Invariant Variational Problems and Invariant Flows via Moving Frames

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Abstract

This paper reviews the moving frame approach to the construction of the invariant variational bicomplex. Applications include explicit formulae for the Euler-Lagrange equations of an invariant variational problem, and for the equations governing the evolution of differential invariants under invariant submanifold flows.

1 Introduction.

This survey paper describes some aspects of the author's recent research, done partly in collaboration with Irina Kogan, [32, 54], into moving frames, the invariant variational bicomplex, and invariant submanifold flows. These results are based on combining two powerful ideas in the modern, geometric approach to differential equations and the variational calculus. The first is the *variational bicomplex*, which is of fundamental importance in the study of the geometry of jet bundles, differential equations and the calculus of variations. Its origins can be found in the work of Dedecker, [15], then developed in full detail by Tulczyjew, [68], and Vinogradov, [69, 70]. Later contributions of Tsujishita, [67], Anderson, [2, 3], and Krupka and Janyška, [33, 34], have amply demonstrated the power of the bicomplex formalism for both local and global problems in the geometric theory of differential equations and the calculus of variations.

The second ingredient is a reformulation of Cartan's method of moving frames, [17, 52]. For a general finite-dimensional transformation group G, a *moving frame* is defined as an equivariant map from an open subset of jet space to the Lie group G. Moving frames are constructed by the process of normalization based on the choice of cross-section to the group orbits. The moving frame then provides a canonical mechanism, called *invariantization*, that allows us to systematically construct the invariant counterparts of all objects of interest in the usual variational bicomplex, including differential invariants, invariant differential forms, invariant differential operators, etc. The key recurrence formulae relate the differentials of ordinary functions and forms to the invariant differentials of invariant functions and forms, and thereby lead to the complete structure of the algebra[1] of differential invariants, including the syzygies and commutation formulae, [27, 26, 28, 53]. The equivariant moving frame method has impacted a remarkable range of subjects, including symmetry methods for partial differential equations, the calculus of variations, classical invariant theory, computer vision, numerical analysis, Hamiltonian systems, integrable soliton equations, materials and micromagnetics, joint invariants, relativity, quantum mechanics, invariants of Lie algebras, Lie pseudo-groups, symbolic methods, and (non-commutative) differential algebra; see [51, 52] for recent surveys of developments in the field.

A key application of the invariant variational bicomplex is the general solution to an outstanding problem in the calculus of variations. Every group-invariant variational problem can be written in terms of the differential invariants. The associated Euler-Lagrange equations inherit the symmetry group, and so can also be written in terms of the differential invariants. The problem is to directly construct the invariant form of the Euler–Lagrange equations from the invariant form of the variational problem. Before the general solution to this problem appeared in [32], only a few specific examples were known, [3, 22]. A striking recent application of these techniques is the work of Starostin and van der Heijden, [66], on equilibrium configurations of flexible Möbius bands.

A second application is to the evolution of differential invariants under invariant submanifold flows. Invariant curve flows and surface flows arise in an impressive range of applications, including geometric optics, [7], elastodynamics, [38], computer vision, [56, 57, 61, 63, 65], visual tracking and control, [46], vortex dynamics, [25, 37], interface motions, [65], thermal grooving, [9], and elsewhere. A celebrated example is the Euclidean invariant curve shortening flow, [18, 20], in which a plane curve moves in its normal direction in proportion to its curvature. In computer vision, Euclidean curve shortening and its equi-affine counterpart have been successfully applied to image denoising and segmentation, [56, 62, 63]. In three dimensional space, Euclidean-invariant curve flows include the integrable vortex filament flow, [25, 37], while mean curvature and Willmore flows of surfaces have been the subject of extensive analysis and applications, [6, 14].

Given an invariant submanifold flow, a key issue is to track the induced evolution of its basic geometric invariants — curvature, torsion and the like. While a number of particular examples have been worked out by direct computation, e.g., in [18, 44], many cases of interest have yet to appear in the literature, owing to their computational complexity. Therefore, it is worth developing general, practical tools to ameliorate this often tedious task. Mansfield and van der Kamp, [40], have developed a method based on the differential invariant syzygies. Here we present a direct approach, applying the invariant variational bicomplex calculus discussed above. As we will see, the same basic invariant differential operators appearing in the construction of invariant Euler–Lagrange equations also play a key role in this context.

2 The Invariant Variational Bicomplex.

In this section, we review the basics of prolonged group actions on submanifold jets, moving frames, and the induced invariant variational bicomplex. Basic references include [49, 50] for jets, contact forms, and prolonged Lie group actions, [3, 67] for the variational bicomplex, [17, 52, 53] for the equivariant approach to moving frames, and [32] for the moving

[1] Technically, because differential invariants may only be locally defined, we should speak of the "sheaf of differential invariants". However, as we work locally on suitable open subsets, this extra level of abstraction is not required; moreover, experts can readily translate our constructions into sheaf-theoretic language, [71].

frame construction of the invariant variational bicomplex. For simplicity, we will only deal with finite-dimensional Lie group actions in this paper, although the general ideas can be straightforwardly adapted to infinite-dimensional pseudo-group actions using more recent extensions of the moving frame technology, [55].

Let G be an r-dimensional Lie group, acting smoothly on a m-dimensional manifold M. We will study the induced action on p-dimensional submanifolds $S \subset M$. For $0 \le n \le \infty$, let $\mathrm{J}^n = \mathrm{J}^n(M,p)$ denote the n-th order (extended) jet bundle for such submanifolds, [50]. The action of G on M naturally prolongs to an action on J^n. Since the prolonged group actions are all mutually compatible under projection $\mathrm{J}^n \to \mathrm{J}^k$, we will avoid explicit reference to the order of prolongation, and just use $g \cdot z^{(n)}$ for the action of $g \in G$ on the jet $z^{(n)} \in \mathrm{J}^n$, rather than the more traditional notation $g^{(n)} \cdot z^{(n)}$.

By definition, a *moving frame* is right-equivariant[2] map[3] $\rho : \mathrm{J}^n \to G$, meaning that $\rho(g \cdot z^{(n)}) = \rho(z^{(n)}) \cdot g^{-1}$ for all $g \in G$ and all $z^{(n)} \in \mathrm{J}^n$ where defined. The existence of a moving frame requires that the prolonged group action be free, meaning the isotropy subgroups of each individual jet are trivial, and regular, meaning the prolonged group orbits form a regular foliation, on an open subset $\mathcal{V} \subset \mathrm{J}^n$. Under these conditions, a moving frame can be algorithmically constructed by a *normalization* process based on the choice of a compatible cross-section $K^n \subset \mathrm{J}^n$ to the group orbits. Specifically, given $z^{(n)} \in \mathrm{J}^n$, we set $g = \rho(z^{(n)})$ to be the unique group element such that $g \cdot z^{(n)} \in K^n$, when defined. Compatibility of moving frames under the jet space projections allows us to also suppress the order in the notation of ρ. We use ι to denote the *invariantization* process induced by the moving frame. The invariantization of a differential form Ω is the unique invariant differential form $\iota(\Omega)$ that agrees with Ω when restricted to the cross-section. In particular, if Ω is an invariant differential form or function, then $\iota(\Omega) = \Omega$. Invariantization defines an (exterior) algebra morphism that projects differential functions and forms on J^n to invariant differential functions and forms.

Let $(x,u) = (x^1,\ldots,x^p,u^1,\ldots,u^q)$ be local coordinates on M. Viewing the x's as independent variables and the u's as dependent variables, we let $u^\alpha_J = \partial^{\#J} u / \partial x^J$ be the usual induced local coordinates on J^n. Separating the local coordinates (x,u) on M into independent and dependent variables naturally splits the differential one-forms on J^∞ into *horizontal forms*, spanned by $dx^1,\ldots,dx^p$, and *vertical forms*, spanned by the *basic contact one-forms*

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i}\, dx^i, \qquad \alpha = 1,\ldots,q, \qquad \#J \ge 0. \tag{1}$$

Let π_H and π_V denote the projections mapping one-forms on J^∞ to their horizontal and vertical (contact) components, respectively. The induced splitting $d = d_H + d_V$ of the differential into horizontal and vertical components results in the *variational bicomplex*[4]. In particular, if $F(x,u^{(n)})$ is any differential function, its horizontal and vertical differentials

[2]All classical moving frames, [23], are left-equivariant, and can be obtained by composing ρ with the group inversion $g \mapsto g^{-1}$. We choose to concentrate on the right-equivariant version to (slightly) simplify some of the calculations.

[3]All maps, differential forms, differential functions, etc., need only be locally defined; thus, the domain of ρ is typically a suitable open subset of J^n.

[4]Since the splitting depends on a choice of independent variables on M, the variational bicomplex is not intrinsic. A more refined version of this construction, known as the $\mathcal{C}$ *spectral sequence*, [69, 70], relies on the contact filtration of the algebra of differential forms. However, since all our calculations take place in local coordinates, we will avoid all the extra complications inherent in this more sophisticated machinery. Experts will be able to readily translate our results as desired.

are

$$d_H F = \sum_{i=1}^{p} (D_i F)\, dx^i, \qquad d_V F = \mathbf{D}_F(\theta) = \sum_{\alpha,J} \frac{\partial F}{\partial u_J^\alpha}\, D_J \theta^\alpha = \sum_{\alpha,J} \frac{\partial F}{\partial u_J^\alpha}\, \theta_J^\alpha, \tag{2}$$

in which $D_i = D_{x^i}$ denote the total derivative operators with respect to the independent variables, $D_J = D_{j_1} \cdots D_{j_k}$ are the higher order total derivatives, $\theta = (\theta^1, \ldots, \theta^q)^T$ is the column vector containing the order zero contact forms, while $\mathbf{D}_F = (\mathbf{D}_{F,1}, \ldots, \mathbf{D}_{F,q})$ is the *Fréchet derivative* or *formal linearization* of the differential function F.

We will employ our moving frame to invariantize the variational bicomplex as follows. First, invariantization of the jet coordinate functions produces the *fundamental differential invariants*:

$$H^i = \iota(x^i), \qquad I_J^\alpha = \iota(u_J^\alpha), \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0. \tag{3}$$

These naturally split into two classes: The $r = \dim G$ combinations defining the cross-section equations will be constant, and are known as the *phantom differential invariants*. The remainder, called the *basic differential invariants*, form a complete system of functionally independent differential invariants. Next, let

$$\varpi^i = \omega^i + \eta^i = \iota(dx^i), \qquad \text{where} \qquad \omega^i = \pi_H(\varpi^i), \quad \eta^i = \pi_V(\varpi^i), \tag{4}$$

denote the *invariantized horizontal one-forms*. Their horizontal components $\omega^1, \ldots, \omega^p$ form, in the language of [50], a contact-invariant coframe for the prolonged group action, while $\eta^1, \ldots, \eta^p$ supply "contact corrections" that make the one-forms $\varpi^1, \ldots, \varpi^p$ fully G-invariant. The corresponding dual invariant total differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are defined so that

$$d_{\mathcal{H}} F = \sum_{i=1}^{p} (\mathcal{D}_i F)\, \varpi^i, \qquad d_{\mathcal{H}}\, \Omega = \sum_{i=1}^{p} \varpi^i \wedge \mathcal{D}_i \Omega, \tag{5}$$

for any differential function F and, more generally, differential form Ω, on which the $\mathcal{D}_i$ act via Lie differentiation. Finally, let

$$\vartheta_J^\alpha = \iota(\theta_J^\alpha), \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0. \tag{6}$$

be the *invariantized basis contact forms*.

As in the usual, non-invariant bicomplex construction, the decomposition of invariant one-forms on J^∞ into invariant horizontal and invariant contact components induces a decomposition of the differential. However, now $d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$ splits into three constituents, where $d_{\mathcal{H}}$ adds an invariant horizontal form, $d_{\mathcal{V}}$ adds a invariant contact form, while $d_{\mathcal{W}}$ replaces an invariant horizontal one-form with a combination of wedge products of two invariant contact forms. In other words, if we let $\widetilde{\mathbf{\Omega}}^{r,s}$ denote the space of differential forms of degree $r+s$ spanned by wedge products of r invariant horizontal one-forms (4) and s invariant contact one-forms (6), then

$$d_{\mathcal{H}} : \widetilde{\mathbf{\Omega}}^{r,s} \longrightarrow \widetilde{\mathbf{\Omega}}^{r+1,s}, \qquad d_{\mathcal{V}} : \widetilde{\mathbf{\Omega}}^{r,s} \longrightarrow \widetilde{\mathbf{\Omega}}^{r,s+1}, \qquad d_{\mathcal{W}} : \widetilde{\mathbf{\Omega}}^{r,s} \longrightarrow \widetilde{\mathbf{\Omega}}^{r-1,s+2}. \tag{7}$$

The resulting *invariant variational quasi-tricomplex* is characterized by the formulae

$$\begin{aligned} d_{\mathcal{H}}^2 &= 0, & d_{\mathcal{H}}\, d_{\mathcal{V}} + d_{\mathcal{V}}\, d_{\mathcal{H}} &= 0, \\ d_{\mathcal{W}}^2 &= 0, & d_{\mathcal{V}}\, d_{\mathcal{W}} + d_{\mathcal{W}}\, d_{\mathcal{V}} &= 0, \end{aligned} \qquad d_{\mathcal{V}}^2 + d_{\mathcal{H}}\, d_{\mathcal{W}} + d_{\mathcal{W}}\, d_{\mathcal{H}} = 0. \tag{8}$$

Fortunately, the third, anomalous component $d_{\mathcal{W}}$ plays no role in the applications; in particular, $d_{\mathcal{W}} F = 0$ for any differential function F.

The most important fact underlying the moving frame construction is that the invariantization map ι does *not* respect the exterior derivative operator. Thus, in general, $d\,\iota(\Omega) \neq \iota(d\Omega)$. The *recurrence formulae*, [17, 32], which we now review, provide the missing "correction terms" $d\iota(\Omega) - \iota(d\Omega)$. Remarkably, these formulas can be explicitly and algorithmically constructed using only linear differential algebra — without knowing the explicit formulas for either the differential invariants or invariant differential forms, the invariant differential operators, or even the moving frame! The only required ingredients are the cross-section equations and the formulae for the prolonged infinitesimal generators of the group action.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be a basis for the infinitesimal generators of our transformation group. We prolong each infinitesimal generator to J^n. For conciseness, we will retain the same notation $\mathbf{v}_\kappa$ for the prolonged vector fields on any J^n which, in local coordinates, take the form

$$\mathbf{v}_\kappa = \sum_{i=1}^{p} \xi^i_\kappa(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{j=\#J=0}^{n} \varphi^\alpha_{J,\kappa}(x,u^{(j)})\frac{\partial}{\partial u^\alpha_J}, \qquad \kappa = 1,\ldots,r. \tag{9}$$

The coefficients $\varphi^\alpha_{J,\kappa} = \mathbf{v}_\kappa(u^\alpha_J)$ can be successively constructed by Lie's recursive prolongation formula, [49, 50]:

$$\varphi^\alpha_{Ji,\kappa} = D_i \varphi^\alpha_{J,\kappa} - \sum_{j=1}^{p} u^\alpha_{Jj}\, D_i \xi^j_\kappa. \tag{10}$$

A straightforward induction establishes the explicit prolongation formula, first written down by the author in [48]:

$$\varphi^\alpha_{J,\kappa} = D_J Q^\alpha_\kappa + \sum_{i=1}^{p} \xi^i_\kappa\, u^\alpha_{J,i}, \qquad \text{where} \qquad Q^\alpha_\kappa = \varphi^\alpha_\kappa - \sum_{i=1}^{p} \xi^i_\kappa\, u^\alpha_i \tag{11}$$

are the components of the *characteristic* of $\mathbf{v}_\kappa$.

Strikingly, all the recurrence relations are consequences of a single *universal recurrence formula* that prescribes the differential of an invariantized differential function or form.

Theorem 1 *If Ω is any differential form on J^∞, then*

$$d\,\iota(\Omega) = \iota(d\,\Omega) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota\,[\mathbf{v}_\kappa(\Omega)], \tag{12}$$

where $\nu^1, \ldots, \nu^r$ are the invariantized Maurer–Cartan forms *dual to the infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_r$, while $\mathbf{v}_\kappa(\Omega)$ denotes the Lie derivative of Ω with respect to the prolonged infinitesimal generator $\mathbf{v}_\kappa$.*

The invariantized Maurer–Cartan forms $\nu^1, \ldots, \nu^r$ are obtained by pulling back the usual dual Maurer–Cartan forms $\mu^1, \ldots, \mu^r$ on G by the moving frame map: $\nu^\kappa = \rho^* \mu^\kappa$. Details would take us too far afield, [32], but, fortunately, are not required thanks to the following marvelous result that allows us to compute them directly without reference to their underlying definition:

Lemma 2 *Let $I_1 = \iota(z_1), \ldots, I_r = \iota(z_r)$ be the phantom differential invariants stemming from our cross-section. Then the corresponding* phantom recurrence formulae

$$0 = dI_\varsigma = d\,\iota(z_\varsigma) = \iota(dz_\varsigma) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota\,[\mathbf{v}_\kappa(z_\varsigma)], \qquad \varsigma = 1, \ldots, r, \tag{13}$$

can be uniquely solved for the invariantized Maurer–Cartan forms $\nu^1, \ldots, \nu^r$.

Having solved the linear system (13) for $\nu^1, \ldots, \nu^r$, we then decompose the resulting invariantized Maurer–Cartan forms into their invariant horizontal and contact components:

$$\nu^\kappa = \gamma^\kappa + \varepsilon^\kappa, \qquad \text{where} \qquad \gamma^\kappa = \sum_{i=1}^{p} R_i^\kappa \, \varpi^i, \qquad \varepsilon^\kappa = \sum_{\alpha,J} S_\alpha^{\kappa,J} \, \vartheta_J^\alpha, \tag{14}$$

where $R_i^\kappa, S_\alpha^{\kappa,J}$ are certain differential invariants. The R_i^κ will be called the *Maurer–Cartan invariants*, [27, 28, 53]. In the case of curves, the R_i^κ appear as the entries of the Frenet–Serret matrix $\mathcal{D}\rho(x, u^{(n)}) \cdot \rho(x, u^{(n)})^{-1}$, in the case $G \subset \mathrm{GL}(N)$ is a matrix Lie group, [23]. Substituting (14) back into the universal formula (12) produces a complete system of explicit recurrence relations for all the differentiated invariants and invariant differential forms.

In particular, taking Ω to be any one of the individual jet coordinate functions x^i, u_J^α, results in the recurrence formulae for the fundamental differential invariants (3):

$$\begin{aligned} dH^i &= \iota(dx^i) + \sum_{\kappa=1}^{r} \nu^\kappa \, \iota\,[\mathbf{v}_\kappa(x^i)] = \varpi^i + \sum_{\kappa=1}^{r} \iota(\xi_\kappa^i) \, \nu^\kappa, \\ dI_J^\alpha &= \iota(du_J^\alpha) + \sum_{\kappa=1}^{r} \nu^\kappa \, \iota\,[\mathbf{v}_\kappa(u_J^\alpha)] = \iota \left(\sum_{i=1}^{p} u_{Ji}^\alpha \, dx^i + \theta_J^\alpha \right) + \sum_{\kappa=1}^{r} \iota(\varphi_{J,\kappa}^\alpha) \, \nu^\kappa \\ &= \sum_{i=1}^{p} I_{Ji}^\alpha \, \varpi^i + \vartheta_J^\alpha + \sum_{\kappa=1}^{r} \iota(\varphi_{J,\kappa}^\alpha) \, \nu^\kappa. \end{aligned} \tag{15}$$

In view of (14), the coefficient of ϖ^i in (15) yields the recurrence relations

$$\mathcal{D}_i H^j = \delta_i^j + \sum_{\kappa=1}^{r} R_i^\kappa \, \iota(\xi_\kappa^i), \qquad \mathcal{D}_i I_J^\alpha = I_{Ji}^\alpha + \sum_{\kappa=1}^{r} R_i^\kappa \, \iota(\varphi_{J,\kappa}^\alpha), \tag{16}$$

where δ_i^j is the usual Kronecker delta. Owing to the functional independence of the basic (non-phantom) differential invariants, these formulae, in fact, serve to completely characterize the structure of the non-commutative differential invariant algebra, [17, 26, 53]. Similarly, the contact components in (15) yield the vertical recurrence formulae

$$d_{\mathcal{V}}\, H^i = \sum_{\kappa=1}^{r} \iota(\xi_\kappa^i) \, \varepsilon^\kappa, \qquad d_{\mathcal{V}}\, I_J^\alpha = \vartheta_J^\alpha + \sum_{\kappa=1}^{r} \iota(\varphi_{K\kappa}^\alpha) \, \varepsilon^\kappa, \tag{17}$$

while, as noted earlier, $d_{\mathcal{W}}\, H^i = d_{\mathcal{W}}\, I_J^\alpha = 0$.

The recurrence formulae (12) for the derivatives of the invariant horizontal forms are

$$\begin{aligned} d\varpi^i = d[\iota(dx^i)] &= \iota(d^2x^i) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota\,[\mathbf{v}_\kappa(dx^i)] \\ &= \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota \left(\sum_{k=1}^{p} D_k \xi_\kappa^i \, dx^k + \sum_{\alpha=1}^{q} \frac{\partial \xi_\kappa^i}{\partial u^\alpha} \theta^\alpha \right) \\ &= \sum_{\kappa=1}^{r} \sum_{k=1}^{p} \iota\left(D_k \xi_\kappa^i\right) \nu^\kappa \wedge \varpi^k + \sum_{\kappa=1}^{r} \sum_{\alpha=1}^{q} \iota \left(\frac{\partial \xi_\kappa^i}{\partial u^\alpha} \right) \nu^\kappa \wedge \vartheta^\alpha. \end{aligned} \tag{18}$$

The resulting two-form can be decomposed into three basic constituents, belonging, respectively, to the invariant summands $\widetilde{\mathbf{\Omega}}^{2,0} \oplus \widetilde{\mathbf{\Omega}}^{1,1} \oplus \widetilde{\mathbf{\Omega}}^{0,2}$. In view of (14), the terms in (18) involving wedge products of two horizontal forms, i.e., in $\widetilde{\mathbf{\Omega}}^{2,0}$, yield

$$d_{\mathcal{H}}\, \varpi^i = - \sum_{j<k} Y_{jk}^i \, \varpi^j \wedge \varpi^k,$$

where

$$Y_{jk}^i = \sum_{\kappa=1}^{r} \sum_{j=1}^{p} R_k^\kappa \, \iota(D_j \xi_\kappa^i) - R_j^\kappa \, \iota(D_k \xi_\kappa^i) \tag{19}$$

are called the *commutator invariants*, since combining (19) with (5) produces the commutation formulae for the invariant differential operators:

$$[\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y_{jk}^i \, \mathcal{D}_i = - \sum_{i=1}^{p} Y_{kj}^i \, \mathcal{D}_i. \tag{20}$$

Next, the terms in (18) involving wedge products of a horizontal and a contact form yield

$$d_{\mathcal{V}}\, \varpi^i = \sum_{\kappa=1}^{r} \left[\sum_{\alpha=1}^{q} \iota \left(\frac{\partial \xi_\kappa^i}{\partial u^\alpha} \right) \gamma^\kappa \wedge \vartheta^\alpha + \sum_{k=1}^{p} \iota(D_k \xi_\kappa^i) \, \varepsilon^\kappa \wedge \varpi^k \right]. \tag{21}$$

Finally, the remaining terms, involving wedge products of two contact forms, provide the formulas for the anomalous third component of the differential:

$$d_{\mathcal{W}}\, \varpi^i = \sum_{\kappa=1}^{r} \sum_{\alpha=1}^{q} \iota \left(\frac{\partial \xi_\kappa^i}{\partial u^\alpha} \right) \varepsilon^\kappa \wedge \vartheta^\alpha. \tag{22}$$

In a similar fashion, we derive the recurrence formulae (12) for the differentiated invariant contact forms:

$$d\vartheta_J^\alpha = d[\iota(\theta_J^\alpha)] = \iota(d\theta_J^\alpha) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota\,[\mathbf{v}_\kappa(\theta_J^\alpha)] = \iota \left(\sum_{i=1}^{p} dx^i \wedge \theta_{Ji}^\alpha \right) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota(\psi_{J\kappa}^\alpha), \tag{23}$$

where

$$\psi_{J\kappa}^\alpha = \mathbf{v}_\kappa(\theta_J^\alpha) = d\varphi_{J\kappa}^\alpha - \sum_{i=1}^{p} \left[\varphi_{Ji\kappa}^\alpha \, dx^i + u_{Ji}^\alpha \, d\xi_\kappa^i \right] = d_V \, \varphi_{J\kappa}^\alpha - \sum_{i=1}^{p} u_{Ji}^\alpha \, d_V \, \xi_\kappa^i \tag{24}$$

is known as the *vertical prolongation coefficient* of the vector field $\mathbf{v}_\kappa$. For our purposes, we only require the component of (23) that involves invariant horizontal forms:

$$d_{\mathcal{H}}\,\vartheta^\alpha_J = \sum_{i=1}^p \varpi^i \wedge \vartheta^\alpha_{Ji} + \sum_{\kappa=1}^r \gamma^\kappa \wedge \iota(\psi^\alpha_{J\kappa}). \tag{25}$$

Since[5]

$$d_{\mathcal{H}}\,\vartheta = \sum_{i=1}^p \varpi^i \wedge \mathcal{D}_i\,\vartheta \tag{26}$$

for any contact form ϑ, we deduce the recurrence formulae

$$\mathcal{D}_i\vartheta^\alpha_J = \vartheta^\alpha_{Ji} + \sum_{\kappa=1}^r R^\kappa_i\,\iota(\psi^\alpha_{J\kappa}) \tag{27}$$

for the invariant (Lie) derivatives of the invariant contact forms. The latter can inductively be solved to express the higher order invariantized contact forms as certain invariant derivatives of those of order 0:

$$\vartheta^\alpha_J = \sum_{\beta=1}^q \mathcal{E}^\alpha_{J,\beta}(\vartheta^\beta) = \mathcal{E}^\alpha_J(\vartheta), \tag{28}$$

in which $\vartheta = (\vartheta^1,\ldots,\vartheta^q)^T$ denotes the column vector containing the order zero invariantized contact forms, while $\mathcal{E}^\alpha_J = (\mathcal{E}^\alpha_{J,1},\ldots,\mathcal{E}^\alpha_{J,q})$ are certain invariant differential operators.

In view of (17, 28), if $K = K(\ldots H^i \ldots I^\alpha_J \ldots)$ is any differential invariant, we can write its invariant vertical derivative in the form

$$d_{\mathcal{V}}\,K = \sum_{i=1}^p \frac{\partial K}{\partial H^i}\,d_{\mathcal{V}}\,H^i + \sum_{\alpha,J} \frac{\partial K}{\partial I^\alpha_J}\,d_{\mathcal{V}}\,I^\alpha_J = \mathcal{A}_K(\vartheta) = \sum_{\alpha=1}^q \mathcal{A}_{K,\alpha}(\vartheta^\alpha), \tag{29}$$

in which $\mathcal{A}_K = (\mathcal{A}_{K,1},\ldots,\mathcal{A}_{K,q})$ is a row vector of invariant differential operators. We view (29) as the invariant version of the vertical differentiation formula $d_V\,F = \mathbf{D}_F(\theta)$, cf. (2), which motivates the following terminology.

Definition 3 The *invariant linearization* of a differential invariant K is the invariant differential operator $\mathcal{A}_K$ that satisfies $d_{\mathcal{V}}\,K = \mathcal{A}_K(\vartheta)$.

Remark: In [32], $\mathcal{A}_K$ was called the *Eulerian operator* associated with K owing to its appearance in the differential invariant form of the Euler–Lagrange equations for an invariant variational problem; see Theorem 5 below.

Similarly, we combine (14), (21), and (28), to produce formulae

$$d_{\mathcal{V}}\,\varpi^i = \sum_{j=1}^p \sum_{\alpha=1}^q \mathcal{B}^i_{j\alpha}(\vartheta^\alpha) \wedge \varpi^j = \sum_{j=1}^p \mathcal{B}^i_j(\vartheta) \wedge \varpi^j \tag{30}$$

for the vertical differentials of the invariant horizontal forms, in which $\mathcal{B}^i_j = (\mathcal{B}^i_{j1},\ldots,\mathcal{B}^i_{jq})$ is a family of p^2 row-vector-valued invariant differential operators, known, collectively, as the *invariant Hamiltonian operator complex*, cf. [59], again stemming from its role in the invariant calculus of variations.

[5] *Warning*: The analogous formula is *not* valid for horizontal forms.

Example 4 The Euclidean geometry of plane curves is governed by the standard action $y = x\cos\phi - u\sin\phi + a$, $v = x\sin\phi + u\cos\phi + b$, of the *proper Euclidean group* $g = (\phi, a, b) \in \mathrm{SE}(2)$ on $M = \mathbb{R}^2$. The prolonged group transformations are constructed by applying the implicit differentiation operator $D_y = (\cos\phi - u_x \sin\phi)^{-1} D_x$ to v, and so

$$v_y = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi}\,, \qquad v_{yy} = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}\,, \qquad \text{etc.}$$

Solving the normalization equations $y = v = v_y = 0$ for the group parameters produces the right moving frame

$$\phi = -\tan^{-1} u_x\,, \qquad a = -\frac{x + u u_x}{\sqrt{1+u_x^2}}\,, \qquad b = \frac{x u_x - u}{\sqrt{1+u_x^2}}\,. \tag{31}$$

(The classical moving frame, [23], is the left counterpart obtained by inverting the group element given in (31).) Invariantization of the coordinate functions, which is done by substituting the moving frame formulae into the prolonged group transformations, produces the fundamental normalized differential invariants

$$\begin{aligned} &\iota(x) = H = 0, && \iota(u) = I_0 = 0, && \iota(u_x) = I_1 = 0,\\ &\iota(u_{xx}) = I_2 = \kappa, && \iota(u_{xxx}) = I_3 = \kappa_s, && \iota(u_{xxxx}) = I_4 = \kappa_{ss} + 3\kappa^3, \end{aligned}$$

and so on. The first three, arising from the normalizations, are called *phantom invariants*. The lowest order non-trivial differential invariant is the Euclidean curvature $I_2 = \kappa = u_{xx}(1+u_x^2)^{3/2}$, while $\kappa_s, \kappa_{ss}, \ldots$ denote the derivatives of κ with respect to the arc-length form $\omega = \sqrt{1+u_x^2}\, dx$. The invariant horizontal one-form

$$\varpi = \iota(dx) = \frac{dx + u_x\, du}{\sqrt{1+u_x^2}} = \sqrt{1+u_x^2}\, dx + \frac{u_x}{\sqrt{1+u_x^2}}\,\theta \tag{32}$$

is a sum of the contact-invariant arc length form along with a contact correction. In the same manner we obtain the basis invariant contact forms

$$\vartheta = \iota(\theta) = \frac{\theta}{\sqrt{1+u_x^2}}\,, \qquad \vartheta_1 = \iota(\theta_x) = \frac{(1+u_x^2)\,\theta_x - u_x u_{xx}\theta}{(1+u_x^2)^2}\,, \qquad \ldots\,. \tag{33}$$

To obtain the explicit recurrence formulae, we begin with the prolonged infinitesimal generators of SE(2):

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_u, \qquad \mathbf{v}_3 = -u\,\partial_x + x\,\partial_u + (1+u_x^2)\,\partial_{u_x} + 3u_x u_{xx}\,\partial_{u_{xx}} + \;\cdots\;.$$

The one-forms $\gamma^\kappa, \varepsilon^\kappa$ governing the correction terms are found by applying the recurrence formulae (12) to the phantom invariants. From the first equation in (12), we obtain

$$\begin{aligned} 0 &= d_{\mathcal{H}}\, H = \iota(d_H x) + \iota(\mathbf{v}_1(x))\,\gamma^1 + \iota(\mathbf{v}_2(x))\,\gamma^2 + \iota(\mathbf{v}_3(x))\,\gamma^3 = \varpi + \gamma^1,\\ 0 &= d_{\mathcal{H}}\, I_0 = \iota(d_H u) + \iota(\mathbf{v}_1(u))\,\gamma^1 + \iota(\mathbf{v}_2(u))\,\gamma^2 + \iota(\mathbf{v}_3(u))\,\gamma^3 = \gamma^2,\\ 0 &= d_{\mathcal{H}}\, I_1 = \iota(d_H u_x) + \iota(\mathbf{v}_1(u_x))\,\gamma^1 + \iota(\mathbf{v}_2(u_x))\,\gamma^2 + \iota(\mathbf{v}_3(u_x))\,\gamma^3 = \kappa\,\varpi + \gamma^3, \end{aligned}$$

and hence $\gamma^1 = -\varpi,\ \gamma^2 = 0,\ \gamma^3 = -\kappa\,\varpi$. Similarly, applying $d_{\mathcal{V}}$ to the phantom invariants and using the second equation in (12) yields $\varepsilon^1 = 0,\ \varepsilon^2 = -\vartheta,\ \varepsilon^3 = -\vartheta_1$. We are now ready to substitute the non-phantom invariants into (12). The horizontal differentials $d_{\mathcal{H}}\, I_k$ of the normalized differential invariants $I_n = \iota(u_n)$ are used to produce the explicit recurrence formulae

$$\kappa = I_2, \qquad \kappa_s = \mathcal{D} I_2 = I_3, \qquad \kappa_{ss} = \mathcal{D} I_3 = I_4 - 3 I_2^3, \qquad \ldots$$

relating them to the differentiated invariants $\mathcal{D}^m \kappa$. Similarly, the second equation in (12) gives the vertical differential

$$d_{\mathcal{V}}\, I_2 = \, d_{\mathcal{V}}\, \kappa = \iota(\theta_2) + \iota(\mathbf{v}_3(u_{xx}))\, \varepsilon^3 = \vartheta_2 = (\mathcal{D}^2 + \kappa^2)\, \vartheta, \tag{34}$$

where the final equation follows from the invariant contact form recurrence formulae $\mathcal{D}\vartheta = \vartheta_1$, $\mathcal{D}\vartheta_1 = \vartheta_2 - \kappa^2\, \vartheta$, which are found by applying $d_{\mathcal{H}}$ to the invariant contact forms and using the first equation in (12). Thus, we deduce the following invariant linearization operators:

$$\begin{gathered} \mathcal{A}_{\kappa} = \mathcal{D}^2 + \kappa^2, \qquad \mathcal{A}_{\kappa_s} = \mathcal{D}^3 + \kappa^2 \mathcal{D} + 3\, \kappa\, \kappa_s, \\ \mathcal{A}_{\kappa_{ss}} = \mathcal{D}^4 + \kappa^2 \mathcal{D}^2 + 5\, \kappa\, \kappa_s \mathcal{D} + 4\, \kappa\, \kappa_{ss} + 3\, \kappa_s^2, \end{gathered} \tag{35}$$

etc. In fact, one can recursively construct the higher order operators starting with $\mathcal{A}_{\kappa}$ via

$$\mathcal{A}_{\kappa_n} = \mathcal{D} \cdot \mathcal{A}_{\kappa_{n-1}} + \kappa\, \kappa_n, \tag{36}$$

where $\kappa_n = \mathcal{D}^n \kappa$. Finally, applying the second formula in (12) to ϖ yields

$$d_{\mathcal{V}}\, \varpi = -\, \kappa\, \vartheta \wedge \varpi,$$

and hence the invariant Hamiltonian operator is

$$\mathcal{B} = -\, \kappa. \tag{37}$$

3 Invariant Variational Problems.

We now apply our construction to derive the formulae for the Euler-Lagrange equations associated with an invariant variational problem. Let us recall the variational bicomplex construction of the Euler-Lagrange equations.

A variational problem $\mathcal{I}[u] = \int L[u]\, d\mathbf{x}$ is determined by the *Lagrangian form* $\lambda = L[u]\, d\mathbf{x} \in \mathbf{\Omega}^{p,0}$. Its differential $d\lambda = d_V\, \lambda \in \mathbf{\Omega}^{p,1}$ defines a form of type $(p, 1)$. We introduce an equivalence relation on such forms, so that $\Theta \sim \Omega$ if and only if $\Theta = \Omega + d_H\, \Psi$ for some $\Psi \in \mathbf{\Omega}^{p-1,1}$. The quotient space $\mathcal{F}^1 = \mathbf{\Omega}^{p,1} / \sim$ is known as the space of *source forms*. Integration by parts proves that every source form has a canonical representative $\sum_{\alpha=1}^{q} \Delta_\alpha(x, u^{(n)})\, \theta^\alpha \wedge d\mathbf{x}$, and so can be identified with a q-tuple of differential functions $\Delta = (\Delta_1, \ldots, \Delta_q)$. In applications, a source form is regarded as defining a system of q differential equations $\Delta_1 = \cdots = \Delta_q = 0$ for the q dependent variables $u = (u^1, \ldots, u^q)$. Composing the differential $d: \mathbf{\Omega}^{p,0} \to \mathbf{\Omega}^{p,1}$ with the projection $\pi_*: \mathbf{\Omega}^{p,1} \to \mathcal{F}^1$ produces the *variational differential* $\delta = \pi_* \circ d$ that takes a Lagrangian form $\lambda = L[u]\, d\mathbf{x}$ to its *variational derivative* source form

$$\delta\, \lambda \;\simeq\; \sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L)\, \theta^\alpha \wedge d\mathbf{x}, \qquad \text{where} \qquad \mathbf{E}_\alpha(L) = \sum_J\, (-D)_J \frac{\partial L}{\partial u_J^\alpha} \tag{38}$$

are the classical Euler-Lagrange expressions for the Lagrangian L.

According to Lie, [39, 50], as long as we work on the open subset $\mathcal{V} \subset \mathrm{J}^n$ where G acts regularly and freely, any G-invariant variational problem is given by an invariant Lagrangian form $\lambda = \widetilde{L}\,\boldsymbol{\omega}$, where $\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ is the contact-invariant volume form, and the invariant Lagrangian $\widetilde{L}$ is an arbitrary differential invariant, and hence a function of the fundamental differential invariants $I^1, \ldots, I^l$ and their invariant derivatives $\mathcal{D}_J I^\alpha$. The associated Euler-Lagrange equations $\mathbf{E}(L) = 0$ admit G as a symmetry group, and so, under suitable nondegeneracy hypotheses, [50, Theorem 6.25], can themselves be written in terms of the differential invariants. The main problem is to go directly from the differential invariant formula for the variational problem to the differential invariant formula for the Euler-Lagrange equations.

Not surprisingly, the required calculations rely on an invariant version of integration by parts. For this purpose, given invariant differential forms α, β, let us write $\alpha \equiv \beta$ whenever $\alpha = \beta + d_{\mathcal{H}}\,\gamma$. If F is a differential function and ψ is a *contact* one-form, then[6]

$$d_{\mathcal{H}}\,F = \sum_{i=1}^{p} (\mathcal{D}_i\,F)\,\varpi^i \qquad\qquad d_{\mathcal{H}}\,\psi = \sum_{i=1}^{p} \varpi^i \wedge \mathcal{D}_i\,\psi. \tag{39}$$

To effect the computation, we begin by replacing our Lagrangian $\lambda = \widetilde{L}\,\boldsymbol{\omega}$ by the fully invariant version $\widetilde{\lambda} = \widetilde{L}\,\boldsymbol{\varpi}$, noting that, since they differ by contact forms, they have identical Euler–Lagrange equations. We then compute

$$d_{\mathcal{V}}\,\widetilde{\lambda} = \sum_{\alpha,K} \frac{\partial \widetilde{L}}{\partial I^\alpha_{,K}}\,d_{\mathcal{V}}\,I^\alpha_{,K} \wedge \boldsymbol{\varpi} + \widetilde{L}\;d_{\mathcal{V}}\,\boldsymbol{\varpi}, \tag{40}$$

Introduce the $(p-1)$–forms

$$\boldsymbol{\varpi}_{(i)} = \mathcal{D}_i \lrcorner\, \boldsymbol{\varpi} = (-1)^{i-1}\,\varpi^1 \wedge \cdots \wedge \varpi^{i-1} \wedge \varpi^{i+1} \wedge \cdots \wedge \varpi^p \in \widetilde{\boldsymbol{\Omega}}^{p-1,0}.$$

If F is any differential function and ψ any contact one-form, then

$$d_{\mathcal{H}}\,(F\,\psi \wedge \boldsymbol{\varpi}_{(i)}) = d_{\mathcal{H}}\,F \wedge \psi \wedge \boldsymbol{\varpi}_{(i)} + F\,d_{\mathcal{H}}\,\psi \wedge \boldsymbol{\varpi}_{(i)} - F\,\psi \wedge d_{\mathcal{H}}\,\boldsymbol{\varpi}_{(i)}. \tag{41}$$

Since $d_{\mathcal{H}}\,\boldsymbol{\varpi}_{(i)} \in \widetilde{\boldsymbol{\Omega}}^{p,0}$, it must be a multiple of the invariant volume form, and we write $d_{\mathcal{H}}\,\boldsymbol{\varpi}_{(i)} = Z_i\,\boldsymbol{\varpi}$, where $Z_1, \ldots, Z_p$ are certain differential invariants, which we will call the *twist invariants*. Using (39) we can rewrite (41) as

$$F\,d_{\mathcal{H}}\,\psi \wedge \boldsymbol{\varpi}_{(i)} = F(\mathcal{D}_i\psi) \wedge \boldsymbol{\varpi} \equiv -\left[\,(\mathcal{D}_i + Z_i)F\,\right]\psi \wedge \boldsymbol{\varpi} = (\mathcal{D}_i^\dagger F)\,\psi \wedge \boldsymbol{\varpi}, \tag{42}$$

where $\mathcal{D}_i^\dagger = -\,(\mathcal{D}_i + Z_i)$ is called the *twisted invariant adjoint* of the invariant differential operator $\mathcal{D}_i$. If we choose $\psi = d_{\mathcal{V}}\,H$ where H is a differential function, then (42) results in the multivariate invariant integration by parts formula

$$F\,d(\mathcal{D}_i H) \wedge \boldsymbol{\varpi} = (\mathcal{D}_i^\dagger F)\,d_{\mathcal{V}}\,H \wedge \boldsymbol{\varpi} \;-\; \sum_{j=1}^{p} F\,(\mathcal{D}_j H)\,d_{\mathcal{V}}\,\varpi^j \wedge \boldsymbol{\varpi}_{(i)}. \tag{43}$$

We use (43) repeatedly to integrate the first term of (40) by parts, leading to

$$d_{\mathcal{V}}\,\widetilde{\lambda} \equiv \sum_{\alpha=1}^{q} \mathcal{E}_\alpha(\widetilde{L})\,d_{\mathcal{V}}\,I^\alpha \wedge \boldsymbol{\varpi} \;-\; \sum_{i=1}^{p} \mathcal{H}^i_j(\widetilde{L})\,d_{\mathcal{V}}\,\varpi^j \wedge \boldsymbol{\varpi}_{(i)}, \tag{44}$$

[6] *Warning*: The second identity *is not* true for a general one-form.

where

$$\mathcal{E}_\alpha(\widetilde{L}) = \sum_K \mathcal{D}_K^\dagger \frac{\partial \widetilde{L}}{\partial I^\alpha_{,K}}, \qquad \mathcal{H}^i_j(\widetilde{L}) = -\,\widetilde{L}\,\delta^i_j + \sum_{\alpha=1}^q \sum_{J,K} I^\alpha_{,J,j}\,\mathcal{D}_K^\dagger \frac{\partial \widetilde{L}}{\partial I^\alpha_{,J,i,K}}, \tag{45}$$

are, respectively, the *invariant Eulerian* and *invariant Hamiltonian tensor* of the invariant Lagrangian $\widetilde{L}$. In (45), we use the twisted adjoints

$$\mathcal{D}_K^\dagger = \mathcal{D}_{k_1}^\dagger \cdots \mathcal{D}_{k_m}^\dagger = (-1)^m (\mathcal{D}_{k_1} + Z_{k_1}) \cdots (\mathcal{D}_{k_m} + Z_{k_m}), \qquad K = (k_1, \ldots, k_m),$$

of the repeated invariant differential operators.

The second phase of the computation requires the vertical differentiation formulae

$$d_{\mathcal{V}}\, I^\alpha = \sum_{\beta=1}^q \mathcal{A}^\alpha_\beta(\vartheta^\beta), \qquad\qquad d_{\mathcal{V}}\, \varpi^j = \sum_{\beta=1}^q \mathcal{B}^j_{i,\beta}(\vartheta^\beta) \wedge \varpi^i, \tag{46}$$

where $\mathcal{A} = \left(\mathcal{A}^\alpha_\beta\right)$ denotes the *Eulerian operator*, which is an $m \times q$ matrix of invariant differential operators whose rows are the invariant linearizations of the fundamental differential invariants $I^1, \ldots, I^l$, while the p^2 row vectors $\mathcal{B}^j_i = \left(\mathcal{B}^j_{i,\beta}\right)$ of invariant differential operators form the *invariant Hamiltonian operator complex*. This allows us to write (44) in the vectorial form

$$d_{\mathcal{V}}\,\widetilde{\lambda} \equiv \mathcal{E}(\widetilde{L})\,\mathcal{A}(\boldsymbol{\vartheta}) \wedge \boldsymbol{\varpi} - \sum_{i,j=1}^p \mathcal{H}^i_j(\widetilde{L})\,\mathcal{B}^j_i(\boldsymbol{\vartheta}) \wedge \boldsymbol{\varpi}.$$

We now apply (42) to further integrate both terms by parts. The final result is written in terms of twisted adjoints of the Eulerlian and Hamiltonian operators,

$$d_{\mathcal{V}}\,\widetilde{\lambda} \equiv \delta\widetilde{\lambda} = \left(\mathcal{A}^\dagger\,\mathcal{E}(\widetilde{L}) - \sum_{i,j=1}^p (\mathcal{B}^j_i)^\dagger\,\mathcal{H}^i_j(\widetilde{L}) \right) \boldsymbol{\vartheta} \wedge \boldsymbol{\varpi}.$$

Theorem 5 *The Euler-Lagrange equations of the invariant Lagrangian form* $\widetilde{\lambda} = \widetilde{L}(I^{(n)})\,\boldsymbol{\varpi}$ *Have the following invariant form:*

$$\mathcal{A}^\dagger\,\mathcal{E}(\widetilde{L}) - \sum_{i,j=1}^p (\mathcal{B}^j_i)^\dagger\,\mathcal{H}^i_j(\widetilde{L}) = 0. \tag{47}$$

In the case of curves, when $p = 1$, there are no twist invariants, and so the general formula (47) reduces to

$$\mathcal{A}^*\mathcal{E}(\widetilde{L}) - \mathcal{B}^*\mathcal{H}(\widetilde{L}) = 0, \tag{48}$$

where $\mathcal{A}^*$ and $\mathcal{B}^*$ are the ordinary formal adjoints of the invariant Eulerian and Hamiltonian operators, respectively.

Example 6 In the context of the Euclidean group acting on plane curves in Example 4, any Euclidean-invariant variational problem corresponds to a contact invariant Lagrangian $\lambda = \widetilde{L}(\kappa, \kappa_s, \kappa_{ss}, \ldots)\,\omega$. Both the Eulerian operator (35) and the Hamiltonian operator (37) are invariantly self-adjoint: $\mathcal{A} = \mathcal{A}^*$ and $\mathcal{B} = \mathcal{B}^*$. Thus, the invariant Euler-Lagrange formula (48) reduces to the known formula, [3, 22],

$$(\mathcal{D}^2 + \kappa^2)\,\mathcal{E}(\widetilde{L}) + \kappa\,\mathcal{H}(\widetilde{L}) = 0$$

for the Euclidean-invariant Euler-Lagrange equation.

Example 7 Consider the standard action of the Euclidean group SE(3) on surfaces $S \subset \mathbb{R}^3$. We assume that the surface is parametrized by $z = (x, y, u(x, y))$, noting that the final formulae are, in fact, parameter-independent. The classical (local) left moving frame $\rho(x, u^{(2)}) = (R, a) \in \mathrm{SE}(3)$ consists of the point on the curve defining the translation component $a = z$, while the columns of the rotation matrix R contain the unit tangent vectors forming the Frenet frame along with the unit normal to the surface. The fundamental differential invariants are the principal curvatures $\kappa^1 = \iota(u_{xx})$, $\kappa^2 = \iota(u_{yy})$. The mean and Gaussian curvature invariants $H = \frac{1}{2}(\kappa^1 + \kappa^2)$, $K = \kappa^1 \kappa^2$, are often used as convenient alternatives, since they eliminate some of the residual discrete ambiguities in the moving frame. Higher order differential invariants are obtained by repeatedly applying the dual invariant differential operators $\mathcal{D}_1, \mathcal{D}_2$ associated with the diagonalizing Frenet coframe $\varpi^1 = \iota(dx^1)$, $\varpi^2 = \iota(dx^2)$. The resulting differentiated invariants are not functionally independent, owing to the Codazzi identity

$$\kappa^1_{,22} - \kappa^2_{,11} + \frac{\kappa^1_{,1}\kappa^2_{,1} + \kappa^1_{,2}\kappa^2_{,2} - 2(\kappa^2_{,1})^2 - 2(\kappa^1_{,2})^2}{\kappa^1 - \kappa^2} - \kappa^1\kappa^2(\kappa^1 - \kappa^2) = 0. \tag{49}$$

The Codazzi syzygy can, in fact, be directly deduced from our infinitesimal moving frame computations by comparing the recurrence formulae for $\kappa^1_{,22}$ and $\kappa^2_{,11}$ with the normalized invariant $\iota(u_{xxyy})$.

Any Euclidean-invariant variational problem has the form

$$\int \widetilde{L}(\kappa^{(n)})\, \omega^1 \wedge \omega^2, \qquad \text{where} \qquad \omega^1 \wedge \omega^2 = \pi_{2,0}(\varpi^1 \wedge \varpi^2)$$

is the usual intrinsic surface area 2-form. The invariant Lagrangian $\widetilde{L}$ is an arbitrary differential invariant, and so can be rewritten in terms of the principal curvature invariants and their derivatives, or, equivalently, in terms of the Gaussian and mean curvatures. The former representation leads to simpler formulae and will be retained. Since

$$d_{\mathcal{H}}\, \boldsymbol{\varpi}_{(1)} = d_{\mathcal{H}}\, \varpi^2 = \frac{\kappa^2_{,1}}{\kappa^1 - \kappa^2}\, \boldsymbol{\varpi}, \qquad d_{\mathcal{H}}\, \boldsymbol{\varpi}_{(2)} = -\, d_{\mathcal{H}}\, \varpi^1 = \frac{\kappa^1_{,2}}{\kappa^2 - \kappa^1}\, \boldsymbol{\varpi},$$

the twist invariants are

$$Z_1 = \frac{\kappa^2_{,1}}{\kappa^1 - \kappa^2}\,, \qquad Z_2 = \frac{\kappa^1_{,2}}{\kappa^2 - \kappa^1}\,.$$

These invariants appear in Guggenheimer's proof of the fundamental existence theorem for Euclidean surfaces, [23, p. 234]. The denominator vanishes at umbilic points on the surface, where the moving frame is not valid. The Codazzi syzygy (49) can be written compactly as

$$K = \kappa^1\kappa^2 = \mathcal{D}_1^\dagger(Z_1) + \mathcal{D}_2^\dagger(Z_2) = -\,(\mathcal{D}_1 + Z_1)Z_1 - (\mathcal{D}_2 + Z_2)Z_2,$$

which expresses the Gaussian curvature K as an invariant divergence. This fact lies at the heart of the Gauss–Bonnet Theorem. The invariant vertical derivatives of the principal curvatures are straightforwardly determined from (12),

$$d_{\mathcal{V}}\, \kappa^1 = \iota(\theta_{xx}) = \left(\mathcal{D}_1^2 + Z_2\, \mathcal{D}_2 + (\kappa^1)^2 \right) \vartheta, \qquad d_{\mathcal{V}}\, \kappa^2 = \iota(\theta_{yy}) = \left(\mathcal{D}_2^2 + Z_1\, \mathcal{D}_1 + (\kappa^2)^2 \right) \vartheta,$$

where $\vartheta = \iota(\theta) = \iota(du - u_x\, dx - u_y\, dy)$ is the fundamental invariant contact form. Therefore, the Eulerian operator is $\mathcal{A} = \begin{pmatrix} \mathcal{D}_1^2 + Z_2\,\mathcal{D}_2 + (\kappa^1)^2 \\ \mathcal{D}_2^2 + Z_1\,\mathcal{D}_1 + (\kappa^2)^2 \end{pmatrix}$. Further,

$$d_{\mathcal{V}}\,\varpi^1 = -\,\kappa^1\,\vartheta\wedge\varpi^1 + \frac{\left(\mathcal{D}_1\mathcal{D}_2 - Z_2\mathcal{D}_1\right)\vartheta\wedge\varpi^2}{\kappa^1-\kappa^2},$$
$$d_{\mathcal{V}}\,\varpi^2 = \frac{\left(\mathcal{D}_2\mathcal{D}_1 - Z_1\mathcal{D}_2\right)\vartheta\wedge\varpi^1}{\kappa^2-\kappa^1} - \kappa^2\,\vartheta\wedge\varpi^2,$$

which yields the Hamiltonian operator complex

$$\mathcal{B}_1^1 = -\,\kappa^1, \qquad \mathcal{B}_2^2 = -\,\kappa^2,$$
$$\mathcal{B}_2^1 = \frac{1}{\kappa^1-\kappa^2}\left(\mathcal{D}_1\mathcal{D}_2 - Z_2\mathcal{D}_1\right) = \frac{1}{\kappa^1-\kappa^2}\left(\mathcal{D}_2\mathcal{D}_1 - Z_1\mathcal{D}_2\right) = -\,\mathcal{B}_1^2.$$

Therefore, according to our formula (47), the Euler-Lagrange equation for a Euclidean-invariant variational problem is

$$\begin{aligned}
0 = {} & \left[\,(\mathcal{D}_1+Z_1)^2 - (\mathcal{D}_2+Z_2)\cdot Z_2 + (\kappa^1)^2\,\right]\mathcal{E}_1(\widetilde{L}) \\
& + \left[\,(\mathcal{D}_2+Z_2)^2 - (\mathcal{D}_1+Z_1)\cdot Z_1 + (\kappa^2)^2\,\right]\mathcal{E}_2(\widetilde{L}) + \kappa^1\,\mathcal{H}_1^1(\widetilde{L}) + \kappa^2\,\mathcal{H}_2^2(\widetilde{L}) \\
& + \left[\,(\mathcal{D}_2+Z_2)(\mathcal{D}_1+Z_1) + (\mathcal{D}_1+Z_1)\cdot Z_2\,\right]\cdot\left(\frac{\mathcal{H}_2^1(\widetilde{L}) - \mathcal{H}_1^2(\widetilde{L})}{\kappa^1-\kappa^2}\right).
\end{aligned}$$

As before, $\mathcal{E}_\alpha(\widetilde{L})$ are the invariant Eulerians with respect to the principal curvatures κ^α, while $\mathcal{H}_j^i(\widetilde{L})$ are the invariant Hamiltonians. In particular, if $\widetilde{L}(\kappa^1,\kappa^2)$ does not depend on any differentiated invariants, the Euler-Lagrange equation reduces to

$$\left[\,(\mathcal{D}_1^\dagger)^2 + \mathcal{D}_2^\dagger\cdot Z_2 + (\kappa^1)^2\,\right]\frac{\partial\widetilde{L}}{\partial\kappa^1} + \left[\,(\mathcal{D}_2^\dagger)^2 + \mathcal{D}_1^\dagger\cdot Z_1 + (\kappa^2)^2\,\right]\frac{\partial\widetilde{L}}{\partial\kappa^2} - (\kappa^1+\kappa^2)\widetilde{L} = 0.$$

For example, the problem of minimizing surface area has invariant Lagrangian $\widetilde{L} = 1$, and so has the well-known Euler-Lagrange equation $\mathbf{E}(L) = -\,(\kappa^1+\kappa^2) = -\,2H = 0$, and hence minimal surfaces have vanishing mean curvature. The mean curvature Lagrangian $\widetilde{L} = H = \frac{1}{2}\,(\kappa^1+\kappa^2)$ has Euler-Lagrange equation

$$\tfrac{1}{2}\left[\,(\kappa^1)^2 + (\kappa^2)^2 - (\kappa^1+\kappa^2)^2\,\right] = -\,\kappa^1\,\kappa^2 = -K = 0.$$

For the Willmore Lagrangian $\widetilde{L} = \frac{1}{2}(\kappa^1)^2 + \frac{1}{2}(\kappa^2)^2$, [3, 8], the Euler-Lagrange equation is

$$0 = \mathbf{E}(L) = \Delta(\kappa^1+\kappa^2) + \tfrac{1}{2}\,(\kappa^1+\kappa^2)(\kappa^1-\kappa^2)^2 = 2\,\Delta H + 4\,(H^2-K)H,$$

where $\Delta = (\mathcal{D}_1+Z_1)\mathcal{D}_1 + (\mathcal{D}_2+Z_2)\mathcal{D}_2 = -\,\mathcal{D}_1^\dagger\cdot\mathcal{D}_1 - \mathcal{D}_2^\dagger\cdot\mathcal{D}_2$ is the Laplace–Beltrami operator on our surface.

4 Invariant Submanifold Flows.

In this section, we shift our attention to invariant submanifold flows. Let us single out the $m = p+q$ invariant one-forms

$$\varpi^1,\ldots,\varpi^p,\ \vartheta^1,\ldots,\vartheta^q \tag{50}$$

consisting of the invariant horizontal forms $\varpi^i = \iota(dx^i)$ and the order 0 invariant contact forms $\vartheta^\alpha = \iota(\theta^\alpha)$. Each is a linear combination of the coordinate one-forms $dx^1, \ldots, dx^p$, $du^1, \ldots, du^q$ on M, whose coefficients are certain $(n+1)$-st order differential functions, where n is the order of the underlying moving frame.

Let $S \subset M$ be a p-dimensional submanifold. Evaluating the coefficients of (50) on the submanifold jet $(x, u^{(n)}) = \mathrm{j}_n S|_z$ produces a basis for the cotangent space $T^*M|_z$ of the ambient manifold at $z = (x, u) \in S$, which we continue to denote by (50). By construction, the resulting cotangent space basis is equivariant under the action of G on $S \subset M$.

Let $\mathbf{t}_1, \ldots, \mathbf{t}_p, \mathbf{n}_1, \ldots, \mathbf{n}_q$, denote the corresponding dual tangent vectors, which form a G–equivariant basis of the bundle $TM \to S$, or *frame* on S. Since the contact forms annihilate the tangent space to S, the vectors $\mathbf{t}_1, \ldots, \mathbf{t}_p$ form a basis for the tangent bundle TS, while $\mathbf{n}_1, \ldots, \mathbf{n}_q$ form a basis for the complementary G–equivariant *normal bundle* $NS \to S$ induced by the moving frame. In classical geometrical situations, [23], they can be identified with the classical moving frame vectors.

Example 8 Let us return to the case of planar Euclidean curves $C \subset M = \mathbb{R}^2$. According to Example 4, the invariant coframe is given by the invariant horizontal form (32) and the order 0 invariant contact form in (33). The corresponding dual frame vectors are the usual (right-handed) Euclidean frame vectors — the unit tangent and unit normal:

$$\mathbf{t} = \frac{1}{\sqrt{1+u_x^2}} \left(\frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} \right), \qquad \mathbf{n} = \frac{1}{\sqrt{1+u_x^2}} \left(-u_x \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right). \tag{51}$$

In general, let

$$\mathbf{V} = \mathbf{V}\,|_S = \mathbf{V}_T + \mathbf{V}_N = \sum_{j=1}^{p} I^j\, \mathbf{t}_j + \sum_{\alpha=1}^{q} J^\alpha\, \mathbf{n}_\alpha \tag{52}$$

be a section of the bundle $TM \to S$, where $\mathbf{V}_T, \mathbf{V}_N$ denote, respectively, its tangential and normal components, while I^j, J^α are differential functions, depending on the submanifold jets. We will, somewhat imprecisely, refer to $\mathbf{V}$ as a *vector field*, even though it is only defined on S. Any such vector field generates a submanifold flow:

$$\frac{\partial S}{\partial t} = \mathbf{V}|_{S(t)}, \tag{53}$$

which forms an n-th order system of partial differential equations, where n refers to the larger of the order of our moving frame and the coefficients I^j, J^α. Assuming local existence and uniqueness, a solution $S(t)$ to the submanifold flow equations (53) defines a smoothly varying family of p-dimensional submanifolds of M. On the other hand, one typically expects singularities to appear if the flow is continued for a sufficiently long time. The submanifold flow (53) is called *G-invariant* if G is a symmetry group of the partial differential equation, which requires that its coefficients $I^j = \langle\, \mathbf{V}\,; \varpi^j \,\rangle$, $J^\alpha = \langle\, \mathbf{V}\,; \vartheta^\alpha \,\rangle$, be differential invariants.

The tangential components $\mathbf{V}_T$ do not affect the extrinsic geometry of the submanifold, but only its internal parametrization. Thus, if we are only interested in the images of $S(t)$ under the flow, and not their underlying parametrizations, we can set $\mathbf{V}_T = 0$ without loss of generality. Therefore, the normal component

$$\mathbf{V}_N = \sum_{\alpha=1}^{q} J^\alpha \mathbf{n}_\alpha \tag{54}$$

serves to characterize the *same* invariant submanifold flow as $\mathbf{V}$, modulo reparametrization. We will say that the vector field $\mathbf{V}_N$ generates a *normal flow*, since it only moves the submanifold in its G-equivariant normal direction — as prescribed by the moving frame.

Example 9 The most well-studied are the Euclidean-invariant curve and surface flows. A plane curve flow is generated by a vector field of the form

$$\mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n} \qquad \text{or, equivalently,} \qquad \mathbf{V}_N = J\,\mathbf{n}, \tag{55}$$

if we are not concerned about the tangential component's effect of the parametrization of the curve. In this case, $\mathbf{n}$ denotes (one of the two) Euclidean normals to the curve; by convention, we use the inwards normal $\mathbf{n}$ when the curve is closed. Particular cases include:

i) $\mathbf{V} = \mathbf{n}$: this induces the geometric optics or grassfire flow, [7, 62];

ii) $\mathbf{V} = \kappa\,\mathbf{n}$: this generates the celebrated curve shortening flow, [18, 20], used to great effect in image processing, [56, 62];

iii) $\mathbf{V} = \kappa^{1/3}\,\mathbf{n}$: the induced flow is equivalent, modulo reparametrization, to the equi-affine invariant curve shortening flow, also effective in image processing, [4, 56, 62];

iv) $\mathbf{V} = \kappa_s\,\mathbf{n}$: this flow induces the modified Korteweg–deVries equation for the curvature evolution, and is the simplest of a large number of soliton equations arising in geometric curve flows, [13, 19, 43];

v) $\mathbf{V} = \kappa_{ss}\,\mathbf{n}$: this flow models thermal grooving of metals, [9].

A second important class are the invariant curve flows that preserve arc length. Remarkably, in many classical geometries, certain basic intrinsic curve flows induce integrable, soliton evolutions for the differential invariants. The prototypical example is the Euclidean–invariant vortex filament flow studied by Hasimoto, [25, 36, 37]. The curvature and torsion invariants of the evolving filament satisfy an integrable dynamical system, which can be mapped to the completely integrable nonlinear Schrödinger equation, [1]. This led Lamb, [35], to draw attention to the surprisingly common, but still poorly understood connection between invariant curve flows and integrable soliton dynamics; since then, many other examples have been found, [5, 12, 13, 16, 19, 24, 29, 41, 42, 43, 45, 58, 60]. By "integrable", we shall mean that the evolution equation possesses a recursion operator, [47], inducing an infinite hierarchy of higher order symmetries. However, not all induced differential invariant evolutions are integrable, and, at present, we do not understand the general conditions on the group action and invariant curve flow needed to guarantee integrability.

When $p = 1$, there is only one independent invariant horizontal one-form

$$\varpi = \omega + \eta = ds + \eta, \tag{56}$$

whose horizontal component $\omega = ds$ can be identified with the G-invariant arc length element. Invariance requires that the Lie derivative $\mathbf{V}(\omega)$ vanishes on the submanifold, which (because Lie derivatives preserve the contact ideal) implies the following:

Lemma 10 *The curve flow induced by*

$$\mathbf{V} = I\,\mathbf{t} + \sum_{\alpha=1}^{q} J^\alpha\,\mathbf{n}_\alpha, \qquad \text{where} \qquad I = \langle\, \mathbf{V}\,; \varpi\,\rangle, \qquad J^\alpha = \langle\, \mathbf{V}\,; \vartheta^\alpha\,\rangle, \tag{57}$$

preserves arc length if and only if the Lie derivative $\mathbf{V}(\varpi)$ *is a contact form.*

Submanifolds of dimension $p \geq 2$ do not have distinguished parametrizations to play the role of the arc length parameter; this is because the invariant horizontal forms are almost never exact on the submanifold. On the other hand, the Lie derivative condition can be straightforwardly mimicked.

Definition 11 The invariant submanifold flow induced by $\mathbf{V}$ is called *intrinsic* if $\mathbf{V}(\varpi^i) \equiv 0$ for all $i = 1, \ldots, p$.

Lemma 12 *If the vector field* $\mathbf{V}$ *defines an intrinsic flow, then it commutes with the invariant differentiations:* $\left[\, \mathbf{V}, \mathcal{D}_i \,\right] = 0$ *for* $i = 1, \ldots, p$. *This holds if and only if*

$$\mathcal{D}_j I^i + \sum_{k=1}^{p} Y^i_{jk} I^k + \sum_{\alpha=1}^{q} \mathcal{B}^i_{j\alpha}(J^\alpha) = 0. \tag{58}$$

In particular, for curve flows generated by (57), the condition (58) guaranteeing arc length preservation reduces to

$$\mathcal{D} I = -\,\mathcal{B}(J) = -\sum_{\alpha=1}^{q} \mathcal{B}_\alpha(J^\alpha), \tag{59}$$

where $\mathcal{D}$ is the arc length derivative, while $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_q)$ is the invariant *Hamiltonian operator*, defined by (30).

Example 13 For the Euclidean group action on plane curves, in view of (30), the condition that a curve flow generated by the vector field $\mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$ be intrinsic is that

$$\mathcal{D} I = \kappa\, J. \tag{60}$$

Most of the curve flows listed in Example 9 have *non-local* intrinsic counterparts owing to the non-invertibility of the arc length derivative operator on $\kappa\, J$. An exception is the modified Korteweg-deVries flow, where $J = \kappa_s$, and so $I = \frac{1}{2}\kappa^2$. In general, the normal flow induced by $\mathbf{V}_N = J\,\mathbf{n}$ has a local intrinsic version if and only if $\mathcal{E}(\kappa\, J) = 0$, where $\mathcal{E}$ is the invariantized Euler–Lagrange operator, [32].

The next result prescribes the evolution of differential invariants under general intrinsic and normal invariant submanifold flows. See [54] for the proof.

Theorem 14 *Let* K *be any differential invariant. If the submanifold flow* (53) *is intrinsic, then*

$$\frac{\partial K}{\partial t} = \mathbf{V}(K) = \mathcal{A}_K(J) + \sum_{i=1}^{p} I^i\, \mathcal{D}_i K. \tag{61}$$

If the submanifold flow (53) *is normal, then*

$$\frac{\partial K}{\partial t} = \mathbf{V}(K) = \mathcal{A}_K(J). \tag{62}$$

Example 15 For any of the Euclidean invariant normal plane curve flows $C_t = J\,\mathbf{n}$ listed in Example 9, we have, according to Example 4,

$$\begin{gathered} \frac{\partial \kappa}{\partial t} = (\mathcal{D}^2 + \kappa^2)\, J, \qquad \frac{\partial \kappa_s}{\partial t} = (\mathcal{D}^3 + \kappa^2 \mathcal{D} + 3\,\kappa\,\kappa_s)\, J, \\ \frac{\partial \kappa_{ss}}{\partial t} = (\mathcal{D}^4 + \kappa^2 \mathcal{D}^2 + 5\,\kappa\,\kappa_s \mathcal{D} + 4\,\kappa\,\kappa_{ss} + 3\,\kappa_s^2)\, J. \end{gathered} \tag{63}$$

For instance, for the grassfire flow $J = 1$, and so

$$\frac{\partial \kappa}{\partial t} = \kappa^2, \qquad \frac{\partial \kappa_s}{\partial t} = 3\,\kappa\,\kappa_s, \qquad \frac{\partial \kappa_{ss}}{\partial t} = 4\,\kappa\,\kappa_{ss} + 3\,\kappa_s^2. \tag{64}$$

The first equation immediately implies finite time blow-up at a caustic for a convex initial curve segment, where $\kappa > 0$. For the curve shortening flow, $J = \kappa$, and

$$\frac{\partial \kappa}{\partial t} = \kappa_{ss} + \kappa^3, \qquad \frac{\partial \kappa_s}{\partial t} = \kappa_{sss} + 4\,\kappa^2 \kappa_s, \qquad \frac{\partial \kappa_{ss}}{\partial t} = \kappa_{ssss} + 5\,\kappa^2\,\kappa_{ss} + 8\,\kappa\,\kappa_s^2, \tag{65}$$

thereby recovering formulas used in Gage and Hamilton's analysis, [18]; see also Mikula and Ševčovič, [44]. Finally, for the mKdV flow, $J = \kappa_s$,

$$\begin{gathered} \frac{\partial \kappa}{\partial t} = \kappa_{sss} + \kappa^2 \kappa_s, \qquad \frac{\partial \kappa_s}{\partial t} = \kappa_{ssss} + \kappa^2 \kappa_{ss} + 3\,\kappa\,\kappa_s^2, \\ \frac{\partial \kappa_{ss}}{\partial t} = \kappa_{sssss} + \kappa^2\,\kappa_{sss} + 9\,\kappa\,\kappa_s \kappa_{ss} + 3\,\kappa_s^3. \end{gathered} \tag{66}$$

Warning: Normal flows *do not preserve arc length*, and so the arc length parameter s will vary in time. Or, to phrase it another way, time differentiation ∂_t and arc length differentiation $\mathcal{D} = D_s$ *do not commute* — as can easily be seen in the preceding examples. Thus, one must be very careful *not* to interpret the resulting evolutions (64–66) as partial differential equations in the usual sense. Rather, one should regard the differential invariants $\kappa, \kappa_s, \kappa_{ss}, \ldots$ as satisfying an infinite dimensional dynamical system of coupled ordinary differential equations.

Turning our attention to the intrinsic, arc length preserving curve flow, the complication alluded to in the preceding paragraph does not arise because, by Lemma 12, time differentiation now commutes with arc length differentiation. Substituting (59) in the formula (61):

Theorem 16 *Under an arc-length preserving flow,*

$$\kappa_t = \mathcal{R}_\kappa(J) \qquad \text{where} \qquad \mathcal{R}_\kappa = \mathcal{A}_\kappa - \kappa_s\,\mathcal{D}^{-1}\mathcal{B} \tag{67}$$

is the characteristic operator *associated with κ. More generally, the time evolution of $\kappa_n = \mathcal{D}^n \kappa$ is given by arc length differentiation: $\partial \kappa_n / \partial t = \mathcal{D}^n \mathcal{R}_\kappa(J)$.*

In this case arc length is preserved, and hence the arc length and time derivatives commute. Thus, unlike (62), the arc-length preserving flow (67) is of a more usual analytical form. However, there is a complication in that the term

$$\kappa_s\,\mathcal{D}^{-1}\mathcal{B}(J) = \kappa_s \int \mathcal{B}(J)\,ds \tag{68}$$

may very well be *nonlocal*, and so (67) is, in general, an integro-differential equation. Note that any integration constant appearing in (68) just adds in a multiple of κ_s, which represents the arc length preserving tangential flow $\kappa_t = \kappa_s$ that just serves to translate the arc length parameter: $s \mapsto s+c$ and so can be effectively ignored. Also, on a closed curve, the integral in (68) need not be periodic in s, and so one may not be able to continuously assign a uniquely determined evolution along the entire curve — although, by the preceding remarks, all such evolutions only differ by an overall translation, by an integer multiple of the total length of the curve, of the arc length parameter.

In certain situations, (67) turns out to be a well-known local integrable evolution equation, and the characteristic operator $\mathcal{R}$ is its recursion operator!

Example 17 In the case of Euclidean plane curves, the evolution of the curvature is given by

$$\kappa_t = \mathcal{R}_\kappa(J), \tag{69}$$

where

$$\mathcal{R}_\kappa = \mathcal{A}_\kappa - \kappa_s \mathcal{D}^{-1}\mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s \mathcal{D}^{-1}\cdot\kappa = D_s^2 + \kappa^2 + \kappa_s\, D_s^{-1}\cdot\kappa \tag{70}$$

is the modified Korteweg-deVries recursion operator, [49]. In particular, for the mKdV flow, $J = \kappa_s$, and (69) becomes

$$\kappa_t = \mathcal{R}_\kappa(\kappa_s) = \kappa_{sss} + \tfrac{3}{2}\kappa^2\kappa_s,$$

which is the modified Korteweg-deVries equation, and $\mathcal{R}$ is its recursion operator, [49]. On the other hand, for the grassfire flow, $J = 1$, and so

$$\kappa_t = \mathcal{R}_\kappa(1) = \kappa^2 + \kappa_s\, D_s^{-1}\kappa.$$

For the curve shortening flow, $J = \kappa$, and so

$$\kappa_t = \mathcal{R}_\kappa(\kappa) = \kappa_{ss} + \kappa^3 + \kappa_s\, D_s^{-1}\kappa^2.$$

Finally, for the thermal grooving flow, $J = \kappa_{ss}$ and so

$$\kappa_t = \mathcal{R}_\kappa(\kappa_{ss}) = \kappa_{ssss} + \kappa^2\kappa_{ss} + \kappa_s\, D_s^{-1}\kappa\,\kappa_{ss}.$$

As noted above, the induced curvature flow (69) is local if and only if $\mathcal{E}(\kappa\, J) = 0$, where $\mathcal{E}$ is the invariantized Euler operator or variational derivative, [49]. Clearly not all these local curvature flows will be integrable.

Example 18 As another example, consider the action

$$(x,u) \longmapsto (\alpha\, x + \beta\, u + a, \gamma\, x + \delta\, u + b), \qquad \alpha\,\delta - \beta\,\gamma = 1, \tag{71}$$

of the equi-affine group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$ on plane curves $C \subset \mathbb{R}^2$. Applications to computer vision can be found, for instance, in [4, 10, 56, 61]. According to [17, 23, 32], the classical equi-affine moving frame arises from the choice of coordinate cross-section $x = u = u_x = 0,\ u_{xx} = 1,\ u_{xxx} = 0$. The fundamental differential invariant is the equi-affine curvature

$$\kappa = \iota(u_{xxxx}) = \frac{u_{xx}u_{xxxx} - \frac{5}{3}\, u_{xxx}^2}{u_{xx}^{8/3}}\,. \tag{72}$$

All higher order differential invariants are obtained by invariant differentiation with respect to the invariant arc length form

$$\varpi = \iota(dx) = \omega + \eta, \qquad \text{where} \qquad \omega = ds = u_{xx}^{1/3}\, dx, \qquad \eta = \frac{u_{xxx}}{3\, u_{xx}^{5/3}}\,\theta, \tag{73}$$

with dual invariant differential operator $\mathcal{D} = u_{xx}^{-1/3}\, D_x$ being the equi-affine arc length derivative. Applying our computational algorithm, but suppressing the details, we obtain

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}_\kappa(\vartheta), \qquad\qquad d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta)\wedge\varpi,$$

where

$$\mathcal{A}_\kappa = \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{5}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2, \qquad\qquad \mathcal{B} = \tfrac{1}{3}\,\mathcal{D}^2 - \tfrac{2}{9}\,\kappa.$$

The characteristic operator is

$$\mathcal{R}_\kappa = \mathcal{A}_\kappa - \kappa_s \mathcal{D}^{-1}\mathcal{B} = \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{4}{3}\,\kappa_s \mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2 + \tfrac{2}{9}\,\kappa_s\, D_s^{-1} \cdot \kappa. \tag{74}$$

As in the Euclidean action, both the Eulerian and Hamiltonian operators are invariantly self-adjoint: $\mathcal{A} = \mathcal{A}^*$ and $\mathcal{B} = \mathcal{B}^*$. Therefore, the Euler-Lagrange equation for an equi-affine invariant Lagrangian $\widetilde{L}(\kappa, \kappa_s, \ldots)\, ds$ takes the invariant form (48), namely,

$$\left(\mathcal{D}^4 + \tfrac{5}{3}\,\kappa\mathcal{D}^2 + \tfrac{5}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2\right)\mathcal{E}(\widetilde{L}) - \left(\tfrac{1}{3}\,\mathcal{D}^2 - \tfrac{2}{9}\,\kappa\right)\mathcal{H}(\widetilde{L}) = 0.$$

The equi-affine arc-length functional $\int ds$ with $\widetilde{L} = 1$ has $\mathcal{E}(\widetilde{L}) = 0$, $\mathcal{H}(\widetilde{L}) = -1$, and hence the Euler-Lagrange equation is

$$\mathcal{A}^*(0) - \mathcal{B}^*(-1) = -\tfrac{2}{9}\,\kappa = 0.$$

We conclude that the minimal equi-affine curves are those with zero equi-affine curvature — the conic sections. As another example, the variational problem $\int \kappa\, ds$ has Euler-Lagrange equation

$$\mathcal{A}^*(1) - \mathcal{B}^*(-\kappa) = \tfrac{2}{3}\kappa_{ss} + \tfrac{2}{9}\kappa^2 = 0,$$

the solution to which, [31], gives κ as an elliptic function of s.

A general equi-affine invariant curve flow takes the form

$$C_t = I\,\mathbf{t} + J\,\mathbf{n}, \tag{75}$$

where $\mathbf{t}, \mathbf{n}$ are, respectively, the equi-affine tangent and normal directions, [23]. The equi-affine curve shortening flow, [4, 62], is the normal flow with $I = 0$, $J = 1$. Under this flow, the equi-affine curvature and its derivative evolves according to

$$\begin{aligned} \frac{\partial\kappa}{\partial t} &= \mathcal{A}_\kappa(1) = \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2, \\ \frac{\partial\kappa_s}{\partial t} &= \mathcal{A}_{\kappa_s}(1) = \mathcal{D}\,\mathcal{A}_\kappa(1) - \kappa_s\mathcal{B}(1) = \tfrac{1}{3}\,\kappa_{sss} + \tfrac{10}{9}\,\kappa\,\kappa_s. \end{aligned} \tag{76}$$

A second example is the intrinsic (arc-length preserving) flow with $J = \kappa_s$. In this case, the curvature evolution arises from the characteristic operator:

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{5s} + \tfrac{5}{3}\,\kappa\,\kappa_{sss} + \tfrac{5}{3}\,\kappa_s\kappa_{ss} + \tfrac{5}{9}\,\kappa^2\kappa_s,$$

which is the integrable Sawada–Kotera equation, [64]. In this case, the characteristic operator $\mathcal{R}$ is closely related to, but *not* the same as the Sawada–Kotera recursion operator, which is given by the following formula, [12]:

$$\widehat{\mathcal{R}} = \mathcal{R} \cdot (\mathcal{D}^2 + \tfrac{1}{3}\,\kappa + \tfrac{1}{3}\,\kappa_s\mathcal{D}^{-1}). \tag{77}$$

Example 19 In the case of space curves $C \subset \mathbb{R}^3$, under the Euclidean group $G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$, there are two generating differential invariants, the curvature κ and torsion τ. According to [32], the relevant moving frame formulae are

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}_\kappa(\vartheta), \qquad d_{\mathcal{V}}\,\tau = \mathcal{A}_\tau(\vartheta), \qquad d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi,$$

where $\vartheta = (\vartheta_1, \vartheta_2)^T$ is the column vector containing the order 0 invariant contact forms, while the characteristic and Hamiltonian operators are:

$$\mathcal{A}_\kappa = \left(D_s^2 + (\kappa^2 - \tau^2),\ -2\tau D_s - \tau_s \right),$$
$$\mathcal{A}_\tau = \left(\frac{2\tau}{\kappa} D_s^2 + \frac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2} D_s + \frac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2}, \right. \qquad \mathcal{B} = \left(-\kappa,\ 0 \right).$$
$$\left. \frac{1}{\kappa} D_s^3 - \frac{\kappa_s}{\kappa^2} D_s^2 + \frac{\kappa^2 - \tau^2}{\kappa} D_s + \frac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \right),$$

Thus, under an intrinsic flow with normal component $\mathbf{V}_N = J\,\mathbf{n}_1 + K\,\mathbf{n}_2$, the curvature and torsion evolve via

$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} J \\ K \end{pmatrix}, \qquad \text{where} \qquad \mathcal{R} = \begin{pmatrix} \mathcal{R}_\kappa \\ \mathcal{R}_\tau \end{pmatrix} = \begin{pmatrix} \mathcal{A}_\kappa \\ \mathcal{A}_\tau \end{pmatrix} - \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix} \mathcal{D}^{-1} \mathcal{B}$$

is the recursion operator for the integrable vortex filament flow, with $J = 0$, $K = \kappa$. This flow can be mapped to the nonlinear Schrödinger equation via the Hasimoto transformation, [25, 37].

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