Equivalence of higher-order Lagrangians. II. The Cartan form for particle Lagrangians

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It is shown how Cartan's method of equivalence may be used to obtain the Cartan form for an r th-order particle Lagrangian on the line by solving the standard equivalence problem under contact transformations on the jet bundle J^{r+k} for $k \geqslant r-1$.

I. INTRODUCTION

This is the second in a series of papers investigating different aspects of the Cartan equivalence problem for higher-order variational problems. In Pt. I,[1] it was shown how each of the basic Lagrangian equivalence problems, in any number of independent and dependent variables, could be formulated as a Cartan equivalence problem, and a fundamental reduction theorem, demonstrating that the equivalence problem for an r th-order Lagrangian could always be reduced to the minimal-order jet bundle J^r, was proved. In this paper, we will be exclusively concerned with r th-order variational problems in one independent and one dependent variable,

$$\mathscr{L}[u] = \int_\Omega L(x,u^{(r)})dx. \tag{1.1}$$

The Lagrangian L depends analytically on the coordinates $(x,u^{(r)}) = (x,u,u_1,...,u_r)$ on the jet bundle $J^r = J^r(\mathbb{R},\mathbb{R})$. Here, the coordinate u_j represents the jth-order derivative of the single dependent variable u with respect to the single independent variable x, so $u_j = D_x^j\, u$, where D_x denotes the total derivative operator. We will be interested in properties of the functional (1.1) that are preserved under change of variables, which we take to mean general contact transformations. In the language of Pt. I,[1] we are dealing with the standard equivalence problem for the particle Lagrangian (1.1) under the pseudogroup of contact transformations. The *contact ideal* on J^r, denoted $\mathscr{I}^{(r)}$, plays a key role; it is generated by the contact forms

$$\theta_j = du_j - u_{j+1}\, dx, \quad 0 \leqslant j < r. \tag{1.2}$$

According to Bäcklund's theorem,[1,2] a transformation $\Psi: J^r \to J^r$ will preserve the contact ideal $\mathscr{I}^{(r)}$ if and only if it is the prolongation of a contact transformation $\Psi_0: J^1 \to J^1$ of the first-order jet bundle, a fact that will play an important role in our discussion.

An important invariant one-form associated to the functional (1.1) is the so-called *Cartan form*,[3]

$$\Theta_C = L\, dx + \sum_{i=1}^{r} \sum_{j=0}^{r-i} (-D_x)^j \left(\frac{\partial L}{\partial u_{i+j}}\right)\cdot \theta_{i-1}. \tag{1.3}$$

It is well known that the Cartan form encodes both the Euler–Lagrange equations for (1.1), and that it plays an important role in the formulation of Noether's theorem relating symmetries and conservation laws. It also figures prominently in the implementation of field theory via the Hamilton–Jacobi equation, which is used to deduce the existence of strong minimizers.[4,5] Note that Θ_C lives on the jet bundle J^{2r-1}, which reflects the fact that the Euler–Lagrange equations for a nondegenerate r th-order Lagrangian are of order $2r$. We will see that the Cartan form remains invariant under contact transformations of the Lagrangian (1.1), i.e., if one Lagrangian is mapped to another via a contact transformation, then the corresponding Cartan forms are mapped to each other by the appropriate prolongation of the same contact transformation. (We remark that the Cartan form is not invariant under the more general operations of transforming and adding a total divergence to the Lagrangian. This explains why we consider the standard equivalence problem and not the divergence equivalence problem in this paper.)

A powerful construction that produces the invariants (functions and differential forms) associated with such a variational problem is Cartan's *Method of Equivalence*,[6–8] a general method for determining if two exterior differential systems generated by one-forms are equivalent under a change of variables belonging to a prescribed pseudogroup. It has been observed by Gardner[7] that, in the first-order case ($r = 1$), the Cartan form is part of an invariant adapted coframe obtained by formulating and solving the equivalence problem for (1.1) as a Cartan equivalence problem on J^1. In the higher-order case ($r > 1$), it is *not* true that the Cartan form can be recovered by solving the equivalence problem on J^r. This had led some researchers, such as Shadwick,[9] to suggest that one should study the equivalence problem for r th-order variational problems on jet bundles J^{r+k}, where k is sufficiently large so as to yield the Cartan form (i.e., $k \geqslant r-1$), but otherwise arbitrary.

In the first paper in this series,[1] it was shown how to formulate the equivalence problem for the Lagrangian (1.1) as a Cartan equivalence problem on the space of $(r+k)$-jets for any $k \geqslant 0$. Moreover, we found that each of these potentially different equivalence problems, on the different bundles J^{r+k}, $k \geqslant 0$, are really the same problem, in that they all encode the same equivalence problem, and hence must have isomorphic solutions. Let us begin by recalling the basic definition and theorem on the equivalence of Lagrangians under point transformations.[1]

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Definition 1: Two r th-order Lagrangians are said to be $(r+k)$*-standard equivalent*, $k\geqslant 0$, if and only if there is a contact map $\Psi: J^{r+k}\rightarrow J^{r+k}$ such that

$$\Psi^*\{\bar{L}\, d\bar{x}\} = L\, dx \bmod \mathscr{I}^{(r+k)}. \tag{1.4}$$

Theorem 1: Two r th-order Lagrangians are $(r+k)$-standard equivalent if and only if they are r-standard equivalent.

From this point of view, one does not gain anything as far as the ultimate solution to the equivalence problem is concerned by increasing the order of the jet bundle to serve as the base space, and, for simplicity, may as well solve the problem on the minimal-order jet bundle, viz. J^r. On the other hand, since the Cartan form (1.3) clearly involves $(2r-1)$st-order derivatives of u, it cannot arise as an invariant one-form if one solves the equivalence problem on a jet bundle of order $r+k$ for any $k<r-1$. For a first-order Lagrangian, this does not present any difficulties, as $1=r=2r-1$; however, for higher-order Lagrangians, difficulties arise since $r<2r-1$. For example, Cartan's solution to the second-order particle Lagrangian equivalence problem[10] does not lead to the Cartan form, as he implements the solution to this problem on the jet bundle J^2, while the relevant Cartan form lives on the bundle J^3.

A resolution of this apparent contradiction has been proposed in Ref. 1, where it was argued that the solution to the r th-order equivalence problem will lead to a purely r th-order differential form, which can be obtained from the Cartan form Θ_C by replacing all derivatives of order greater than r by the associated "derivative covariants," which are certain universal r th-order functions that can be constructed from the Lagrangian and its derivatives, with the remarkable property that they transform precisely like the higher-order derivatives of u. In a subsequent paper in this series, we hope to illustrate explicitly this point in the case of a second-order Lagrangian, but for now we will content ourselves with this rather general statement, and refer the reader to Ref. 1 for the details on this point. See also remarks in Sec. III.

II. THE CARTAN FORM

Our goal now is to prove the main result, that by setting up the equivalence problem for the variational problem (1.1) as a Cartan equivalence problem on J^{r+k}, where $k\geqslant r-1$, one obtains, after several iterations of Cartan's reduction procedure, the Cartan form Θ_C given by (1.3) as part of an adapted coframe. We will not attempt to make the complete reduction here (this is too hard to do for general r and k), but will discuss the second-order case in more detail in a future publication.

We begin by recalling how the standard equivalence problem for r th-order Lagrangians was encoded in terms of certain differential one-forms on the jet bundle J^{r+k}. The base coframe is given by the one-forms

$$\theta_0, \theta_1, \ldots, \theta_{r+k-1}, \quad \omega_0 = L\, dx, \quad \pi_0 = du_{r+k}, \tag{2.1}$$

where the θ_j are the contact forms given in (1.2). We assemble these into a column vector $\boldsymbol{\theta}_0 = (\theta_0, \theta_1, \ldots, \theta_{r+k-1})^T$, and use $\boldsymbol{\eta}_0 = (\boldsymbol{\theta}_0, \omega_0, \pi_0)^T = (\theta_0, \theta_1, \ldots, \theta_{r+k-1}, \omega_0, \pi_0)^T$ to denote the complete column vector of coframe elements.

Given any non-negative integer $m\leqslant r+k$, we define a $\{\frac{1}{2}(r+k+3)(r+k)+m+2\}$-dimensional matrix Lie group $G^{(m)}$. It consists of all lower triangle matrices of the form

$$g = \begin{pmatrix} A & 0 & 0 \\ B & 1 & 0 \\ C & D & E \end{pmatrix}, \tag{2.2}$$

where $A=(A^i_j)$ is an invertible $(r+k)\times(r+k)$ lower triangular matrix, D and E are scalars, $E\neq 0$, and $B=(B_1, B_2, \ldots, B_{r+k})$ and $C=(C_1, C_2, \ldots, C_{r+k})$ are row vectors, with

$$B = (B_1, B_2, \ldots, B_m, 0, \ldots, 0). \tag{2.3}$$

Note that $G^{(l)}\subset G^{(m)}$ for $l<m$. In Pt. I,[1] we showed how the structure groups $G^{(m)}$ can be used to encode our equivalence problem in Cartan form.

Theorem 2: Let $r>1$, $k\geqslant 0$, and let $2\leqslant m\leqslant r+k$. Two rth-order Langrangians L and $\bar{L}$ are $(r+k)$-standard equivalent under the pseudogroup of contact transformations if and only if there is a diffeomorphism $\Psi: J^{r+k}\rightarrow J^{r+k}$ that satisfies

$$\Psi^*(\bar{\boldsymbol{\eta}}_0) = g\cdot\boldsymbol{\eta}_0, \tag{2.4}$$

where $\bar{\boldsymbol{\eta}}_0$ and $\boldsymbol{\eta}$ are the respective coframes associated with the two Lagrangians and g is a $G^{(m)}$-valued function on J^{r+k}.

According to Bäcklund's theorem,[1,2] since any transformation preserving the contact ideal on J^{r+k} is the prolongation of a contact transformation on J^1, we could take the minimal value of $m=2$ to encode the equivalence problem; however, as we shall see, this would not lead us directly to the Cartan form. (The cases $m=1$ and $m=0$ will further restrict the allowable change of variables to the pseudogroups of point and fiber-preserving transformations, respectively.) The case $r=1$ is special, since it can be shown that equivalence of first-order Lagrangians under contact transformations automatically reduces to equivalence under point transformations,[7,11,12] so $m=1$ anyway. From here on, we will leave this case aside as it is already well understood.

The main result to be proved in this paper can now be stated as follows:

Theorem 3: Let L be an r th-order Lagrangian for $r\geqslant 2$. The Cartan form Θ_C, given by (1.3), appears naturally among the invariant adapted coframe elements resulting from an application of the Cartan method of equivalence to the equivalence problem (2.4) on the jet bundle J^{r+k} under the structure group $G^{(m)}$ provided $k\geqslant r-1$ and $m\geqslant r$.

Proof: The restrictions on k and m both follow from the ultimate form (1.3) for the Cartan form Θ_C; more on this later. To prove the general result, it suffices to start with the largest of the possible structure groups, so we assume that we are working in J^{r+k}, $k\geqslant r-1$, and using the structure group G^{r+k}. In accordance with the Cartan algorithm, we begin by lifting the problem to $J^{r+k}\times G^{r+k}$, and use $\boldsymbol{\eta}=g^{-1}\cdot\boldsymbol{\eta}_0$, i.e.,

$$\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix}=\begin{pmatrix}A & 0 & 0\\ B & 1 & 0\\ C & D & E\end{pmatrix}^{-1}\begin{pmatrix}\theta_0\\ \omega_0\\ \pi_0\end{pmatrix}, \tag{2.5}$$

as our lifted coframe. (The exponent -1 in the group element parametrization is introduced solely for computational convenience.) In particular,

$$\omega = L\,dx + \sum_{j=0}^{r+k-1} Z_j\theta_j, \tag{2.6}$$

where the coefficients Z_j are the entries of the row vector

$$Z = -BA^{-1}. \tag{2.7}$$

In the first loop of the algorithm implementing Cartan's method of equivalence,[6,7,8] we are supposed to compute the exterior derivatives of the lifted coframe and rewrite the result in terms of the right-invariant one-forms on the structure group $G^{(r+k)}$, i.e., the entries of the matrix differential $g^{-1}\cdot dg$, and the lifted coframe elements $\boldsymbol{\eta}$. It turns out that, for the purposes of recovering the Cartan form, we need only look at the formulas for the differential $d\omega$, and so we will concentrate on this single component of the structure equations throughout. Using (2.5) and (2.6), we find that

$$d\omega = \sum_{i=0}^{r+k-1} (\beta_{i+1}\wedge\theta_i + T_i\omega\wedge\theta_i + T_i'\pi\wedge\theta_i) + T^*\pi\wedge\omega,$$

where T_i, T_i', T^*, are certain *torsion coefficients*, depending on the group parameters and the base coframe, and where the β_j are the right-invariant one-forms on the structure group $G^{(r+k)}$ corresponding to the group parameters B_j. After performing an obvious Lie algebra compatible absorption of torsion,[7,8] we are left with the structure equation

$$d\omega = \sum_{i=0}^{r+k-1} \tilde\beta_{i+1}^{(r+k)}\wedge\theta_i + T^*\pi\wedge\omega,$$

where the $\tilde\beta_j^{(r+k)}$ are congruent modulo the lifted coframe to the right-invariant one-forms β_j. Thus we readily deduce that only the coefficient

$$T^* = -(E/L)Z_{r+k},$$

is essential torsion. Clearly, $G^{(r+k)}$ acts on the essential torsion coefficient T^* by translation, and we can normalize this torsion coefficient to 0 by setting $Z_{r+k}=0$, or, equivalently, by setting $B_{r+k}=0$. Thus, at this stage, the algorithm automatically tells us to reduce the structure group to the subgroup $G^{(r+k-1)}$.

Thanks to the reduction theorem for the Cartan equivalence problem,[7,8] we know that the reduced problem with the same base coframe (2.1) and reduced structure group $G^{(r+k-1)}$ has the *same* set of solutions as the original equivalence problem. We proceed to analyze this reduced equivalence problem. Since $k\geqslant r-1\geqslant 1$, a second Lie algebra compatible absorption of torsion in the recomputed structure equation for $d\omega$ will yield

$$d\omega = \sum_{i=0}^{r+k-2} \tilde\beta_{i+1}^{(r+k-1)}\wedge\theta_i + T_{r+k-1}^*\,\theta_{r+k-1}\wedge\omega,$$

where the $\tilde\beta_j^{(r+k-1)}$ are congruent modulo the lifted coframe to the right-invariant one-forms β_j. Again, $G^{(r+k-1)}$ acts on the essential torsion coefficient

$$T_{r+k-1}^* = -(A_{r+k}^{r+k}/L)Z_{r+k-1}$$

by translation, and we can normalize the torsion coefficient to 0 by setting $Z_{r+k-1}=0$, or, equivalently, $B_{r+k-1}=0$, further reducing to the structure group $G^{(r+k-2)}$.

Clearly, this procedure continues until the derivatives of the Lagrangian L start contributing to the essential torsion in $d\omega$. This will occur when we have reduced our original problem to an equivalence problem with the same base coframe (2.1), and reduced structure group $G^{(r)}$. We now indicate how the above analysis changes at this point. After Lie algebra compatible absorption of torsion, we find

$$d\omega = \sum_{i=0}^{r-1} \tilde\beta_{i+1}^{(r)}\wedge\theta_i + T_r^*\,\theta_r\wedge\omega$$

as before, but where the essential torsion is now given by

$$T_r^* = -\frac{A_{r+1}^{r+1}}{L}\left(Z_r - \frac{\partial L}{\partial u_r}\right).$$

The structure group $G^{(r)}$ still acts on the essential torsion by translation, but there is an additional inhomogeneous term. Consequently, we can normalize this torsion coefficient to 0 by setting

$$Z_r = \frac{\partial L}{\partial u_r}. \tag{2.8}$$

Now, plugging (2.8) into the formula (2.6) for ω (with earlier normalizations $Z_{r+1}=\cdots=Z_{r+k}=0$ also taken into account) has the effect of (a) reducing the structure group to $G^{(r-1)}$, just as before, and (b) to incorporate the inhomogeneity, changing the base coframe so as to replace our original one-form $\omega_0 = L\,dx$ by the new one-form

$$\omega_0^{(r-1)} = L\,dx + \frac{\partial L}{\partial u_r}(du_{r-1} - u_r\,dx).$$

This new base coframe element constitutes our first "approximation" to the Cartan form. The corresponding lifted one-form coincides with (2.6) taking (2.8) into account, i.e.,

$$\omega = L\,dx + \frac{\partial L}{\partial u_r}\theta_r + \sum_{i=1}^{r-1} Z_i\theta_{i-1}.$$

We continue our reduction procedure by again recomputing the basic structure equation and reabsorbing. Now we find

$$d\omega = \sum_{i=0}^{r-2} \tilde\beta_{i+1}^{(r-1)}\wedge\theta_i + T_{r-1}^*\,\theta_{r-1}\wedge\omega + \cdots,$$

where the dots stand for other essential torsion terms that we will try not to deal with here. As usual, we normalize the torsion coefficient

$$T_{r-1}^* = -\frac{A_r^r}{L}\left(-Z_{r-1} + \frac{\partial L}{\partial u_{r-1}} - D_x\frac{\partial L}{\partial u_r}\right)$$

to 0 by setting

$$Z_{r-1} = \frac{\partial L}{\partial u_{r-1}} - D_x\frac{\partial L}{\partial u_r}.$$

Now we have reduced the structure group to $G^{(r-2)}$, and also modified the base coframe so as to replace $\omega_0^{(r-1)}$ by

$$\omega_0^{(r-2)} = L\,dx + \frac{\partial L}{\partial u_r}\theta_r + \left(\frac{\partial L}{\partial u_{r-1}} - D_x\frac{\partial L}{\partial u_r}\right)\theta_{r-1},$$

giving the next approximation to the Cartan form.

Clearly, we can continue in this manner, and a simple inductive argument will show that we end up normalizing all the entries of the vector Z, cf. (2.7), as

$$Z_i = \sum_{j=0}^{r-1} (-D_x)^j\left(\frac{\partial L}{\partial u_{i+j}}\right), \quad i=1,\ldots,r. \tag{2.9}$$

The structure group has finally been reduced to $G^{(0)}$, which consists of all invertible matrices of the form (2.2) with $B=0$. Substituting all the normalizations (2.9) into (2.6), we see that the base coframe element replacing ω_0 is now the Cartan form (1.3). Moreover, since the corresponding row of the structure group matrix consists of all 0s save for a 1 in the diagonal position, the Cartan form Θ_C is invariant under contact transformations (for the standard equivalence problem), and will be part of the invariant adapted coframe resulting from the full implementation of the Cartan algorithm. The general reduction theorem completes the proof of Theorem 3.

III. DISCUSSION

We now return to a more detailed discussion of our initial formulation of the equivalence problem. What we have shown is that, if we formulate the basic Lagrangian equivalence problem on the jet bundle J^{r+k} for $k \geqslant r-1$, *and* use the group $G^{(m)}$ for $m \geqslant r$ as our structure group, then the Cartan reduction procedure will naturally lead us to the Cartan form as discussed in Sec. II. There are two obvious objections to this formulation: First, according to the reduction theorem of Ref. 1, we are really working on too high an order jet bundle, and second, according to Bäcklund's theorem, we are using too large a structure group. Let us discuss the latter difficulty first.

As was presented in Pt. I,[1] Bäcklund's theorem[2] tells us that any contact transformation on J^{r+k} is just the prolongation of a contact transformation on the first jet bundle J^1. In particular, the base transformation of the independent variable x depends on at most first-order derivatives of u, $\bar{x} = \varphi(x,u,u_1)$, so the pull back $\Psi^*(d\bar{x}) = d\varphi$ will only involve the form dx and the first two contact forms θ,θ_1. This means that the structure group will naturally reduce to a subgroup of the group $G^{(2)}$, and we could have begun our reduction procedure with $G^{(2)}$ as the starting structure group without losing anything as far as the final solution to our equivalence problem is concerned. However, it is easy to see that, for $r \geqslant 3$, the $G^{(2)}$ equivalence problem can never lead to the Cartan form Θ_C as an adapted invariant coframe element. Indeed, in this case the lifted coframe element corresponding to the base form ω_0 just depends on the first two contact forms:

$$\omega = \omega_0 + B_1\theta + B_2\theta_1. \tag{3.1}$$

Barring prolongation, the Cartan reduction algorithm will eventually normalize the group parameters B_1, B_2, to be certain combinations of the Lagrangian and its derivatives, leading to an adapted coframe element of the same form (3.1). If $r \geqslant 3$, this cannot be the Cartan form (1.3) since it does not involve enough contact forms!

We seem to be left with a paradox: If we reduce the equivalence problem using the larger structure group $G^{(m)}$ for $m \geqslant r$, we are naturally led to the Cartan form, whereas if we reduce using the more reasonable structure group $G^{(2)}$, which mathematically encodes the *same* equivalence problem, we cannot obtain the Cartan form directly. This state of affairs appears to be contradictory, especially considering that all these problems are the same, and must therefore lead to the same necessary and sufficient conditions for equivalence of the two variational problems. The resolution of the difficulty is to realize that the Cartan solution to the $G^{(2)}$ equivalence problem will lead to additional adapted invariant coframe elements that will be certain particular linear combinations of the contact forms alone. Since any linear combination of invariant one-forms, whose coefficients are scalar invariants, is itself an invariant one-form, we conclude that the Cartan form (1.3) *must* appear in this version of the equivalence problem, but in disguised form. Namely, we deduce that there *is* an invariant one-form of the form

$$\omega^* = \omega_0 + B_1^*\theta + B_2^*\theta_1,$$

where B_1^* and B_2^* will be certain combinations of L and its derivatives. Moreover, there exist *additional* invariant one-forms that are certain combinations of contact forms

$$\theta_j^* = \sum_{i=0}^{r} A_{ij}^*\,\theta_i,$$

with the property that Θ_C is the sum of these component pieces,

$$\Theta_C = \omega^* + \sum I_j\,\theta_j^*, \tag{3.2}$$

where the I_j are either constants, or, perhaps, invariants of the problem. Thus the Cartan form does appear as an invariant one-form for the $G^{(2)}$, but in the disguised form (3.2), *not* directly as an adapted coframe element. (In a subsequent paper, we will illustrate this point in some special cases.) It would be interesting to find the formulas for the "reduced" invariant one-form ω^* and determine its geometric or analytic significance for the original Lagrangian.

However, as we have demonstrated, the Cartan form appears much more directly if we "artificially" expand the original structure group to be $G^{(r+k)}$ (or even just $G^{(r)}$) even though we know that this is ultimately not necessary for the solution of the equivalence problem. A key lesson of this exercise appears to be that the use of different (larger) structure groups to encode the self-same equivalence problem can lead to different adapted coframe elements, even though all the different possible invariant coframes must be related to each other according to a formula like (3.2).

There is another way to interpret our results. We could begin the entire reduction procedure by using the reduced structure group $G^{(2)}$ initially, as would be warranted by the form of the contact transformations. However, we would need to compensate by replacing our original base coframe element $\omega_0 = L\,dx$ by a slightly different one-form having the form

$$\tilde{\omega}_0 = L\,dx + \sum_{j=0}^{r+k-1} \lambda_j \theta_j,$$

where the coefficients $\lambda_j : J^{r+k} \to \mathbb{R}$ are arbitrary, to be determined during the course of the application of Cartan's method. However, as the reader can verify, these two approaches are essentially the same and lead to the same conclusion.

The second difficulty with our original formulation is that we were forced to use a higher-order jet bundle, namely, J^{2r-1}, than is really necessary for solving the equivalence problem. Indeed, if we do solve the Cartan equivalence problem on the minimal-order jet bundle J^r, then, barring prolongation, we will be led to a complete set of r th-order invariants and invariant one-forms. How does the Cartan form arise here? The answer is provided by the "derivative covariants," which are certain combinations of the Lagrangian and its derivatives of u. (See Ref. 1 for the details.) If we replace all the derivatives of u of order higher than r that appear in the Cartan form (1.2) by their corresponding derivative covariants, we will be led to a purely r th-order one-form, which incorporates all the transformation properties of the Cartan form, even though the explicit formula for it will be quite a bit more complicated than (1.3). (For instance, it will depend nonlinearly on the Lagrangian.) Thus there is purely r th-order invariant one-form that corresponds to the Cartan form, and hence will appear in the equivalence problem on J^r, either directly as an adapted coframe element, or more probably, in disguised form similar to (3.2).

In a subsequent paper in this series, we will illustrate all these matters with a concrete problem—the equivalence problem for a second-order particle Lagrangian. Also, we hope to extend these techniques to higher dimensional Lagrangians, where the nonuniqueness of the Cartan form becomes an issue.[13,14]

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