(1+u_x^2)\,\theta_x - u_x u_{xx}\theta}{(1+u_x^2)^2}\,, \qquad \ldots\,. \tag{2.15}$$

3. Recurrence.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be a basis for the infinitesimal generators of our transformation group. We prolong each infinitesimal generator to J^n. For conciseness, we will retain the same notation $\mathbf{v}_\kappa$ for the prolonged vector fields which, in local coordinates, take the form

$$\mathbf{v}_\kappa = \sum_{i=1}^{p} \xi^i_\kappa(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\;\sum_{j=\#J=0}^{n} \varphi^\alpha_{J,\kappa}(x,u^{(j)})\,\frac{\partial}{\partial u^\alpha_J}\,, \qquad \kappa = 1, \ldots, r. \tag{3.1}$$

[†] Actually, this moving frame is only locally equivariant, since there remains an ambiguity of π in the prescription of the rotation angle. For simplicity (and in accord with most treatments of this example), we shall ignore this technicality, referring to [**48**] for a detailed discussion.

The coefficients $\varphi^\alpha_{J,\kappa} = \mathbf{v}_\kappa(u^\alpha_J)$ can be successively constructed by Lie's recursive prolongation formula, [**46**, **47**]:

$$\varphi^\alpha_{Ji,\kappa} = D_i \varphi^\alpha_{J,\kappa} - \sum_{j=1}^{p} u^\alpha_{Jj} \, D_i \xi^j_\kappa. \tag{3.2}$$

A straightforward induction establishes the explicit version, first written down by the author in [**45**]:

$$\varphi^\alpha_{J,\kappa} = D_J Q^\alpha_\kappa + \sum_{i=1}^{p} \xi^i_\kappa \, u^\alpha_{J,i}, \qquad \text{where} \qquad Q^\alpha_\kappa = \varphi^\alpha_\kappa - \sum_{i=1}^{p} \xi^i_\kappa \, u^\alpha_i \tag{3.3}$$

are the components of the *characteristic* of $\mathbf{v}_\kappa$.

Given a moving frame, by a *recurrence relation*, we mean an equation that expresses an invariantly differentiated invariant in terms of the basic differential invariants (2.1). Strikingly, all such relations are consequences of a single *universal recurrence formula* that governs the differentials of all invariantized differential functions and forms on J^∞.

Theorem 3.1. *If Ω is any differential form on J^∞, then*

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota[\mathbf{v}_\kappa(\Omega)], \tag{3.4}$$

where $\nu^1, \ldots, \nu^r$ are the invariantized Maurer–Cartan forms *dual to the infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_r$, while $\mathbf{v}_\kappa(\Omega)$ denotes the Lie derivative of Ω with respect to the prolonged infinitesimal generator $\mathbf{v}_\kappa$.*

The invariantized Maurer–Cartan forms $\nu^1, \ldots, \nu^r$ are obtained by pulling back the usual dual Maurer–Cartan forms $\mu^1, \ldots, \mu^r$ on G by the moving frame map: $\nu^\kappa = \rho^* \mu^\kappa$. Details would take us too far afield, [**29**], but, fortunately, are superfluous thanks to the following wonderful result that allows us to directly compute them:

Lemma 3.2. *Let $I_1 = \iota(z_1), \ldots, I_r = \iota(z_r)$ be the phantom differential invariants stemming from our cross-section. Then the corresponding* phantom recurrence formulae

$$0 = dI_\varsigma = d\,\iota(z_\varsigma) = \iota(dz_\varsigma) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota[\mathbf{v}_\kappa(z_\varsigma)], \qquad \varsigma = 1, \ldots, r, \tag{3.5}$$

can be uniquely solved for the invariantized Maurer–Cartan forms $\nu^1, \ldots, \nu^r$.

Having solved the linear system (3.5) for $\nu^1, \ldots, \nu^r$, we then decompose the resulting invariantized Maurer–Cartan forms into their invariant horizontal and contact components:

$$\nu^\kappa = \gamma^\kappa + \varepsilon^\kappa, \qquad \text{where} \qquad \gamma^\kappa = \sum_{i=1}^{p} R^\kappa_i \, \varpi^i, \qquad \varepsilon^\kappa = \sum_{\alpha, J} S^{\kappa,J}_\alpha \, \vartheta^\alpha_J, \tag{3.6}$$

where $R^\kappa_i, S^{\kappa,J}_\alpha$ are certain differential invariants. The R^κ_i will be called the *Maurer–Cartan invariants*, [**26**, **27**, **50**]. In the case of curves, the R^κ_i appear as the entries of the Frenet–Serret matrix $\mathcal{D}\rho(x, u^{(n)}) \cdot \rho(x, u^{(n)})^{-1}$, assuming $G \subset \mathrm{GL}(N)$ is a matrix Lie group, [**23**].

Substituting (3.6) back into the universal formula (3.4) produces a complete system of explicit recurrence relations for all the differentiated invariants and invariant differential forms.

In particular, successively setting Ω to be each of the jet coordinate functions x^i, u^α_J, results in the recurrence formulae for the fundamental differential invariants (2.1):

$$
\begin{aligned}
dH^i &= \iota(dx^i) + \sum_{\kappa=1}^{r} \nu^\kappa\, \iota[\mathbf{v}_\kappa(x^i)] = \varpi^i + \sum_{\kappa=1}^{r} \iota(\xi^i_\kappa)\, \nu^\kappa, \\
dI^\alpha_J &= \iota(du^\alpha_J) + \sum_{\kappa=1}^{r} \nu^\kappa\, \iota[\mathbf{v}_\kappa(u^\alpha_J)] = \iota\left(\sum_{i=1}^{p} u^\alpha_{Ji}\, dx^i + \theta^\alpha_J \right) + \sum_{\kappa=1}^{r} \iota(\varphi^\alpha_{J,\kappa})\, \nu^\kappa \\
&= \sum_{i=1}^{p} I^\alpha_{Ji}\, \varpi^i + \vartheta^\alpha_J + \sum_{\kappa=1}^{r} \iota(\varphi^\alpha_{J,\kappa})\, \nu^\kappa.
\end{aligned} \tag{3.7}
$$

In view of (3.6), the coefficient of ϖ^i in (3.7) yields the recurrence relations

$$
\mathcal{D}_i H^j = \delta^j_i + \sum_{\kappa=1}^{r} R^\kappa_i\, \iota(\xi^i_\kappa), \qquad\qquad \mathcal{D}_i I^\alpha_J = I^\alpha_{Ji} + \sum_{\kappa=1}^{r} R^\kappa_i\, \iota(\varphi^\alpha_{J,\kappa}), \tag{3.8}
$$

where δ^j_i is the usual Kronecker delta. Owing to the functional independence of the basic (non-phantom) differential invariants, these formulae, in fact, serve to completely characterize the structure of the non-commutative differential algebra of differential invariants, [**15**, **50**]. Similarly, the contact components in (3.7) yield the vertical recurrence formulae

$$
d_{\mathcal{V}}\, H^i = \sum_{\kappa=1}^{r} \iota(\xi^i_\kappa)\, \varepsilon^\kappa, \qquad\qquad d_{\mathcal{V}}\, I^\alpha_J = \vartheta^\alpha_J + \sum_{\kappa=1}^{r} \iota(\varphi^\alpha_{K\kappa})\, \varepsilon^\kappa, \tag{3.9}
$$

while $d_{\mathcal{W}}\, H^i = d_{\mathcal{W}}\, I^\alpha_J = 0$.

Next, the recurrence formulae (3.4) for the derivatives of the invariant horizontal forms are

$$
\begin{aligned}
d\varpi^i = d[\iota(dx^i)] &= \iota(d^2 x^i) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota[\mathbf{v}_\kappa(dx^i)] \\
&= \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota\left(\sum_{k=1}^{p} D_k \xi^i_\kappa\, dx^k + \sum_{\alpha=1}^{q} \frac{\partial \xi^i_\kappa}{\partial u^\alpha}\, \theta^\alpha \right) \\
&= \sum_{\kappa=1}^{r} \sum_{k=1}^{p} \iota\left(D_k \xi^i_\kappa\right) \nu^\kappa \wedge \varpi^k + \sum_{\kappa=1}^{r} \sum_{\alpha=1}^{q} \iota\left(\frac{\partial \xi^i_\kappa}{\partial u^\alpha} \right) \nu^\kappa \wedge \vartheta^\alpha.
\end{aligned} \tag{3.10}
$$

The resulting two-form can be decomposed into three basic constituents, belonging, respectively, to the invariant summands $\widetilde{\boldsymbol{\Omega}}^{2,0} \oplus \widetilde{\boldsymbol{\Omega}}^{1,1} \oplus \widetilde{\boldsymbol{\Omega}}^{0,2}$. In view of (3.6), the terms in (3.10) involving wedge products of two horizontal forms, i.e., in $\widetilde{\boldsymbol{\Omega}}^{2,0}$, yield

$$
d_{\mathcal{H}}\, \varpi^i = -\sum_{j<k} Y^i_{jk}\, \varpi^j \wedge \varpi^k, \quad \text{where} \quad Y^i_{jk} = \sum_{\kappa=1}^{r} \sum_{j=1}^{p} \left[\, R^\kappa_k\, \iota(D_j \xi^i_\kappa) - R^\kappa_j\, \iota(D_k \xi^i_\kappa) \,\right] \tag{3.11}
$$

are called the *commutator invariants*, since combining (3.11) with (2.5) produces the commutation formulae for the invariant differential operators:

$$[\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y^i_{jk}\, \mathcal{D}_i = - \sum_{i=1}^{p} Y^i_{kj}\, \mathcal{D}_i. \tag{3.12}$$

Next, the terms in (3.10) involving wedge products of a horizontal and a contact form yield

$$d_{\mathcal{V}}\, \varpi^i = \sum_{\kappa=1}^{r} \left[\sum_{\alpha=1}^{q} \iota\left(\frac{\partial \xi^i_\kappa}{\partial u^\alpha}\right) \gamma^\kappa \wedge \vartheta^\alpha + \sum_{k=1}^{p} \iota(D_k \xi^i_\kappa)\, \varepsilon^\kappa \wedge \varpi^k \right]. \tag{3.13}$$

Finally, the remaining terms, involving wedge products of two contact forms, provide the formulas for the anomalous third component of the differential:

$$d_{\mathcal{W}}\, \varpi^i = \sum_{\kappa=1}^{r} \sum_{\alpha=1}^{q} \iota\left(\frac{\partial \xi^i_\kappa}{\partial u^\alpha}\right) \varepsilon^\kappa \wedge \vartheta^\alpha. \tag{3.14}$$

In a similar fashion, we derive the recurrence formulae (3.4) for the differentiated invariant contact forms:

$$d\vartheta^\alpha_J = d[\iota(\theta^\alpha_J)] = \iota(d\theta^\alpha_J) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota[\mathbf{v}_\kappa(\theta^\alpha_J)] = \iota\left(\sum_{i=1}^{p} dx^i \wedge \theta^\alpha_{Ji}\right) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota(\psi^\alpha_{J\kappa}), \tag{3.15}$$

where

$$\psi^\alpha_{J\kappa} = \mathbf{v}_\kappa(\theta^\alpha_J) = d\varphi^\alpha_{J\kappa} - \sum_{i=1}^{p} \left[\varphi^\alpha_{Ji\kappa}\, dx^i + u^\alpha_{Ji}\, d\xi^i_\kappa \right] = d_V\, \varphi^\alpha_{J\kappa} - \sum_{i=1}^{p} u^\alpha_{Ji}\, d_V\, \xi^i_\kappa \tag{3.16}$$

are known as the *vertical prolongation coefficients* of the vector field $\mathbf{v}_\kappa$. For our purposes, we only require the component of (3.15) that involves invariant horizontal forms:

$$d_{\mathcal{H}}\, \vartheta^\alpha_J = \sum_{i=1}^{p} \varpi^i \wedge \vartheta^\alpha_{Ji} + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \iota(\psi^\alpha_{J\kappa}), \tag{3.17}$$

Since[†]

$$d_{\mathcal{H}}\, \vartheta = \sum_{i=1}^{p} \varpi^i \wedge \mathcal{D}_i\, \vartheta \tag{3.18}$$

for any contact form ϑ, we deduce the recurrence formulae

$$\mathcal{D}_i \vartheta^\alpha_J = \vartheta^\alpha_{Ji} + \sum_{\kappa=1}^{r} R^\kappa_i\, \iota(\psi^\alpha_{J\kappa}) \tag{3.19}$$

[†] *Warning*: The analogous formula is *not* valid for horizontal forms.

for the invariant (Lie) derivatives of the invariant contact forms. The latter can inductively be solved to express the higher order invariantized contact forms as certain invariant derivatives of those of order 0:

$$\vartheta^\alpha_J = \sum_{\beta=1}^{q} \mathcal{E}^\alpha_{J,\beta}(\vartheta^\beta) = \mathcal{E}^\alpha_J(\vartheta), \tag{3.20}$$

in which $\vartheta = (\vartheta^1, \ldots, \vartheta^q)^T$ denotes the column vector containing the order zero invariantized contact forms, while $\mathcal{E}^\alpha_J = (\mathcal{E}^\alpha_{J,1}, \ldots, \mathcal{E}^\alpha_{J,q})$ are certain invariant differential operators of order $\#J$.

In view of (3.9, 20), if $K = K(\ldots H^i \ldots I^\alpha_J \ldots)$ is any differential invariant, we can write its invariant vertical derivative in the form

$$d_{\mathcal{V}}\, K = \sum_{i=1}^{p} \frac{\partial K}{\partial H^i}\ d_{\mathcal{V}}\, H^i + \sum_{\alpha,J} \frac{\partial K}{\partial I^\alpha_J}\ d_{\mathcal{V}}\, I^\alpha_J = \mathcal{A}_K(\vartheta) = \sum_{\alpha=1}^{q} \mathcal{A}_{K,\alpha}(\vartheta^\alpha), \tag{3.21}$$

in which $\mathcal{A}_K = (\mathcal{A}_{K,1}, \ldots, \mathcal{A}_{K,q})$ is a row vector of invariant differential operators. We view (3.21) as the invariant version of the vertical differentiation formula $d_V\, F = \mathbf{D}_F(\theta)$, cf. (2.3), which motivates the following terminology.

Definition 3.3. The *invariant linearization* of a differential invariant K is the invariant differential operator $\mathcal{A}_K$ that satisfies $d_{\mathcal{V}}\, K = \mathcal{A}_K(\vartheta)$.

Remark: In [**29**], $\mathcal{A}_K$ was called the *Eulerian operator* associated with K owing to its appearance in the differential invariant form of the Euler–Lagrange equations for an invariant variational problem.

Similarly, we combine (3.6, 13, 20) to produce formulae

$$d_{\mathcal{V}}\, \varpi^i = \sum_{j=1}^{p} \sum_{\alpha=1}^{q} \mathcal{B}^i_{j\alpha}(\vartheta^\alpha) \wedge \varpi^j = \sum_{j=1}^{p} \mathcal{B}^i_j(\vartheta) \wedge \varpi^j \tag{3.22}$$

for the vertical differentials of the invariant horizontal forms, in which $\mathcal{B}^i_j = (\mathcal{B}^i_{j1}, \ldots, \mathcal{B}^i_{jq})$ is a family of p^2 row-vector-valued invariant differential operators, known, collectively, as the *invariant Hamiltonian operator complex*, again stemming from its role in the invariant calculus of variations. (See [**55**] for the original, non-invariant version.)

Example 2.1. (*continued*) The vector fields

$$\begin{gathered} \mathbf{v}_1 = \partial_x, \qquad\qquad \mathbf{v}_2 = \partial_u, \\ \mathbf{v}_3 = -u\,\partial_x + x\,\partial_u + (1+u_x^2)\,\partial_{u_x} + 3\,u_x u_{xx}\,\partial_{u_{xx}} + (4\,u_x u_{xxx} + 3\,u_{xx}^2)\,\partial_{u_{xxx}} + \ \cdots\ , \end{gathered} \tag{3.23}$$

form a basis for the prolonged infinitesimal generators of the planar Euclidean group action on $\mathbb{R}^2$. To establish the recurrence formulae, the initial step is to determine the invariantized Maurer–Cartan forms ν^1, ν^2, ν^3 dual to the generators (3.23), by solving the phantom recurrence relations

$$\begin{aligned}
0 = dH = d\,\iota(x) &= \iota(dx) + \nu^1\,\iota[\mathbf{v}_1(x)] + \nu^2\,\iota[\mathbf{v}_2(x)] + \nu^3\,\iota[\mathbf{v}_3(x)] = \varpi + \nu^1, \\
0 = dI_0 = d\,\iota(u) &= \iota(u_x\,dx + \theta) + \nu^1\,\iota[\mathbf{v}_1(u)] + \nu^2\,\iota[\mathbf{v}_2(u)] + \nu^3\,\iota[\mathbf{v}_3(u)] = \vartheta + \nu^2, \\
0 = dI_1 = d\,\iota(u_x) &= \iota(u_{xx}\,dx + \theta_x) + \nu^1\,\iota[\mathbf{v}_1(u_x)] + \nu^2\,\iota[\mathbf{v}_2(u_x)] + \nu^3\,\iota[\mathbf{v}_3(u_x)] \\
&= \kappa\,\varpi + \vartheta_1 + \nu^3.
\end{aligned}$$

Therefore,

$$\nu^1 = -\,\varpi, \qquad \nu^2 = -\,\vartheta, \qquad \nu^3 = -\,\kappa\,\varpi - \vartheta_1. \tag{3.24}$$

The Maurer–Cartan invariants

$$R^1 = -1, \qquad R^2 = 0, \qquad R^3 = -\,\kappa = -\,I_2. \tag{3.25}$$

can then be read off as the coefficients of the invariant horizontal one-form ϖ. Substituting (3.24) into the higher order recurrence relations

$$\begin{aligned}
dI_k = d\,\iota(u_k) &= \iota(u_{k+1}\,dx + \theta_k) + \nu^1\,\iota[\mathbf{v}_1(u_k)] + \nu^2\,\iota[\mathbf{v}_2(u_k)] + \nu^3\,\iota[\mathbf{v}_3(u_k)] \\
&= I_{k+1}\,\varpi + \vartheta_k - \iota(\varphi_3^k)(\kappa\,\varpi + \vartheta_1),
\end{aligned}$$

will prescribe their invariant horizontal differentials

$$d_{\mathcal{H}}\,I_k = (\mathcal{D}I_k)\,\varpi = \big(\,I_{k+1} - \iota(\varphi_3^k)\,\kappa\,\big)\,\varpi.$$

In particular,

$$\mathcal{D}I_2 = I_3, \qquad \mathcal{D}I_3 = I_4 - 3\,I_2^3, \qquad \mathcal{D}I_4 = I_5 - 10\,I_2^2 I_3, \tag{3.26}$$

and so on. These can be iteratively solved to produce the explicit formulae

$$I_2 = \kappa, \qquad I_3 = \kappa_s, \qquad I_4 = \kappa_{ss} + 3\,\kappa^3, \qquad I_5 = \kappa_{sss} + 19\,\kappa^2\kappa_s, \tag{3.27}$$

for the normalized differential invariants. Similarly,

$$d_{\mathcal{V}}\,I_2 = \vartheta_2, \qquad d_{\mathcal{V}}\,I_3 = \vartheta_3 - 3\,\kappa^2\,\vartheta_1, \qquad d_{\mathcal{V}}\,I_4 = \vartheta_4 - 10\,\kappa\,\kappa_s\,\vartheta_1. \tag{3.28}$$

We next use (3.19) and (3.25) to compute the arc length derivatives of the invariant contact forms

$$\mathcal{D}\,\vartheta_k = \vartheta_{k+1} + R^1\,\iota(\psi_{k,1}) + R^2\,\iota(\psi_{k,2}) + R^3\,\iota(\psi_{k,3}) = \vartheta_{k+1} - \iota(\psi_{k,1}) - \kappa\,\iota(\psi_{k,3}), \tag{3.29}$$

where the vertical prolongation coefficients $\psi_{k,\nu} = \mathbf{v}_\nu(\theta_k)$ are given by

$$\psi_{k,1} = \psi_{k,2} = 0, \qquad \text{while} \qquad \begin{aligned} \psi_{0,3} &= \mathbf{v}_3(\theta) = u_x\,\theta, \\ \psi_{1,3} &= \mathbf{v}_3(\theta_x) = 2\,u_x\,\theta_x + u_{xx}\,\theta, \\ \psi_{2,3} &= \mathbf{v}_3(\theta_{xx}) = 3\,u_x\theta_{xx} + 3\,u_{xx}\,\theta_x + u_{xxx}\,\theta, \end{aligned}$$

and so on. In particular,

$$\begin{aligned}
\mathcal{D}\,\vartheta &= \vartheta_1, & \mathcal{D}\,\vartheta_2 &= \vartheta_3 - 3\,\kappa^2\,\vartheta_1 - \kappa\,\kappa_s\,\vartheta, \\
\mathcal{D}\,\vartheta_1 &= \vartheta_2 - \kappa^2\,\vartheta, & \mathcal{D}\,\vartheta_3 &= \vartheta_4 - 6\,\kappa^2\,\vartheta_2 - 4\,\kappa\,\kappa_s\,\vartheta_1 - (\kappa\,\kappa_{ss} + 3\,\kappa^4)\,\vartheta,
\end{aligned} \tag{3.30}$$

which can be recursively solved for

$$\begin{aligned} &\vartheta_1 = \mathcal{D}\,\vartheta, && \vartheta_3 = (\mathcal{D}^3 + 4\,\kappa^2\mathcal{D} + 3\,\kappa\,\kappa_s)\,\vartheta, \\ &\vartheta_2 = (\mathcal{D}^2 + \kappa^2)\,\vartheta, && \vartheta_4 = (\mathcal{D}^4 + 10\,\kappa^2\mathcal{D}^2 + 15\,\kappa\,\kappa_s\mathcal{D} + 4\,\kappa\,\kappa_{ss} + 3\,\kappa_s^2 + 9\,\kappa^4)\,\vartheta. \end{aligned} \tag{3.31}$$

Substituting the latter formulae into (3.28) yields

$$\begin{gathered} d_{\mathcal{V}}\,\kappa = d_{\mathcal{V}}\,I_2 = (\mathcal{D}^2 + \kappa^2)\,\vartheta, \qquad d_{\mathcal{V}}\,\kappa_s = d_{\mathcal{V}}\,I_3 = (\mathcal{D}^3 + \kappa^2\mathcal{D} + 3\,\kappa\,\kappa_s)\,\vartheta, \\ d_{\mathcal{V}}\,I_4 = (\mathcal{D}^4 + 10\,\kappa^2\mathcal{D}^2 + 5\,\kappa\,\kappa_s\mathcal{D} + 4\,\kappa\,\kappa_{ss} + 3\,\kappa_s^2 + 9\,\kappa^4)\,\vartheta, \end{gathered} \tag{3.32}$$

and hence, in view of (3.27),

$$d_{\mathcal{V}}\,\kappa_{ss} = d_{\mathcal{V}}\,I_4 - 9\,\kappa^2\,d_{\mathcal{V}}\,\kappa = (\mathcal{D}^4 + \kappa^2\mathcal{D}^2 + 5\,\kappa\,\kappa_s\mathcal{D} + 4\,\kappa\,\kappa_{ss} + 3\,\kappa_s^2)\vartheta.$$

Thus, we deduce the following invariant linearization operators:

$$\begin{gathered} \mathcal{A}_\kappa = \mathcal{D}^2 + \kappa^2, \qquad \mathcal{A}_{\kappa_s} = \mathcal{D}^3 + \kappa^2\mathcal{D} + 3\,\kappa\,\kappa_s, \\ \mathcal{A}_{\kappa_{ss}} = \mathcal{D}^4 + \kappa^2\mathcal{D}^2 + 5\,\kappa\,\kappa_s\mathcal{D} + 4\,\kappa\,\kappa_{ss} + 3\,\kappa_s^2, \end{gathered} \tag{3.33}$$

etc. In fact, one can recursively construct the higher order operators starting with $\mathcal{A}_\kappa$ via

$$\mathcal{A}_{\kappa_n} = \mathcal{D}\cdot\mathcal{A}_{\kappa_{n-1}} + \kappa\,\kappa_n, \tag{3.34}$$

where $\kappa_n = \mathcal{D}^n\kappa$; this will be proved below. Finally, specializing (3.10) and using (3.24), we find

$$\begin{aligned} d\varpi = d[\iota(dx)] &= \iota(d^2x) + \nu^1 \wedge \iota[\mathbf{v}_1(dx)] + \nu^2 \wedge \iota[\mathbf{v}_2(dx)] + \nu^3 \wedge \iota[\mathbf{v}_3(dx)] \\ &= (-\,\kappa\,\varpi - \vartheta_1) \wedge (-\vartheta) = -\,\kappa\,\vartheta \wedge \varpi + \vartheta_1 \wedge \vartheta. \end{aligned}$$

The first summand in the final expression is $d_{\mathcal{V}}\,\varpi$ (the second is $d_{\mathcal{W}}\,\varpi$), and hence the invariant Hamiltonian operator is

$$\mathcal{B} = -\,\kappa. \tag{3.35}$$

Formula (3.34) is, in fact, a special case of the following result.

Lemma 3.4. *If K is any differential invariant, then*

$$\mathcal{A}_{\mathcal{D}_j K} = \mathcal{D}_j \cdot \mathcal{A}_K - \sum_{i=1}^{p} (\mathcal{D}_i K)\,\mathcal{B}^i_j. \tag{3.36}$$

Proof: First, we have

$$d_{\mathcal{H}}\,d_{\mathcal{V}}\,K = d_{\mathcal{H}}\left[\,\mathcal{A}_K(\vartheta)\,\right] = \sum_{j=1}^{p} \varpi^j \wedge \mathcal{D}_j\mathcal{A}_K(\vartheta).$$

On the other hand, according to (2.8), (3.22),

$$
\begin{aligned}
d_{\mathcal{H}}\, d_{\mathcal{V}}\, K = -\, d_{\mathcal{V}}\, d_{\mathcal{H}}\, K = -\, d_{\mathcal{V}} & \left[\, \sum_{j=1}^{p} (\mathcal{D}_j K)\, \varpi^j \,\right] \\
&= -\sum_{j=1}^{p} \big[\, d_{\mathcal{V}}(\mathcal{D}_j K) \wedge \varpi^j + (\mathcal{D}_j K)\ d_{\mathcal{V}}\, \varpi^j \,\big] \\
&= -\sum_{j=1}^{p} \left[\, \mathcal{A}_{\mathcal{D}_j K}(\vartheta) + \sum_{i=1}^{p} (\mathcal{D}_i K)\, \mathcal{B}^i_j(\vartheta) \,\right] \wedge \varpi^j .
\end{aligned}
$$

Comparison of the individual coefficients of ϖ^j completes the proof. *Q.E.D.*

Formula (3.36) is reminiscent of the recursive prolongation formula (3.2) for vector field coefficients. In the case of curves, the analogy is exact, and one can establish an explicit "prolongation formula", as in (3.3), for the invariant linearization operators associated with the higher order differential invariants.

Corollary 3.5. *For $p = 1$–dimensional submanifolds (curves), given a differential invariant K,*

$$
\mathcal{A}_{\mathcal{D}^n K} = \mathcal{D}^n \cdot \mathcal{R}_K + (\mathcal{D}^{n+1} K)\, \mathcal{D}^{-1} \mathcal{B}, \tag{3.37}
$$

for all $n \geq 0$, where

$$
\mathcal{R}_K = \mathcal{A}_K - (\mathcal{D} K)\, \mathcal{D}^{-1} \mathcal{B} \tag{3.38}
$$

will be called the characteristic operator *associated with the differential invariant K.*

Remark: The non-local terms (involving $\mathcal{D}^{-1}$) in (3.37) cancel out, and so $\mathcal{A}_{\mathcal{D}^n K}$ is an honest differential operator.

Proof: The case $n = 0$ is a tautology. We then use induction on n and (3.36) to check

$$
\begin{aligned}
\mathcal{A}_{\mathcal{D}^{n+1} K} &= \mathcal{D} \cdot \mathcal{A}_{\mathcal{D}^n K} - (\mathcal{D}^{n+1} K)\, \mathcal{B} = \mathcal{D}\big[\, \mathcal{D}^n \cdot \mathcal{R}_K + (\mathcal{D}^{n+1} K)\, \mathcal{D}^{-1} \mathcal{B} \,\big] - (\mathcal{D}^{n+1} K)\, \mathcal{B} \\
&= \mathcal{D}^{n+1} \cdot \mathcal{R}_K + (\mathcal{D}^{n+2} K)\, \mathcal{D}^{-1} \mathcal{B},
\end{aligned}
$$

establishing the induction step. *Q.E.D.*

Strikingly, the characteristic operator $\mathcal{R}_K$ will reappear in the following section as the recursion operator for certain integrable nonlinear evolution equations arising from invariant curve flows. Unfortunately, there does not appear to be any counterpart to this explicit formula for higher dimensional submanifolds, e.g., surfaces. The difficulty stems from the non-commutativity of the invariant differential operators coupled with the fact that the Hamiltonian operator complex is not, in general, a total Jacobian. On the other hand, recursion operators for higher dimensional evolution equations are also more involved, [**18**, **57**, **19**]. An intriguing question is whether the Fokas–Santini formalism can be adapted to the present framework.

4. Invariant Submanifold Flows.

In this section, we shift our attention to invariant submanifold flows. Let us single out the $m = p + q$ invariant one-forms

$$\varpi^1, \ldots, \varpi^p, \ \vartheta^1, \ldots, \vartheta^q \tag{4.1}$$

consisting of the invariant horizontal forms (2.4) and the order 0 invariant contact forms (2.6). Each is a linear combination of the coordinate one-forms $dx^1, \ldots, dx^p, du^1, \ldots, du^q$ on M, whose coefficients are certain $(n+1)^{\text{st}}$ order differential functions, where n is the order of the underlying moving frame.

Let $S \subset M$ be a p-dimensional submanifold. Evaluating the coefficients of (4.1) on the submanifold jet $(x, u^{(n+1)}) = \mathrm{j}_{n+1} S|_z$ produces a basis for the cotangent space $T^*M|_z$ of the ambient manifold, which we continue to denote by (4.1). By construction, the resulting coframe is equivariant under the action of G on $S \subset M$.

Warning: The resulting moving coframe forms are *not* obtained by simply pulling back the one-forms (4.1) to S; the latter are sections of its cotangent bundle $T^*S \to S$, whereas our construction produces a basis for the sections of the larger vector bundle $T^*M \to S$. Indeed, any pulled-back contact form automatically vanishes on S itself. As a result, the one-forms ϑ^α span the tangent annihilator bundle $(TS)^\perp \subset T^*M$ at each point of S.

Let $\mathbf{t}_1, \ldots, \mathbf{t}_p, \mathbf{n}_1, \ldots, \mathbf{n}_q$, denote the corresponding dual tangent vectors, which form a G–equivariant basis of the bundle $TM \to S$, or *frame* on S. Since the contact forms annihilate the tangent space to S, the vectors $\mathbf{t}_1, \ldots, \mathbf{t}_p$ form a basis for the tangent bundle TS, while $\mathbf{n}_1, \ldots, \mathbf{n}_q$ form a basis for the complementary G–equivariant *normal bundle* $NS \to S$ induced by the moving frame. In geometrical situations, they can be identified with the classical moving frame vectors, [**23**].

Example 4.1. Let us return to the case of planar Euclidean curves $C \subset M = \mathbb{R}^2$. According to Example 2.1, the invariant coframe (4.1) is

$$\varpi = \frac{dx + u_x\, du}{\sqrt{1+u_x^2}} = \sqrt{1+u_x^2}\; dx + \frac{u_x}{\sqrt{1+u_x^2}}\, \theta, \qquad \vartheta = \frac{du - u_x\, dx}{\sqrt{1+u_x^2}} = \frac{\theta}{\sqrt{1+u_x^2}}\,. \tag{4.2}$$

The corresponding dual frame vectors satisfy $\langle\, \mathbf{t}\,; \varpi \,\rangle = \langle\, \mathbf{n}\,; \vartheta \,\rangle = 1$, $\langle\, \mathbf{n}\,; \varpi \,\rangle = \langle\, \mathbf{t}\,; \vartheta \,\rangle = 0$, and hence

$$\mathbf{t} = \frac{1}{\sqrt{1+u_x^2}} \left(\frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} \right), \qquad \mathbf{n} = \frac{1}{\sqrt{1+u_x^2}} \left(-u_x \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right), \tag{4.3}$$

are the usual (right-handed) Euclidean frame vectors — the unit tangent and unit normal.

In general, let

$$\mathbf{V} = \mathbf{V}\,|_S = \mathbf{V}_T + \mathbf{V}_N = \sum_{j=1}^{p} I^j\, \mathbf{t}_j + \sum_{\alpha=1}^{q} J^\alpha\, \mathbf{n}_\alpha \tag{4.4}$$

be a section of the bundle $TM \to S$, where $\mathbf{V}_T, \mathbf{V}_N$ denote, respectively, its tangential and normal components, while I^j, J^α are differential functions, depending on the submanifold jets. We will, somewhat imprecisely, refer to $\mathbf{V}$ as a *vector field*, even though it is only defined on S. Any such vector field generates a submanifold flow:

$$\frac{\partial S}{\partial t} = \mathbf{V}|_{S(t)}. \tag{4.5}$$

The flow (4.5) constitutes an n^{th} order system of partial differential equations, where n refers to the maximum order among our moving frame and the coefficients I^j, J^α. Assuming local existence and uniqueness, a solution $S(t)$ to the submanifold flow equations (4.5) defines a smoothly varying family of p-dimensional submanifolds of M. On the other hand, one typically expects singularities to appear if the flow is continued for a sufficiently long time.

A submanifold flow (4.5) is called *G-invariant* if G is a symmetry group of the partial differential equation. A general characterization of invariant submanifold flows is readily established.

Lemma 4.2. *The vector field* $\mathbf{V}$ *generates an invariant submanifold flow if and only if its coefficients* $I^j = \langle\, \mathbf{V}\,; \varpi^j \,\rangle$, $J^\alpha = \langle\, \mathbf{V}\,; \vartheta^\alpha \,\rangle$, *are differential invariants.*

The tangential components $\mathbf{V}_T$ do not affect the extrinsic geometry of the submanifold, but only affect its internal parametrization. Thus, if we are only interested in the images of $S(t)$ under the flow, and not their underlying parametrizations, we can set $\mathbf{V}_T = 0$ without loss of generality. Therefore, the normal component

$$\mathbf{V}_N = \sum_{\alpha=1}^{q} J^\alpha \mathbf{n}_\alpha \tag{4.6}$$

serves to characterize the *same* invariant submanifold flow as $\mathbf{V}$, modulo reparametrization. We will say that the vector field $\mathbf{V}_N$ generates a *normal flow*, since it only moves the submanifold in its G-equivariant normal direction — as prescribed by the moving frame.

Example 4.3. The most well-studied are the Euclidean-invariant curve and surface flows. A plane curve flow is generated by a vector field of the form

$$\mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n} \qquad \text{or, equivalently,} \qquad \mathbf{V}_N = J\,\mathbf{n}, \tag{4.7}$$

if we are not concerned about the tangential component's effect on the parametrization. Here, $\mathbf{n}$ denotes (one of the two) Euclidean normals to the curve; by convention, we use the inwards normal $\mathbf{n}$ when the curve is closed. Particular cases include:

(*i*) $\mathbf{V} = \mathbf{n}$: this induces the geometric optics or grassfire flow, [**6**, **59**];

(*ii*) $\mathbf{V} = \kappa\,\mathbf{n}$: this generates the celebrated curve shortening flow, [**20**, **21**], used to great effect in image processing, [**52**, **59**];

(*iii*) $\mathbf{V} = \kappa^{1/3}\,\mathbf{n}$: the induced flow is equivalent, modulo reparametrization, to the equi-affine invariant curve shortening flow, also effective in image processing, [**3**, **52**, **59**];

(iv) $\mathbf{V} = \kappa_s\, \mathbf{n}$: this flow induces the modified Korteweg–deVries equation for the curvature evolution, and is the simplest of a large number of soliton equations arising in geometric curve flows, [**12**, **22**, **37**];

(v) $\mathbf{V} = \kappa_{ss}\, \mathbf{n}$: this flow models thermal grooving of metals, [**7**].

A second important class are the invariant curve flows that preserve arc length, [**12**, **37**]. When $p = 1$, there is only one independent invariant horizontal one-form

$$\varpi = \omega + \eta = ds + \eta, \tag{4.8}$$

whose horizontal component $\omega = ds$ can be identified with the G-invariant arc length element. Invariance requires that the Lie derivative $\mathbf{V}(\omega)$ vanishes on the submanifold, which (because Lie derivatives preserve the contact ideal) implies the following:

Lemma 4.4. *The curve flow induced by*

$$\mathbf{V} = I\, \mathbf{t} + \sum_{\alpha=1}^{q} J^\alpha\, \mathbf{n}_\alpha, \qquad \text{where} \qquad I = \langle\, \mathbf{V}\,;\varpi\,\rangle, \qquad J^\alpha = \langle\, \mathbf{V}\,;\vartheta^\alpha\,\rangle, \tag{4.9}$$

preserves arc length if and only if the Lie derivative $\mathbf{V}(\varpi)$ is a contact form.

Submanifolds of dimension $p \geq 2$ do not have distinguished parametrizations to play the role of the arc length parameter; this is because the invariant horizontal forms are almost never exact on the submanifold. On the other hand, the Lie derivative condition can be straightforwardly mimicked.

Definition 4.5. The invariant submanifold flow induced by $\mathbf{V}$ is called *intrinsic* if $\mathbf{V}(\varpi^i) \equiv 0$ for all $i = 1, \ldots, p$.

Lemma 4.6. *If the vector field $\mathbf{V}$ defines an intrinsic flow, then it commutes with the invariant differentiations: $\left[\, \mathbf{V}, \mathcal{D}_i \,\right] = 0$ for $i = 1, \ldots, p$.*

Proof: For one-forms α, β, we will write $\alpha \equiv \beta$ if $\alpha - \beta$ is a contact form. This implies that α and β assume the same value when pulled back to a submanifold S. If F is any differential function, then, by (2.5),

$$\begin{aligned}\sum_{i=1}^{p} \mathcal{D}_i(\mathbf{V}(F))\varpi^i &= d_{\mathcal{H}}\,(\mathbf{V}(F)) \equiv d(\mathbf{V}(F)) = \mathbf{V}(dF) \equiv \mathbf{V}(\,d_{\mathcal{H}}\,F) = \mathbf{V}\left(\sum_{i=1}^{p} \mathcal{D}_i(F)\varpi^i\right) \\ &= \sum_{i=1}^{p} \Big(\mathbf{V}\big[\,\mathcal{D}_i(F)\,\big]\varpi^i + \mathcal{D}_i(F)\mathbf{V}(\varpi^i)\Big) \equiv \sum_{i=1}^{p} \mathbf{V}\big[\,\mathcal{D}_i(F)\,\big]\varpi^i,\end{aligned}$$

because $\mathbf{V}$ is assumed to be intrinsic. *Q.E.D.*

Lemma 4.7. *If $\mathbf{V}$ is an intrinsic flow, and $\mathcal{A}$ is any invariant differential operator, then $\mathbf{V} \mathbin{\lrcorner} \mathcal{A}(\vartheta) = \mathcal{A}(\mathbf{V} \mathbin{\lrcorner} \vartheta)$ for any invariant contact form ϑ.*

Proof: Since $\mathbf{V}$ preserves the contact ideal, by Cartan's formula for the Lie derivative,

$$0 \equiv \mathbf{V}(\vartheta) = \mathbf{V} \,\lrcorner\, d\vartheta + d(\mathbf{V} \,\lrcorner\, \vartheta) \equiv \mathbf{V} \,\lrcorner\, \left(\sum_{i=1}^{p} \varpi^i \wedge \mathcal{D}_i(\vartheta) \right) + \sum_{i=1}^{p} \mathcal{D}_i(\mathbf{V} \,\lrcorner\, \vartheta)\, \varpi^i$$
$$\equiv \sum_{i=1}^{p} \left[-\mathbf{V} \,\lrcorner\, \mathcal{D}_i(\vartheta) + \mathcal{D}_i(\mathbf{V} \,\lrcorner\, \vartheta) \right] \varpi^i,$$

whence $\mathbf{V} \,\lrcorner\, \mathcal{D}_i(\vartheta) = \mathcal{D}_i(\mathbf{V} \,\lrcorner\, \vartheta)$ for all $i = 1, \ldots, p$. The general result follows by iteration. *Q.E.D.*

Let us establish explicit conditions for a submanifold flow to be intrinsic. We apply Cartan's formula, along with Lemma 4.6 to compute

$$\begin{aligned} \mathbf{V}(\varpi^i) &= \mathbf{V} \,\lrcorner\, d\varpi^i + d(\mathbf{V} \,\lrcorner\, \varpi^i) \\ &= \mathbf{V} \,\lrcorner\, \left(-\sum_{j<k} Y^i_{jk}\, \varpi^j \wedge \varpi^k + \sum_{j=1}^{p} \sum_{\alpha=1}^{q} \mathcal{B}^i_{j\alpha}(\vartheta^\alpha) \wedge \varpi^j + d_{\mathcal{W}}\, \varpi^i \right) + dI^i \\ &\equiv \sum_{j,k=1}^{p} Y^i_{jk}\, I^k\, \varpi^j + \sum_{j=1}^{p} \sum_{\alpha=1}^{q} \mathcal{B}^i_{j\alpha}(J^\alpha)\, \varpi^j + \sum_{j=1}^{p} \mathcal{D}_j I^i\, \varpi^j, \end{aligned}$$

where we used Lemma 4.7 on the middle summation, and the final expression omits all contact components. This implies:

Theorem 4.8. *The flow induced by the vector field* (4.4) *is intrinsic if and only if*

$$\mathcal{D}_j I^i + \sum_{k=1}^{p} Y^i_{jk} I^k + \sum_{\alpha=1}^{q} \mathcal{B}^i_{j\alpha}(J^\alpha) = 0. \tag{4.10}$$

In particular, for curve flows generated by (4.9), there are no commutator invariants, and so the condition (4.10) guaranteeing arc length preservation reduces to

$$\mathcal{D}I = -\mathcal{B}(J) = -\sum_{\alpha=1}^{q} \mathcal{B}_\alpha(J^\alpha), \tag{4.11}$$

where $\mathcal{D}$ is the arc length derivative, while $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_q)$ is the invariant *Hamiltonian operator*, defined by (3.22), which, in the case of curves, becomes

$$d_{\mathcal{V}}\, \varpi = \mathcal{B}(\vartheta) \wedge \varpi = \sum_{\alpha=1}^{q} \mathcal{B}_\alpha(\vartheta^\alpha) \wedge \varpi. \tag{4.12}$$

Example 4.9. For the Euclidean group action on plane curves, in view of (3.35), the condition that a curve flow generated by the vector field $\mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$ be intrinsic is that

$$\mathcal{D}I = \kappa\, J. \tag{4.13}$$

Most of the curve flows listed in Example 4.3 have *non-local* intrinsic counterparts owing to the non-invertibility of the arc length derivative operator on $\kappa\, J$. One exception is the modified Korteweg-deVries flow, where $J = \kappa_s$, with $I = \frac{1}{2}\kappa^2$. In general, the normal flow induced by $\mathbf{V}_N = J\,\mathbf{n}$ has a local intrinsic version if and only if $\mathcal{E}(\kappa\, J) = 0$, where $\mathcal{E}$ is the invariantized Euler–Lagrange operator, [**29**].

5. Evolution of Invariants.

A key issue appearing in many applications is to determine the time evolution of differential invariants as the submanifold $S(t)$ varies according to an invariant submanifold flow (4.5). In this section, we derive general formulas that answer this question.

Let us first look at the case when the vector field $\mathbf{V}$ generates an intrinsic flow. Let K be any differential invariant. Its time variation under the submanifold flow induced by $\mathbf{V}$ is found by computing the Lie derivative:

$$\mathbf{V}(K) = \mathbf{V} \,\lrcorner\, dK = \mathbf{V} \,\lrcorner \left(\mathcal{A}_K(\vartheta) + \sum_{i=1}^{p} \mathcal{D}_i K\, \varpi^i \right) = \mathcal{A}_K(J) + \sum_{i=1}^{p} I^i\, \mathcal{D}_i K,$$

where $J = (J^1, \ldots, J^q)^T$, and we used Lemma 4.7.

Theorem 5.1. *If the submanifold flow* (4.5) *is intrinsic, and* K *is any differential invariant, then*

$$\frac{\partial K}{\partial t} = \mathbf{V}(K) = \mathcal{A}_K(J) + \sum_{i=1}^{p} I^i\, \mathcal{D}_i K. \tag{5.1}$$

The summation on the right hand side of (5.1) is exactly the tangential evolution of K due to the reparametrization:

$$\sum_{i=1}^{p} I^i\, \mathcal{D}_i K = \sum_{i=1}^{p} (\mathcal{D}_i K)\, \mathbf{V} \,\lrcorner\, \varpi^i = \mathbf{V} \,\lrcorner\, d_{\mathcal{H}} K.$$

Thus, we immediately deduce the corresponding result for normal flows, obtained by eliminating the tangential component:

Theorem 5.2. *If the submanifold flow* (4.5) *is normal, and* K *is any differential invariant, then*

$$\frac{\partial K}{\partial t} = \mathbf{V}(K) = \mathcal{A}_K(J). \tag{5.2}$$

Example 5.3. For any of the Euclidean invariant normal plane curve flows $C_t = J\,\mathbf{n}$ listed in Example 4.3, we have, according to (3.33),

$$\begin{gathered} \frac{\partial \kappa}{\partial t} = (\mathcal{D}^2 + \kappa^2)\, J, \qquad \frac{\partial \kappa_s}{\partial t} = (\mathcal{D}^3 + \kappa^2 \mathcal{D} + 3\,\kappa\,\kappa_s)\, J, \\ \frac{\partial \kappa_{ss}}{\partial t} = (\mathcal{D}^4 + \kappa^2 \mathcal{D}^2 + 5\,\kappa\,\kappa_s \mathcal{D} + 4\,\kappa\,\kappa_{ss} + 3\,\kappa_s^2)\, J. \end{gathered} \tag{5.3}$$

For instance, for the grassfire flow $J = 1$, and so

$$\frac{\partial \kappa}{\partial t} = \kappa^2, \qquad \frac{\partial \kappa_s}{\partial t} = 3\,\kappa\,\kappa_s, \qquad \frac{\partial \kappa_{ss}}{\partial t} = 4\,\kappa\,\kappa_{ss} + 3\,\kappa_s^2. \tag{5.4}$$

The first equation immediately implies finite time blow-up at a caustic for a convex initial curve segment, where $\kappa > 0$. For the curve shortening flow, $J = \kappa$, and

$$\frac{\partial \kappa}{\partial t} = \kappa_{ss} + \kappa^3, \qquad \frac{\partial \kappa_s}{\partial t} = \kappa_{sss} + 4\,\kappa^2\kappa_s, \qquad \frac{\partial \kappa_{ss}}{\partial t} = \kappa_{ssss} + 5\,\kappa^2\,\kappa_{ss} + 8\,\kappa\,\kappa_s^2, \tag{5.5}$$

thereby recovering formulas used in Gage and Hamilton's analysis, [**20**]; see also Mikula and Ševčovič, [**39**, **40**, **41**]. Finally, for the mKdV flow, $J = \kappa_s$,

$$\begin{gathered} \frac{\partial \kappa}{\partial t} = \kappa_{sss} + \kappa^2\kappa_s, \qquad \frac{\partial \kappa_s}{\partial t} = \kappa_{ssss} + \kappa^2\kappa_{ss} + 3\,\kappa\,\kappa_s^2, \\ \frac{\partial \kappa_{ss}}{\partial t} = \kappa_{sssss} + \kappa^2\,\kappa_{sss} + 9\,\kappa\,\kappa_s\kappa_{ss} + 3\,\kappa_s^3. \end{gathered} \tag{5.6}$$

Warning: Normal flows *do not preserve arc length*, and so the arc length parameter s will vary in time. Or, to phrase it another way, time differentiation ∂_t and arc length differentiation $\mathcal{D} = D_s$ *do not commute* — as can be observed in the preceding examples. Thus, one must be very careful *not* to interpret the resulting evolutions (5.4–6) as partial differential equations in the usual sense. Rather, one should regard the differential invariants $\kappa, \kappa_s, \kappa_{ss}, \ldots$ as satisfying an infinite-dimensional dynamical system of coupled ordinary differential equations. Later we will see how the potentially infinite hierarchy of ordinary differential equations can be closed off at a finite order through the use of signatures.

Turning our attention to the intrinsic, arc length preserving curve flow, the complication alluded to in the preceding paragraph does not arise because, by Lemma 4.6, time differentiation now commutes with arc length differentiation. Substituting (4.11) in the formula (5.1):

Theorem 5.4. *Under an arc-length preserving flow,*

$$\kappa_t = \mathcal{R}_\kappa(J) \qquad \text{where} \qquad \mathcal{R}_\kappa = \mathcal{A}_\kappa - \kappa_s\,\mathcal{D}^{-1}\mathcal{B} \tag{5.7}$$

is the characteristic operator (3.38) *associated with* κ*. More generally, the time evolution of* $\kappa_n = \mathcal{D}^n\kappa$ *is given by arc length differentiation:*

$$\frac{\partial \kappa_n}{\partial t} = \mathcal{R}_{\kappa_n}(J) = \mathcal{D}^n\mathcal{R}_\kappa(J). \tag{5.8}$$

In this case arc length is preserved, and hence the arc length and time derivatives commute. Thus, unlike (5.2), the arc-length preserving flow (5.7) is of the more usual analytical form. However, there is a complication in that the term

$$\kappa_s\,\mathcal{D}^{-1}\mathcal{B}(J) = \kappa_s \int \mathcal{B}(J)\,ds \tag{5.9}$$

may very well be *nonlocal*, and so (5.7) is, in general, an integro-differential equation. Note that any integration constant appearing in (5.9) just adds in a multiple of κ_s, which represents the arc length preserving tangential flow $\kappa_t = \kappa_s$ that just serves to translate the arc length parameter: $s \mapsto s + c$ and so can be effectively ignored. Also, on a closed curve, the integral in (5.9) need not be periodic in s, and so one may not be able to continuously assign a uniquely determined evolution along the entire curve — although, by the preceding remarks, all such evolutions only differ by an overall translation of the arc length parameter by an integer multiple of the total length of the curve.

In certain situations, (5.7) turns out to be a well-known local integrable evolution equation, and the characteristic operator $\mathcal{R}$ is its recursion operator!

Example 5.5. In the case of Euclidean plane curves, the evolution of the curvature is given by

$$\kappa_t = \mathcal{R}_\kappa(J), \tag{5.10}$$

where

$$\mathcal{R}_\kappa = \mathcal{A}_\kappa - \kappa_s \mathcal{D}^{-1}\mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s \mathcal{D}^{-1} \cdot \kappa = D_s^2 + \kappa^2 + \kappa_s\, D_s^{-1} \cdot \kappa \tag{5.11}$$

is the modified Korteweg-deVries recursion operator, [**46**]. In particular, for the mKdV flow, $J = \kappa_s$, and (5.10) becomes

$$\kappa_t = \mathcal{R}_\kappa(\kappa_s) = \kappa_{sss} + \tfrac{3}{2}\kappa^2\kappa_s,$$

which is the modified Korteweg-deVries equation, and $\mathcal{R}$ is its recursion operator. On the other hand, for the grassfire flow, $J = 1$, and so

$$\kappa_t = \mathcal{R}_\kappa(1) = \kappa^2 + \kappa_s\, D_s^{-1}\kappa.$$

For the curve shortening flow, $J = \kappa$, and so

$$\kappa_t = \mathcal{R}_\kappa(\kappa) = \kappa_{ss} + \kappa^3 + \kappa_s\, D_s^{-1}\kappa^2.$$

Finally, for the thermal grooving flow, $J = \kappa_{ss}$ and so

$$\kappa_t = \mathcal{R}_\kappa(\kappa_{ss}) = \kappa_{ssss} + \kappa^2\kappa_{ss} + \kappa_s\, D_s^{-1}\kappa\,\kappa_{ss}.$$

As noted above, the induced curvature flow (5.10) is local if and only if $\mathcal{E}(\kappa\, J) = 0$, where $\mathcal{E}$ is the invariantized Euler operator or variational derivative, [**46**]. Clearly not all these local curvature flows will be integrable.

Example 5.6. Let us treat a different example. Consider the action

$$(x,u) \;\longmapsto\; (\alpha\, x + \beta\, u + a, \gamma\, x + \delta\, u + b), \qquad \alpha\,\delta - \beta\,\gamma = 1, \tag{5.12}$$

of the equi-affine group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$ on plane curves $C \subset \mathbb{R}^2$. Applications to computer vision can be found, for instance, in [**3**, **8**, **52**, **58**]. According to [**15**, **23**, **29**], the classical equi-affine moving frame arises from the choice of coordinate cross-section

$x = u = u_x = 0$, $u_{xx} = 1$, $u_{xxx} = 0$. The fundamental differential invariant is the equi-affine curvature

$$\kappa = \iota(u_{xxxx}) = \frac{u_{xx} u_{xxxx} - \frac{5}{3} u_{xxx}^2}{u_{xx}^{8/3}} \,. \tag{5.13}$$

All higher order differential invariants are obtained by invariant differentiation with respect to the invariant arc length form

$$\varpi = \iota(dx) = \omega + \eta, \qquad \text{where} \qquad \omega = ds = u_{xx}^{1/3}\, dx, \qquad \eta = \frac{u_{xxx}}{3\, u_{xx}^{5/3}}\, \theta, \tag{5.14}$$

with dual invariant differential operator $\mathcal{D} = u_{xx}^{-1/3}\, D_x$ being the equi-affine arc length derivative. Applying our computational algorithm, but suppressing the details, we obtain

$$d_{\mathcal{V}}\, \kappa = \mathcal{A}_\kappa(\vartheta), \qquad\qquad d_{\mathcal{V}}\, \varpi = \mathcal{B}(\vartheta) \wedge \varpi,$$

where

$$\mathcal{A}_\kappa = \mathcal{D}^4 + \tfrac{5}{3}\, \kappa\, \mathcal{D}^2 + \tfrac{5}{3}\, \kappa_s \mathcal{D} + \tfrac{1}{3}\, \kappa_{ss} + \tfrac{4}{9}\, \kappa^2, \qquad\qquad \mathcal{B} = \tfrac{1}{3}\, \mathcal{D}^2 - \tfrac{2}{9}\, \kappa.$$

The characteristic operator is

$$\mathcal{R}_\kappa = \mathcal{A}_\kappa - \kappa_s \mathcal{D}^{-1} \mathcal{B} = \mathcal{D}^4 + \tfrac{5}{3}\, \kappa\, \mathcal{D}^2 + \tfrac{4}{3}\, \kappa_s \mathcal{D} + \tfrac{1}{3}\, \kappa_{ss} + \tfrac{4}{9}\, \kappa^2 + \tfrac{2}{9}\, \kappa_s\, D_s^{-1} \cdot \kappa. \tag{5.15}$$

A general equi-affine invariant curve flow takes the form

$$C_t = I\, \mathbf{t} + J\, \mathbf{n}, \tag{5.16}$$

where $\mathbf{t}, \mathbf{n}$ are, respectively, the equi-affine tangent and normal directions, [**23**]. The equi-affine curve shortening flow, [**3, 59**], is the normal flow with $I = 0, J = 1$. Under this flow, the equi-affine curvature and its derivative evolves according to

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(1) = \tfrac{1}{3}\, \kappa_{ss} + \tfrac{4}{9}\, \kappa^2, \qquad \frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(1) = \mathcal{D}\, \mathcal{A}_\kappa(1) - \kappa_s \mathcal{B}(1) = \tfrac{1}{3}\, \kappa_{sss} + \tfrac{10}{9}\, \kappa\, \kappa_s. \tag{5.17}$$

A second example is the intrinsic (arc-length preserving) flow with $J = \kappa_s$. In this case, the curvature evolution arises from the characteristic operator:

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{5s} + \tfrac{5}{3}\, \kappa\, \kappa_{sss} + \tfrac{5}{3}\, \kappa_s \kappa_{ss} + \tfrac{5}{9}\, \kappa^2 \kappa_s,$$

which is the integrable Sawada–Kotera equation, [**61**]. In this case, the characteristic operator $\mathcal{R}$ is closely related to, but *not* the same as the Sawada–Kotera recursion operator, which is given by the following formula, [**10**]:

$$\widehat{\mathcal{R}} = \mathcal{R} \cdot (\mathcal{D}^2 + \tfrac{1}{3}\, \kappa + \tfrac{1}{3}\, \kappa_s \mathcal{D}^{-1}). \tag{5.18}$$

Example 5.7. In the case of space curves $C \subset \mathbb{R}^3$, under the Euclidean group $G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$, there are two generating differential invariants, the curvature κ and torsion τ. According to [**29**], the relevant moving frame formulae are

$$d_{\mathcal{V}}\, \kappa = \mathcal{A}_\kappa(\vartheta), \qquad\qquad d_{\mathcal{V}}\, \tau = \mathcal{A}_\tau(\vartheta), \qquad\qquad d_{\mathcal{V}}\, \varpi = \mathcal{B}(\vartheta) \wedge \varpi,$$

where $\vartheta = (\vartheta_1, \vartheta_2)^T$ is the column vector containing the order 0 invariant contact forms, while the invariant linearization and Hamiltonian operators are:

$$\mathcal{A}_\kappa = \big(D_s^2 + (\kappa^2 - \tau^2), \; -2\tau D_s - \tau_s \big),$$

$$\mathcal{A}_\tau = \left(\frac{2\tau}{\kappa} D_s^2 + \frac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2} D_s + \frac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2}, \right. \qquad \mathcal{B} = \big(-\kappa, \; 0 \big).$$

$$\left. \frac{1}{\kappa} D_s^3 - \frac{\kappa_s}{\kappa^2} D_s^2 + \frac{\kappa^2 - \tau^2}{\kappa} D_s + \frac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \right),$$

Thus, under an intrinsic flow with normal component $\mathbf{V}_N = J\,\mathbf{n}_1 + K\,\mathbf{n}_2$, the curvature and torsion evolve via

$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} J \\ K \end{pmatrix}, \qquad \text{where} \qquad \mathcal{R} = \begin{pmatrix} \mathcal{R}_\kappa \\ \mathcal{R}_\tau \end{pmatrix} = \begin{pmatrix} \mathcal{A}_\kappa \\ \mathcal{A}_\tau \end{pmatrix} - \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix} \mathcal{D}^{-1} \mathcal{B}$$

is the complete characteristic operator. In particular, the flow with $J = 0$, $K = \kappa$ induces the vortex filament flow

$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} 0 \\ \kappa \end{pmatrix}$$

which is integrable and can be mapped to the nonlinear Schrödinger equation via the Hasimoto transformation, [**25**, **32**, **37**]. Similarly, the flow with $J = \kappa_s$, $K = \kappa\,\tau$, maps to the integrable complex modified Korteweg-deVries equation in the nonlinear Schrödinger hierarchy.

6. Signature Evolution.

In the preceding section, we showed how to directly determine the time evolution of differential invariants under an invariant submanifold flow. With this in hand, we are able to find differential equations governing the evolution of their differential invariant signatures, [**8**, **49**]. A particularly important case is the behavior of the signature under invariant smoothing, e.g., the Euclidean-invariant curve shortening flow. The methods are completely general, but, for brevity, we will only discuss the case of signatures of plane curves in this section.

For a plane curve $C \subset \mathbb{R}^2$, the differential invariant signature is the curve $\Sigma \subset \mathbb{R}^2$ parametrized by the first two differential invariants, i.e., (κ, κ_s). The signature curve uniquely prescribes the original curve up to a group transformation. Thus, it provides an effective means of object recognition and symmetry detection, [**8**].

Suppose that, locally, the signature is given as the graph of a function

$$\kappa_s = \Phi(\kappa). \tag{6.1}$$

For the moment we ignore singularities. Also, the curve is assumed to have at most discrete symmetries, and so the signature does not degenerate to a point. Once we know the functional dependence (6.1) between κ and κ_s, the relations for the higher order derivatives follow. For instance,

$$\begin{aligned}\kappa_{ss} &= \Phi_\kappa(\kappa)\,\kappa_s = \Phi(\kappa)\,\Phi_\kappa(\kappa),\\ \kappa_{sss} &= \Phi(\kappa)^2\,\Phi_{\kappa\kappa}(\kappa) + \Phi(\kappa)\,\Phi_\kappa(\kappa)^2,\\ \kappa_{ssss} &= \Phi(\kappa)^3\,\Phi_{\kappa\kappa\kappa}(\kappa) + 4\,\Phi(\kappa)^2\,\Phi_\kappa(\kappa)\,\Phi_{\kappa\kappa}(\kappa) + \Phi(\kappa)\,\Phi_\kappa(\kappa)^3\end{aligned} \tag{6.2}$$

and so on.

Now suppose we have a parametrized family of curves $C(t)$ evolving according to an invariant curve flow, which will be taken in normal form (4.6). (The signature is independent of reparametrization, and this avoids the nonlocalities introduced in the intrinsic form.) Our goal is to determined the induced signature curve flow, $\Sigma(t)$. We assume that the family of signatures is, locally, given by

$$\kappa_s = \Phi(t,\kappa). \tag{6.3}$$

Applying the chain rule and (3.36), we find

$$\frac{\partial\Phi}{\partial t} = \frac{\partial\kappa_s}{\partial t} - \frac{\partial\Phi}{\partial\kappa}\,\frac{\partial\kappa}{\partial t} = \mathcal{A}_{\kappa_s}(J) - \Phi_\kappa\,\mathcal{A}_\kappa(J) = \big(\mathcal{D} - \Phi_\kappa\big)\mathcal{A}_\kappa(J) - \kappa_s\,\mathcal{B}(J). \tag{6.4}$$

Thus, to specify the time evolution of the signature function Φ, we replace the derivatives of κ appearing in (6.4) by their expressions (6.1–2).

Example 6.1. Consider the Euclidean signature curve, parameterized by the curvature and its derivative with respect to arc length. First, let's look at the grassfire flow. Substituting (5.4) into the signature flow equation (6.4), we find

$$\frac{\partial\Phi}{\partial t} = 3\,\kappa\,\kappa_s - \kappa^2\,\frac{\partial\Phi}{\partial\kappa} = 3\,\kappa\,\Phi - \kappa^2\,\frac{\partial\Phi}{\partial\kappa}\ , \tag{6.5}$$

which is a first order linear transport equation, and hence easily solved by the method of characteristics. For the curve shortening flow, we substitute (5.5) into (6.4) and then use (6.2), whence

$$\begin{aligned}\frac{\partial\Phi}{\partial t} &= \kappa_{sss} + 4\,\kappa^2\kappa_s - (\kappa_{ss} + \kappa^3)\,\frac{\partial\Phi}{\partial\kappa} = \Phi^2\,\Phi_{\kappa\kappa} + \Phi\,\Phi_\kappa^2 + 4\,\kappa^2\Phi - (\Phi\,\Phi_\kappa + \kappa^3)\,\Phi_\kappa\\ &= \Phi^2\,\Phi_{\kappa\kappa} - \kappa^3\Phi_\kappa + 4\,\kappa^2\Phi,\end{aligned} \tag{6.6}$$

leading to an nonlinear parabolic equation for $\Phi(t,\kappa)$ that has the flavor of a one-dimensional porous medium equation, [**64**]. Finally, for the modified Korteweg-deVries flow with (5.6),

$$\begin{aligned}\frac{\partial\Phi}{\partial t} &= \kappa_{ssss} + \kappa^2\kappa_{ss} + 3\,\kappa\,\kappa_s^2 - (\kappa_{sss} + \kappa^2\kappa_s)\Phi_\kappa\\ &= \Phi^3\,\Phi_{\kappa\kappa\kappa} + 4\,\Phi^2\,\Phi_\kappa\,\Phi_{\kappa\kappa} + \Phi\,\Phi_\kappa^3 + \kappa^2\Phi\,\Phi_\kappa + 3\,\kappa\,\Phi^2 - (\Phi^2\,\Phi_{\kappa\kappa} + \Phi\,\Phi_\kappa^2 + \kappa^2\Phi)\Phi_\kappa\\ &= \Phi^3\,\Phi_{\kappa\kappa\kappa} + 3\,\Phi^2\,\Phi_\kappa\,\Phi_{\kappa\kappa} + 3\,\kappa\,\Phi^2.\end{aligned} \tag{6.7}$$

Example 6.2. The equi-affine signature curve is parametrized by κ, κ_s, where κ denotes the equi-affine curvature (5.13) and s the equi-affine arc length (5.14). According to Example 5.6, under the *equi-affine curve shortening flow* $C_t = \mathbf{n}$, the fundamental equi-affine differential invariants evolve according to (5.17). Therefore, applying the preceding algorithm, we conclude that the equi-affine signature $\kappa_s = \Phi(\kappa)$ evolves according to

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= \mathcal{A}_{\kappa_s}(1) - \Phi_\kappa \, \mathcal{A}_\kappa(1) = \tfrac{1}{3}\, \kappa_{sss} + \tfrac{10}{9}\, \kappa\, \kappa_s - \Phi_\kappa \big(\tfrac{1}{3}\, \kappa_{ss} + \tfrac{4}{9}\, \kappa^2 \big) \\ &= \tfrac{1}{3} \left(\Phi^2\, \Phi_{\kappa\kappa} + \Phi\, \Phi_\kappa^2 \right) + \tfrac{10}{9}\, \kappa\, \Phi - \Phi_\kappa \big(\tfrac{1}{3}\, \Phi\, \Phi_\kappa + \tfrac{4}{9}\, \kappa^2 \big) \\ &= \tfrac{1}{3}\, \Phi^2\, \Phi_{\kappa\kappa} + \tfrac{10}{9}\, \kappa\, \Phi - \tfrac{4}{9}\, \kappa^2 \Phi_\kappa, \end{aligned} \tag{6.8}$$

again of porous medium type.

Further analysis of signature flows, including space curves and surfaces, and applications to image processing, tracking and control, will be discussed elsewhere.

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