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EQUIVALENCE OF HIGHER ORDER LAGRANGIANS I. FORMULATION AND REDUCTION

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ABSTRACT. — It is shown that the general equivalence problem for an r-th order variational problem (with or without the addition of a divergence) can always be formulated as a Cartan equivalence problem on the jet bundle J^r. Moreover, equivalence on any higher order jet bundle automatically reduces to equivalence on J^r. As a consequence, we deduce the existence of "derivative covariants", which are certain functions of the partial derivatives of a suitably nondegenerate r-th order Lagrangian whose transformation rules are the same as those of the n-th order derivatives for any $n > r$. This implies that any such Lagrangian determines an invariantly defined system of n-th order differential equations for any $n > r$, generalizing the Euler-Lagrange equations.

1. Introduction

The most basic equivalence problem in the calculus of variations is to determine when two variational problems can be transformed into each other by a suitable change of variables. The solution to this problem would have many important consequences, and significant applications in a wide range of physical problems where variational methods are of importance. Elie Cartan, *cf.* [4], developed a powerful method which produces necessary and sufficient conditions for the solution of such equivalence problems, and, in [5], [6], began direct investigations into specific equivalence problems from the calculus of variations. Subsequent research ([2], [3], [7], [8], [11], [12]), has almost exclusively concentrated on first order Lagrangians. With this paper, we initiate a series of papers on various aspects of the equivalence problem for higher order Lagrangians, with particular emphasis on novel phenomena that do not appear in the first order case. Applications to the determination of Cartan forms, [13], and new invariant differential equations beyond the classical Euler-Lagrange equations, [20], will be among the immediate results of this investigation.

The first part of this paper is concerned with the formulation of the various equivalence problems for r-th order Lagrangians, both with and without the addition of divergence

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terms. Clearly, for a nondegenerate r-th order Lagrangian, the jet bundle J^r is the minimal order jet bundle for which we can reasonably expect a Cartan formulation of these equivalence problems. However, there have been suggestions that, since the Euler-Lagrange equation for a regular r-th order Lagrangian is a $2r$-th order system of differential equations, and, also since the Cartan form proposed by Shadwick, [17], naturally lives on the jet bundle J^{2r-1}, the Cartan equivalence problem for regular r-th order Lagrangians should itself be properly cast on the jet bundle J^{2r-1}. We begin by addressing this general "reduction problem", and prove that this is not actually the case; although one can formulate the equivalence problem for an r-th order Lagrangian on J^{r+k} for any $k \geqq 0$, these equivalence problems all have the same solution, and one might as well begin by reducing the equivalence problem to the minimal order jet bundle, viz. J^r. We then show how to generalize the results of [12] for first order Lagrangians so as to formulate the Lagrangian equivalence problems as Cartan equivalence problems on the jet bundle J^r. The standard equivalence problem is fairly straightforward, but the divergence equivalence problem requires a much more complicated construction and structure group.

Application of the Cartan construction to the equivalence problem on J^r will lead to a complete set of r-th order invariants which provide the necessary and sufficient conditions for equivalence. Consequently, any $(r+k)$-th order invariant, $k>0$, (e.g. the Euler-Lagrange equations), or invariant differential form on J^{r+k}, $k>0$, (e.g. the Cartan form) must be expressible in terms of the fundamental r-th order invariants of the given Lagrangian. Indeed, as we will show, provided the equivalence problem finally reduces to an $\{e\}$-structure on the base (even after perhaps requiring a prolongation), it is possible to construct a series of "derivative covariants", which will be certain r-th order functions constructed from the various partial derivatives of the Lagrangian, whose transformation rules are identical with those of the partial derivatives of the dependent variables of order n for any $n>r$. Consequently, given any invariant, invariant equation (e.g. the Euler-Lagrange equations), or invariant form (e.g. the Cartan form) which depends on derivatives of the dependent variables of order greater than r, one can replace all such derivatives by the associated derivative covariants so as to produce a corresponding r-th order invariant of the same type, which has precisely the same transformation rules as the original invariant object. Thus, invariants produced from the Cartan method for the Lagrangian equivalence problem formulated on a higher order jet bundle will all have corresponding r-th order counterparts arising from the J^r equivalence problem. We argue that it is in this way that the higher order invariant quantities such as the Euler-Lagrange equations and Cartan form associated with an r-th order Lagrangian are "hiding" in the solution to the Cartan equivalence problem over J^r.

Moreover, for any $n>r$, the n-th order derivative covariants will naturally lead to invariantly defined n-th order systems of differential equations associated with an r-th order Lagrangian, of which the $2r$-th order Euler-Lagrange system of equations and its covariant derivatives are but some of the examples. As a particularly striking example, in a subsequent paper in this series, [20], we shall exhibit an invariant third order ordinary differential equation associated with any second order particle Lagrangian (whose Euler-Lagrange equation is of order 4)! This stands in contrast to a commonly accepted "folk theorem" that the Euler-Lagrange equation is the *only* invariant differential equation which can be associated with a variation problem. However, as shown by Anderson [1], the Euler-Lagrange equation *is* the only invariant differential equation which is a linear function of the Lagrangian, so these new expressions must be nonlinear combinations of the Lagrangian and its derivatives.

2. Contact transformations and contact forms

In this section, we introduce the basic notation. As the equivalence calculations are purely local, it suffices to work in Euclidean space throughout. Let $x \in X = \mathbb{R}^p$ denote the independent variables, and $u \in U = \mathbb{R}^q$ the dependent variables, so that functions $u = f(x)$ can be viewed as sections of the trivial bundle $Z = X \times U$. (As our considerations are primarily local throughout, it suffices to work on a trivial bundle for simplicity, although these constructions can readily be put into global form, in which Z would be a more general vector bundle over the base manifold X.) We use the notation $u_J^\alpha = \partial_J u^\alpha$, where $\alpha = 1, \ldots, q$, and $J = (j_1, j_2, \ldots, j_m)$ is an *unordered* multi-index with $1 \leqq j_\nu \leqq p$, for the partial derivatives of u of order $m = \# J$. We let (J, i) denote the multi-index $(j_1, j_2, \ldots, j_m, i)$, so that $u_{J,i}^\alpha = \partial_i u_J^\alpha$. Further let $u_m = (u_J^\alpha)$, $\alpha = 1, \ldots, q$, $\# J = m$, denote all the m-th order derivatives, which serve as coordinates on the space

$$U_m \cong U \otimes \odot^m X^* \cong U_{m-1} \otimes X^*; \tag{2.1}$$

here $\odot^m X^*$ denotes the m-th symmetric power of the dual space to X. Further, let $u^{(r)} = (u, u_1, \ldots, u_r)$ denote all the derivatives up to order r. The variables $(x, u^{(r)})$ are coordinates on the jet space

$$J^r = J^r Z = X \times U^{(r)} = X \times (U \times U_1 \times \ldots \times U_r). \tag{2.2}$$

A function $f : J^r \to \mathbb{R}$ will be called a r-th order function; in general the *order* of a function (or differential form) will indicate the highest order derivative of u upon which it depends.

There are various changes of variables which are to be considered, of which the fiber-preserving, point, and contact transformations (*cf.* [11]) are the most important. Throughout this paper, we will use the index κ to refer to the class of transformations allowed, with $\kappa = 0$ corresponding to fiber-preserving transformations, $\kappa = 1$ corresponding to point transformations, and $\kappa = 2$ corresponding to contact transformations. According to Bäcklund's theorem ([9], p. 202]) contact transformations only generalize point transformations when $q = 1$, and we shall accordingly reserve the index $\kappa = 2$ for these cases. By a κ-*map* $\Psi : J^r \to J^r$ we mean the r-th prolongation of a fiber-preserving, point or contact transformation according to the value of κ. In particular, on the base bundle $J^0 = Z = X \times U$ our transformations all take the form

$$\bar{x} = \varphi(x, u^{(1)}), \qquad \bar{u} = \psi(x, u^{(1)}). \tag{2.3}$$

The functions φ, ψ actually depend on first order derivatives of u only in the case of contact transformations ($\kappa=2$); for point transformations ($\kappa=1$), φ, ψ depend on just (x, u), while for fiber-preserving transformations ($\kappa=0$), φ is additionally restricted to just depend on x alone. A reason for our convention on the index κ is the fact that, for a general κ-map, the Jacobian matrix $D\varphi=(D_i\varphi^j)$ of total derivatives of the base transformation, which we can regard as a linear map on X, has order κ:

$$\operatorname{order} D\varphi=\kappa. \tag{2.4}$$

Given a κ-map Ψ, let

$$\bar{u}_m=\psi_m(x, u^{(m)}) \tag{2.5}$$

denote the coordinate transformation on the m-th order derivatives, and

$$\bar{u}^{(r)}=\psi^{(r)}(x, u^{(r)})=(\psi_0(x, u), \ldots, \psi_r(x, u^{(r)})) \tag{2.6}$$

the complete fiber transformation on J^r. The functions ψ_m can be constructed inductively using the chain rule formula

$$\begin{aligned}\bar{u}_m&=\psi_m(x, u^{(m)})\\ &=(1\otimes D\varphi^{-T})D\psi_{m-1}\\ &=A_m u_m+(1\otimes D\varphi^{-T})\tilde{D}\psi_{m-1}.\end{aligned} \tag{2.7}$$

Here 1 denotes the identity map on U_{m-1}, and $D\varphi^{-T}$ is the inverse transpose of the Jacobian matrix (2.4), which we are regarding as a linear map on X^*, so that $1\otimes D\varphi^{-T}$ determines a linear map on $U_{m-1}\otimes X^*=U_m$. Further, $D\psi_{m-1}$ denotes the matrix of total derivatives $(D_i\psi^\alpha_J)$ of ψ_{m-1}, regarded as an element of U_m in the obvious manner, while the $(m-1)$-st order matrix $\tilde{D}\psi_{m-1}$ consists of total derivatives of the entries of ψ_{m-1} restricted to J^{m-1}, *i.e.* without the leading order terms which depend on u_m. Finally, the "matrix" $A_m\in GL(U_m)$, which, for $m\geqq\kappa+1$, gives the complete dependence of $\bar{u}_m$ on the highest order derivatives u_m, is given explicitly by the formula

$$A_m=A_0\otimes\odot^m(D\varphi^{-T}), \tag{2.8}$$

where

$$A_0=\partial_u\psi-\bar{u}_1\,\partial_u\varphi=\partial_u\psi-D\psi\,.\,D\varphi^{-1}\,.\,\partial_u\varphi, \tag{2.9}$$

is a κ-th order function, with $\partial_u\psi$ referring to the Jacobian matrix of ψ with respect to the dependent variables u.

The *contact ideal* on J^r, $r\geqq 1$, denoted $\mathscr{I}^{(r)}$, is generated by the contact forms

$$\theta^\alpha_J=du^\alpha_J-\sum_{i=1}^p u^\alpha_{J,i}\,dx^i,\qquad \alpha=1, \ldots, q,\quad 0\leqq \# J<r. \tag{2.10}$$

DEFINITION 2. — Two r-th order Lagrangians L and $\bar{\mathrm{L}}$ are said to be *$(r+k)$-standard equivalent*, $k \geqq 0$, if and only if there is a κ-map $\Psi: \mathrm{J}^{r+k} \to \mathrm{J}^{r+k}$ such that

$$\Psi^* \{\bar{\mathrm{L}}\, d\bar{x}\} \equiv \mathrm{L}\, dx \qquad \mathrm{mod}\, \mathcal{I}^{(r+k)}. \tag{3.2}$$

An easy result from the form of the transformations (2.3) is that this equivalence immediately reduces to J^r, except in the special case of first order Lagrangians under contact transformations, *i.e.* the case $r=1$, $\kappa=2$. This case is anomalous, since the transformation (2.3) will necessarily reduce to a point transformation ([7], [11]).

PROPOSITION 3. — *Suppose $k \geqq 0$ and either $r>1$, or $\kappa \neq 2$. If two Lagrangians* L *and* $\bar{\mathrm{L}}$ *are $(r+k)$-standard equivalent, then they are r-standard equivalent, and satisfy the change of variables formula*

$$\Psi^* \{\bar{\mathrm{L}}(\bar{x}, \bar{u}^{(r)})\} . \det \mathrm{D}\varphi = \mathrm{L}(x, u^{(r)}). \tag{3.3}$$

Proof. — Note first that by Bäcklund's Theorem ([9], p. 202]) since Ψ preserves the contact ideal, its restriction φ to the base space X depends on at most first order derivatives of the u's, *cf.* (2.3). Thus, according to (2.4), the pull-back $\Psi^*(d\bar{x}^i)$ can depend on contact forms of order at most κ. Therefore, upon expanding the left hand side of (3.2), we find

$$\left\{\begin{aligned} &\Psi^* \{\bar{\mathrm{L}}\, d\bar{x}\} = \bar{\mathrm{L}} . \det \mathrm{D}\varphi\, dx, && \kappa=0,\\ &\Psi^* \{\bar{\mathrm{L}}\, d\bar{x}\} = \bar{\mathrm{L}} . \det \mathrm{D}\varphi\, dx + \sum_\alpha \theta^\alpha \wedge \xi^\alpha, && \kappa=1,\\ &\Psi^* \{\bar{\mathrm{L}}\, d\bar{x}\} = \bar{\mathrm{L}} . \det \mathrm{D}\varphi\, dx + \sum_\alpha \left(\theta^\alpha \wedge \xi^\alpha + \sum_{j=1}^{p} \theta^\alpha_j \wedge \xi^\alpha_j\right), && \kappa=2,\end{aligned}\right. \tag{3.4}$$

for certain $(p-1)$-forms ξ^α, ξ^α_j. This demonstrates that the only possible case in which derivatives of u of order strictly greater than r can occur in the pull-back $\Psi^*(\bar{\mathrm{L}}\, d\bar{x})$ is the exceptional case $r=1$, $\kappa=2$. Apart from this case, we can clearly take $k=0$ in (3.2).

We now turn to the divergence equivalence problem.

DEFINITION 4. — Two r-th order Lagrangians are said to be *$(r+k)$-divergence equivalent*, $k \geqq 0$, if and only if there is a κ-map $\Psi: \mathrm{J}^{r+k} \to \mathrm{J}^{r+k}$, and a $(p-1)$-form Ω on J^{r+k} such that

$$\Psi^* \{\bar{\mathrm{L}}\, d\bar{x}\} \equiv \mathrm{L}\, dx + d\Omega \qquad \mathrm{mod}\, \mathcal{I}^{(r+k+1)}. \tag{3.5}$$

The reason we take the $(r+k+1)$-st order contact ideal is because the total divergence of a p-tuple of functions of order $r+k$ is, in general, of order $r+k+1$. An alternative approach is to replace the contact ideal $\mathcal{I}^{(r+k)}$ by its differential closure, noting that this is contained in $\mathcal{I}^{(r+k+1)}$ (*cf.* [12]).

PROPOSITION 5. — *Two r-th order Lagrangians* L *and* $\bar{\mathrm{L}}$ *are $(r+k)$-divergence equivalent for any κ, r, k if and only if*

$$\Psi^* \{\bar{\mathrm{L}}(\bar{x}, \bar{u}^{(r)})\} . \det \mathrm{D}\varphi = \mathrm{L}(x, u^{(r)}) + \mathrm{Div}\, \mathrm{F}, \tag{3.6}$$

where $F(x, u^{(r+k)})$ *is a p-tuple of* $(r+k)$-th *order functions.*

Proof. — By the same argument as in the previous proof, we obtain the expressions (3.4) for the pull back of the Lagrangian form. Also, we can write the $(p-1)$-form Ω in (3.5) as

$$\Omega = \sum_{i=1}^{p} F_i \, dx^{\hat{\imath}} \qquad \operatorname{mod} \mathscr{I}^{(r+k)},$$

where the F_i are functions on J^{n+r}, and $dx^{\hat{\imath}} = \partial_i \lrcorner\, dx$. Then

$$\begin{aligned} d\Omega &= \operatorname{Div} F \, dx \qquad \operatorname{mod} \mathscr{I}^{(r+k+1)}, \\ &= \operatorname{Div} F \, dx + \sum_{\# J \leq r+k} \sum_{\alpha} \theta_J^\alpha \wedge \zeta_J^\alpha, \end{aligned}$$

for certain $(p-1)$-forms ζ_J^α. Plugging this expression and (3.4) into (3.5), we conclude that the two Lagrangians are related by formula (3.6), proving the proposition.

We now proceed to analyze the change of variables formula (3.6) closer. The first remark is that, since the two Lagrangians are assumed to be of order r, the divergence term $\operatorname{Div} F$ is restricted so as to preserve this order. According to (2.4), except in the exceptional case $r=1$, $\kappa=2$, both of the terms L and $\Psi^* \bar{L} . \det D\varphi$ in (3.6) depend on at most r-th order derivatives of u, hence $\operatorname{Div} F$ must be an r-th order function. The case $r=1$, $\kappa=2$ is special, since, for instance, for $p=q=1$ all Lagrangians are divergence equivalent under contact transformations [11]. One further remark: if L is actually of order $r' < r$, but the divergence term $\operatorname{Div} F$ in (3.6) depends on r-th order derivatives of u, then the resulting r-th order Lagrangian $\bar{L}$ will be degenerate, and thus not of great intrinsic interest. Indeed, the Cartan method will inevitably exclude degenerate Lagrangians, as they can themselves always be reduced to a lower order jet bundle by adding in a suitable divergence. The most interesting case, then, is when both L and $\bar{L}$ are nondegenerate r-th order Lagrangians.

Now, the key point here is that the requirement that $\operatorname{Div} F$ be an r-th order function does *not* necessarily imply that F itself has order r. An elementary counter-example is provided by the third order 2-tuple $F = (u\, u_{xyy},\; u_y u_{xx} - u\, u_{xxy} - u_x u_{xy})$, which has second order divergence:

$$\operatorname{Div} F = D_x(u\, u_{xyy}) + D_y(u_y u_{xx} - u\, u_{xxy} - u_x u_{xy}) = u_{xx} u_{yy} - u_{xy}^2.$$

However, a fundamental result, proved in [14], states that if $\operatorname{Div} F$ is of order r, then one can always find some r-th order p-tuple having the same divergence.

LEMMA 6. — *Let r and k be positive integers. Suppose* $F(x, u^{(k)})$ *is a p-tuple of k-th order functions with the property that its total divergence* $\operatorname{Div} F = A(x, u^{(r)})$ *is of order r. Then there is an "equivalent" r-th order p-tuple* $\tilde{F}(x, u^{(r)})$ *having the same divergence:* $A = \operatorname{Div} \tilde{F} = \operatorname{Div} F$,

For instance, in the above example, we can take $\tilde{F} = (u_x u_{yy},\; -u_x u_{xy})$ as our equivalent second order 2-tuple, although this is not the only possibility. Thus we can replace F

Note that the order of θ_J^α is one more than the order of J, which is the reason for the strict inequality in (2.10). It will be convenient to assemble these forms into a column vectors $\theta_m = (\theta_J^\alpha)$, $\alpha = 1, \ldots, q$, $\# J = m$, of $(m+1)$-st order contact forms, which we can view as lying in the tensor product space $T^* J^{m+1} \otimes U_m$, the second factor indexing the various entries of θ_m. Similarly let $\theta^{(r)} = (\theta_0, \ldots, \theta_{r-1})$ be the complete collection of contact forms generating the ideal $\mathcal{I}^{(r)}$, which we regard as an element of $T^* J^r \otimes U^{(r-1)}$. We note here the fundamental result that any κ-map preserves the contact ideal.

LEMMA 1. — *Let* $\Psi : J^r \to J^r$ *be any* κ*-map. Then*

$$\Psi^* \theta^{(r)} = A^{(r)} . \theta^{(r)},$$

where the matrix $A^{(r)} \in GL(U^{(r)})$ is an invertible linear transformation, which is block lower triangular with respect to the Cartesian product decomposition (2.2), and the diagonal blocks $A_m : U_m \to U_m$, $m \leqq r$, are given explicitly by the formula (2.8), (2.9). (There are also many structural relations among the various off-diagonal blocks of $A^{(r)}$ but we wll not require these here.)

This is a consequence of the elementary fact that the transformation rules for contact forms and for derivatives are essentially the same. Indeed, we have the basic formula

$$\Psi^* (du_m) = A_m \, du_m + \tilde{D}\psi_m \, dx \qquad \operatorname{mod} \mathcal{I}^{(m)}, \tag{2.11}$$

which is a consequence of the chain rule formula (2.7).

3. Equivalence of Lagrangians

We begin by setting up the basic equivalence problem for a general variational problem

$$\mathcal{L}[u] = \int_\Omega L(x, u^{(r)}) \, dx, \tag{3.1}$$

whose Lagrangian L is a smooth function of order r. We regard (3.1) as an *oriented integral*, so the integrand is a p-form $L\,dx = L\,dx^1 \wedge \ldots \wedge dx^p$.

In formulating the equivalence problem for such a variational integral, we must first choose the appropriate class of allowable changes of variables and the precise notion of equivalence. Throughout this section, the index $\kappa = 0, 1, 2$ will correspond to fiber-preserving, point or general contact transformations, as above. In the *standard equivalence problem* the two variational integrals are mapped directly to each other under the prolonged transformation. The *divergence equivalence problem* only requires that the two sets of Euler-Lagrange expressions are mapped into each other, which is the same as the requirement that the two Lagrangians differ by a divergence ([3], [11]). We begin our analysis with the standard equivalence problem, which is much easier to formulate.

in (3.6) by any equivalent r-th order divergence $\tilde{F}: J^r \to \mathbb{R}^p$, and still preserve the basic equivalence of the two Lagrangians. Putting the above remarks together, we have proved the following reduction theorem for the divergence equivalence problem.

THEOREM 7. – *Suppose $r > 1$, or $\kappa \neq 2$. Then two r-th order Lagrangians are $(r+k)$-divergence equivalent for some $k > 0$ if and only if they are r-divergence equivalent.*

This has still not quite reduced the divergence equivalence problem for r-th order Lagrangians back down to the jet bundle J^r, since (3.5) (with $k=0$) still requires equivalence modulo the contact ideal $\mathcal{I}^{(r+1)}$, which contains forms depending on $(r+1)$-st order derivatives. The full reduction to J^r is harder, and requires a more complete discussion of the Cartan formulation for this problem, which we present in the following section.

4. Cartan formulation of the Equivalence Problem

We now turn to the formulation of our basic equivalence problems in a form amenable to the Cartan equivalence method. First we indicate how to set up the standard equivalence problem on J^r. Assume for simplicity that we are on a domain where $L > 0$. (The points where the Lagrangian vanishes play a distinguished role in the standard equivalence problem, and we will avoid them. If $L < 0$, then we perform an orientation reversing transformation before beginning). We need to construct a suitable coframe on the base space, which in this case will be J^r, and then determine the appropriate structure group so as to encode the Lagrangian equivalence problem. We begin with the independent variables x, and define the column vector of one-forms $\omega \in T^* J^r \otimes X$, with entries

$$(4.1) \qquad \omega_i = \sqrt[p]{L}\, dx^i, \qquad i=1, \ldots, p.$$

Note that $\omega_1 \wedge \ldots \wedge \omega_p = L\, dx$ is just the Lagrangian p-form. The coframe elements corresponding to the coordinates $u^{(r-1)}$ will be rescaled contact forms, which we assemble into a column vector

$$\tilde{\theta} = (\theta_0, L^{-1/p}\theta_1, L^{-2/p}\theta_2, \ldots, L^{-(r-1)/p}\theta_{r-1})^T \in T^* J^r \otimes U^{(r-1)}.$$

(The scaling factors are introduced for later convenience.) To complete the elements $\tilde{\theta}$, ω to be a coframe on $J^r = X \times U^{(r)} = X \times U^{(r-1)} \times U_r$, we introduce the column vector of one-forms $\pi \in T^* J^r \otimes U_r$, with entries

$$(4.2) \qquad \pi^\alpha_J = L^{-r/p}\, du^\alpha_J, \qquad \alpha=1, \ldots, q, \quad \# J = r.$$

Finally let $\Omega = (\tilde{\theta}, \omega, \pi)^T \in T^* J^r \otimes J^r$ be the column vector representing the entire base coframe. Vectors and matrices will be represented in block form relative to the components $\tilde{\theta}$, ω, π of the full coframe Ω.

For the standard Lagrangian equivalence problem, the relevant structure group will be the matrix Lie group $G_\kappa = G_\kappa^{(r)}$, where $\kappa = 0, 1, 2$, corresponds to the class of transformations allowed, consisting of all block lower triangular matrices of the form

$$\begin{pmatrix} A & 0 & 0 \\ B & J & 0 \\ C & D.J & E \end{pmatrix}. \tag{4.3}$$

where, using the notation introduced in section 2,

(i) $J \in GL(X)$, $\det J = 1$,

(ii) $A \in GL(U^{(r-1)})$ is block lower triangular, with diagonal blocks $A_k = A_0 \otimes \odot^k J^{-T}$,

(iii) $B \in \mathrm{Hom}(U^{(r-1)}, X)$, where $B = (B_0, B_1, \ldots, B_{r-1})$, $B_j \in \mathrm{Hom}(U_j, X)$, and

$$B_j = 0 \qquad \text{for} \quad j \geqq \kappa,$$

(iv) $C \in \mathrm{Hom}(U^{(r-1)}, U_r)$

(v) $D \in \mathrm{Hom}(X, U_r)$, $D^\alpha_{J,i} = D^\alpha_{J',i'}$ whenever $(J, i) = (J', i')$,

(vi) $E = A_0 \otimes \odot^r J^{-T} \in GL(U_r)$.

Of these conditions, (i) will ensure that the Lagrangians match up exactly according to (3.2), (ii) are the contact conditions given in lemma 1, and (iii) reflects the form of the allowed transformations, *cf.* (3.4). Indeed, according to Bäcklund's Theorem, if we are dealing with contact transformations, we could even enlarge our structure group to allow all the entries B_j to be nonzero initially, since the ultimate form (2.3) of the base transformations will ultimately require us to reduce down to $B_j = 0$ for $j \geqq 2$. (However, see also the discussion in [13].) Condition (v) reflects the equality of mixed partial derivatives. [Here, recall that (J, i) represents an unordered multi-index, so the condition is that (J', i') is some permutation of (J, i).] Finally, (vi) reflects the chain rule formula for the r-th order derivatives given in (2.11).

THEOREM 8. — *Two Lagrangians* L *and* $\bar{L}$ *are r-standard equivalent if and only if there is a diffeomorphism* $\Psi : J^r \to J^r$ *which satisfies*

$$\Psi^*(\bar{\Omega}) = g.\Omega, \tag{4.4}$$

where g is a G_κ*-valued function on* J^r.

The proof is fairly straightforward, and is left to the reader. Note that we did not *a priori* assume that the map Ψ be a κ-map; this follows as a consequence of (4.4). An important remark is that, although we know that we can always reduce to J^r, we could equally well set up the $(r+k)$-standard equivalence problem, using the analogous base coframe on the higher order jet bundle J^{r+k} for any $k \geqq 0$ (with r replaced by $r+k$ in the definitions of θ, π), and the corresponding structure group $G_\kappa^{(r+k)}$. Proposition 3 implies that the solutions to these two equivalence problems must be the same. (Although, as we will see in [13], for a specific example, the explicit necessary and sufficient conditions for equivalence, while certainly isomorphic, may not have identical forms.)

The divergence equivalence problem is a bit more tricky to encode in Cartan form, since in proposition 4, we still have not entirely eliminated all the $(r+1)^{st}$ order derivatives of u. However, by mimicking the constructions presented in [12] for first order Lagrangians, we will still be able to formulate the problem as a Cartan equivalence problem on the jet bundle J^r. As discussed in [12], we will still assume that $L > 0$, as this can always be arranged by adding in a suitable divergence. (This restriction is a little discomforting, as it is not an intrinsic property of the problem. Robert Bryant (personal communication) has been able to give a formulation of the problem which avoids this condition in the case of first order Lagrangians (*see* [3] for the case $p=1$), but his constructions do not appear to generalize to higher order Lagrangians.) As in [12], we need to append some additional variables $w=(w_1, \ldots, w_p)$ to take care of the additional divergence term. These are essentially the coordinates on the space $W=\Lambda^{p-1}T^*X \cong \mathbb{R}^p$. The base space will now be $J^r \times W$, and we complete our earlier coframe on J^r by letting ν denote the column vector of one-forms

$$\nu_i = L^{(1/p)-1}\, dw_i, \qquad i=1, \ldots, p. \tag{4.5}$$

Finally let $\Xi = (\tilde{\theta}, \omega, \pi, \nu)^T$ be the column vector representing the base coframe.

The structure group for the divergence equivalence problem will be the matrix Lie group $H_\kappa = H_\kappa^{(r)}$, $\kappa = 0, 1, 2$, corresponding to the class of transformations allowed, consisting of all block lower triangular matrices of the form

$$\begin{pmatrix} A & 0 & 0 & 0 \\ B & J & 0 & 0 \\ C & D.J & E & 0 \\ Q & T.R & T.M & T \end{pmatrix}, \tag{4.6}$$

subject to the following conditions

(i') $J \in GL(X)$,

(ii) $A \in GL(U^{(r-1)})$ is block lower triangular, with diagonal blocks $A_k = A_0 \otimes \odot^k J^{-T}$,

(iii) $B \in \mathrm{Hom}(U^{(r-1)}, X)$, where $B=(B_0, B_1, \ldots, B_{r-1})$, $B_j \in \mathrm{Hom}(U_j, X)$, and

$$B_j = 0 \qquad \text{for} \quad j \geqq \kappa,$$

(iv) $C \in \mathrm{Hom}(U^{(r-1)}, U_r)$

(v) $D \in \mathrm{Hom}(X, U_r)$, $D^\alpha_{J,i} = D^\alpha_{J',i'}$ whenever $(J, i) = (J', i')$,

(vi) $E = A_0 \otimes \odot^r J^{-T} \in GL(U_r)$.

(vii) $Q \in \mathrm{Hom}(U^{(r-1)}, W)$

(viii) $R \in \mathrm{Hom}(X, W) \cong \mathrm{Hom}(\mathbb{R}^p, \mathbb{R}^p)$, and $\det J = 1 + \mathrm{tr}\, R$,

(ix) $M \in \mathrm{Hom}(U_r, W)$, $\sum_{(J,i)=K} M^\alpha_{i,J} = 0$ for each K with $\# K = r$,

(x) $T = \dfrac{J}{\det J} \in GL(W)$.

Here, as with the first order case presented in [12], conditions (i′) and (viii) replace the condition $\det J = 1$ in the standard equivalence problem, and ensure that the Lagrangians match up as in (3.6), with $\operatorname{tr} R$ corresponding to the total divergence $\operatorname{Div} F$. Conditions (ii)-(vi), are exactly the same as above. Condition (ix), which reduces to a skew symmetry condition in the first order Lagrangian case (where the matrix M is square) ensures that both F and $\operatorname{Div} F$ are functions on J^r, with the $(r+1)$-st order derivative terms in the divergence cancelling out. (Here is where we finally get rid of the $(r+1)$-st order derivatives in our problem!) Finally, condition (x) reflects the induced transformation rules for the new variables w, which are really coordinates on the $(p-1)$-st exterior power of the cotagent bundle, $\Lambda^{p-1} T^* X$. It is a remarkable fact that these conditions do indeed define a Lie group.

THEOREM 9. — *Two Lagrangians* L *and* $\bar{L}$ *are divergence equivalent if and only if there is a diffeomorphism* $\Psi : J^r \times W \to J^r \times W$ *which satisfies*

$$\Psi^* (\bar{\Xi}) = h . \Xi, \tag{4.7}$$

where h *is an* $H_\varkappa$*-valued function on* $J^r \times W$.

The proof of this result is far less straightforward than the standard equivalence problem, but follows as in the first order case discussed in [12]. For brevity it will be omitted. This completes the preliminary encoding of the Lagrangian equivalence problems. At this point, one can lift the coframes and commence the arduous task of implementing the equivalence algorithm; however, this will be taken up in future papers in this series. (*See* [20] for the case $p=q=1$, $r=2$.)

5. Derivative covariants and invariant differential equations

One of the important consequences of the reduction theorem for the Lagrangian equivalence problem is the existence of "derivative covariants". Loosely speaking, a derivative covariant of order $m \geqq r$, will be a certain function of the Lagrangian and its partial derivatives (hence an r-th order function) with the surprising property that, taking into account the transformation properties of the Lagrangian itself, it transforms in exactly the same way as the m-th order derivatives of the variables u. Thus, as far as the changes of variables are concerned, derivative covariants and derivatives are "isomorphic" quantities, even though the derivative covariants only depend on r-th order derivatives of u. The goal of this section is to make these statements precise, and prove a fundamental result on the existence of derivative covariants. We begin by presenting a framework for general equivalence problems which, while somewhat novel, is in a form amenable to the development of these ideas.

In any equivalence problem, there is a collection of functions $\mathscr{F}$, which we regard as the coordinates on a fiber bundle E with structure group G and fiber F, over our base manifold M. (*See* [18] for the basic theory of fiber bundles.) Thus, for the standard r-th order Lagrangian equivalence problem, $M = J^r$, and the only function is the

Lagrangian L, which we regard as the fiber coordinate in a one-dimensional vector bundle (line bundle) over M; the transformation rules on the fibers (which determine the structure group of the bundle) are, in this case, the Lagrangian change of variables formula (3.3). Note especially that the base manifold may come equipped with extra structure restricting the allowable changes of coordinates, so we only need define transition rules on the fibers of E for the allowable coordinate changes. (In the case under consideration, the base manifold is J^r, and we are only allowing changes of coordinates induced by the r-jets of κ-maps on $X \times U$.)

In the original formulation of a specific equivalence problem, we are given two local coordinate expressions for an object, e. g. a Lagrangian, and want to determine whether there is an allowable coordinate transformation from one to the other. In our fiber bundle language, by an E-*equivalence problem* we mean the problem of determining whether two given local coordinate expressions for sections of the fiber bundle E are actually the same section. Now, local coordinates $\chi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ on M provide a local trivialization

$$\chi_\alpha^* : \quad E|_{U_\alpha} \to V_\alpha \times F.$$

The transition functions on the overlap of the two coordinate charts take the form

$$\chi_{\alpha\beta}^* = \chi_\beta^* \circ \chi_\alpha^{*-1} : V_\alpha \times F \to V_\beta \times F,$$
$$\chi_{\alpha\beta}^*(x_\alpha, f) = (\chi_{\alpha\beta}(x_\alpha), g_{\alpha\beta}(x_\alpha).f), \qquad x_\alpha \in V_\alpha, \quad f \in F, \quad g_{\alpha\beta}(x_\alpha) \in G,$$

where $\chi_{\alpha\beta} = \chi_\beta \circ \chi_\alpha^{-1}$ is the transition function on the base (which may be restricted by the extra structure on M). For a specific E-equivalence problem, we are provided with two different local coordinate expressions

$$s_\alpha = \{(x_\alpha, f_\alpha(x_\alpha))\} \subset V_\alpha \times F, \qquad s_\beta = \{(x_\beta, f_\beta(x_\beta))\} \subset V_\beta \times F,$$

for possible sections of E, defined over open subsets V_α, $V_\beta \subset \mathbb{R}^n$. The equivalence problem is to determine whether they determine the same section s of E, *i.e.* whether we can find local coordinate maps χ_α, χ_β with the property that $s_\alpha = \chi_\alpha(s)$, $s_\beta = \chi_\beta(s)$ represent the same section s of E. A second way of stating this problem is that we are required to find an allowable transition function $\chi_{\alpha\beta}$ such that $s_\beta = \chi_{\alpha\beta}^*(s_\alpha)$, *i.e.*

$$(x_\beta, f_\beta(x_\beta)) = \chi_{\alpha\beta}^*(x_\alpha, f_\alpha(x_\alpha)) = (\chi_{\alpha\beta}(x_\alpha), g_{\alpha\beta}(x_\alpha).f_\alpha(x_\alpha)), \tag{5.1}$$

on the overlap of the two coordinate systems.

Any invariant I of an equivalence problem will, in particular, depend on the functions in the collection $\mathscr{F}$ and their derivatives, and, possibly, the base variables themselves. The derivatives of the functions in $\mathscr{F}$ will coordinatize the jet bundle $J^k E$, which is a fiber bundle whose structure group is determined completely by the structure group of E. Thus I will be a function from $J^k E$ (or possibly just some open subset thereof) to some manifold Y, the image space for I. (In particular, $Y = \mathbb{R}$ for a scalar invariant.) The key remark is that the invariance of I is equivalent to the requirement that it define a bundle map from $J^k E$ to the trivial fiber bundle $M \times Y$, having structure group $\{e\}$. Thus, we are led to the following abstract definition of an invariant.

DEFINITION 10. – Let $E \to M$ be a fiber bundle, which determines a corresponding equivalence problem. Let Y be a smooth manifold. A Y-*valued invariant* for the E-equivalence problem is a bundle map

$$\hat{I}: \ J^k E \to M \times Y,$$

where $M \times Y$ is the trivial fiber bundle with structure group $\{e\}$.

In local coordinates, $\hat{I}$ will take the form

$$\hat{I}(x_\alpha, f^{(k)}) = (x_\alpha, I(x_\alpha, f^{(k)})), \qquad x_\alpha \in V_\alpha, f^{(k)} \in F^{(k)},$$

$f^{(k)}$ denoting the k-jet coordinates corresponding to the fiber coordinates f, and where $I(x_\alpha, f^{(k)}) \in Y$ is the usual formula for the invariant. The fact that $\hat{I}$ maps to a fiber bundle with trivial structure group $\{e\}$ means that whenever two sections s_α, s_β, are related by an allowable coordinate change as in (5.1), then, on the overlap of the two coordinate charts,

$$(5.2) \qquad I(x_\beta, f_\beta^{(k)}(x_\beta)) = I(\chi_{\alpha\beta}(x_\alpha), g_{\alpha\beta}^{(k)}(x_\alpha).f_\beta^{(k)}(\chi_{\alpha\beta}(x_\alpha))) = I(x_\alpha, f_\alpha^{(k)}(x_\alpha)).$$

This now agrees with the traditional definition of an invariant.

We can also reformulate these concepts in terms of more familiar objects from the geometric theory of the equivalence method, *cf.* [19]. We first note that, once a given equivalence problem has been reformulated in Cartan form, the functions in $\mathscr{F}$ which are of interest will re-appear as the coefficients of the coframe elements over the base manifold M. As such, they naturally fit into the framework of G-structures. Accordingly, let G_* be a structure group (which, in examples, is usually not the same as the structure group G of the fiber bundle E used above). Let B_{G_*} and $\bar{B}_{G_*}$ be G_*-structures on manifolds M and $\bar{M}$ respectively. The Cartan equivalence problem for these structures is the problem of finding all the local diffeomorphisms $f: M \to \bar{M}$ such that the diagram

$$(5.3) \qquad \begin{array}{ccc} B_{G_*} & \xrightarrow{f_*} & \bar{B}_{G_*} \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_{\bar{M}}} \\ M & \xrightarrow{f} & \bar{M} \end{array}$$

commutes (*cf.* [19], p. 313). Locally, the G_*-structure B_{G_*} will be coordinatized by the lifted coframes $\Theta = g.\Omega$, where Ω is a prescribed coframe on the base manifold M, and g is an element of the structure group G_*. Since the invariants can depend on higher order jets of the coefficients of the coframe elements, we are led to introduce the higher order structure bundle $B_{G_*}^{(k)}$, *cf.* [16], §II.3, whose structure group is the prolongation $G_*^{(k)}$. Locally the bundle $B_{G_*}^{(k)}$ will be coordinatized by the higher order lifted coframes $\Theta^{(k)} = g^{(k)}.\Omega^{(k)}$, where $g^{(k)} \in G_*^{(k)}$, and $\Theta^{(k)} = j_k \Omega$ is the higher order frame determined by Ω.

In this language, a Y-valued invariant of the Cartan equivalence problem (5.3) will be given by a map $I: B^{(k)}_{G_*} \to Y$ such that the diagram

$$\begin{tikzcd} B^{(k)}_{G_*} \arrow[rr, "f^{(k)}_*"] \arrow[dr, "I"'] & & B^{(k)}_{G_*} \arrow[dl, "I"] \\ & Y & \end{tikzcd}$$

commutes for all maps f which solve the self-equivalence problem

$$\begin{tikzcd} B_{G_*} \arrow[r, "f_*"] \arrow[d, "\pi_M"'] & B_{G_*} \arrow[d, "\pi_M"] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

In this framework, once we've managed to encode our equivalence problem in Cartan form, the original fiber bundle E will correspond to a certain G_*-invariant subbundle of the vector bundle associated to the structure bundle B_{G_*}. For instance, in the standard Lagrangian equivalence problem, rather than consider the Lagrangian line bundle over J^r as constructed above, we could look at the subbundle generated by the base forms (4.1), and the corresponding contact forms (2.10) required to preserve invariance of E under the structure group. (In the fiber-preserving standard equivalence problem, no contact forms are required.) Moreover, the k-jet bundle $J^k E$ will then naturally appear as an invariant subbundle of the vector bundle associated to the higher order bundle $B^{(k)}_{G_*}$. Thus we can work entirely within the standard differential-geometric framework of the equivalence problem. However, for the present discussion, we find it more convenient to work directly with the fiber bundle E so as to avoid extra complications.

We now generalize the definition of a Y-valued invariant to include quantities that, while not invariant, obey some prescribed transformation rule. Referring back to definition 10, we find that it suffices to replace the trivial bundle $M \times Y$ by some other fiber bundle.

DEFINITION 11. – Let $E \to M$ be a fiber bundle, which determines a corresponding equivalence. Let $K \to M$ be another fiber bundle. A K-*covariant* for the E-equivalence problem is a bundle map

$$\hat{F}: \; J^k E \to K.$$

In local coordinates, $\hat{F}$ will take the form

$$\hat{F}(x_\alpha, f^{(k)}) = (x_\alpha, F(x_\alpha, f^{(k)})) \in V_\alpha \times Y,$$

where Y now denotes the fiber of K. Covariance of $\hat{F}$ means that, on the overlap of the two coordinate charts, the function f obeys the transformation rule

$$(5.4) \qquad F(x_\beta, f^{(k)}_\beta(x_\beta)) = F(\chi_{\alpha\beta}(x_\alpha), g^{(k)}_{\alpha\beta}(x_\alpha) . f^{(k)}_\beta(\chi_{\alpha\beta}(x_\alpha))) = h_{\alpha\beta}(x_\alpha) . F(x_\alpha, f^{(k)}_\alpha),$$

where the fiber transition functions $h_{\alpha\beta}(x_\alpha)$ take values in the structure group H of the bundle K. In the case when $H=\{e\}$ is trivial, then (5.4) reduces to (5.2), and we are back to the classical case of an invariant. We now illustrate this general definition by some simple examples taken from the equivalence problem for a first order Lagrangian on the line (*cf.* [6], [7], [11]).

Example 12. – For the standard Lagrangian equivalence problem, the Lagrangian itself is trivially an E-covariant, where E is the line bundle determined by L, and $\hat{F}: E \to E$ is the identity map. Powers of the Lagrangian will similarly define E^m-covariants, where E^m denotes the corresponding m-th tensor power of the line bundle E.

Less trivial examples can be constructed as follows: Consider a scalar first order Lagrangian $L(x, u, p)$, where $p=u_x$, under the pseudogroup of point transformations $\bar{x}=\varphi(x, u)$, $\bar{u}=\psi(x, u)$, where the barred and unbarred variables refer to the two different local coordinate systems [called (x_β, f_β) and (x_α, f_α) respectively above]. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \varphi_x & \varphi_u \\ \psi_x & \psi_u \end{pmatrix}$$

denote the Jacobian matrix of the base transformation. The transition rules on the fibers of the Lagrangian line bundle E are given by

$$\bar{L} = D\varphi^{-1} L = (bp+a)^{-1} L. \tag{5.5}$$

Let $\pi: J^1 \to Z = X \times U$ denote the trivial projection, and define the determinant line bundle $\Delta = \pi^* \Lambda^2 T^* Z$, which has fiber transition rules

$$\bar{f} = (ad-bc) f.$$

Then the Hessian $H = L_{pp}$ determines a $\Delta^{-2} \otimes E^{-3}$-covariant because it transforms according to the rule

$$\bar{H} = (ad-bc)^{-2} (bp+a)^3 H.$$

Similarly, the differential polynomial

$$K = LL_{ppp} + 3 L_p L_{pp}$$

is a $\Delta^{-3} \otimes E^{-5}$-covariant because

$$\bar{K} = (ad-bc)^{-3} (bp+a)^5 K,$$

as the reader can readily check using the chain rule. (The individual summands of K are *not* covariants.) The product

$$I = K^2 . L^{-1} . H^{-3} = \frac{(LL_{ppp} + 3 L_p L_{pp})^2}{L . L_{pp}^3}$$

is thus an invariant, being a section of the trivial product bundle

$$(\Delta^{-3} \otimes E^{-5})^2 \otimes E \otimes (\Delta^{-2} \otimes E^{-3})^{-3} = M \times \mathbb{R}.$$

This expression is the well-known first invariant for the standard Lagrangian equivalence problem (*cf.* [6], [11]).

In accordance with this example, we note that the functions which determine the different branches of a given Lagrangian equivalence problem turn out to be certain particular covariants associated with the problem. However, the precise form of the associated image bundle K in the context of the Cartan procedure is not clear to us at present.

DEFINITION 13. – Assume that our base space has the form $M = J^l = J^l Z$, $Z = X \times U$. An m-th order *derivative covariant* of an equivalence problem is a $\pi_l^* J^m$-covariant. Here $\pi_l : J^l \to X$ is the natural projection, so that $\pi_l^* J^m$ denotes the pull-back of the m-jet bundle J^m to the l-jet bundle J^l.

Locally, what does a derivative covariant look like? For simplicity, let us restrict attention to the case under consideration, the r-th order Lagrangian equivalence problem on the jet bundle J^l, $l \geqq r$. Thus, $M = J^l$, and the line bundle $E \to M$ will have base coordinates $(x, u^{(l)})$ and the single fiber coordinate L, while $J^k E$ will have the same base coordinates, and fiber coordinates $L^{(k)}$, meaning all the partial derivatives of L with respect to all the base variables $(x, u^{(l)})$ up to order k. (However, note that L depends only on $(x, u^{(r)})$, so the partial derivatives of L with respect to the derivatives of u of order greater than r will automatically vanish.) On the pull-back bundle $K = \pi_l^* J^m$, the base coordinates are $(x, u^{(l)})$ and the fiber coordinates $v^{(m)} = (v_K^\alpha)$, for $\alpha = 1, \ldots, q$, $\# J \leqq m$. The transformation rules on K are the same as those of the m-th order derivatives of u,

$$\bar{v}^{(m)} = \psi^{(m)}(x, v^{(m)}),$$

cf. (2.6). An m-th order derivative covariant will then be a collection of functions

$$v^{(m)} = F(x, u^{(l)}, L^{(k)}) = (F_K^\alpha(x, u^{(l)}, L^{(k)})), \qquad \alpha = 1, \ldots, q, \quad \# J \leqq m.$$

According to the local form (5.4) of definition 11, these functions must transform exactly the same way as the k-th order derivatives of u do:

$$\bar{F} = \psi^{(m)}(x, F). \tag{5.6}$$

A trivial example of an m-th order derivative covariant is provided by

$$F(x, u^{(l)}, L^{(k)}) = u^{(m)}, \tag{5.7}$$

for any $m \leqq l$.

In all the interesting examples of m-th order derivative covariants associated with an r-th order Lagrangian that we know so far, we have $m > r$, and the components of F of order $< r$ agree with the trivial example (5.7); it is only the higher order components that involve L and its derivatives. In fact, if

$$\mathrm{F}=(\mathrm{F}_0, \mathrm{F}_1, \ldots, \mathrm{F}_m), \tag{5.8}$$

is any m-th order derivative covariant, then its homogeneous components $\mathrm{F}_j=(\mathrm{F}_\mathrm{J}^\alpha)$, $\alpha=1, \ldots, q$, $\# \mathrm{J}=j$, called *homogeneous derivative covariants*, can be re-assembled individually to form further derivative covariants; for example, we can replace any k-th order homogeneous piece of any derivative covariant by the k-th order piece of any other derivative covariant [e. g. the trivial covariant (5.7)], and still preserve the covariance of the function. For instance, if F is as in (5.8) with $m \leq l$, then

$$\tilde{\mathrm{F}}=(u^{(m-1)}, \mathrm{F}_m)$$

is also a derivative covariant, with trivial $(m-1)$-st order components, but non-trivial m-th order component. Thus, in practice it makes sense to concentrate on the homogeneous derivative covariants from now on. Also note that we can multiply any component of a derivative covariant by any invariant of the equivalence problem without affecting the basic covariance, so we can readily produce many derivative covariants from any given one.

Example 14. – Consider a first order Lagrangian $\mathrm{L}(x, u, p)$ on the line. The rational function

$$\mathrm{Q}(x, u, p, \mathrm{L}^{(2)})=\frac{\mathrm{L}_u-\mathrm{L}_{xp}-p\,\mathrm{L}_{up}}{\mathrm{L}_{pp}} \tag{5.9}$$

is a second order homogeneous derivative covariant. In other words, if $\bar{x}$, $\bar{u}$, are related to x, u, by a point transformation, whose prolongation to J^2 is given by

$$\bar{p}=\psi_1(x, u, p)=\frac{\mathrm{D}\psi}{\mathrm{D}\varphi}, \qquad \bar{q}=\psi_2(x, u, p, q)=\frac{\mathrm{D}\psi_1}{\mathrm{D}\varphi},$$

where $q=u_{xx}$, and $\bar{\mathrm{L}}$ is the new Lagrangian as in (5.5), then a short calculation shows that Q satisfies the derivative covariant transformation rule

$$\mathrm{Q}(\bar{x}, \bar{u}, \bar{p}, \bar{\mathrm{L}}^{(2)})=\psi_2(x, u, p, \mathrm{Q}(x, u, p, \mathrm{L}^{(2)})).$$

Thus the rational function Q obeys the same transformation rules as the second order derivative q, and hence Q determines a homogeneous second order derivative covariant for L. By the above remarks, we can combine Q with the trivial first order derivative covariant to get a full second order derivative covariant

$$\mathrm{F}(x, u, p, \mathrm{L}^{(2)})=(u, p, \mathrm{Q}(x, u, p, \mathrm{L}^{(2)})).$$

One of the main consequences of the existence of derivative covariants is that they determine invariantly defined systems of differential equations associated with the given equivalence problem. The class of solutions of such a system will therefore play a distinguished role. For example, the solutions of the Euler-Lagrange equations determine the critical points of the functional.

PROPOSITION 15. – *Let* $\mathbf{F}_m$ *be a homogeneous* m-th *order derivative covariant. Then the* m-th *order system of differential equations*

$$u_m = \mathbf{F}_m(x, u^{(l)}, \mathrm{L}^{(k)}), \tag{5.10}$$

are intrinsically defined in the sense that if Ψ *is any* κ*-map taking the Lagrangian* L *to* $\bar{\mathrm{L}}$, *then* Ψ *will map the system* (5.10) *to the corresponding system in the barred variables. In particular, the transformation will map solutions of one system to solutions of the other.*

For instance, in example 14, the differential equation corresponding to the covariant (5.9) is

$$q = \mathrm{Q}(x, u, p, \mathrm{L}^{(2)}),$$

which is nothing but the Euler-Lagrange equations for the first order Lagrangian, *cf.* (5.9). Indeed, in the case of one independent variable, $p=1$, solving the Euler-Lagrange equations for the highest order derivative of u will always produce a derivative covariant, similar to the one described above, whose associated invariant system of differential equations (5.10) is just the Euler-Lagrange equations themselves. Similarly, solving for thė highest order derivative of u in any derivative of the Euler-Lagrange equations will produce higher order derivative covariants. One might conjecture at this point that these are the only invariantly defined differential equations associated with a given variational problem. Surprisingly, this naïve guess is not correct, and we will see that there are other such intrinsically defined systems.

The key result that allows one to readily determine derivative covariants is the following.

PROPOSITION 16. – *Let* $m \geq r$, *and suppose* $\mathrm{E} = (\mathrm{E}^{\alpha\,\mathrm{K}}_{\mathrm{J}\,\beta}) \in \mathrm{GL}(\mathrm{U}_m)$,

$$\mathrm{F} = (\mathrm{F}^{\alpha}_{\mathrm{J},\,i}),\ \alpha = 1, \ldots, q,\ \#\,\mathrm{J} = \#\,\mathrm{K} = m,\ i = 1, \ldots, p,$$

are functions depending on $(x, u^{(m)})$ *and partial derivatives of* L. *Define a corresponding collection of one-forms*

$$\zeta = \mathrm{E}\,.\,(du_m - \mathrm{F}\,dx) \in \mathrm{T}^*\,\mathrm{J}^m \otimes \mathrm{J}^m, \tag{5.11}$$

with components

$$\zeta^{\alpha}_{\mathrm{J}} = \sum_{\#\,\mathrm{K}=m} \mathrm{E}^{\alpha\,\mathrm{K}}_{\mathrm{J}\,\beta}\left(du^{\beta}_{\mathrm{K}} - \sum_{i=1}^{p} \mathrm{F}^{\beta}_{\mathrm{K},\,i}\,dx^i\right), \qquad \alpha = 1, \ldots, q, \quad \#\,\mathrm{J} = m, \tag{5.12}$$

such that the coefficients F *obey the symmetry condition*

$$\mathrm{F}^{\alpha}_{\mathrm{J},\,i} = \mathrm{F}^{\alpha}_{\mathrm{J}',\,i'} \quad \textit{whenever} \ (\mathrm{J}, i) = (\mathrm{J}', i'). \tag{5.13}$$

Suppose the one-forms ζ are invariant modulo the contact ideal $\mathcal{I}^{(m)}$ under κ-maps, i.e.

$$\Psi^*(\bar{\zeta})=\zeta \qquad \operatorname{mod} \mathcal{I}^{(m)}.$$

Then F *is an* $(m+1)$*-st order homogeneous derivative covariant.*

Proof. — This turns out to be a simple consequence of the fact, noted in lemma 1, that the transformation rules for contact forms and for derivatives are essentially the same. Since ζ is invariant modulo $\mathcal{I}^{(m)}$, we have $\Psi^*\bar{\zeta}=\zeta \operatorname{mod} \mathcal{I}^{(m)}$, which we expand using (2.11):

$$\Psi^*\bar{\zeta}=\Psi^*\{\bar{\mathrm{E}}.(d\bar{u}_m-\bar{\mathrm{F}}\,dx)\}=\Psi^*\bar{\mathrm{E}}.\{\mathrm{A}_m\,du_m+[\tilde{\mathrm{D}}\psi_{m-1}-(1\otimes \mathrm{D}\varphi^{\mathrm{T}})\Psi^*\bar{\mathrm{F}}]\,dx\} \qquad \operatorname{mod} \mathcal{I}^{(m)}.$$

Comparing this expression with (5.11), we find that

$$\Psi^*\bar{\mathrm{E}}.\mathrm{A}_m=\mathrm{E},$$
$$\Psi^*\bar{\mathrm{E}}.\{-\tilde{\mathrm{D}}\psi_{m-1}+(1\otimes \mathrm{D}\varphi^{\mathrm{T}})\Psi^*\bar{\mathrm{F}}]=\mathrm{E}.\mathrm{F}.$$

Therefore

$$\Psi^*\bar{\mathrm{E}}\{(1\otimes \mathrm{D}\varphi^{\mathrm{T}})\Psi^*\bar{\mathrm{F}}-\mathrm{A}_m.\mathrm{F}-\tilde{\mathrm{D}}\psi_{m-1}]=0.$$

Finally, since $\bar{\mathrm{E}}$ is invertible, we use (2.7) to conclude that

$$\Psi^*\bar{\mathrm{F}}=(1\otimes \mathrm{D}\varphi^{-\mathrm{T}})\{\mathrm{A}_m.\mathrm{F}-\tilde{\mathrm{D}}\psi_{m-1}\}=\psi_m(x,u^{(m)},\mathrm{F})$$

obeys the same transformation rules as the m-th order derivatives of u. This suffices to prove that F is an $(m+1)$-st order homogeneous derivative covariant.

Suppose we have managed to solve the J^{r+k} equivalence problem (either standard or divergence) for some $k \geqq 0$, and that, as a result of applying Cartan's method, we have managed to reduce the equivalence problem to an $\{e\}$-structure on the base manifold J^{r+k}. We are allowing the possibility of prolonging the problem (which may be inevitable, *cf.* [12]), as long as the problem ultimately reduces to an $\{e\}$-structure on the base. It is then a simple matter to then apply this result to find derivative covariants for any such Lagrangian. Consider the resulting invariant coframe elements

$$\zeta=\mathbf{E}.\pi+\mathbf{D}.\mathbf{J}.\omega+\mathbf{C}.\theta,$$

corresponding to the base coframe elements $\pi=(\pi_{\mathrm{J}}^{\alpha})$, $\alpha=1,\ldots q$, $\#\mathrm{J}=r+k$, *cf.* (4.2). The coefficients **E**, **D**, **J**, **C**, will be certain functions on J^{r+k}, depending on the coordinates $(x, u^{(r+k)})$ and the partial derivatives of L; they are explicitly determined by the Cartan method, and result from the normalizations of essential torsion terms. These invariant one-forms will be of the form (5.12), where the functions **F** are related to the normalized group parameters according to the system of equations

$$\mathbf{E}.\mathbf{F}=\mathbf{D}.\mathbf{J}. \tag{5.14}$$

We need also check that the coefficients F obey the symmetry condition (5.13). However, this follows directly from (5.14), and the symmetry conditions (v), (vi) placed on the elements of structure group G_x. Since each one-form ζ^α_J is, by construction, invariant, proposition 16 implies that the coefficients $F = (F^\alpha_{J,i})$ form an $(r+k+1)$-st order homogeneous derivative covariant.

COROLLARY 17. — *Let* L *be a nondegenerate r-th order Lagrangian with the property that the corresponding Cartan equivalence problem reduces to an $\{e\}$-structure on the base* J^r, *and let m be any integer with $m > r$. Then* L *possesses a set of m-th order derivative covariants* $F = (F^\alpha_K)$. *Consequently there exists an invariantly-defined system of m-th order differential equations*

$$u^\alpha_K = F^\alpha_K(x, u^{(l)}, L^{(k)}) \tag{5.15}$$

associated with any such Lagrangian.

(It would be interesting to see if one could eliminate the hypothesis that the Cartan solution to the equivalence problem reduce to an $\{e\}$-structure on the base in this result. The method introduced in [20] has some bearing on this question.)

Example 18. — For the case of a first order Lagrangian on the line, formulated as an equivalence problem on the jet bundle J^1, there is just one lifted form corresponding to the base coframe element $\pi = dp$. According to the calculations in [11], for each of the equivalence problems (standard or divergence), the corresponding invariant adapted coframe element takes the form

$$\zeta = A(dp - Q\,dx) + C\theta,$$

where Q is the derivative covariant described above, $\theta = du\text{-}p\,dx$ is the contact form, and A depends on L and its derivatives in a complicated manner, which depends on the particular equivalence problem under consideration. Since ζ is invariant, we conclude immediately from proposition 16 that Q is a homogeneous second order derivative covariant, as we observed earlier. The corresponding differential equation (5.15) is just the Euler-Lagrange equation.

It appears that, for a first order Lagrangian, the only homogeneous derivative covariants are essentially the Euler-Lagrange equation, its derivatives, and invariant multiples thereof. This is no longer true for a second order Lagrangian on the line. Indeed, the solution of the equivalence problem on the minimal order jet bundle J^2 will lead to a third order homogeneous derivative covariant, and a corresponding invariantly defined *third* order differential equation

$$u''' = F(x, u, u', u'', L^{(k)}(x, u, u', u'')). \tag{5.16}$$

The Euler-Lagrange equation is, of course, of order 4 for a nondegenerate Lagrangian, and is probably some "covariant derivative" of the third order equation (5.16), although we do not know this at present. An explicit example of such an invariant equation is the third order equation

$$D_x(LL_q^2) = LL_p L_q. \tag{5.17}$$

As will be proved in a later paper in this series, this differential equation is invariant under the pseudogroup of fiber-preserving transformations of the second order Lagrangian $L(x, u, p, q)$, provided L satisfies the nondegeneracy condition

$$2LL_{qq} + L_q^2 \neq 0, \quad \text{i.e. } L \neq (Aq+B)^{2/3}, \tag{5.18}$$

for A, B functions of x, u, p. This means that if L is mapped to $\bar{L}$ under a transformation of the form $\bar{x} = \varphi(x)$, $\bar{u} = \psi(x, u)$, then the equation (5.17) is mapped to the analogous equation for $\bar{L}$. Therefore, the solutions to the equation (5.17) play a distinguished role for the associated variational problems. The geometrical or analytical significance of such an equation remains to be determined. In a subsequent paper, [20], we will develop these and other parametric formulas coming from Cartan's analysis of the second order particle Lagrangian on both J^2, [5], and also the corresponding problem on J^3, which is the bundle the Cartan form lives on, and compare the results. In particular, this will lead to explicit formulas for the invariants and derivative covariants of this problem, and the corresponding invariant differential equations. This will be necessary to pursue the tantalizing problem of the interpretation and application of these new invariant differential equations associated with variational problems.

We also remark that even for first order Lagrangians in several independent variables, the system of invariant differential equations (5.15) is new. For instance, in the case of a Lagrangian in one dependent and two independent variables treated by Gardner and Shadwick, [8], their solution to the equivalence problem will lead to an invariantly defined second order system of the form

$$\begin{cases} u_{xx} = F_1(x, u^{(1)}, L^{(k)}(x, u^{(1)})), \\ u_{xy} = F_2(x, u^{(1)}, L^{(k)}(x, u^{(1)})), \\ u_{yy} = F_3(x, u^{(1)}, L^{(k)}(x, u^{(1)})). \end{cases} \tag{5.19}$$

It would be of interest to determine the explicit parametric formulas for these equations, and discuss their interpretation. In particular, we conjecture that the Euler-Lagrange equation for such a Lagrangian will be built up of a suitable invariant combination of the three component invariant parts given in (5.17), although this remains to be verified explicitly.

Finally, we remark on the correspondence between the invariants of a higher order version of the Lagrangian equivalence problem and the r-th order invariants coming from the solution to the minimal order J^r version of the same equivalence problem. Suppose we have a Lagrangian with the property that the associated Cartan equivalence problem reduces to an $\{e\}$-structure on the base manifold. Suppose $I(x, u^{(r+k)}, L^{(j)})$ is any invariant associated with some version of the Lagrangian equivalence problem, obtained from the Cartan solution to the J^{r+k} formulation of the problem, and depending not only on the Lagrangian and its derivatives, but also higher order derivatives of the u's. The question is, is there a corresponding r-th order invariant quantity which will appear the J^r equivalence problem? The answer to this question is provided naturally by the derivative covariants associated with L. Indeed, for $r < m \leqq r+k$, we can replace each m-th order derivative of u appearing in the formula for I by its corresponding derivative covariant. In this way we will deduce an "equivalent" invariant $\hat{I}(x, u^{(r)}, L^{(j)})$ which *does* just depend on at most r-th order derivatives of the u's, and therefore must be a combination of the r-th order invariants coming from the J^r equivalence problem and their covariant derivatives. Thus, as far as the transformation properties of the Lagrangian are concerned, I and $\hat{I}$ determine "isomorphic" invariants, even though their precise formulas are quite different. Similar remarks apply to other invariant quantities, including the Euler-Lagrange equations and the Lagrangian form. For each of these invariant objects depending on higher order derivatives of the u's there will be a corresponding r-th order invariant object with the same transformation rules as the higher order object. Note that we are not implying that these objects are the same (One certainly can't reduce the $2r$-th order Euler-Lagrange equations to an r-th order equation with the same solutions!) – only that they transform in identical manners under the appropriate pseudogroup, and therefore from an equivalence problem standpoint describe "isomorphic" objects. Again, we hope to make these ideas clearer with specific examples in a later in this series.

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