Moving Coframes

II. Regularization and Theoretical Foundations

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Abstract. The primary goal of this paper is to provide a rigorous theoretical justification of Cartan's method of moving frames for arbitrary finite-dimensional Lie group actions on manifolds. The general theorems are based a new regularized version of the moving frame algorithm, which is of both theoretical and practical use. Applications include a new approach to the construction and classification of differential invariants and invariant differential operators on jet bundles, as well as equivalence, symmetry, and rigidity theorems for submanifolds under general transformation groups. The method also leads to complete classifications of generating systems of differential invariants, explicit commutation formulae for the associated invariant differential operators, and a general classification theorem for syzygies of the higher order differentiated differential invariants. A variety of illustrative examples demonstrate how the method can be directly applied to practical problems arising in geometry, invariant theory, and differential equations.

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1. Introduction.

This paper is the second in a series devoted to the analysis and applications of the method of moving frames and its generalizations. In the first paper, [**9**], we introduced the method of moving coframes, which can be used to practically compute moving frames and differential invariants, and is applicable to finite-dimensional Lie transformation groups as well as infinite-dimensional pseudo-group actions. In this paper, we introduce a second method, called regularization, that not only provides, in a simple manner, the theoretical justification for the method of moving frames in the case of finite-dimensional Lie group actions, but also gives an alternative, practical approach to their construction. The regularized method successfully bypasses many of the complications inherent in traditional approaches by completely avoiding the usual process of normalization during the general computation. In this way, the issues of branching and regularity do not arise. Once a moving frame and coframe, along with the complete system of invariants, are constructed in the regularized framework, one can easily restrict these invariants to particular classes of submanifolds, producing (in nonsingular cases) the standard moving frame. Perhaps Griffiths is the closest in spirit to our guiding philosophy; we fully agree with his statement, [**12**; p. 777], that "The effective use of frames ... goes far beyond the notion that 'frames are essentially the same as studying connections in the principal bundle of the tangent bundle'." Indeed, by de-emphasizing the group theoretical basis for the method, which, in the past, has hindered the theoretical foundations from covering all the situations to which the practical algorithm could be applied, our formulation of the framework goes beyond what Griffiths envisioned, and successfully realizes Cartan's original vision, [**5, 7**]. The regularized method can be readily used to compute all classical, known examples of moving frames, as well as a vast array of other, non-traditional Lie group actions. Indeed, the method is not restricted to transitive group actions on homogeneous spaces, although these form an important subclass of transformation groups that can be handled by our general procedure.

In general, given a finite-dimensional Lie group G acting on a manifold M, a moving frame (of order zero) is defined as a G-equivariant map $\rho: M \to G$. Moving frames on submanifolds $N \subset M$ are then obtained by restriction. This general definition appears in Griffiths, [**12**], Green, [**11**], and Jensen, [**14**], and can be readily reconciled with classical geometrical constructions, [**9**]. It is not hard to see that an order zero moving frame can only exist when the group action is free and regular. Consequently, the first part of this paper will be devoted to developing the theory of moving frames in the simple context of free group actions on manifolds. We show how a moving frame and a complete system of invariants can be constructed via the process of normalization. Normalization amounts to choosing a cross-section $K \subset M$ to the group orbits, and computing the group element $g = \rho(z)$ which maps a point $z \in M$ in the manifold to the chosen cross-section, so $g \cdot z \in K$. The resulting map $\rho: z \mapsto g$ from the manifold to the group is the moving frame. With this data in hand, the group action can be characterized as the local diffeomorphisms which preserve a system of invariant functions and one-forms that are prescribed by the choice of cross-section and the pull-back of the Maurer-Cartan forms on the group via the moving frame. By restricting the invariant functions and one-forms to a submanifold, the solution to the basic congruence and symmetry problems follow directly from Cartan's solution to the general equivalence problem for coframes, [**8**, **18**]. That is, the invariants and the derived invariants of a submanifold serve to parameterize a classifying manifold that uniquely characterizes the equivalence class and symmetries of the submanifold under the action of the group.

If the prescribed group action is not free on M, then an order zero moving frame cannot be determined. The strategy then is to prolong the group action to the jet bundles $\mathrm{J}^n = \mathrm{J}^n(M,p)$ of n-jets of p-dimensional submanifolds of the underlying manifold M. Assuming that the group G acts effectively on subsets[†] then the prolonged transformation group will act locally freely on an open subset of J^n for n sufficiently large, and hence one can use the moving frame construction described in the previous paragraph to determine a moving frame of order n for regular submanifolds. In general, the invariants and derived invariants associated with such a moving frame can be identified with a complete system of n^{th} order differential invariants for the transformation group. Thus, the congruence and symmetry theorems for regular submanifolds are easily restated in terms of differential invariants and their associated classifying manifold. Moreover, our methods have the widest range of generality possible; by using sufficiently high order jets, we are able to establish moving frames for all submanifolds except those which are "totally singular". The latter can be geometrically characterized as submanifolds whose isotropy subgroup does not act freely thereon, and hence cannot be endowed with fully determined moving frames. For example, in equi-affine geometry, the straight lines are totally singular, and do not possess equi-affine moving frames. In this manner, the regularized procedure also sheds light on a comment of Weyl, [**27**; p. 600], on the desirability of investigating "special classes of manifolds by imposing conditions on the invariants", using the example of minimal curves in Euclidean geometry where the usual normalization procedure breaks down. A related idea of I. Anderson (personal communication) involves the regularization of differential invariants for transformation groups by introducing additional parametric coordinates in order to avoid "phantom" singularities in jet space. The regularized moving frame method provides a general construction that allows one to rigorously implement the ideas of Weyl and Anderson in practical situations.

A key idea that underlies our theory of regularization is to replace any complicated group action on a manifold by a "lifted action" of the group on the trivial principal bundles $\mathcal{B}^{(n)} = G \times \mathrm{J}^n$ over the original manifold and its associated jet spaces. Once the action of the group is free on a particular jet space, the moving frame map is nothing but an equivariant section of the principal bundle $\mathcal{B}^{(n)}$ under the lifted action. The equivariant section so obtained allows one to pull back invariant objects on the principal bundle to the base. Fortunately, all the invariant objects on the principal bundle are trivial to construct, and so the particularities of the construction are all embodied in the chosen moving frame section, and can thereby be systematically analyzed.

The regularization approach to moving frames provides new, effective tools for understanding the geometry of submanifolds and their jets under a transformation group.

[†] This condition is very mild. See Section 2 for the precise definition, and a demonstration that it always holds, without loss of generality, in the analytic category.

Applications include a new and more general proof of the fundamental theorem on classification of differential invariants, a general classification theorem for syzygies of differential invariants, as well as new explicit commutation formulae for the associated invariant differential operators. We demonstrate a simple but striking generalization of a “replacement theorem” due to T.Y. Thomas, [**24**]. Two types of general rigidity theorems, uniquely characterizing congruent submanifolds by finite order jets, are proved, thereby extending known results for submanifolds of homogeneous spaces. We also give a new basis for Ovsiannikov's theory of partially invariant solutions of partial differential equations, [**22**]. All of our theoretical results are provided in a form that can be applied to practical examples, which we illustrate with several explicit examples of independent interest in Section 15. This wide range of both theoretical and practical applications clearly demonstrates the power of our approach to the general theory of moving frames.

2. Lie Transformation Groups.

Let us begin by collecting some basic terminology associated with finite-dimensional transformation groups. See [**18**] for details. Throughout this paper, G will denote an r-dimensional Lie group acting smoothly on an m-dimensional manifold M.

Definition 2.1. The *isotropy subgroup* of a subset $S \subset M$ is

$$G_S = \{\, g \in G \mid \; g \cdot S = S \,\}. \tag{2.1}$$

The *global isotropy subgroup* is the subgroup

$$G_S^* = \bigcap_{x \in S} G_x = \{\, g \in G \mid \; g \cdot s = s \text{ for all } s \in S \,\}$$

consisting of those group elements which fix *all* points in S.

Definition 2.2. The group G acts

(i) *freely* if $G_z = \{e\}$ for all $z \in M$,

(ii) *locally freely* if G_z is a discrete subgroup of G for all $z \in M$,

(iii) *effectively* if $G_M^* = \{e\}$,

(iv) *effectively on subsets* if $G_U^* = \{e\}$ for every open $U \subset M$,

(v) *locally effectively* if G_M^* is a discrete subgroup of G,

(vi) *locally effectively on subsets* if G_U^* is a discrete subgroup of G for every open $U \subset M$.

If the group G does not act effectively, one can, without any loss of generality, replace G by the effectively acting quotient group G/G_M^*, which acts in essentially the same manner as G does, cf. [**18**]. Clearly, if G acts effectively on subsets, then G acts effectively. Analytic continuation demonstrates that the converse is true in the analytic category. However, it does not hold for more general smooth actions as the following elementary example shows.

Example 2.3. Let $h(x)$ be any C^∞ function such that $h(x) > 0$ for $x > 0$, but $h(x) = 0$ for $x \leq 0$. Let $G \simeq \mathbb{R}^2$ be the two-parameter abelian transformation group acting on $M = \mathbb{R}^2$ via $(x, u) \mapsto (x, u + a h(x) + b h(-x))$, where $(a, b) \in G$ and $(x, u) \in M$. Then G acts effectively on M, but not effectively on any open subset that is contained in either the right or left half plane.

Since they do not arise in usual applications, we will not attempt to analyze pathological smooth actions which are effective but not subset effective. Thus we shall, without significant loss of generality, only consider transformation groups that act effectively on subsets.

Definition 2.4. A group G acts *semi-regularly* on M if all its orbits have the same dimension. A semi-regular group action is *regular* if, in addition, each point $x \in M$ has arbitrarily small neighborhoods whose intersection with each orbit is a connected subset thereof.

Proposition 2.5. *An r-dimensional Lie group G acts locally freely on M if and only if its orbits all have dimension r.*

Definition 2.6. Suppose G acts semi-regularly on the m-dimensional manifold M with s-dimensional orbits. A (local) *cross-section* is a $(m-s)$-dimensional submanifold $K \subset M$ such that K intersects each orbit transversally. The cross-section is *regular* if K intersects each orbit at most once.

If G acts semi-regularly, then the Implicit Function Theorem guarantees the existence of local cross-sections at any point of M. Regular actions admit regular local cross-sections.

Example 2.7. The following simple construction, based on the Frobenius Theorem, cf. [**18**], is of fundamental importance for the theoretical justification of the method of moving frames. Suppose G acts freely and regularly on M. Then we can introduce *flat local coordinates*

$$z = (x, y) = (x^1, \ldots, x^r, y^1, \ldots, y^{m-r}), \qquad x \in G, \quad y \in Y, \tag{2.2}$$

that locally identify M with a subset of the Cartesian product $G \times Y$, with $Y \simeq \mathbb{R}^{m-r}$, and such that the action of G reduces to the trivial left action $g \cdot z = (g \cdot x, y)$. The y coordinates provide a complete system of functionally independent invariants for the group action. In these coordinates, a general cross-section is given by the graph $K = \{(a(y), y)\}$ of a smooth map $a: Y \to G$. When we use flat coordinates, we shall always assume, without loss of generality, that the identity cross-section $\{e\} \times Y$, i.e., when $a(y) \equiv e$, belongs to the flat coordinate chart.

Remark: In practice, of course, the determination of the flat coordinates for a given transformation group action may be extremely difficult. A significant achievement of the method of moving frames is that it allows one to compute invariants without having to find the flat coordinates, or integrate any differential equations.

Throughout this paper, we shall let $\mathfrak{g}$ denote the *right* Lie algebra of G consisting of right-invariant vector fields on G. The map $\psi: \mathbf{v} \mapsto \widehat{\mathbf{v}}$ that associates a Lie algebra element $\mathbf{v} \in \mathfrak{g}$ to the corresponding infinitesimal generator $\widehat{\mathbf{v}} = \psi(\mathbf{v})$ of the associated one-parameter subgroup forms a Lie algebra homeomorphism from $\mathfrak{g}$ to the space of vector fields on M. The kernel of ψ coincides with the Lie algebra of the global isotropy subgroup G_M^*, thereby identifying the Lie algebra of infinitesimal generators $\widehat{\mathfrak{g}} = \psi(\mathfrak{g})$ with the quotient Lie algebra of the effectively acting quotient group G/G_M^*. In particular, G acts locally effectively if and only if $\ker \psi = \{0\}$.

3. Regularization.

Our approach to the theory of moving frames is based on the following simple but remarkably powerful device. In general, any complicated transformation group action can be "regularized" by lifting it to a suitable bundle sitting over the original manifold. The construction is reminiscent of the regularization procedure based on universal bundles used to compute equivariant cohomology, cf. [**3**], [**13**; § 4.11], although our method is considerably simpler in that we only require finite-dimensional bundles.

Let G be a smooth transformation group acting on a manifold M. Let $\mathcal{B} = G \times M$ denote the trivial left† principal G bundle over M.

Definition 3.1. The *left regularization* of the action of G on M is the diagonal action of G on $\mathcal{B} = G \times M$ provided by the maps

$$\widehat{L}_g(h,z) = \widehat{L}(g,(h,z)) = (g\cdot h, g\cdot z), \qquad g\in G, \quad (h,z)\in \mathcal{B}. \tag{3.1}$$

The *right regularization* of G is given by

$$\widehat{R}_g(h,z) = \widehat{R}(g,(h,z)) = (h\cdot g^{-1}, g\cdot z), \qquad g\in G, \quad (h,z)\in \mathcal{B}. \tag{3.2}$$

We will also refer to the regularized actions (3.1), (3.2), as the left or right *lifted action* of G since either projects back to the given action on M via the G equivariant projection $\pi_M : \mathcal{B} \to M$. In the sequel, the left (respectively right) regularization of a group action will lead to left (right) moving frames associated with submanifolds of M. The key, elementary result is that regularizing any group action immediately eliminates all singularities and irregularities, e.g., lower dimensional orbits, non-embedded orbits, etc. Moreover, the orbits of G in M are the projections of their lifted counterparts in $\mathcal{B}$; all of the lifted orbits have the same dimension as G itself.

Theorem 3.2. *The right and left regularizations of any transformation group G define regular, free actions on the bundle $\mathcal{B} = G \times M$.*

Thus, lifting the action of G on M to the bundle $\mathcal{B}$ has the effect of completely eliminating any irregularities appearing in the original action.

Definition 3.3. A *lifted invariant* is a (locally defined) smooth function $L : \mathcal{B} \to N$ which is invariant with respect to the (either left or right) lifted action of G on $\mathcal{B}$.

Both regularized actions admit a complete system of globally defined, functionally independent lifted invariants.

Definition 3.4. The *fundamental right lifted invariant* is the multiplication function $w : \mathcal{B} \to M$ given by

$$w = g\cdot z. \tag{3.3}$$

† Modern treatments of principal bundles, e.g., [**13**, **23**], tend to concentrate on right principal bundles. However, we find the left version more convenient for our purposes.

The *fundamental left lifted invariant* is the function $\widetilde{w}: \mathcal{B} \to M$ given by

$$\widetilde{w} = g^{-1} \cdot z. \tag{3.4}$$

From the point of view of invariants and moving frames, right regularization is the simpler of the two because its fundamental invariant does not require the computation of the inverse transformation g^{-1}. On the other hand, in the literature, most examples are constructed using the left regularization. Moreover, the final formulae for the moving frame are typically simpler if the left regularization is used. However, the theoretical and practical aspects of our regularized moving frame method underline the primacy of the right version. Therefore, from now on, the terms "regularization" or "lift" without qualification will always mean the *right* versions of these objects. All results will automatically have a left counterpart, typically found by applying the group inversion $g \mapsto g^{-1}$.

Proposition 3.5. *The fundamental lifted invariant $w = g \cdot z$ is invariant with respect to the regularized action* (3.2) *of G on $\mathcal{B}$. Moreover, given $z \in M$, the corresponding level set $w^{-1}\{z\}$ coincides with the orbit of G through the point $(e, z) \in \mathcal{B}$.*

If we introduce local coordinates on M, then the components of w form a complete system of $m = \dim M$ functionally independent invariants on $\mathcal{B}$.

Proposition 3.6. *Any lifted invariant $L: \mathcal{B} \to N$ can be locally written as a function of the fundamental lifted invariants, $L(g,z) = F[w(g,z)]$, so that $L = F \circ w$ for some $F: M \to N$.*

In particular, if $F(z)$ is any function on M, then we can produce a lifted invariant $F \circ w$ on $\mathcal{B}$ by replacing z by $w = g \cdot z$ in the formula for F. The ordinary invariants $I: M \to N$ of the group action are particular cases of lifted invariants, where we identify I with its composition $I \circ \pi_M$ with the standard projection. Therefore, Proposition 3.6 indicates that ordinary invariants are particular functional combinations of lifted invariants that happen to be independent of the group parameters. For such functions, a simple but striking "replacement theorem" provides an explicit formula expressing an ordinary invariant in terms of the lifted invariants.

Theorem 3.7. *If $I(z) = F(w(g,z)) = F(g \cdot z)$ is an ordinary invariant, then $F(z) = I(z)$.*

Proof: In other words, replacing z by w in the formula for the invariant does not change its value, i.e., $I(z) = I(w)$. To prove this result, we use the invariance of I and the fact that at the identity $g = e$, the lifted invariant reduces to $w = z$. *Q.E.D.*

Example 3.8. Let $G = \mathrm{SO}(2)$ be the rotation group acting on $M = \mathbb{R}^2$ via

$$(x, u) \longmapsto (x \cos\theta - u \sin\theta, x \sin\theta + u \cos\theta). \tag{3.5}$$

The (right) regularized action on the cylinder $\mathcal{B} = \mathrm{SO}(2) \times \mathbb{R}^2$ is given by supplementing the planar transformation rules (3.5) with the group law $\phi \mapsto (\phi - \theta) \bmod 2\pi$. Note that the action on $\mathcal{B}$ is regular, so we have effectively replaced the singular orbit at the origin by a regular orbit $\{(0,0)\} \times \mathrm{SO}(2) \subset \mathcal{B}$. There are two fundamental right lifted invariants:

$$y = x\cos\phi - u\sin\phi, \qquad v = x\sin\phi + u\cos\phi. \tag{3.6}$$

Note that

$$r^2 = y^2 + v^2 = x^2 + u^2$$

is an invariant for the lifted action which reduces to the ordinary radial invariant for the action back on M. The fact that r has the same formula in terms of x, u as it does in y, v is a simple manifestation of the general Replacement Theorem 3.7.

A differential form ω on the principal bundle $\mathcal{B} = G \times M$ is (right) G-invariant if it satisfies $(\widehat{R}_g)^*\omega = \omega$ for every $g \in G$. Of particular importance are the (pulled-back) Maurer–Cartan forms associated with the Lie group G. We introduce a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ for the (right) Lie algebra $\mathfrak{g}$ of G. The corresponding dual basis $\boldsymbol{\mu} = \{\mu^1, \ldots, \mu^r\}$ for the right-invariant differential forms on G are known as the *Maurer–Cartan forms*. We shall also use $\boldsymbol{\mu}$ to denote the corresponding Maurer–Cartan one-forms on $\mathcal{B}$, namely the pull-backs $(\pi_G)^*\boldsymbol{\mu}$ of the forms on G under the standard projection $\pi_G\colon \mathcal{B} \to G$. The Maurer–Cartan forms $\boldsymbol{\mu}$ on $\mathcal{B}$ are invariant under the right regularized action of G.

Since $\mathcal{B} = G \times M$ is a Cartesian product, its differential d naturally splits into a group and manifold components: $d = d_G + d_M$. Moreover, since the regularized action (3.2) is a Cartesian product action, the splitting is G-invariant.

Proposition 3.9. *If ω is any G-invariant differential form on $\mathcal{B}$, then both $d_M\omega$ and $d_G\omega$ are invariant forms. In particular, if L is any lifted invariant, then $d_M L$ and $d_G L$ are invariant one-forms on $\mathcal{B}$.*

In particular, the differential dw of the fundamental lifted invariant $w = g \cdot z$ will split into two sets of invariant one-forms on $\mathcal{B}$, namely $d_M w = g \cdot dz$ and the group component $d_G w$. The notation $g \cdot dz$ is meant suggestively; in terms of local coordinates $(z^1, \ldots, z^m)$ on M, the components of $g \cdot dz$ are the pull-backs $g^*\, dz^i$ of the coordinate differentials via the group transformation g. There is a beautiful explicit formula that expresses group components $d_G w$ as invariant linear combinations of the Maurer–Cartan forms $\boldsymbol{\mu}$ on $\mathcal{B}$.

Theorem 3.10. *Let G act on M. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be a basis for the Lie algebra $\mathfrak{g}$, and let*

$$\widehat{\mathbf{v}}_\kappa = \sum_{i=1}^m f^i_\kappa(z)\, \frac{\partial}{\partial z^i}\,, \qquad \kappa = 1, \ldots, r, \tag{3.7}$$

be the corresponding infinitesimal generators on M, written in local coordinates $z = (z^1, \ldots, z^m)$. Let $\boldsymbol{\mu} = \{\mu^1, \ldots, \mu^r\}$ be the dual Maurer–Cartan forms, pulled back to $\mathcal{B}$. Let $w = (w^1, \ldots, w^m)$ be the components of the the fundamental lifted invariant $w = g \cdot z$, expressed in the same local coordinates. Then the group differential of the components of w are given by

$$d_G w^i = \sum_{\kappa=1}^r f^i_\kappa(w)\, \mu^\kappa\,, \qquad i = 1, \ldots, m. \tag{3.8}$$

In other words, the coefficients of the Maurer–Cartan forms in (3.8) *are the lifted invariant counterparts of the coefficients of the infinitesimal generators* (3.7), *obtained by replacing* z *by the lifted invariant* w.

Proof: Let $\mathbf{v} \in \mathfrak{g}$ correspond to the infinitesimal generator $\widehat{\mathbf{v}}$ on M. For simplicity, we use the same notation for the corresponding vertical and horizontal vector fields on $\mathcal{B}$, which generate the actions[†] $(h,z) \mapsto (\exp(t\mathbf{v})\cdot h, z)$ and $(h,z) \mapsto (h, \exp(t\widehat{\mathbf{v}})\cdot z)$ respectively. (The infinitesimal generators of the left regularization (3.1), then, are the sums $\mathbf{v} + \widehat{\mathbf{v}}$ of these vector fields.) We then notice that

$$\mathbf{v}(w) = \frac{d}{dt}\left[\big(\exp(t\mathbf{v})\cdot g\big)\cdot z\right]\bigg|_{t=0} = \frac{d}{dt}\left[\exp(t\widehat{\mathbf{v}})\cdot w\right]\bigg|_{t=0}.$$

The latter expression is equal to the value of the vector field $\widehat{\mathbf{v}}$ at the point $w = g\cdot z$; therefore, in local coordinates,

$$\mathbf{v}_\kappa(w^i) = f^i_\kappa(w), \qquad i = 1,\ldots,m, \quad \kappa = 1,\ldots,r.$$

On the other hand, duality[‡] of the Maurer–Cartan forms implies that

$$d_G w^i = \sum_{\kappa=1}^{r} \mathbf{v}_\kappa(w^i)\,\mu^\kappa = \sum_{\kappa=1}^{r} f^i_\kappa(w)\,\mu^\kappa,$$

completing the proof. *Q.E.D.*

Remark: Theorem 3.10 justifies the method for computing Maurer–Cartan forms directly from the group transformations introduced in part I, [**9**].

Example 3.11. Return to the rotation group acting on $M = \mathbb{R}^2$ as in (3.5). Applying d_M and d_G to the lifted invariants (3.6) will produce four lifted invariant one-forms on $\mathcal{B}$. The manifold components are

$$d_M y = (\cos\phi)\,dx - (\sin\phi)\,du, \qquad d_M v = (\sin\phi)\,dx + (\cos\phi)\,du.$$

On the other hand, the group components can be written as invariant multiples of the Maurer–Cartan form $\mu = d\phi$, namely

$$d_G y = -(x\sin\phi + u\cos\phi)\,d\phi = -v\,d\phi, \qquad d_G v = (x\cos\phi - u\sin\phi)\,d\phi = y\,d\phi.$$

Equation (3.8) implies that the coefficients $(-v, y)$ can be computed directly as the invariant counterparts of the coefficients $(-u, x)$ of the infinitesimal generator $\widehat{\mathbf{v}} = -u\partial_x + x\partial_u$.

Remark: A lifted invariant $L(g,z) = F(w)$ is independent of all group parameters, and hence reduces to an ordinary invariant as in Theorem 3.7 if and only if $d_G L = 0$. In view of (3.8), the equation $d_G L(g,z) = d_G F(w) = 0$ is equivalent to the usual Lie infinitesimal invariance conditions $\mathbf{v}_\kappa(F(z)) = 0$, $\kappa = 1,\ldots,r$, rewritten in terms of w instead of z.

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† Recall that the right-invariant vector fields generate the left action of G on itself.

‡ See Example 5.13 below for details.

4. Moving Frames.

Let us now define moving frames in the context of a Lie group acting on a manifold. The justification for this definition appears in part I, [**9**], and is based on the earlier work of Green, [**11**], Griffiths, [**12**], and Jensen, [**14**].

Definition 4.1. Given a transformation group G acting on a manifold M, a *moving frame* is a smooth G-equivariant map

$$\rho : M \longrightarrow G. \tag{4.1}$$

In (4.1), we can use either the right or the left action of G on itself, and thus speak of right and left moving frames. As in the usual method of moving frames, we shall only be interested in their local existence and construction. Thus, we can relax our condition and only require local G-equivariance of the moving frame map, i.e., for group elements near the identity. There is an elementary correspondence between right and left moving frames.

Lemma 4.2. *If $\widetilde{\rho}(z)$ is a left moving frame on M, then $\rho(z) = \widetilde{\rho}(z)^{-1}$ is a right moving frame.*

Example 4.3. An important example is when G is a Lie group acting on itself, so $M = G$, by left multiplication $h \mapsto g \cdot h$. If $a \in G$ is any fixed element, then the map $\widetilde{\rho}(g) = g \cdot a$ clearly defines a (left) moving frame. Moreover, every (left) moving frame necessarily has this form, with $a = \widetilde{\rho}(e)$. Similarly, every right moving frame is provided by a map $\rho(g) = a \cdot g^{-1}$ for some fixed $a \in G$.

Not every group action admits a moving frame. The key condition is that the action be both free and regular.

Theorem 4.4. *If G acts on M, then a moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.*

Proof: To see the necessity of freeness, suppose $z \in M$, and let $g \in G_z$ belong to its isotropy subgroup. Let $\widetilde{\rho} : M \to G$ be a left moving frame. Then, by left equivariance of $\widetilde{\rho}$,

$$\widetilde{\rho}(z) = \widetilde{\rho}(g \cdot z) = g \cdot \widetilde{\rho}(z).$$

Therefore $g = e$, and hence $G_z = \{e\}$ for all $z \in M$. To prove regularity, suppose that $z \in M$ and that there exist points $z_\kappa = g_\kappa \cdot z$ belonging to the orbit of z such that $z_\kappa \to z$ as $\kappa \to \infty$. Thus, by continuity,

$$\widetilde{\rho}(z_\kappa) = \widetilde{\rho}(g_\kappa \cdot z) = g_\kappa \cdot \widetilde{\rho}(z) \longrightarrow \widetilde{\rho}(z) \qquad \text{as} \qquad \kappa \to \infty,$$

which implies that $g_\kappa \to e$ in G. This suffices to ensure regularity of the orbit through z.

To prove sufficiency, we use the flat local coordinates $z = (x, y) \in G \times Y$ introduced in Example 2.7. A general local cross-section $K \subset M$ is given by a graph $x = a(y)$. Then the map

$$\widetilde{\rho}(x, y) = x \cdot a(y) \tag{4.2}$$

is clearly G-equivariant under left multiplication on G, and hence defines a left moving frame. Moreover, every left moving frame has this form, provided we define the cross-section via $a(y) = \widetilde{\rho}(e, y)$. *Q.E.D.*

Remark: If G acts only semi-regularly and/or locally freely[†], then the preceding proof can be easily adapted to find a locally G-equivariant moving frame.

Theorem 4.5. *If $\rho(z)$ is a right moving frame, then the components of the map $I: M \to M$ defined by $I(z) = \rho(z) \cdot z$ provide a complete system of invariants for the group.*

Proof: Using our flat local coordinates, Lemma 4.2 implies that the right moving frame corresponding to (4.2) is

$$\rho(z) = a(y)^{-1} \cdot x^{-1}, \qquad z = (x, y). \tag{4.3}$$

Therefore

$$\rho(z) \cdot z = (a(y)^{-1}, y) \in K. \tag{4.4}$$

In particular, the last $m - r$ components of (4.4) provide the invariants y, while the first r components are functions of the invariants. *Q.E.D.*

The proof of Theorem 4.4 shows that the determination of a moving frame is intimately connected to the process of choosing a cross-section to the group orbits. Example 4.3 is a particular case of this construction since a cross-section to a transitive group action is just a single point. Equation (4.4) shows that the group element $g = \rho(z)$ given by the right moving frame map can be geometrically characterized as the unique group transformation that moves the point z onto the cross-section K. Moreover, $I(z) = \rho(z) \cdot z$ is the point on the cross-section K that lies on the G orbit passing through z.

Remark: In fact, any map $\rho: M \to G$ that satisfies $I(z) = \rho(z) \cdot z \in K$ will produce invariants by a choice of local coordinates on K. The action of G need not be free and the map ρ need not be equivariant; moreover, the group can equally well be a pseudo-group.

Theorem 4.5 implies that if $J(z)$ is any other invariant function, then, locally, we can write $J(z) = H(I(z))$ in terms of the moving frame invariants I. As noted in the proof, the components of I are not necessarily functionally independent, but one can always locally choose a set of $m - r$ components which do provide a complete system of functionally independent invariants, or, equivalently, a system of local coordinates on the quotient manifold M/G.

An alternative way of understanding the moving frame construction presented above is to view the regularization of a group action as giving rise to the double fibration

$$\begin{array}{ccc} & G \times M & \\ {}^{\pi_M}\swarrow & & \searrow^{w} \\ M & & M \end{array} \tag{4.5}$$

of the regularized bundle $\mathcal{B}$ over M. Given a cross-section K to the G orbits, the set

$$\mathcal{L} = w^{-1}(K) \subset \mathcal{B} = G \times M$$

[†] A (locally) free action is automatically semi-regular.

forms an m-dimensional submanifold of $\mathcal{B}$ that is invariant with respect to the lifted action of G on $\mathcal{B}$. Projection onto M defines a locally equivariant diffeomorphism $\pi_M: \mathcal{L} \xrightarrow{\sim} M$ and hence $\mathcal{L}$ is the graph of a local section $\sigma = (\pi_M \mid \mathcal{L})^{-1}$, called the *moving frame section*. It is not hard to see that σ defines the graph of the moving frame, so $\sigma(z) = (\rho(z), z)$ for $z \in M$, i.e.,

$$\rho = \pi_G \circ \sigma: M \longrightarrow G.$$

Since $\sigma: M \to \mathcal{L}$ is G-equivariant, any invariant object on $\mathcal{B}$ pulls-back, via σ, to an invariant object on M. In particular, the invariant $I(z) = \rho(z) \cdot z$ constructed in Theorem 4.5 is given by

$$I = \sigma^*(w) = w \circ \sigma: M \longrightarrow K.$$

As noted above, given any function $F: M \to \mathbb{R}$, the composition $F \circ w: \mathcal{B} \to \mathbb{R}$ defines a lifted invariant, $L(g, z) = F(g \cdot z)$. Moreover, pulling back L via the moving frame section $\sigma: M \to \mathcal{B}$, defines an ordinary invariant $J(z) = F(w(\sigma(z))) = F(\rho(z) \cdot z)$. Thus a moving frame provides a natural way to construct invariants from arbitrary functions!

Definition 4.6. The *invariantization* of a function $F: M \to N$ with respect to a moving frame $\rho: M \to G$ is the composition $J = F \circ w \circ \sigma = F \circ I$.

Invariantization *does* depend on the choice of moving frame. Geometrically, $J(z)$ equals the value of F at the point on the cross-section that lies on the G orbit through z. Theorem 3.7 says that if F itself is an invariant, then $F \circ w$ is independent of the group parameters, and hence $J = F$, i.e., the invariantization process leaves invariants unchanged. Thus, one can view invariantization as a projection operator from the space of functions to the space of invariants.

Example 4.7. Consider the usual action (3.5) of SO(2), which is regular on $M = \mathbb{R}^2 \setminus \{0\}$. The positive u axis defines a cross-section $K = \{ (0, v) \mid v > 0 \}$ to the orbits. The map $g = \rho(x, u): M \to K$ which rotates the point (x, u) to the point $(0, r) \in K$, where $r = \sqrt{x^2 + u^2}$, is clearly SO(2)–equivariant. The moving frame $\rho: M \to \mathrm{SO}(2)$ induced by this choice of cross-section is therefore given by the equivariant map $\phi = \tan^{-1}(x/u)$ that determines the the rotation angle needed to map (x, u) to K. The corresponding moving frame section $\sigma: M \to \mathcal{B} = \mathrm{SO}(2) \times M$ is given by $\sigma(x, u) = (\tan^{-1}(x/u), x, u)$. Pulling back the lifted invariants (3.6) produces the invariants $\sigma^* y = 0$, $\sigma^* v = r$. If $F(x, u)$ is any function, then $L(\phi, x, u) = F(y, v)$ is its lifted counterpart, and so its invariantization is the radial invariant $J = F(0, r)$. The reader should try computing other moving frames and the corresponding invariants by choosing other cross sections, e.g., $\{ (v, v) \mid v > 0 \}$, or $\{ (v, v^2) \mid v > 0 \}$.

Our construction is intimately tied to the Cartan procedure of normalization of group parameters, which is, traditionally, the basic process used in the practical construction of moving frames, [**5**, **7**]. Normalization can be interpreted as the restriction of the regularized group action to an invariant submanifold of the regularized bundle $\mathcal{B}$. In particular, when G acts freely on M, we can restrict to a local section of $\mathcal{B}$ and thereby uniquely specify all of the group parameters.

Definition 4.8. A lifted invariant $L: \mathcal{B} \to N$ is *regular* provided its group differential $d_G L$ has maximal rank $n = \dim N$ at every point in its domain of definition.

The essence of the normalization procedure that appears both in the method of moving frames, as well as the Cartan equivalence method, is captured by the following simple definition.

Definition 4.9. A *normalization* of the regularized group action consists of its restriction to a nonempty level set $\mathcal{L}_c = L^{-1}\{c\}$ of a regular lifted invariant $L: \mathcal{B} \to N$.

Every level set of a lifted invariant forms a G-invariant submanifold of the regularized action. Note that the regularity assumption on the lifted invariant implies that the projection $\pi_M: \mathcal{L}_c \to M$ maps $\mathcal{L}_c$ onto an open subset of M. Thus, regularity ensures that the normalization does not introduce any dependencies among the z coordinates, since that would introduce unacceptable constraints on the original manifold M.

In local coordinates, if $L(g, z) = (L_1(g, z), \ldots, L_n(g, z))$ is a regular lifted invariant and $c = (c_1, \ldots, c_n) \in N$ belongs to the image of L, then the implicit function theorem says that we can (locally) solve the system of n equations

$$L_1(g, z) = c_1, \qquad \ldots \qquad L_n(g, z) = c_n, \tag{4.6}$$

for n of the group parameters, say $\widehat{g} = (g^1, \ldots, g^n)$, in terms of the remaining $r - n$ group parameters, which we denote by $h = (h^1, \ldots, h^{r-n}) = (g^{n+1}, \ldots, g^r)$, and the z coordinates:

$$g^1 = \gamma^1(h, z), \qquad \ldots \qquad g^n = \gamma^n(h, z), \tag{4.7}$$

or, simply, $\widehat{g} = \gamma(h, z)$. The coordinates h and z serve to parametrize the G-invariant level set $\mathcal{L}_c = L^{-1}\{c\}$. The remaining group parameters h can be interpreted as parametrizing the isotropy subgroup of the submanifold $\{ z \mid L(e, z) = c \}$.

Proposition 4.10. *If G acts freely and regularly on M, then we can completely normalize all of the group parameters by choosing a regular lifted invariant $L: \mathcal{B} \to N$ having maximal* rank $d_G L = r = \dim N = \dim G$ *everywhere.*

Definition 4.11. Let $K \subset M$ be a local cross-section to the G orbits. The *normalization equations* associated with K are the system of equations

$$w = g \cdot z = k, \qquad \text{where} \qquad k \in K. \tag{4.8}$$

Remark: The normalization equations (4.8) are the same as the compatible lift equations discussed in part I, [**9**].

If we assume that G acts freely and K is a regular cross-section, then there is a unique solution $g = \rho(z)$ to the normalization equations, determining the right moving frame associated with K. More explicitly, if we choose the flat local coordinates $z = (x, y) \in G \times Y$ from Example 2.7, then the fundamental lifted invariant has the form $w = g \cdot z = (g \cdot x, y)$. Choosing a cross-section $x = a(y)$ reduces the normalization equations (4.8) to $g \cdot x = a(y)$, with G-equivariant solution $g = \rho(x, y) = a(y) \cdot x^{-1}$, which agrees with the right moving frame (4.3) after applying the group inversion to the cross-section map $a(y)$.

In practice, one constructs a "standard" cross-section by solving the normalization equations in the following manner. Locally we choose r components of the fundamental lifted invariant $w = g \cdot z$, say $w^1, \ldots, w^r$, which satisfy the regularity condition

$$\frac{\partial(w^1, \ldots, w^r)}{\partial(g^1, \ldots, g^r)} \neq 0. \tag{4.9}$$

Solving the equations

$$w^1(g, z) = c_1, \qquad \ldots \qquad w^r(g, z) = c_r, \tag{4.10}$$

where the constants $c_1, \ldots, c_r$ are chosen to lie in the range of the w's, leads to a complete system of normalizations $g = \rho(z)$ for the group parameters. The resulting map determines a moving frame, and corresponds to the local cross-section $K = \{z^1 = c_1, \ldots, z^r = c_r\}$. Furthermore, Theorem 4.5 implies that if we substitute the normalization formulae $g = \rho(z)$ into the remaining lifted invariants $\widetilde{w} = (w^{r+1}, \ldots, w^m)$, we obtain a complete system of $m - r$ functionally independent invariants for the group action on M:

$$I^{r+1}(z) = w^{r+1}(\rho(z), z), \qquad \ldots \qquad I^m(z) = w^m(\rho(z), z). \tag{4.11}$$

Thus, barring algebraic complications, the normalization procedure provides a simple direct method for determining the invariants of free group actions. Note particularly that, unlike Lie's infinitesimal method, cf. [**17**], we do *not* have to integrate[†] any differential equations in order to compute invariants.

Remark: If $L(g, z)$ is any other regular lifted invariant of rank r, then we can introduce local coordinates on M to make L agree with the first r components of $w = g \cdot z$ when written in the new coordinates. Thus changing the normalized invariants is equivalent to changing coordinates on M.

Example 4.12. Let $G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ be the planar Euclidean group, parametrized by (ϕ, a, b). Consider the free, local action of SE(2) on $M = \mathbb{R}^4$ that maps a point $(x, u, p, q) \in M$ to

$$\left(x\cos\phi - u\sin\phi + a, x\sin\phi + u\cos\phi + b, \frac{\sin\phi + p\cos\phi}{\cos\phi - p\sin\phi}, \frac{q}{(\cos\phi - p\sin\phi)^3} \right). \tag{4.12}$$

The fundamental lifted invariants are the individual components (y, w, r, s) of (4.12). Let us normalize the first three lifted invariants to all be zero, leading to the normalization equations

$$y = x\cos\phi - u\sin\phi + a = 0, \qquad v = x\sin\phi + u\cos\phi + b = 0, \qquad r = \frac{\sin\phi + p\cos\phi}{\cos\phi - p\sin\phi} = 0. \tag{4.13}$$

[†] In a sense, though, we have integrated the differential equations by being able to explicitly write down the group transformation formulae for $w = g \cdot z$. However, it is rare that one can integrate the ordinary differential equations for the invariants without being able to find the group transformations!

This corresponds to choosing the cross-section $K = \{ (0,0,0,\kappa) \mid \kappa \in \mathbb{R} \}$ to the three-dimensional orbits of SE(2). The solution to (4.13) is

$$\phi = -\tan^{-1} p, \qquad a = -\frac{x+up}{\sqrt{1+p^2}}, \qquad b = \frac{xp-u}{\sqrt{1+p^2}}, \tag{4.14}$$

which defines the right moving frame $\rho: M \to \mathrm{SE}(2)$. The left moving frame is obtained by inverting the group element parametrized by (4.14), whereby

$$\widetilde{\phi} = \tan^{-1} p, \qquad \widetilde{a} = x, \qquad \widetilde{b} = u. \tag{4.15}$$

Finally, if we substitute (4.14) into the final lifted invariant $s = (\cos\phi - p\sin\phi)^{-3} q$, we recover the fundamental invariant

$$\kappa = \frac{s}{(1+r^2)^{3/2}} = \frac{q}{(1+p^2)^{3/2}}. \tag{4.16}$$

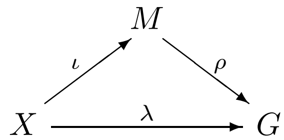
Note again the common functional dependency on the coordinates on M and the associated lifted invariants, in accordance with Theorem 3.7. If we identify $p = u_x$, $q = u_{xx}$, then (4.12) coincides with the second prolongation of the standard action of SE(2) on curves in the plane, (4.15) agrees with the classical left moving frame for Euclidean curves, cf. [**9**], and the invariant (4.16) is, of course, the Euclidean curvature. See Example 10.4 below.

Example 4.13. Consider the joint action $(x,y) \mapsto (Rx+a, Ry+a)$ of the Euclidean group $(R,a) \in \mathrm{SE}(2)$ on $(x,y) \in M = \mathbb{R}^2 \times \mathbb{R}^2$. The action is free on $M \setminus D$, where $D = \{x = y\}$ is the diagonal. The lifted invariants are the components of $z = Rx + a$, $w = Ry + a$. We normalize $z = 0$ by setting $a = -Rx$. The remaining normalized invariant now reduces to $w = R(y-x)$. Away from the diagonal, we can further normalize the second component of w to be zero by specifying the rotation matrix R to have angle $\phi = -\arg(y-x)$; this amounts to picking the cross-section $K = \{(0,0,d,0)\}$. The resulting normalizations specify the right moving frame for the joint Euclidean action. The first component of w then reduces to the distance $|\, y - x \,| = d$, which forms the fundamental joint invariant for the Euclidean group. A similar construction for the n-dimensional Euclidean group $\mathrm{E}(n)$ provides a simple proof of an analytical version of a theorem in Weyl, [**28**], that the only joint Euclidean invariants are functions of the distances between points. Extensions to joint invariants for other transformation groups are straightforward. See [**9**] for recent results on joint differential invariants.

In applications to equivalence problems, one restricts the moving frame to a submanifold of the underlying space M. The resulting maps from the submanifold to the group agree with the traditional definition of a moving frame in classical geometrical situations. Assume that $S = \iota(X)$ is an immersed submanifold parametrized by a smooth map $\iota: X \to M$ of maximal rank equal to the dimension of X.

Definition 4.14. A *moving frame* on a submanifold $S = \iota(X)$ is a map $\lambda: X \to G$ that factors through a G-equivariant map $\rho: M \to G$, so that $\lambda = \rho \circ \iota$.

In other words, the moving frame λ on S can be realized by the following commutative diagram



(4.17)

The moving frame ρ must, of course, be defined in a neighborhood of S. As before, we can consider either left or right moving frames on the submanifold S. Lemma 4.2 still applies and shows that they are merely inverses of each other.

5. Equivalence Problems for Coframes.

We now turn to the applications of moving frames to equivalence problems for submanifolds. In preparation, we first review a very particular equivalence problem, that of coframes on a manifold. The goal of both the Cartan equivalence method and the moving coframe method is to produce, via the normalization and reduction process, an invariant coframe, and thereby reduce the original equivalence problem to an equivalence problem for coframes. Thus it is essential that we understand the known solution to this particular equivalence problem before proceeding further. We refer the reader to [**8**, **10**, **18**] for more details on the basic theory as well as numerous examples.

Let M and $\overline{M}$ be m-dimensional manifolds, and let $\boldsymbol{\omega} = \{\omega^1, \ldots, \omega^m\}$ and $\overline{\boldsymbol{\omega}} = \{\overline{\omega}^1, \ldots, \overline{\omega}^m\}$ be respective coframes thereon. The basic *coframe equivalence problem* is to determine when there exists a (local) diffeomorphism $\psi\colon M \to \overline{M}$ such that

$$\psi^* \, \overline{\omega}^i = \omega^i, \qquad i = 1, \ldots, m. \tag{5.1}$$

More generally, one might also include a collection of smooth scalar-valued functions $I_\nu\colon M \to \mathbb{R}$ and $\bar{I}_\nu\colon \overline{M} \to \mathbb{R}$, where $\nu = 1, \ldots, l$, that are required to be mapped to each other, meaning that $\bar{I}_\nu(\overline{x}) = I_\nu(x)$ whenever $\overline{x} = \psi(x)$, or, equivalently,

$$\psi^* \, \bar{I}_\nu = I_\nu, \qquad \nu = 1, \ldots, l. \tag{5.2}$$

We formalize this as follows.

Definition 5.1. An *extended coframe* on a manifold M is a collection $\Omega = \{\boldsymbol{\omega}, I\}$ consisting of a coframe $\boldsymbol{\omega}$ along with a collection $I = (I_1, \ldots, I_l)$ of smooth scalar functions.

Definition 5.2. A local diffeomorphism $\psi\colon M \to \overline{M}$ is an *equivalence* between extended coframes $\Omega = \{\boldsymbol{\omega}, I\}$ on M, and $\overline{\Omega} = \{\overline{\boldsymbol{\omega}}, \bar{I}\}$, on $\overline{M}$ if and only if ψ satisfies (5.1), (5.2), which we abbreviate as $\psi^* \, \overline{\Omega} = \Omega$. In particular, a *symmetry* of an extended coframe Ω is a self-equivalence, i.e., a local diffeomorphism $\psi\colon M \to M$ such that $\psi^* \Omega = \Omega$.

The *symmetry group* G of an extended coframe $\Omega = \{\boldsymbol{\omega}, I\}$ is the local transformation group consisting of all symmetries. The functions I_ν in Ω are then invariants for the group G, hence their common level sets are G-invariant subsets of M. In view of this remark, we shall often refer to the functions I in an extended coframe Ω as its *invariants*.

Two equivalent extended coframes *must* have the same number of invariants. Moreover, if there is a functional dependency $I_l = H(I_1, \ldots, I_{l-1})$ among the invariants of Ω, then the corresponding invariants of any equivalent coframe $\overline{\Omega}$ must satisfy an *identical* functional relation: $\bar{I}_l = H(\bar{I}_1, \ldots, \bar{I}_{l-1})$. The function $H(y_1, \ldots, y_{l-1})$ in such a functional relation is known as a *classifying function* for the extended coframe. As argued in [**18**], the most natural way to keep track of such functional dependencies between the structure invariants is to introduce the associated classifying manifold.

Definition 5.3. The *classifying manifold* $\mathcal{C}(\Omega)$ of an extended coframe $\Omega = \{\boldsymbol{\omega}, I\}$ is the subset $I(M) \subset Z = Z(\Omega)$ of the *classifying space* $Z \simeq \mathbb{R}^l$ that is parametrized by the invariant functions $I = (I_1, \ldots, I_l)\colon M \to Z$.

Definition 5.4. An extended coframe Ω is called *semi-regular* of *rank* t if its invariants have constant rank $t = \operatorname{rank} dI$. Note that t equals the number of functionally independent invariants near any point. An extended coframe Ω is called *regular* if its classifying manifold is an embedded submanifold of its ambient classifying space. In this case, the rank of the coframe equals the dimension of $\mathcal{C}(\Omega)$.

Lemma 5.5. *If $\Omega = \psi^* \overline{\Omega}$ are equivalent extended coframes, then their classifying manifolds are identical, $\mathcal{C}(\overline{\Omega}) = \mathcal{C}(\Omega)$.*

Remark: If the equivalence map ψ is only locally defined, then one must restrict the classifying manifolds to the open subsets $U = \operatorname{dom} \psi \subset M$ and $\overline{U} = \psi(U) \subset \overline{M}$.

The converse to Lemma 5.5 is not true in general — one must impose an additional "involutivity condition" on the extended coframes in order to prove sufficiency of the classifying manifold condition. In preparation, we note that one can (simultaneously) perform two elementary operations on extended coframes that preserve their symmetry and equivalence constraints.

Definition 5.6. Two regular extended coframes $\Omega = \{\boldsymbol{\omega}, I\}$ and $\Theta = \{\boldsymbol{\theta}, J\}$ on M are said to be *invariantly related* if

(a) There exists a local diffeomorphism $\varphi\colon \mathcal{C}(\Omega) \to \mathcal{C}(\Theta)$ such that $J = \varphi \circ I$, and

(b) There is a smooth map $A\colon \mathcal{C}(\Omega) \to \mathrm{GL}(m, \mathbb{R})$ such that $\boldsymbol{\theta} = (A \circ I)\, \boldsymbol{\omega}$.

We shall write $\Theta = \Phi(\Omega)$, where $\Phi = (\varphi, A)$, for such an invariant relation.

Note that the first condition means that the two classifying manifolds have the same dimension, and so both coframes contain the same number of functionally independent invariants. Moreover, each invariant in Θ is functionally dependent on the invariants in Ω, i.e., $J_\nu = \varphi_\nu(I_1, \ldots, I_l)$, and conversely. The second condition means that the one-forms in the two coframes are invariant linear combinations of each other, so

$$\theta^i = \sum_{j=1}^{m} (A^i_j \circ I)\, \omega^j, \qquad i = 1, \ldots, m. \tag{5.3}$$

Proposition 5.7. *If $\psi^* \overline{\Omega} = \Omega$ are equivalent extended coframes, and $\Theta = \Phi(\Omega)$ and $\overline{\Theta} = \Phi(\overline{\Omega})$ are invariantly related coframes for the* same *functions $\Phi = (\varphi, A)$, then $\psi^* \overline{\Theta} = \Theta$ are also equivalent.*

The proof is immediate. Note that this allows us to always assume (at least locally) that the functions occurring in our extended coframe are functionally independent; for example, we can use those corresponding to a consistent choice of local coordinates on the classifying manifold.

Corollary 5.8. *Two invariantly related extended coframes* $\Theta = \Phi(\Omega)$ *have the same symmetry group.*

The complete solution to the extended coframe equivalence problem (5.1), (5.2), is based on the Frobenius Theorem for closed differential ideals, [**4**, **18**]. One effectively determines a complete system of functionally independent invariants by successive differentiation, and adjoins them to the original invariance conditions (5.2). There are two ways in which additional scalar invariants can arise. First of all, since the one-forms $\boldsymbol{\omega}$ form a coframe, we can re-express their differentials in terms of wedge products thereof, leading to the structure equations

$$d\omega^k = -\sum_{i<j} I^k_{ij}\, \omega^i \wedge \omega^j, \qquad k = 1, \ldots, m. \tag{5.4}$$

The structure coefficients I^k_{ij} are readily seen to be invariants of the problem, i.e., satisfy (5.2), and hence should be included in our list of invariants. Thus, even if we began with no additional invariants, the structure equations automatically produce some for us, whose invariance must be taken into account in the resolution of the problem. Secondly, the coefficients $I_{\nu,k} = \partial I_\nu / \partial \omega^k$ of the differential

$$dI_\nu = \sum_{k=1}^m I_{\nu,k}\, \omega^k = \sum_{k=1}^m \frac{\partial I_\nu}{\partial \omega^k}\, \omega^k, \tag{5.5}$$

of any invariant are also automatically invariant, and are known as the (first order) *derived invariants* corresponding to the original invariant I_ν. The invariant differential operators $\partial/\partial\omega^k$ are known as *coframe* (or *covariant*) *derivatives*; these coincide with the dual frame vector fields to $\boldsymbol{\omega}$.

Remark: The coframe derivative operators do not necessarily commute. Applying d to (5.5) and comparing with (5.4) produces the basic commutation formulae:

$$\left[\frac{\partial}{\partial \omega^i}, \frac{\partial}{\partial \omega^j} \right] = \sum_{k=1}^m I^k_{ij}\, \frac{\partial}{\partial \omega^k}\,. \tag{5.6}$$

If the one-forms $\omega^k = df^k$ are all (locally) exact, then all the structure coefficients vanish, and so the coframe derivatives do commute in this particular case.

Definition 5.9. The *derived invariants* of an extended coframe $\{\boldsymbol{\omega}, I\}$ with l invariants $I = (I_1, \ldots, I_l)$ are the $l(m+1) + \frac{1}{2}m^2(m-1)$ functions

$$I^{(1)} = (\ldots, I_\nu, \ldots, I_{\nu,k}, \ldots, I^k_{ij}, \ldots)$$

consisting of

(a) the original invariants $I_1, \ldots, I_l$,
(b) their first order coframe derivatives $I_{\nu,k} = \partial I_\nu / \partial \omega^k$, $\nu = 1, \ldots, l$, $k = 1, \ldots, m$, and
(c) the coefficients I^k_{ij}, $k = 1, \ldots, m$, $1 \le i < j \le m$, in the structure equations (5.4) for each $d\omega^k$.

Definition 5.10. The *derived coframe* associated with an extended coframe $\Omega = \{\boldsymbol{\omega}, I\}$ is the extended coframe $\Omega^{(1)} = \{\boldsymbol{\omega}, I^{(1)}\}$ consisting of the original coframe along with all its derived invariants.

Lemma 5.11. *A map $\psi: M \to \overline{M}$ determines an equivalence between extended coframes Ω and $\overline{\Omega}$ if and only if it determines an equivalence between their corresponding derived coframes $\Omega^{(1)}$ and $\overline{\Omega}^{(1)}$.*

In this manner, one recursively defines the higher order derived coframes $\Omega^{(k)} = (\Omega^{(k-1)})^{(1)}$ by computing the higher order derived invariants. Lemma 5.11 shows that all such higher order derived coframes are also equivalent under the given map. The process will terminate whenever the set of first order derived invariants arising from the current list of invariants fails to produce any new, meaning functionally independent, invariants.

Definition 5.12. An extended coframe $\Omega = \{\boldsymbol{\omega}, I\}$ is called *involutive* if it is regular and is invariantly related to its derived coframe $\Omega^{(1)}$.

Thus, a regular extended coframe is involutive if and only if $\operatorname{rank} \Omega = \operatorname{rank} \Omega^{(1)}$, which occurs if and only if its derived invariants are functionally dependent on the original invariants: $I^{(1)} = H \circ I$.

Example 5.13. The most familiar example of an involutive coframe is the Maurer–Cartan coframe $\boldsymbol{\mu} = \{\mu^1, \ldots, \mu^r\}$ on an r-dimensional Lie group G. The symmetry group of the Maurer–Cartan coframe coincides with the right action of G on itself. Involutivity follows from the basic Maurer–Cartan structure equations

$$d\mu^k = -\sum_{i<j} C^k_{ij}\, \mu^i \wedge \mu^j, \qquad k = 1, \ldots, r, \tag{5.7}$$

where C^k_{ij} are the structure constants for the dual basis $\mathbf{v}_i = \partial/\partial \mu^i$ of the Lie algebra $\mathfrak{g}$. Since all the derived invariants are constant, the Maurer–Cartan coframe has rank 0. In fact, any rank 0 coframe is locally equivalent to a Maurer–Cartan coframe; see [**19**] for global versions of this result, based on the theory of "non-associative local Lie groups".

Lemma 5.14. *Let Ω be an extended coframe. If the derived coframe $\Omega^{(s)}$ is involutive, then so are all higher order derived coframes $\Omega^{(k)}$, $k \ge s$. Moreover,* $\operatorname{rank} \Omega^{(k)} = \operatorname{rank} \Omega^{(s)}$ *for all $k \ge s$.*

Proof: Any functional dependency among invariants $I = H(I_1, \ldots, I_l)$ automatically induces a functional dependency among the corresponding derived invariants:

$$\frac{\partial I}{\partial \omega^i} = \sum_{\nu=1}^{l} \frac{\partial H}{\partial I_\nu}\, \frac{\partial I_\nu}{\partial \omega^i}\,, \qquad i = 1, \ldots, p. \tag{5.8}$$

This observation suffices to prove the result. *Q.E.D.*

Remark: Equation (5.8) implies that if an invariant in $\Omega^{(k)}$ is functionally dependent on the others, then one does not need to include its derived invariants in the higher order derived coframes $\Omega^{(k+1)}$ since their associated functional dependencies are automatic. In other words, we can reduce the number of invariants in $\Omega^{(k+1)}$ by a well-determined invariant relation, as in Definition 5.6. Therefore, at each step, one really only needs compute the coframe derivatives of the independent invariants.

Definition 5.15. The *order* of an extended coframe Ω is the minimal integer s such that $\Omega^{(s)}$ is regular and involutive. We call $t = \operatorname{rank} \Omega^{(s)}$ the *involutivity rank* of Ω.

Remark: Our definition of order is slightly different than that in [**18**]. If we start with an ordinary coframe $\boldsymbol{\omega}$, under the present construction the structure invariants I^i_{jk} will appear at order 1, and hence, unless the coframe has rank 0, involutivity will not occur until at least order 2.

Let us call an extended coframe Ω *fully regular* if it and its derived coframes $\Omega^{(k)}$, $k = 0, 1, 2, \ldots$, are regular. In the fully regular case, the ranks $t_k = \operatorname{rank} \Omega^{(k)}$ are nondecreasing, $t_0 \leq t_1 \leq t_2 \leq \cdots \leq m$ and bounded by the dimension of M. Moreover, if $t_s = t_{s+1}$, then $\Omega^{(s)}$ is involutive, and hence Ω has order s and involutivity rank $t = t_s = t_{s+1} = \cdots$. In particular, a fully regular coframe has order $s \leq m$. Coframes of order greater than m can occur if singularities are present, but can be resolved at some higher order.

The fundamental equivalence and symmetry theorems for coframes can now be stated. Both are direct consequences of Frobenius' Theorem; details can be found in [**18**].

Theorem 5.16. *Let M and $\overline{M}$ be m-dimensional manifolds. Two finite order extended coframes Ω on M and $\overline{\Omega}$ on $\overline{M}$ are locally equivalent if and only if they have the same order s, and their $(s+1)^{\text{st}}$ order classifying manifolds are identical: $\mathcal{C}(\overline{\Omega}^{(s+1)}) = \mathcal{C}(\Omega^{(s+1)})$. In this case, if $z_0 \in M$ and $\overline{z}_0 \in \overline{M}$ map to the same point $\overline{I}^{(s+1)}(\overline{z}_0) = I^{(s+1)}(z_0)$ in the common classifying manifold, then there is a unique local diffeomorphism $\Phi\colon M \to \overline{M}$ with $\Phi(z_0) = \overline{z}_0$ and $\Phi^*\overline{\Omega} = \Omega$.*

Remark: One can replace the order s by any higher order $k \geq s$ in the theorem. Thus, in fully regular cases, one can always determine the equivalence of two extended coframes on an m-dimensional manifold by comparing the $(m+1)^{\text{st}}$ order classifying manifolds.

Remark: Regularity relies on two conditions: first, the invariants have constant rank, and, second, they parametrize an embedded submanifold of the classifying space. The latter can clearly be weakened to include immersed classifying manifolds, since the result is local anyway, and so one can restrict to a subdomain where the classifying manifold is embedded. In fact, one can resolve singularities and self-intersections of the classifying manifold by going to a yet higher order coframe. Indeed, if $\Omega^{(s)}$ is "semi-involutive", meaning that it is semi-regular and of the same rank as $\mathcal{C}(\Omega^{(s+1)})$, then formula (5.8) implies that one can identify the classifying manifold $\mathcal{C}(\Omega^{(k)})$ for any $k > s$ with the $k-s-1$ jet of $\mathcal{C}(\Omega^{(s+1)})$. Thus, if $\mathcal{C}(\Omega^{(s+1)})$ intersects itself transversally, then $\mathcal{C}(\Omega^{(s+2)})$ will not intersect itself at all, and can be used instead. Thus, in the analytic category, one can eliminate all such singularities and self-intersections by going to a classifying manifold of sufficiently high order.

Theorem 5.17. *The symmetry group of an extended coframe* Ω *of order* s *is a freely acting local Lie group of transformations of dimension* $r = m - t$*, where* $t = \dim \mathcal{C}(\Omega^{(s)})$ *is the involutivity rank of* Ω*. The orbits of* G *are the common level sets of the* $(s+1)^{\mathrm{st}}$ *order invariants* $I^{(s+1)}$*.*

This completes our survey of the basic equivalence problem for (extended) coframes. One can also investigate the equivalence of more general "extended one-form systems" $\Omega = \{\boldsymbol{\omega}, I\}$ containing of a collection of one-forms that do not necessarily form a coframe. The *overdetermined* case, where the one-forms $\boldsymbol{\omega}$ span the cotangent space, is easily reduced to the case of an extended coframe. One can locally choose a coframe, say $\{\omega^1, \ldots, \omega^m\}$ from among the one-forms in Ω. Any additional one-forms in Ω can be written as linear combinations of the given coframe,

$$\omega^k = \sum_{i=1}^{m} J_i^k \, \omega^i, \qquad k > m. \tag{5.9}$$

The coefficients J_i^k will be invariant functions for the problem, and should be included among the functions in an invariantly related extended coframe. Thus, the overdetermined equivalence problem reduces to an extended coframe equivalence problem (5.1), (5.2), where the invariant functions include all the original invariants I_ν as well as the coefficients J_i^k stemming from the linear dependencies (5.9). The *underdetermined* case, when the one-forms fail to span the relevant cotangent spaces, can be treated by the Cartan equivalence method until it is reduced to either a coframe equivalence problem, or to an involutive differential system defining a Lie pseudo-group via the Cartan–Kähler Theorem, cf. [**4**, **18**]. For brevity, we will not discuss the latter more complicated theory here.

6. Moving Coframes.

The method of moving coframes was introduced in [**9**] as a practical means of determining moving frames for general transformation groups, and will now be incorporated into our regularized approach. The following definition is inspired by Cartan's approach to equivalence problems, which always begins by characterizing the (pseudo-)group of allowable transformations by a suitable collection of differential forms.

Definition 6.1. Let G be a finite-dimensional Lie group acting on a manifold M. A G*–coframe* is, by definition, a regular, involutive extended coframe $\Omega = \{\boldsymbol{\omega}, I\}$ on M, whose symmetry group coincides with the transformation group G.

In other words, if $\Omega = \{\boldsymbol{\omega}, I\}$ is a G–coframe, then a local diffeomorphism $\psi: M \to M$ satisfies the symmetry conditions

$$\psi^* \boldsymbol{\omega} = \boldsymbol{\omega}, \qquad \psi^* I = I, \tag{6.1}$$

if and only if $\psi(z) = g \cdot z$ coincides with the action of a group element $g \in G$. For example, the right Maurer–Cartan coframe on a Lie group forms a G–coframe for the right action of G on itself. Since G–coframes are always assumed to be involutive, the solution to the equivalence problem for coframes implies that they are essentially unique.

Proposition 6.2. *Let Ω be a G–coframe on M. An extended coframe Θ is also a G–coframe if and only if Ω and Θ are invariantly related.*

Proof: Corollary 5.8 implies that if the two extended coframes are invariantly related, then their symmetry groups are the same. Conversely, according to Theorem 5.17, the orbits of the symmetry group of an involutive extended coframe are the level sets of its invariants. Since the two collections of invariants have the same level sets, they are necessarily functionally related, as in part a) of Definition 5.6. Moreover, since the symmetry groups coincide, the coefficients A^i_j relating the coframes, as in (5.3), must also be invariants, proving the result. *Q.E.D.*

Theorem 5.17 implies that the symmetry group of an involutive extended coframe acts locally freely. This condition also turns out to be sufficient; see Theorem 6.5 below. The moving frame method provides a simple mechanism for constructing G–coframes. Suppose G acts freely and regularly on the m-dimensional manifold M. Let $\rho: M \to G$ be a (right) moving frame. We let $\boldsymbol{\zeta} = \rho^* \boldsymbol{\mu}$ denote the pull-back of the Maurer–Cartan forms to M. If G acts transitively on M, whence $m = r$, then $\boldsymbol{\zeta}$ forms a coframe on M, called the *moving coframe* associated with the given moving frame. The coframe $\boldsymbol{\zeta}$ has the same structure equations (5.7) as the Maurer–Cartan coframe on G, and hence forms an involutive coframe of rank zero on M.

Remark: The pull-back of the left Maurer–Cartan coframe $\widetilde{\boldsymbol{\mu}}$ on G under the left moving frame map $\widetilde{\rho}$ leads, up to sign, to the same collection of moving coframe forms: $\widetilde{\rho}^* \widetilde{\boldsymbol{\mu}} = -\rho^* \boldsymbol{\mu} = \boldsymbol{\zeta}$. This is because the inversion $g \mapsto g^{-1}$ maps the right Maurer–Cartan forms on G to minus their left-invariant counterparts.

If G does not act transitively, then the one-forms $\boldsymbol{\zeta} = \rho^* \boldsymbol{\mu}$ only form a coframe when restricted to the orbits, and we need to supplement them by an additional $m - r$ one-forms to construct a full coframe. Locally, if we choose a complete system of functionally independent invariants $y = (y^1, \ldots, y^{m-r})$, then the m one-forms

$$\{\boldsymbol{\zeta}, dy\} = \{\zeta^1, \ldots, \zeta^r, dy^1, \ldots, dy^{m-r}\} \tag{6.2}$$

form a coframe on M.

Definition 6.3. The *moving coframe* associated with a given moving frame map $\rho: M \to G$ is the extended coframe $\Sigma = \{\boldsymbol{\zeta}, dy, y\}$ consisting of the pulled-back Maurer–Cartan forms $\boldsymbol{\zeta} = \rho^* \boldsymbol{\mu}$, along with the invariant functions y and their differentials.

Lemma 6.4. *The moving coframe Σ forms an involutive G–coframe on M.*

Proof: Involutivity is immediate, since the Maurer–Cartan structure equations (5.7) along with the equations $d(dy^i) = 0$ imply that all the derived invariants for the moving coframe are constant. To prove that the only symmetries are the group transformations $z \mapsto g \cdot z$, we note that, in the flat local coordinates of Example 2.7, the associated moving coframe consists of the Maurer–Cartan forms $\boldsymbol{\mu}$ pulled back to the orbits $G \times \{y_0\}$, along with the invariants and their differentials. Invariance of the y's implies that any symmetry of the moving coframe must have the form $\psi(x, y) = (\varphi(x), y)$, where $\varphi: G \to G$ is a symmetry of the Maurer–Cartan coframe, and hence agrees with right multiplication by a group element. *Q.E.D.*

We have thus proved the following basic existence theorem.

Theorem 6.5. *Let G be a Lie group acting on a manifold M. Then the following are equivalent*:

(*i*) *G acts freely and regularly on M.*

(*ii*) *G admits a moving frame in a neighborhood of each point $z \in M$.*

(*iii*) *There exists a G–coframe in a neighborhood of each point $z \in M$.*

There is a second important method that can be used to construct an alternative G–coframe for a free group action without appealing to the Maurer–Cartan forms. First, the invariants $I(z) = \rho(z) \cdot z$ were earlier interpreted as the pull-back, via $\sigma: M \to \mathcal{B}$, of the fundamental lifted invariants $w = g \cdot z$. Second, by applying Proposition 3.9, the differential dw of the fundamental lifted invariant will split into two sets of invariant forms on $\mathcal{B}$, namely $d_M w = g \cdot dz$ and the group component $d_G w$. Theorem 3.10 implies that the latter are invariant linear combination of the Maurer–Cartan forms $\boldsymbol{\mu}$ on $\mathcal{B}$. Therefore

$$dw = d_M w + d_G w = g \cdot dz + F(w)\boldsymbol{\mu}, \tag{6.3}$$

where the coefficients $F(w)$ are explicitly determined by (3.8). We now pull back $d_M w$ via our moving frame section σ to construct a system of G-invariant one-forms on M.

Theorem 6.6. *Let G act freely on M. Let $\rho: M \to G$ be a* right *moving frame. Then the extended coframe $\Gamma = \{\boldsymbol{\gamma}, I\}$ consisting of the invariant functions $I(z) = \rho(z) \cdot z$ along with the one-forms $\boldsymbol{\gamma} = \rho(z) \cdot dz$ forms a G–coframe on M.*

Proof: The fact that $I = \sigma^* w$ include a complete system of functionally independent invariants was given in Theorem 4.5. Applying σ^* to (6.3), we find

$$dI = \sigma^*(d_M w) + \sigma^*(F)\boldsymbol{\zeta} = \boldsymbol{\gamma} + (F \circ I)\boldsymbol{\zeta}.$$

Therefore, the one-forms $\boldsymbol{\gamma} = \sigma^*(d_M w)$ are invariantly related to the moving coframe forms $\{\boldsymbol{\zeta}, dI\}$, as in (5.3). It is not hard to see that the $\boldsymbol{\gamma}$ define a coframe on M, and so the result follows from Corollary 5.8. *Q.E.D.*

Formula (3.8) provides an explicit local coordinate formula relating the normalized coframe forms $\boldsymbol{\gamma}$ with the Maurer–Cartan forms:

$$\gamma^k = dI^k - \sum_{\kappa=1}^{r} (f_\kappa^k \circ I)\, \zeta^\kappa, \qquad k = 1, \ldots, m, \tag{6.4}$$

where $I^k(z)$ is the k^{th} component of $I(z) = \rho(z) \cdot z$. Note that the coefficients in (6.4) are obtained by invariantization, as in Definition 4.6, of the coefficients $f_\kappa^k(z)$ of the infinitesimal generators (3.7) with respect to the moving frame ρ. In particular, if we normalize $w^k = c^k$ to be constant, then $I^k = c^k$ is constant also, and the dI^k term in (6.4) disappears.

Example 6.7. Consider the action (4.12) of the planar Euclidean group on $\mathbb{R}^4$. The right Maurer–Cartan forms on SE(2) are

$$\mu^1 = d\phi, \qquad \mu^2 = da + b\, d\phi, \qquad \mu^3 = db - a\, d\phi. \tag{6.5}$$

The corresponding components of the right moving coframe $\boldsymbol{\zeta} = \rho^* \boldsymbol{\mu}$ are obtained by pulling back the Maurer–Cartan forms (6.5) using the right moving frame (4.14), so

$$\zeta^1 = -\frac{dp}{1+p^2}, \qquad \zeta^2 = -\frac{dx + p\,du}{\sqrt{1+p^2}}, \qquad \zeta^3 = \frac{p\,dx - du}{\sqrt{1+p^2}}. \tag{6.6}$$

In order to complete (6.6) to a G–coframe on $M = \mathbb{R}^4$, we must supplement the forms (6.5) by the fundamental invariant (4.16) and its differential

$$\zeta^4 = d\kappa = \frac{(1+p^2)\,dq - 3pq\,dp}{(1+p^2)^{5/2}}, \qquad \kappa = \frac{q}{(1+p^2)^{3/2}}. \tag{6.7}$$

The complete extended coframe (6.6), (6.7) forms a SE(2) coframe on M — its symmetries coincide with the group transformations (4.12).

On the other hand, computing the G–coframe $\{\boldsymbol{\gamma}, I\}$ as in Theorem 6.6, we only need compute the differentials of the fundamental lifted invariants, and then pull-back via the moving frame map, thereby avoiding explicit determination of the Maurer–Cartan forms. Differentiating (4.12) with respect to the coordinates (x,u,p,q) on M leads to the one-forms

$$\begin{aligned} d_M y &= \cos\phi\,dx - \sin\phi\,du, \qquad & d_M v &= \sin\phi\,dx + \cos\phi\,du, \\ d_M r &= \frac{dp}{(\cos\phi - p\sin\phi)^2}, & d_M s &= \frac{(\cos\phi - p\sin\phi)\,dq + 3q\sin\phi\,dp}{(\cos\phi - p\sin\phi)^4}, \end{aligned} \tag{6.8}$$

which, along with the Maurer–Cartan forms (6.5) form a SE(2) coframe for the lifted action on $\mathcal{B} = \mathrm{SE}(2) \times M$. The corresponding SE(2) coframe on M is found by pulling back (6.8) via the right moving frame (4.14); the result is

$$\begin{aligned} \gamma^1 &= \sigma^*(d_M y) = -\zeta^2, \qquad & \gamma^3 &= \sigma^*(d_M r) = -\zeta^1, \\ \gamma^2 &= \sigma^*(d_M v) = -\zeta^3, & \gamma^4 &= \sigma^*(d_M s) = \zeta^4. \end{aligned} \tag{6.9}$$

The formulae (6.9) relating the two coframes can be deduced from (6.4), as we now explicitly show. The group components of the differentials are

$$\begin{aligned} d_G y &= d_G(x\cos\phi - u\sin\phi + a) = -v\,\mu^1 + \mu^2, \\ d_G v &= d_G(x\sin\phi + u\cos\phi + b) = y\,\mu^1 - \mu^3, \\ d_G r &= d_G\left(\frac{\sin\phi + p\cos\phi}{\cos\phi - p\sin\phi}\right) = (1+r^2)\,\mu^1, \\ d_G s &= d_G\left(\frac{q}{(\cos\phi - p\sin\phi)^3}\right) = 3rs\mu^1. \end{aligned} \tag{6.10}$$

The lifted invariant coefficients in (6.10) follow directly from (3.8) and the formulae

$$\mathbf{v}_1 = -u\partial_x + x\partial_u + (1+p^2)\partial_p + 3pq\partial_q, \qquad \mathbf{v}_2 = \partial_x, \qquad \mathbf{v}_3 = \partial_u, \tag{6.11}$$

for the infinitesimal generators of the Euclidean action (4.12) that are dual to the chosen Maurer–Cartan form basis (6.5). Indeed, if we write down the coefficient matrix

$$\begin{pmatrix} -u & 1 & 0 \\ x & 0 & 1 \\ 1+p^2 & 0 & 0 \\ 3pq & 0 & 0 \end{pmatrix} \tag{6.12}$$

for the vector fields (6.11), then (6.10) can be written in matrix form

$$\begin{pmatrix} d_G y \\ d_G v \\ d_G r \\ d_G s \end{pmatrix} = \begin{pmatrix} -v & 1 & 0 \\ y & 0 & 1 \\ 1+r^2 & 0 & 0 \\ 3rs & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu^1 \\ \mu^2 \\ \mu^3 \end{pmatrix}, \tag{6.13}$$

and the coefficient matrix is the lifted version of (6.12), obtained by replacing the coordinates on M by their corresponding lifted counterparts. Formula (6.9) then follows from (6.4):

$$\begin{pmatrix} \gamma^1 \\ \gamma^2 \\ \gamma^3 \\ \gamma^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ d\kappa \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta^1 \\ \zeta^2 \\ \zeta^3 \end{pmatrix}. \tag{6.14}$$

In (6.14), the first term is the pull-back of the lifted coordinate differentials (dy, dv, dr, ds) via the normalization map (4.13), while the coefficient matrix in the second is the invariantization of the infinitesimal generator coefficient matrix (6.12) with respect to the given moving frame.

7. Equivalence of Submanifolds.

We now apply our general results to the equivalence problem for submanifolds under a freely acting transformation group. Given submanifolds $S, \overline{S} \subset M$, we want to know whether or not they are *congruent* under a group transformation, meaning that $g \cdot S = \overline{S}$ for some $g \in G$. In this section, we review the solution to this problem in the case when G acts regularly and freely on M. Actually, we shall only consider the local problem here, so that the congruence condition is only required to hold in a neighborhood of a point. Global questions can be handled by continuation processes (e.g., analytic continuation).

Definition 7.1. Let $S = \iota(X)$ and $\overline{S} = \overline{\iota}(\overline{X})$ be two embedded p-dimensional submanifolds parametrized by maps $\iota: X \to M$, and $\overline{\iota}: \overline{X} \to M$. The submanifolds are said to be (locally) *congruent* under a transformation group G provided there exists a group element $g \in G$ and a (local) diffeomorphism $\psi: X \to \overline{X}$ such that

$$\overline{\iota}(\psi(x)) = g \cdot \iota(x). \tag{7.1}$$

for all x in the domain of ψ.

In the case when G acts freely, the solution to the congruence problem follows directly from the theorem for submanifolds embedded in Lie groups, [**12**].

Theorem 7.2. *Let G be a free, regular Lie transformation group acting on M. Let $\Omega = \{\boldsymbol{\omega}, I\}$ be a G–coframe on M. Then two embedded p-dimensional submanifolds $S = \iota(X)$ and $\overline{S} = \overline{\iota}(\overline{X})$ are locally congruent under G if and only if the pulled-back extended coframes $\Xi = \{\boldsymbol{\xi}, J\} = \iota^*\Omega$ on X and $\overline{\Xi} = \{\overline{\boldsymbol{\xi}}, \overline{J}\} = \overline{\iota}^*\overline{\Omega}$ on $\overline{X}$ are locally equivalent.*

The diffeomorphism $\psi: X \to \overline{X}$ determining the reparametrization part in the correspondence (7.1) must satisfy $\psi^* \overline{\Xi} = \Xi$. In other words, in terms of the defining coframe and invariants,

$$(\overline{\iota} \circ \psi)^* \boldsymbol{\omega} = \iota^* \boldsymbol{\omega}, \qquad (\overline{\iota} \circ \psi)^* I = \iota^* I. \tag{7.2}$$

Remark: For the fixed parameter equivalence problem, we do not allow the reparametrization map ψ in the equivalence condition (7.1), which thus reduces to $\overline{\iota}(x) = g \cdot \iota(x)$, $x \in X$. The solution to this problem follows from Theorem 7.2: the pulled-back G–coframes must now be identical: $\overline{\Xi} = \Xi$.

The original G–coframe Ω is involutive; in particular, if $\Omega = \Sigma$ is the moving coframe, it will have constant derived invariants. The pull-back $\Xi = \iota^*\Omega$ will, of course, have the same structure equations as Ω. However, if the submanifold S has strictly smaller dimension than M, i.e., $\dim S = p < m$, the one-forms $\boldsymbol{\xi} = \iota^*\boldsymbol{\omega}$ are *not* a coframe on X, because there are too many of them. Thus, Ξ will constitute an overdetermined one-form system, as discussed at the end of Section 5. In order to apply our general equivalence theorems, we need to reduce Ξ to an extended coframe by eliminating the linear dependencies among the pulled-back one-forms. Near each point $x \in X$ we can choose[†] p linearly independent one-forms $\boldsymbol{\varpi} = \{\varpi^1, \ldots, \varpi^p\}$ from among the pulled-back forms $\boldsymbol{\xi}$. The choice of $\boldsymbol{\varpi}$ is governed by a transversality condition on the submanifold S.

Definition 7.3. Let $\widetilde{\boldsymbol{\omega}} = \{\omega^1, \ldots, \omega^p\}$ be a collection of p pointwise linearly independent one-forms on an m-dimensional manifold M. A p-dimensional submanifold $S = \iota(X)$ is *transverse* with respect to $\widetilde{\boldsymbol{\omega}}$ if and only if the one-forms $\boldsymbol{\varpi} = \iota^*\widetilde{\boldsymbol{\omega}}$ forms a coframe on the parameter space X, and so S satisfies the *independence condition*

$$\varpi^1 \wedge \cdots \wedge \varpi^p = \iota^*\big(\omega^1 \wedge \cdots \wedge \omega^p\big) \neq 0. \tag{7.3}$$

We refer the reader to [**4**] for a detailed discussion of the role of independence conditions and transversality in the context of exterior differential systems. Thus, given an extended coframe $\Omega = \{\boldsymbol{\omega}, I\}$, we shall impose an independence condition on p-dimensional submanifolds $S \subset M$ by choosing p of the one-forms in $\boldsymbol{\omega}$. Since we can rearrange the forms in $\boldsymbol{\omega}$ (or, more generally, take constant coefficient linear combinations) without affecting the symmetry properties of Ω, we shall, without loss of generality, assume that the independence condition (7.3) is *always* with respect to the first p one-forms in Ω. With such a choice, we can define transversality of a submanifold with respect to an extended coframe Ω.

[†] As we shall see below, this is effected by a choice of independent and dependent variables on the original manifold M.

Definition 7.4. A p-dimensional submanifold $S = \iota(X)$ is *transverse* with respect to an extended coframe $\Omega = \{\boldsymbol{\omega}, I\}$ where $\boldsymbol{\omega} = \{\omega^1, \ldots, \omega^m\}$, if and only if it is transverse with respect to the first p one-forms $\widetilde{\boldsymbol{\omega}} = \{\omega^1, \ldots, \omega^p\}$.

On a transverse submanifold, $\boldsymbol{\varpi} = \iota^* \widetilde{\boldsymbol{\omega}}$ forms a coframe on the parameter space X. Therefore, we can write the remaining pulled back one-forms as linear combinations of them,

$$\xi^k = \sum_{j=1}^{p} K^k_j(x)\, \varpi^j, \qquad k = p+1, \ldots, m.$$

The coefficients $K = (\ldots K^k_j \ldots)$ provide additional invariants for the overdetermined one-form system Ξ. Replacing the extra one-forms by these invariants reduces Ξ to an invariantly related extended coframe, $\Upsilon = \{\boldsymbol{\varpi}, J, K\}$ on X, having the same symmetry and equivalence properties as Ξ does. We shall call Υ the *restricted G–coframe* on the submanifold S. If $\overline{S}$ is also transverse† we can similarly construct the extended coframe $\overline{\Upsilon} = \{\overline{\boldsymbol{\varpi}}, \overline{J}, \overline{K}\}$ on $\overline{X}$ using the *same* choice of coframe basis $\overline{\boldsymbol{\varpi}} = \overline{\iota}^* \widetilde{\boldsymbol{\omega}}$ relative to the given G–coframe.

Lemma 7.5. *Let Ω be a G–coframe on M. Two transverse submanifolds $S = \iota(X)$ and $\overline{S} = \overline{\iota}(\overline{X})$ are locally G congruent if and only if the corresponding restricted G–coframes Υ on X and $\overline{\Upsilon}$ on $\overline{X}$ are equivalent.*

Now, even though the original G–coframe Ω is involutive, the restricted G–coframe Υ will almost never be involutive. Thus, one will typically need to replace Υ by its involutive counterpart $\Upsilon^{(s)}$, where s is the *order* of Υ.

Definition 7.6. A submanifold $S \subset M$ is called *regular* of *order* s with respect to the G–coframe Ω if S is transverse and the restricted G–coframe Υ has order s. The *classifying manifold* of S is defined as $\mathcal{C}(S) = \mathcal{C}(\Upsilon^{(s+1)})$. The rank t of S is the dimension of its classifying manifold: $t = \dim \mathcal{C}(S)$.

Using this construction, Theorem 5.16 then gives a complete solution to the congruence problem for submanifolds when the group acts freely.

Theorem 7.7. *Let G be a free, regular Lie transformation group acting on M, and let Ω be a G–coframe. Let $S = \iota(X)$ and $\overline{S} = \overline{\iota}(\overline{X})$ be regular p-dimensional submanifolds. Then S and $\overline{S}$ are locally G equivalent if and only if they have the same order, s, and their classifying manifolds $\mathcal{C}(\overline{S}) = \mathcal{C}(S)$ are identical.*

The symmetry group of S is, by definition, its isotropy subgroup $G_S \subset G$. Theorem 7.7 demonstrates that the action of G_S on S can be identified with the symmetry group of the restricted G–coframe Υ on the parameter space X.

Theorem 7.8. *Let $S \subset M$ be a regular p-dimensional submanifold of order s and rank t with respect to the transformation group G. Then its isotropy group G_S has dimension $p - t = \dim S - \operatorname{rank} S$ and acts freely on S.*

† This can always be arranged locally by a suitable choice of the one-forms $\widetilde{\boldsymbol{\omega}}$.

In particular, S has maximal symmetry if and only if it has rank 0, meaning that all the restricted invariants J, K are constant on S. In this case, the dimension of the isotropy group equals the dimension of S and hence G_S acts transitively on (the connected components) of S. In particular, S must lie in a single G orbit of M.

Remark: Later we shall see that the invariants K arising from linear dependencies among the one-forms in Ξ can be identified with the first order differential invariants for the group action. Moreover, the derived invariants correspond to suitable higher order differential invariants. Thus, the classifying manifolds used to solve the equivalence problem are identified with those parameterized by the differential invariants for G.

Example 7.9. Consider the abelian Lie group $G = \mathbb{R}^3$ acting by translations on $M = \mathbb{R}^3$. A G–coframe is given by the coordinate one-forms $\Omega = \{dx, dy, du\}$. The surface $S = \{\, x^2 + 2yu = 0 \mid y \neq 0 \,\}$ satisfies the transversality condition $dx \wedge dy \mid S \neq 0$. Parametrizing S by $\iota\colon (x, y) \mapsto (x, y, -\frac{1}{2}x^2 y^{-1})$, we see that the restricted one-forms $\Xi = \iota^*\Omega$ satisfy the linear dependency $du = -(x/y)\, dx + \frac{1}{2}(x/y)^2\, dy$, leading to the functionally dependent invariants $-x/y$ and $x^2/(2y^2)$. Therefore, the restricted coframe on S is

$$\Upsilon = \left\{ dx, dy, -\frac{x}{y}, \frac{x^2}{2y^2} \right\}.$$

However, Υ is not involutive since $d(x/y) = (1/y)\, dx - (x/y^2)\, dy$, so that the derived coframe†

$$\Upsilon^{(1)} = \left\{ dx, dy, -\frac{x}{y}, \frac{x^2}{2y^2}, -\frac{1}{y}, \frac{x}{y^2}, \frac{x}{y^2}, -\frac{x^2}{y^3} \right\}$$

is involutive. Therefore S is a surface having rank 2 and order 1, and hence admits at most a discrete translation symmetry group; in fact, the isotropy subgroup of S is trivial. The classifying manifold is the surface parametrized by the twelve invariants in

$$\Upsilon^{(2)} = \left\{ dx, dy, -\frac{x}{y}, \frac{x^2}{2y^2}, -\frac{1}{y}, \frac{x}{y^2}, \frac{x}{y^2}, -\frac{x^2}{y^3}, 0, \frac{1}{y^2}, -\frac{2x}{y^3}, \frac{1}{y^2}, -\frac{2x}{y^3}, \frac{3x^2}{y^4} \right\}$$

so that

$$\mathcal{C}(S) = \left\{ (a_1, \ldots, a_{12}) \in Z = \mathbb{R}^{12} \;\middle|\; \begin{array}{l} a_2 = \frac{1}{2}a_1^2, a_4 = a_5 = -a_1 a_3, a_6 = a_1^2 a_3, a_7 = 0 \\ a_8 = a_{10} = a_3^2, a_9 = a_{11} = 2a_1 a_3^2, a_{12} = 3a_1^2 a_3^2 \end{array} \right\}.$$

Any translationally equivalent surface $\overline{S}$ must have the same classifying manifold, so that $\overline{S}$ also has order 1 and rank 2, and has the same functional relationships among its corresponding twelve invariants.

We are now ready to discuss the role of the regularized action in the equivalence problem for general group actions. Here we no longer need to assume that G acts freely

† Actually, since the second invariant $x^2/2y^2$ is a function of the first, its derived invariants are redundant, as their functional dependencies are automatically determined, cf. (5.8).

on M, but we replace it by its freely acting regularization on $\mathcal{B} = G \times M$. Associated with an embedded submanifold $S = \iota(X) \subset M$ is the submanifold $S_G = \iota_G(G \times X) \subset \mathcal{B}$ parametrized by

$$\iota_G(g, x) = (g, \iota(x)), \qquad g \in G, \quad x \in X.$$

The bundle $G \times X$ is the pull-back under ι of $G \times M$. On $\mathcal{B}$, we consider the G–coframe $\Omega = \{\boldsymbol{\mu}, dw, w\}$. As a direct consequence of Theorem 7.2, we obtain the following result.

Proposition 7.10. *Two embedded submanifolds S_G and $\overline{S}_G$ parametrized by maps $\iota_G: G \times X \to G \times M$, and $\overline{\iota}_G: G \times \overline{X} \to G \times M$ are locally G congruent if and only if the pulled-back extended coframes $\Xi = \{\boldsymbol{\xi}, J\} = \iota^*\Omega$ on X and $\overline{\Xi} = \{\overline{\boldsymbol{\xi}}, \overline{J}\} = \overline{\iota}^*\overline{\Omega}$ are locally equivalent.*

Suppose that $S = \iota(X)$ satisfies the transversality condition specified by $\widetilde{\boldsymbol{\omega}}$. Then $S_G = \iota_G(X)$ satisfies the transversality condition defined by $(\pi_M)^*\widetilde{\boldsymbol{\omega}} \cup \boldsymbol{\mu}$. It is clear that we can construct a coframe $\widetilde{\Omega}$ invariantly related to Ω such that the one forms $\omega^1, \ldots, \omega^p \in \widetilde{\Omega}$ generate $(\pi_M)^*\widetilde{\boldsymbol{\omega}}$. Following the procedure above Lemma 7.5 we have the restricted G–coframe Υ on $G \times X$ where $\boldsymbol{\varpi} = \{\varpi^1, \ldots, \varpi^p, \mu^1, \ldots, \mu^r\}$ such that $\varpi^1, \ldots, \varpi^p$ annihilate the tangent space to the fibers of $\pi: G \times X \to X$. Denoting by similar barred expressions using the map $\overline{\iota}_G: G \times \overline{X} \to G \times M$, the equivalence theorem takes the following form.

Proposition 7.11. *Two embedded submanifolds $\iota: X \to M$ and $\overline{\iota}: \overline{X} \to M$ which satisfy the transversality condition $\widetilde{\boldsymbol{\omega}}$, are equivalent if and only if the extended coframes Υ on $G \times X$ and $\overline{\Upsilon}$ on $G \times \overline{X}$ are equivalent.*

Proof: First suppose X and $\overline{X}$ are equivalent. Thus $\overline{\iota} = g \cdot (\iota \circ \psi^{-1})$, where $g \in G$ and $\psi: X \to \overline{X}$. Define the diffeomorphism $\Psi: G \times X \to G \times \overline{X}$ by

$$\Psi(h, x) = (g \cdot h, \psi(x)), \qquad h \in G, \quad x \in X. \tag{7.4}$$

Clearly, $\overline{\iota}_G \circ \Psi = \widehat{R}_g \cdot \iota_G$, where $\widehat{R}_g$ denotes the right regularized action (3.2) of G. Then

$$(\overline{\iota}_G \circ \Psi)^*\boldsymbol{\mu} = (\widehat{R}_g \cdot \iota_G)^*\boldsymbol{\mu} = (\iota_G)^*\boldsymbol{\mu}, \qquad (\overline{\iota}_G \circ \Psi)^* w = (\widehat{R}_g \cdot \iota_G)^* w = (\iota_G)^* w.$$

Conversely, if there exists such a Ψ, then Proposition 7.10 implies that $\overline{\iota}_G \circ \Psi = \widehat{R}_g \cdot \iota_G$ for some $g \in G$. In order to finish the proof we need to check that Ψ splits as in (7.4). The conditions on Ψ in the theorem then imply $\Psi^*\overline{\boldsymbol{\mu}} = \boldsymbol{\mu}$ and $\Psi^*\overline{\boldsymbol{\varpi}} = \boldsymbol{\varpi}$, and hence Ψ has the form in (7.4). *Q.E.D.*

Let $L: G \times M \to N$ be a regular lifted invariant. Let c be in the image of L and let $\mathcal{L}_c = L^{-1}\{c\}$ be the corresponding invariant level set. Denote the restriction of $\boldsymbol{\mu}$ and w to $\mathcal{L}_c$ by $\widetilde{\boldsymbol{\mu}}$ and $\widetilde{w}$. The submanifold $\mathcal{L}_c$ is G invariant and the local diffeomorphisms of $\mathcal{L}_c$ which preserve the restricted invariants $\widetilde{w}$ and forms $\widetilde{\boldsymbol{\mu}}$ coincide with the action of G on $\mathcal{L}_c$. That is $\{\widetilde{\boldsymbol{\mu}}, \widetilde{w}\}$ forms an (overdetermined) G–coframe on $\mathcal{L}_c$. Now let $R = (\iota_G)^{-1}(\mathcal{L}_c) \subset G \times X$, and similarly for $\overline{R} \subset G \times \overline{X}$.

Proposition 7.12. *Under the above hypothesis, the embedded submanifolds X and $\overline{X}$ are equivalent if and only if there exists a diffeomorphism $\widetilde{\Psi}: R \to \overline{R}$ such that*

$$\widetilde{\Psi}^* \bar{I} = I, \qquad \widetilde{\Psi}^* \,\overline{\boldsymbol{\omega}} = \boldsymbol{\omega},$$

where $I, \bar{I}$ and $\boldsymbol{\omega}$, $\overline{\boldsymbol{\omega}}$ are the pull-backs of the restricted invariants $\widetilde{w}$ and forms $\widetilde{\boldsymbol{\mu}}$ by ι_G and $\bar{\iota}_G$ respectively.

The proof is similar to that in Proposition 7.11.

Remark: If the function L defining the invariant submanifold $\mathcal{L}_c$ is of rank $r = \dim G$ in the vertical direction for the projection $\pi_M: G \times M \to M$ then Proposition 7.12 is resolved by Theorem 7.7.

Remark: The transformations in G_S determine symmetries of the restricted coframe on $G \times X$. However since at least p of the invariants I are automatically functionally independent, $\dim G_S \leq \dim G$, as it should be.

Therefore, regularization can be used to replace the equivalence of p-dimensional submanifolds $S \subset M$ under a non-free action of G by equivalence of $(p+r)$-dimensional submanifolds $S_G \subset \mathcal{B}$ under the free regularized G action. This approach avoids the use of differential invariants, and will also take care of singular submanifolds, since the lifted submanifold R is always regular. Incidentally, Proposition 7.12 can be used to justify partial normalization, as discussed in Section 16 below, while the preceding remark can be used to justify complete normalization. This alternative method certainly warrants further investigation.

8. Jet Bundles.

The results in the preceding sections lead to a complete construction of a moving frame in the case when the group acts freely on the underlying manifold. If the group does not act freely, then an ordinary moving frame does not exist, and one needs to prolong to some jet space of suitably high order before the procedure can be applied. In such cases, the higher order moving frame will naturally lead to the differential invariants for the transformation group. We begin by reviewing the basics of jet bundles, cf. [**17**, **18**].

Given a manifold M, we let $\mathrm{J}^n = \mathrm{J}^n(M,p)$ denote the n^{th} order (extended) jet bundle consisting of equivalence classes of p-dimensional submanifolds $S \subset M$ under the equivalence relation of n^{th} order contact, cf. [**16**], [**17**; Chapter 3]. In particular, $\mathrm{J}^0 = M$. We let $\mathrm{j}_n S \subset \mathrm{J}^n$ denote the n-jet of the submanifold S; more explicitly, the parametrization map $\iota: X \to S \subset M$ induces a parametrization $\mathrm{j}_n \iota: X \to \mathrm{j}_n S \subset M$. The fibers of $\pi^n_0: \mathrm{J}^n \to M$ are generalized Grassmann manifolds, [**16**]. A *differential function* of order n is a scalar-valued function $F: \mathrm{J}^n \to \mathbb{R}$. Sometimes, it is convenient to work with the infinite jet bundle $\mathrm{J}^\infty = \mathrm{J}^\infty(M,p)$, which is defined as the inverse limit of the finite order jet bundles under the standard projections $\pi^k_n: \mathrm{J}^k \to \mathrm{J}^n$, $k > n$. Functions and differential forms on J^∞ are obtained from their finite order counterparts, where we identify a form ω on J^n with its pull-backs $(\pi^k_n)^*\omega$ on J^k for any $k > n$, and hence with a differential form on J^∞. For further details on infinite jet bundles, see [**1**, **26**].

We introduce local coordinates $z = (x, u)$ on M, considering the first p components $x = (x^1, \ldots, x^p)$ as independent variables, and the latter $q = m - p$ components $u = (u^1, \ldots, u^q)$ as dependent variables. Splitting the coordinates into independent and dependent variables has the effect of locally identifying M with an open subset of a bundle $E = X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$. Sections $u = f(x)$ of E correspond to p-dimensional submanifolds S that are transverse with respect to the horizontal forms $dx = \{dx^1, \ldots, dx^p\}$, as in Definition 7.3. The induced coordinates on the jet bundle J^n are denoted by $z^{(n)} = (x, u^{(n)})$, with components u^α_J representing the partial derivatives of the dependent variables with respect to the independent variables up to order n. Here $J = (j_1, \ldots, j_k)$ is a symmetric multi-index of order $k = \#J$, with $1 \leq j_\nu \leq p$. The $(x, u^{(n)})$ define local coordinates on the open, dense subbundle $\mathrm{J}^n E \subset \mathrm{J}^n(M, p)$ determined by the jets of transverse submanifolds, or, equivalently, local sections $u = f(x)$. In the limit, we let $z^{(\infty)} = (x, u^{(\infty)})$ denote the corresponding coordinates on $\mathrm{J}^\infty E \subset \mathrm{J}^\infty(M, p)$, consisting of independent variables x^i, dependent variables u^α, and their derivatives u^α_J, $\alpha = 1, \ldots, q$, $0 \leq \#J$, of arbitrary order.

The intrinsic geometry of jet space is governed by a fundamental collection of differential forms.

Definition 8.1. A differential form θ on the jet space $\mathrm{J}^n(M, p)$ is called a *contact form* if it is annihilated by all jets, so that $(\mathrm{j}_n \iota)^* \theta = 0$ for every p-dimensional submanifold $S = \iota(X) \subset M$.

The subbundle of the cotangent bundle $T^* \mathrm{J}^n$ spanned by the contact one-forms will be called the n^{th} order *contact bundle*, denoted by $\mathcal{C}^{(n)}$. The infinite contact bundle $\mathcal{C}^{(\infty)} \subset T^* \mathrm{J}^\infty$ is a codimension p subbundle of $T^* \mathrm{J}^\infty$. (This result is not true for finite order contact subbundles, which is one of the main reasons for going to infinite order.) In terms of local coordinates $(x, u^{(\infty)})$, every contact one-form can be written as a linear combination of the *basic contact forms*

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^p u^\alpha_{J,i}\, dx^i, \qquad \alpha = 1, \ldots, q, \quad 0 \leq \#J. \tag{8.1}$$

Combining the horizontal coordinate one-forms dx^i with the basic contact forms θ^α_J produces the local coordinate coframe on J^∞:

$$dx = \{dx^1, \ldots, dx^p\}, \qquad \boldsymbol{\theta}^{(\infty)} = \{\ldots, \theta^\alpha_J, \ldots\}. \tag{8.2}$$

Therefore, choosing local coordinates on M induces a splitting $T^* \mathrm{J}^\infty = \mathcal{H} \oplus \mathcal{C}^{(\infty)}$ of the cotangent bundle into *horizontal* and contact or *vertical* subbundles, with $\mathcal{H}$ spanned by the horizontal one-forms dx. Let $\pi_H : T^* \mathrm{J}^\infty \to \mathcal{H}$ and $\pi_V : T^* \mathrm{J}^\infty \to \mathcal{C}^{(\infty)}$ be the induced projections, so that any one-form $\omega = \omega_H + \vartheta$ splits into uniquely defined horizontal and vertical components, where

$$\omega_H = \pi_H(\omega) = \sum_{i=1}^p P_i(x, u^{(n)})\, dx^i \tag{8.3}$$

is a horizontal one-form, and

$$\vartheta = \pi_V(\omega) = \sum_{\alpha,J} Q_J^\alpha(x, u^{(n)})\,\theta_J^\alpha \tag{8.4}$$

is a contact form. If ω is a one-form on J^n then, typically, its horizontal component ω_H is a one-form on J^{n+1}.

The splitting of $T^*\mathrm{J}^\infty$ induces a bi-grading of the differential forms on J^∞. The differential d on J^∞ naturally splits into horizontal and vertical components, $d = d_H + d_V$, where d_H increases horizontal degree and d_V increases vertical degree. Closure, $d \circ d = 0$, implies that $d_H \circ d_H = 0 = d_V \circ d_V$, while $d_H \circ d_V = -d_V \circ d_H$. In particular, the horizontal or *total differential* of a differential function $F\colon \mathrm{J}^n \to \mathbb{R}$ is the horizontal one-form

$$d_H F = \sum_{i=1}^{p} (D_i F)\,dx^i \tag{8.5}$$

on J^{n+1}, where

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_J u_{J,i}^\alpha\,\frac{\partial}{\partial u_J^\alpha} \tag{8.6}$$

denotes the usual total derivative with respect to x^i, which can be viewed as a vector field on $\mathrm{J}^\infty E$. Similarly, the vertical differential of a function $F(x, u^{(n)})$ is the contact form

$$d_V F = \sum_{i=1}^{p}\sum_K \frac{\partial F}{\partial u_K^\alpha}\;\theta_K^\alpha. \tag{8.7}$$

Definition 8.2. A *total differential operator* is a vector field on J^∞ which lies in the annihilator of the contact bundle $\mathcal{C}^{(\infty)}$.

Proposition 8.3. *Every total differential operator has the form*

$$\mathcal{D} = \sum_{i=1}^{p} Q_i(x, u^{(n)})\,D_i, \tag{8.8}$$

where $Q_1, \ldots, Q_p$ are differential functions.

The preceding construction forms the foundation of the variational bicomplex that is of fundamental importance in the study of the geometry of jet bundles, differential equations and the calculus of variations; see [**1**, **26**, **29**] for details.

9. Prolonged Transformation Groups.

Any transformation group G acting on M preserves the order of contact between submanifolds. Therefore, there is an induced action of G on the n^{th} order jet bundle $\mathrm{J}^n(M,p)$ known as the n^{th} *prolongation* of G. Alternatively, one can characterize the prolonged group transformations as the unique lifted maps on the jet bundle that preserve the space of contact forms.

Definition 9.1. A map $\Psi: \mathrm{J}^n \to \mathrm{J}^n$ is a *contact transformation* if it preserves the order n contact subbundle: $\Psi^* \mathcal{C}^{(n)} \subset \mathcal{C}^{(n)}$.

Proposition 9.2. *If $\psi: M \to M$ is a local diffeomorphism, then its n^{th} prolongation is the unique contact transformation $\psi^{(n)}: \mathrm{J}^n \to \mathrm{J}^n$ that satisfies $\psi \circ \pi_0^n = \pi_0^n \circ \psi^{(n)}$.*

We denote the prolonged group action on J^n by $G^{(n)}$. Note that if G acts globally on M, then its prolonged action $G^{(n)}$ is also a global transformation group on $\mathrm{J}^n(M,p)$, but, generally only a local transformation group on the coordinate subbundles $\mathrm{J}^n E$ since G may not preserve transversality.

Remark: Our methods also apply, with minor modifications, to more general contact transformation groups. Bäcklund's Theorem, cf. [**18**], implies that these reduce to prolonged point transformation groups on M, or, in the codimension 1 case, prolonged first order contact transformation groups.

Let us choose a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ for the Lie algebra $\mathfrak{g}$ of infinitesimal generators on M, and let $\{\mathrm{pr}^{(n)}\,\mathbf{v}_1, \ldots, \mathrm{pr}^{(n)}\,\mathbf{v}_r\}$ denote the corresponding the infinitesimal generators of the prolonged group action $G^{(n)}$. In terms of local coordinates $(x, u^{(\infty)})$ on J^∞, we obtain $\mathrm{pr}^{(n)}\,\mathbf{v}_\kappa$ by truncating the infinitely prolonged vector field

$$\mathrm{pr}\,\mathbf{v}_\kappa = \sum_{i=1}^{p} \xi^i_\kappa(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{j=\#J\geq 0} \varphi^\alpha_{J,\kappa}(x,u^{(j)})\,\frac{\partial}{\partial u^\alpha_J}\,, \qquad \kappa = 1, \ldots, r, \tag{9.1}$$

at order n. The coefficients of (9.1) are explicitly determined by the standard prolongation formula, [**18**]:

$$\varphi^\alpha_{J,\kappa} = D_J Q^\alpha_\kappa + \sum_{i=1}^{p} \xi^i_\kappa\, u^\alpha_{J,i}, \tag{9.2}$$

where

$$Q^\alpha_\kappa(x,u^{(1)}) = \varphi^\alpha_\kappa(x,u) - \sum_{i=1}^{p} \xi^i_\kappa(x,u)\, u^\alpha_i \tag{9.3}$$

is the *characteristic* of $\mathbf{v}_\kappa$.

The moving frame construction in Section 4 can be applied to the prolonged group action $G^{(n)}$ provided it acts (locally) freely on J^n. Therefore, we need to understand the basic geometry of the prolonged action in order to understand the full range of applicability of the higher order moving frame construction.

Definition 9.3. Given G acting on M, we let s_n denote the maximal orbit dimension of the prolonged action $G^{(n)}$ on J^n. The *stable orbit dimension* $s = \max s_n$ is the maximum prolonged orbit dimension. The *stabilization order* of G is the minimal n such that $s_n = s$.

A fundamental stabilization theorem due to Ovsiannikov, [**22**], completely characterizes the stable orbit dimension; see also [**18, 20**] for further details.

Theorem 9.4. *A Lie group G acts locally effectively on subsets of M if and only if its stable orbit dimension equals its dimension, $s = r = \dim G$.*

Definition 9.5. The *regular subset* $\mathcal{V}^n \subset \mathrm{J}^n$ is the open subset consisting of all prolonged group orbits of dimension equal to the stable orbit dimension. The *singular subset* is the remainder, $\mathcal{S}^n = \mathrm{J}^n \setminus \mathcal{V}^n$, which is the union of all $G^{(n)}$ orbits of less than maximal dimension.

Note that, by this definition, $\mathcal{V}^n = \varnothing$ and $\mathcal{S}^n = \mathrm{J}^n$ if n is less than the stabilization order of $G^{(n)}$. If G acts analytically, then $\mathcal{V}^n$ is a dense open subset of J^n for n greater than or equal to the stabilization order. The singular subset $\mathcal{S}^n$ can be characterized by the vanishing of the Lie determinant or its generalizations, cf. [**18**; chap. 6]. A point $z^{(n)} \in \mathrm{J}^n$ will be called a *regular jet* provided $z^{(n)} \in \mathcal{V}^n$ or, equivalently, the prolonged orbit passing through $z^{(n)}$ has dimension $r = \dim G$, assuming G acts locally effectively on subsets. The stabilization Theorem 9.4 combined with Proposition 2.5 immediately implies the freeness of the prolonged action on the regular subset of jet space.

Proposition 9.6. *If G acts locally effectively on subsets, then G acts locally freely on the regular subset $\mathcal{V}^n \subset \mathrm{J}^n$.*

Remark: It would be nice to know that $G^{(n)}$ acts freely on (a dense open subset of) $\mathcal{V}^n$ provided n is sufficiently large. We do not know a general theorem that guarantees the freeness of prolonged group actions, although it seems highly unlikely, particularly in the analytic category, that a group acts only locally freely on all of $\mathcal{V}^n$ when n is large.

Definition 9.7. A submanifold $S \subset M$ is *order n regular* if $\mathrm{j}_n S \subset \mathcal{V}^n$. A submanifold $S \subset M$ is *totally singular* if $\mathrm{j}_n S \subset \mathcal{S}^n$ for all $n = 0, 1, \ldots$.

The characterization of submanifolds which are singular to all orders is of importance for understanding the range of validity of the moving frame method. The following theorem can be found in [**20**].

Theorem 9.8. *A submanifold $S \subset M$ is totally singular if and only if its isotropy subgroup G_S does not act locally freely on S itself.*

Example 9.9. Consider the special affine group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$ acting on the plane $M = \mathbb{R}^2$ via $z \mapsto Az+b$, where $\det A = 1$. The totally singular curves are the straight lines, the isotropy subgroup consists of translations, shears, and unimodular scalings in the direction of the line. In terms of the coordinates $z = (x, u)$, the singular subset of J^n is given by

$$\mathcal{S}^n = \left\{ \, (x, u^{(n)}) \;\middle|\; u_{xx} = u_{xxx} = \cdots = u_n = 0 \, \right\}, \tag{9.4}$$

where $u_n = d^n u/dx^n$. A curve $u = f(x)$ is totally singular at a point $(x_0, f(x_0))$ if and only if $f^{(n)}(x_0) = 0$ for all $n \geq 2$. In particular, an analytic curve that is totally singular at a point is necessarily a straight line.

The full affine group $\mathrm{A}(2)$ is more interesting. Here the totally singular curves are the parabolas and the straight lines. The isotropy group of a parabola, say $u = x^2$, is the two-dimensional nonabelian subgroup $(x, u) \mapsto (\lambda x + \mu, \lambda^2 u + 2\lambda\mu x + \mu^2)$. In this case the singular subset of J^n is also determined by the total derivatives of the Lie determinant

$$\mathcal{S}^n = \left\{ \, (x, u^{(n)}) \;\middle|\; D_x^{n-4} \left[u_{xx} u_{xxxx} - \tfrac{5}{3} u_{xxx}^2 \right] = 0 \, \right\}. \tag{9.5}$$

The parabolas and straight lines form the general solution to the affine Lie determinant equation $u_{xx}u_{xxxx} = \frac{5}{3}u_{xxx}^2$.

Let us now quickly review the standard theory of differential invariants for Lie transformation groups; see [**18**; Chapter 5] for details.

Definition 9.10. A *differential invariant* is a (locally defined) scalar differential function $I: \mathrm{J}^n \to \mathbb{R}$ which is invariant under the action of $G^{(n)}$, so that $I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)})$ for all $g \in G$ and all $z^{(n)} \in \mathrm{J}^n$ where the equation is defined.

Assuming G acts locally effectively on subsets, there are

$$i_n = \dim \mathrm{J}^n - \dim G = p + q\binom{p+n}{n} - r \tag{9.6}$$

functionally independent differential invariants of order $\leq n$ near any point $z^{(n)} \in \mathcal{V}^n$.

Remark: If n is less that the stabilization order, then we replace r in (9.6) by the maximal orbit dimension of G on J^n and restrict $z^{(n)}$ to lie in the open subset of J^n where the prolonged orbits of $G^{(n)}$ have maximal dimension.

The basic method, due to Lie and Tresse, [**25**], for constructing a complete system of differential invariants is to use invariant differential operators. A total differential operator (8.8) is said to be *G-invariant* if it commutes with the prolonged action of G. The most effective way to construct such operators relies on a suitably G-invariant basis for the horizontal one-forms on the jet space.

Definition 9.11. A differential one-form ω on J^n is called *contact-invariant* if and only if, for every $g \in G$, we have $(g^{(n)})^*\omega = \omega + \theta_g$ for some contact form θ_g. A *horizontal contact-invariant coframe* on J^n is a collection of p linearly independent horizontal one-forms which are contact-invariant under the prolonged action of $G^{(n)}$.

For brevity we shall usually drop the adjective "horizontal" in the description of contact-invariant coframes. Contact-invariant coframes are the jet space counterparts of the differential geometric coframes discussed in Section 5. Note that a contact-invariant coframe only forms a coframe on the horizontal subbundle $\mathcal{H} \subset T^*\mathrm{J}^\infty$. A full coframe on J^∞ requires additional contact forms; see below.

Proposition 9.12. *If I is any differential invariant, its horizontal differential $d_H I$ is a contact-invariant one-form.*

Thus, if we know p suitably independent differential invariants, we can construct a horizontal contact-invariant coframe. However, this approach is usually not the best method for determining such coframes. If $F(x, u^{(n)})$ is any differential function, we can rewrite its horizontal differential in terms of the horizontal coframe as

$$d_H F = \sum_{j=1}^{p} (\mathcal{D}_j F)\, \omega^j. \tag{9.7}$$

The resulting G-invariant total differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_k$ are the jet space counterparts of the usual coframe derivatives, cf. (5.5). In local coordinates, suppose

$$\omega^i = \sum_{j=1}^{p} P_j^i(x, u^{(n)})\, dx^j, \qquad i = 1, \ldots, p, \tag{9.8}$$

where the coefficient matrix $P = (P_j^i)$ is nonsingular. The corresponding invariant differential operators are then given by

$$\mathcal{D}_j = \sum_{i=1}^{p} Q_j^i(x, u^{(n)})\, D_i, \qquad j = 1, \ldots, p, \tag{9.9}$$

with inverse coefficient matrix $Q = (Q_j^i) = P^{-1}$. If we consider the coordinate one-forms $dx = (dx^1, \ldots, dx^p)^T$ and total derivatives $\mathbf{D} = (D_1, \ldots, D_p)^T$ as column vectors, then (9.8) is written as $\boldsymbol{\omega} = P \cdot dx$, while (9.9) becomes $\mathcal{D} = Q^T \cdot \mathbf{D} = P^{-T} \cdot \mathbf{D}$. The invariant differential operators form an invariant "horizontal frame" on J^∞, cf. [**15**].

Any invariant differential operator maps differential invariants to higher order differential invariants, and thus, by iteration, produces hierarchies of differential invariants of arbitrarily large order. In this way, a complete list of differential invariants can be produced by successively differentiating a finite number of differential invariants, which we call a *generating system* of differential invariants. We use the notation $\mathcal{D}_J = \mathcal{D}_{j_1} \cdots \mathcal{D}_{j_k}$, $1 \leq j_\nu \leq p$, denote the corresponding k^{th} order invariant differential operators.

Theorem 9.13. *Suppose that G is a transformation group, and let n be its order of stabilization. Then, in a neighborhood of any regular jet $z^{(n)} \in \mathcal{V}^n$, there exists a contact-invariant coframe $\{\omega^1, \ldots, \omega^p\}$, and corresponding invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$. If $I(z^{(k)})$ is a differential invariant, then so is $\mathcal{D}_J I$ for any multi-index J. Moreover, there exists* a generating system *of functionally independent differential invariants $I_1, \ldots, I_l$, of order at most $n+1$, such that, locally, every differential invariant can be written as a function of the differentiated invariants $\{\, \mathcal{D}_J I_\nu \mid \nu = 1, \ldots, l,\ \#J \geq 0 \,\}$.*

See [**22**; p. 320] and [**18**; Theorem 5.49] for more details. The theorem is mis-stated in [**18**] — the order of the fundamental differential invariants should be at most $n+1$, not $n+2$. Except in the case of curves, where $p = 1$, the precise number of differential invariants required in a generating system is not known. A refinement of Theorem 9.13 will be proved below; see Theorem 13.1.

The invariant differential operators coming from a general contact-invariant coframe do not necessarily commute. The commutation formulae (5.6) for ordinary coframe derivatives are an immediate consequence of the closure identity $d^2 = 0$. Similarly, if $\boldsymbol{\omega}$ is the contact-invariant coframe, then

$$d_H \omega^k = -\sum_{i<j} A_{ij}^k\, \omega^i \wedge \omega^j, \qquad k = 1, \ldots, p, \tag{9.10}$$

where the coefficients $A_{ij}^k = -A_{ji}^k$ are differential invariants. Thus, applying d_H to (9.7) produces the commutation formulae

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{i=1}^{p} A_{ij}^k \, \mathcal{D}_k, \qquad i, j = 1, \ldots, p, \tag{9.11}$$

for the associated invariant differential operators. If all the coframe forms are constructed from differential invariants, i.e., $\omega^k = d_H I^k$, $k = 1, \ldots, p$, then $d_H \omega^k = 0$, and hence the invariant differential operators all commute in this particular case. The commutation formula (9.11) implies that a complete system of higher order differential invariants can be obtained by only including the differentiated invariants $\mathcal{D}_J I_\nu$ corresponding to nondecreasing multi-indices $1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq p$. However, even with this proviso, the differentiated invariants $\mathcal{D}_J I_\nu$ are not necessarily functionally independent.

Definition 9.14. Given a functionally independent generating system of differential invariants $I_1, \ldots, I_k$, a *syzygy* is a functional dependency among the differentiated invariants: $H(\ldots \mathcal{D}_J I_\nu \ldots) \equiv 0$.

There are two types of syzygies, the first arising from the commutation rules for the invariant differential operators, and the second "essential syzygies" are where the function H only depends on the differentiated invariants $\mathcal{D}_J I_\nu$ having nondecreasing multi-indices J. In Theorem 13.4 below, we shall provide a precise classification of all such syzygies.

Example 9.15. An elementary example is provided by the three parameter group $(x^1, x^2, u) \mapsto (\lambda x^1 + a, \lambda x^2, u + b)$ acting on $M = \mathbb{R}^3$. The one-forms $\omega^1 = (x^2)^{-1} dx^1$, $\omega^2 = (x^2)^{-1} dx^2$ form a contact-invariant coframe, with invariant differential operators $\mathcal{D}_1 = x^2 D_1$, $\mathcal{D}_2 = x^2 D_2$. We note the commutation formula $[\mathcal{D}_1, \mathcal{D}_2] = -\mathcal{D}_2$. The first order differential invariants $I_1 = x^2 u_1$ and $I_2 = x^2 u_2$ form a generating system, and $I_{ijk} = (\mathcal{D}_1)^j (\mathcal{D}_2)^k I_i$, $i = 1, 2$, $j + k \geq 0$, form a complete system of differential invariants. In this case there is a single essential syzygy, $\mathcal{D}_2 I_1 = \mathcal{D}_1 I_2 - I_1$, from which all higher order syzygies can be deduced by invariant differentiation.

Remark: For curves in an m-dimensional manifold, one requires $k = m - 1$ generating differential invariants, and a single invariant differential operator $\mathcal{D}$. Moreover, in this case, there are no syzygies among the differentiated invariants $\mathcal{D}^k I_\nu$, cf. [**11, 18**].

10. Higher Order Regularization.

We are now in a position to describe the general version of our moving frame construction. The key idea is to apply the regularization technique to the prolonged group action on the extended jet bundles over the manifold M. All of our earlier constructions (which describe the order zero case) can be immediately applied to this particular type of transformation group action. Moving frames can be computed provided the prolonged action is (locally) free, i.e., on the regular subset of J^n. In this manner, we shall be able to construct a higher order moving frame associated with all but the totally singular submanifolds of the original space.

We assume that G acts locally effectively on subsets of M. For simplicity, we only discuss the right regularization of the prolonged group action on the jet bundle $\mathrm{J}^n = \mathrm{J}^n(M,p)$ corresponding to p-dimensional submanifolds of M. The left counterparts can be simply obtained by applying the group inversion.

Definition 10.1. The n^{th} order *regularized jet bundle* is the trivial left principal bundle $\pi_n \colon \mathcal{B}^{(n)} = G \times \mathrm{J}^n \to \mathrm{J}^n$. The n^{th} order (right) *regularization* of the prolonged group action on J^n is the action of G on $\mathcal{B}^{(n)}$ given by

$$R_g^{(n)}(h, z^{(n)}) = R^{(n)}(g,(h,z^{(n)})) = (h \cdot g^{-1}, g^{(n)} \cdot z^{(n)}), \qquad g \in G, \quad (h, z^{(n)}) \in \mathcal{B}^{(n)}. \tag{10.1}$$

Theorem 3.2 implies that the regularized action on $\mathcal{B}^{(n)}$ is both free and regular.

Definition 10.2. A *lifted differential invariant* is a (locally defined) invariant function $L \colon \mathcal{B}^{(n)} \to N$.

A complete system of functionally independent lifted differential invariants is provided by the components of the order n evaluation map

$$w^{(n)} = g^{(n)} \cdot z^{(n)}. \tag{10.2}$$

Clearly $w^{(n)} \colon \mathcal{B}^{(n)} \to \mathrm{J}^n$ is invariant under the lifted action (10.1). As in Section 3, every lifted differential invariant can be locally written as a function of the fundamental lifted differential invariants $w^{(n)}$. In particular, an ordinary differential invariant $I \colon \mathrm{J}^n \to \mathbb{R}$ also defines a lifted differential invariant $L = I \circ \pi_n$, and hence can also be locally expressed as a function of the w's; conversely, any lifted invariant $L(g, x, u^{(n)})$ that does not depend on the g coordinates automatically defines an ordinary differential invariant. Our simple replacement Theorem 3.7 immediately applies to the construction of differential invariants from their lifted counterparts.

Theorem 10.3. *Let $I(z^{(n)})$ be an ordinary differential invariant. Then we can write $I(z^{(n)}) = I(g^{(n)} \cdot z^{(n)})$ as the* same *function of the lifted differential invariants.*

Remark: In Riemannian geometry, Theorem 10.3 reduces to the striking Thomas Replacement Theorem, [**24**; p. 109], which is proved by appealing to normal coordinates. See [**2**] for recent applications of Thomas' result.

Example 10.4. Consider the (standard) action of the Euclidean group SE(2) on $M = \mathbb{R}^2$. Introducing local coordinates (x,u), the second order prolongation maps a point $(x, u, u_x, u_{xx}) \in \mathrm{J}^2$ to

$$\left(x\cos\phi - u\sin\phi + a,\, x\cos\phi + u\sin\phi + b,\, \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi},\, \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3} \right), \tag{10.3}$$

reproducing the action (4.12). The second order lifted invariants (10.2), which we denote as $w^{(2)} = (y, v, v_y, v_{yy})$, are the components of the transformation formulae (10.3). The Euclidean curvature differential invariant can be constructed in terms of the lifted invariants:

$$\kappa = \frac{v_{yy}}{(1+v_y^2)^{3/2}} = \frac{u_{xx}}{(1+u_x^2)^{3/2}}. \tag{10.4}$$

The Replacement Theorem 10.3 guarantees that the formula for κ in terms of the usual jet coordinates (x, u, u_x, u_{xx}) is the same functional relation as its formula in terms of the lifted invariants (y, v, v_y, v_{yy}).

The regularization construction extends to the infinite order regularized jet bundle $\pi_\infty : \mathcal{B}^{(\infty)} = G \times \mathrm{J}^\infty \to \mathrm{J}^\infty$ in the obvious manner. The pull-back of the contact bundle $\mathcal{C}^{(\infty)} \subset T^*\mathrm{J}^\infty$ defines the contact subbundle† $\mathcal{C}^{(\infty)} \subset T^*\mathcal{B}^{(\infty)}$. Similarly, the pull-back via $\pi_G : \mathcal{B}^{(\infty)} \to G$ of the cotangent bundle of G, spanned by its Maurer–Cartan forms, define a second intrinsic subbundle of $T^*\mathcal{B}^{(\infty)}$, which we also denote by T^*G. The product bundle $T^*G \times \mathcal{C}^{(\infty)}$ forms a codimension p subbundle of the cotangent bundle $T^*\mathcal{B}^{(\infty)}$. Since $\mathcal{B}^{(\infty)} = G \times \mathrm{J}^\infty$ is a Cartesian product, the differential on $\mathcal{B}^{(\infty)}$ naturally splits into jet and group components, $d = d_J + d_G$.

Proposition 10.5. *If ω is a G-invariant differential form on $\mathcal{B}^{(\infty)}$, then so are both $d_J\omega$ and $d_G\omega$. In particular, if $L(g, z^{(n)})$ is a lifted differential invariant, then its jet and group differentials, $d_J L$ and $d_G L$, are invariant one-forms on $\mathcal{B}^{(\infty)}$.*

Let us now discuss the local coordinate expressions for the regularized action and its invariants. As above, the introduction of local coordinates $(x, u^{(\infty)})$ on J^∞ produces a local coframe on $\mathcal{B}^{(\infty)}$ consisting of the horizontal forms dx, the system of basic contact forms $\boldsymbol{\theta}$, along with the right-invariant Maurer–Cartan forms $\boldsymbol{\mu}$, all pulled back to $\mathcal{B}^{(\infty)}$. The choice of horizontal complement produces a splitting of the differential on $\mathcal{B}^{(\infty)}$ into horizontal, vertical, and group components, so that we have the more refined decomposition

$$d\omega = d_J\omega + d_G\omega = d_H\omega + d_V\omega + d_G\omega \tag{10.5}$$

for any differential form on $\mathcal{B}^{(\infty)}$. Note also that

$$\begin{aligned} &d_H \circ d_H = 0, && d_V \circ d_V = 0, && d_G \circ d_G = 0, \\ &d_H \circ d_V = -d_V \circ d_H, && d_H \circ d_G = -d_G \circ d_H, && d_V \circ d_G = -d_G \circ d_V. \end{aligned} \tag{10.6}$$

Remark: We have not investigated the topological and variational aspects of the induced "regularized variational tricomplex" governed by the differentials d_H, d_V and d_G.

In particular, the horizontal and the vertical differentials of a function $F(g, x, u^{(n)})$ have the same formulae (8.5), (8.7), as before, where the total derivatives D_i have their usual coordinate formulae, i.e., there are no derivatives with respect to the g coordinates. Note that the horizontal and vertical differentials of a lifted invariant are *not*, in general, G-invariant one-forms on $\mathcal{B}^{(\infty)}$. However, the horizontal differential does satisfy the weaker, but very important, invariance property of Definition 9.11.

Proposition 10.6. *If $L(g, z^{(n)})$ is a lifted invariant, then its horizontal differential $d_H L$ is a contact-invariant one-form.*

† For simplicity, we drop explicit reference to the pull-back via the projection map.

The standard jet space coordinates $(x, u^{(\infty)})$ are not well adapted to the lifted group action on $\mathcal{B}^{(\infty)}$, and we shall replace them by a fundamental system of invariant coordinates based on the fundamental lifted differential invariants. The introduction of local independent and dependent variable coordinates $z = (x, u)$ on M induces a local identification with a trivial bundle $E = X \times U$. This induces a splitting of the fundamental zero$^{\text{th}}$ order lifted invariants $w = (w^1, \ldots, w^m) = g \cdot z$ into two components. In the (x, u) coordinates, we write‡ $w = (y, v)$, where $y = (y^1, \ldots, y^p)$ will be considered as "lifted independent variables", and $v = (v^1, \ldots, v^q)$, as "lifted dependent variables". Let

$$\eta^i = d_H y^i = \sum_{j=1}^{p} (D_j y^i)\, dx^j, \qquad i = 1, \ldots, p, \tag{10.7}$$

denote the horizontal differentials of lifted independent variables. The coefficient matrix $\mathbf{D}y = \big(D_j y^i\big)$ is obtained by total differentiation of the lifted invariants y treating the group parameters as constants, so the lifted horizontal forms $\boldsymbol{\eta} = (\mathbf{D}y) \cdot dx$ are defined on the first order regularized jet space $\mathcal{B}^{(1)}$. Since the functions y^i are lifted invariants, Proposition 10.6 implies that the one-forms $\boldsymbol{\eta}$ are contact-invariant under the lifted action of G on $\mathcal{B}^{(1)}$. The η's are linearly independent if and only if the y's have nonvanishing total Jacobian determinant:

$$\det \mathbf{D}y = \frac{\mathbf{D}(y^1, \ldots, y^p)}{\mathbf{D}(x^1, \ldots, x^p)} = \det\big(D_j y^i\big) \neq 0. \tag{10.8}$$

This condition can be geometrically characterized as follows.

Proposition 10.7. *The horizontal one-forms $\boldsymbol{\eta} = (\mathbf{D}y) \cdot dx$ are linearly independent, $\eta^1 \wedge \cdots \wedge \eta^p \neq 0$, on the open subset $\mathcal{W}^{(1)} \subset \mathcal{B}^{(1)}$ determined by the* 1*-jets of submanifolds S such that both S and $g \cdot S$ are transverse with respect to the given coordinates on M. Thus, $\mathcal{W}^{(1)} = \{\, (g, z^{(1)}) \in G \times \mathrm{J}^1 E \mid g^{(1)} \cdot z^{(1)} \in \mathrm{J}^1 E \,\}$. At such points, we call $\boldsymbol{\eta} = (\eta^1, \ldots, \eta^p)$ the* lifted (horizontal) contact-invariant coframe *for the given coordinate chart.*

The corresponding invariant differential operators are readily found. As in the usual (unlifted) version, (9.7), we write the total differential of any scalar function $F\colon \mathcal{B}^{(n)} \to \mathbb{R}$ in invariant form

$$d_H F = \sum_{j=1}^{p} (\mathcal{E}_j F)\, \eta^j \tag{10.9}$$

with respect to the prescribed contact-invariant coframe. The corresponding total differential operators are $\mathcal{E} = (\mathbf{D}y)^{-T} \cdot \mathbf{D}$, or, explicitly,

$$\mathcal{E}_j F = \frac{\mathbf{D}(y^1, \ldots, y^{j-1}, F, y^{j+1}, \ldots y^p)}{\mathbf{D}(y^1, \ldots, y^p)} = \sum_{i=1}^{p} Z_j^i(g, x, u^{(1)})\, D_i, \tag{10.10}$$

‡ For simplicity, we have chosen to split the lifted invariants into independent and dependent components in the same way as we split the unlifted variables. Actually, one can choose alternative splittings of w into p independent and q dependent components, although one must then accordingly modify the required transversality conditions.

where $Z = (Z^i_j) = (\mathbf{D}y)^{-1}$. Thus, we can identify the *lifted invariant differential operator* $\mathcal{E}_j = D_{y^j}$ with total differentiation with respect to the lifted invariant y^j; in particular, $\mathcal{E}_j y^i = \delta^i_j$. Note that the lifted invariant differential operators do not involve differentiation with respect to the group parameters. A very important point is that, unlike the usual invariant differential operators, the lifted invariant differential operators *always* mutually commute:

$$\left[\mathcal{E}_i, \mathcal{E}_j\right] = 0. \tag{10.11}$$

This follows from the closure of the horizontal differential, $d_H \circ d_H = 0$, and is an immediate consequence of the fact that the lifted contact-invariant coframe (10.7) is the horizontal derivative of lifted invariants; see the discussion following (9.11). We let $\mathcal{E}_K = \mathcal{E}_{k_1} \cdot \ldots \cdot \mathcal{E}_{k_l}$ denote the associated higher order invariant differential operator; equation (10.11) shows that the order of the invariant differentiation is irrelevant.

The lifted invariant differential operators can be used to compute higher order lifted differential invariants. The basic result follows immediately from (10.9), the contact-invariance of the forms $\boldsymbol{\eta}$, along with the fact that the prolonged group transformations preserve the contact ideal.

Proposition 10.8. *If $L\colon \mathcal{B}^{(n)} \to \mathbb{R}$ is any lifted differential invariant, then so are its invariant derivatives $\mathcal{E}_K L\colon \mathcal{B}^{(n+k)} \to \mathbb{R}$, where $k = \#K \geq 0$.*

If we successively apply the invariant differential operators associated with the first p lifted invariants $y = (y^1, \ldots, y^p)$ to the remaining zero$^{\text{th}}$ order invariants $v = (v^1, \ldots, v^q)$, we recover all the higher order lifted invariants $v^{(n)}$. Since $w^{(n)} = g^{(n)} \cdot z^{(n)}$, an alternative way of viewing this result is that the process of lifted invariant differentiation produces the explicit formulae for the prolonged group transformations, thereby implementing the standard process of implicit differentiation, cf. [**17**].

Lemma 10.9. *The components of the n^{th} order lifted invariant $w^{(n)}$ consist of the basic invariants $w = (y, v)$ together with the higher order lifted differential invariants*

$$v^\alpha_K = \mathcal{E}_K v^\alpha, \qquad \alpha = 1, \ldots, q, \quad \#K \leq n. \tag{10.12}$$

Proof: For fixed g, the map $w^{(n)}\colon \mathrm{J}^n \to \mathrm{J}^n$ is a contact transformation on the jet bundle, hence the pull-back $(w^{(n)})^*$ maps contact forms to contact forms. Now,

$$(w^{(n)})^* \theta^\alpha_K = d_J v^\alpha_K - \sum_{i=1}^p v^\alpha_{K,i}\, d_J y^i, \qquad \alpha = 1, \ldots, q, \quad \#K \leq n-1. \tag{10.13}$$

The right hand side will be a contact form if and only if its horizontal component vanishes, so that

$$d_H v^\alpha_K = \sum_{i=1}^p v^\alpha_{K,i}\, \eta^i, \qquad \alpha = 1, \ldots, q, \quad \#K \leq n-1. \tag{10.14}$$

Comparing (10.14) with (10.9) completes the proof. *Q.E.D.*

Example 10.10. For the (standard) action of the Euclidean group SE(2) on $M = \mathbb{R}^2$, the zero$^{\text{th}}$ order lifted invariants $w = (y, v)$ are just the group transformation formulae:

$$y = x\cos\phi - u\sin\phi + a, \qquad v = x\cos\phi + u\sin\phi + b.$$

The lifted horizontal contact-invariant form is

$$\eta = d_H y = (\cos\phi - u_x \sin\phi)\, dx,$$

which is well-defined provided ϕ does not rotate the curve to have a vertical tangent. Therefore

$$\mathcal{E} = D_y = \frac{1}{\cos\phi - u_x\sin\phi}\, D_x$$

is the lifted invariant differential operator. The higher order lifted invariants are obtained by successively applying $\mathcal{E}$ to the other zero$^{\text{th}}$ order lifted invariant v. The first two are

$$v_y = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi}, \qquad v_{yy} = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}. \tag{10.15}$$

As remarked above, the one-forms $\eta^i = d_H y^i$ are not strictly invariant under the prolonged group action. However, we can use their invariant counterparts

$$\tau^i = d_J y^i = d_H y^i + d_V y^i = \eta^i + \chi^i, \qquad i = 1, \ldots, p, \tag{10.16}$$

to define a fully invariant coframe on the regularized jet bundle $\mathcal{B}^{(\infty)}$. In (10.16), the χ^i are contact forms that are not invariant under the lifted action of G.

There are two methods for constructing invariant contact forms. First, since the horizontal component of the invariant one-form on the right hand side of (10.13) vanishes by virtue of (10.14), its vertical component

$$\vartheta^\alpha_K = d_J v^\alpha_K - \sum_{k=1}^{p} v^\alpha_{K,i}\, d_J y^i = d_V v^\alpha_K - \sum_{k=1}^{p} v^\alpha_{K,i}\, d_V y^i, \qquad \alpha = 1, \ldots, q, \tag{10.17}$$

is an invariant contact form. The resulting collection $\boldsymbol{\vartheta} = \{\ldots, \vartheta^\alpha_K, \ldots\}$ forms a complete set of lifted invariant contact forms on $\mathcal{B}^{(\infty)}$. The forms $\boldsymbol{\tau}, \boldsymbol{\vartheta}$ are the pull-backs of the canonical coframe (8.2) by the map $w^{(\infty)}$ modulo the Maurer–Cartan forms $\boldsymbol{\mu}$.

Proposition 10.11. *The collection of one-forms*

$$\boldsymbol{\tau} = \{\tau^1, \ldots, \tau^p\}, \qquad \boldsymbol{\vartheta} = \{\ldots, \vartheta^\alpha_K, \ldots\}, \qquad \boldsymbol{\mu} = \{\mu^1, \ldots, \mu^r\}, \tag{10.18}$$

provide an invariant local coframe on $\mathcal{B}^{(\infty)} = G \times \mathrm{J}^\infty$.

Invariant contact forms can also be found via the process of invariant differentiation.

Theorem 10.12. *Let ϑ^α define the complete system of invariant zero$^{\text{th}}$ order contact forms, as in* (10.17) *with $K = \varnothing$. The higher order contact forms*

$$\vartheta^\alpha_K = \mathcal{E}_K \vartheta^\alpha, \qquad \alpha = 1, \ldots, q, \quad \#K > 0, \tag{10.19}$$

obtained by Lie differentiating the zero$^{\text{th}}$ order contact forms provide a complete list of lifted invariant contact forms.

Proof: Applying d_H to (10.17) and using (10.6), we find

$$d_H \vartheta^\alpha_K = -\sum_{j=1}^p d_V(v^\alpha_{K,j}\eta^j) - \sum_{i,j=1}^p v^\alpha_{K,i,j}\,\eta^j \wedge d_V y^i + \sum_{i=1}^p v^\alpha_{K,i} d_V \eta^i = -\sum_{j=1}^p \vartheta^\alpha_{K,j} \wedge \eta^j \,. \tag{10.20}$$

The identity (10.19) follows by pairing (10.20) with the total vector field $\mathcal{E}_j$. *Q.E.D.*

11. Higher Order Moving Frames.

The construction of higher order moving frames proceeds in direct analogy with the zero$^{\text{th}}$ order version. As usual, for simplicity, we only explicitly treat the right versions.

Definition 11.1. An n^{th} order (right) *moving frame* is a map

$$\rho^{(n)} \colon \mathrm{J}^n \longrightarrow G \tag{11.1}$$

which is (locally) G-equivariant with respect to the prolonged action $G^{(n)}$ on J^n, and the right multiplication action $h \mapsto h \cdot g^{-1}$ on G itself.

The corresponding left moving frame of order n is merely $\widetilde{\rho}^{\,(n)}(z^{(n)}) = \rho^{(n)}(z^{(n)})^{-1}$. Note that an n^{th} order moving frame automatically defines a moving frame on the higher order jet bundles, namely $\rho^{(n)} \circ \pi^k_n \colon \mathrm{J}^k \to G$, $k \geq n$. The fundamental existence theorem for moving frames is an immediate consequence of Theorem 4.4.

Theorem 11.2. *If G acts on M, then an n^{th} order moving frame exists in a neighborhood of any point $z^{(n)} \in \mathcal{V}^n$ in the regular component of J^n.*

Remark: Proposition 9.6 only guarantees the local G-equivariance of the moving frame; global equivariance requires that $G^{(n)}$ act freely on $\mathcal{V}^n$.

In particular, the minimal order at which any moving frame can be constructed is the stabilization order of the group. Indeed, according to the construction in Section 4, the choice of a cross-section $K^{(n)} \subset \mathrm{J}^n$ to the prolonged group orbits serves to define a moving frame $\rho^{(n)}$ in a neighborhood of any point $z^{(n)} \in K^{(n)}$. The set $\mathcal{L}^{(n)} = (w^{(n)})^{-1} K^{(n)}$ forms the graph of a local G-equivariant section $\sigma^{(n)} \colon \mathrm{J}^n \to \mathcal{B}^{(n)}$, whose moving frame is $\rho^{(n)} = \pi_G \circ \sigma^{(n)}$. Moreover, composing $\sigma^{(n)}$ with $w^{(n)}$ produces the corresponding differential invariants.

Definition 11.3. The *fundamental n^{th} order normalized differential invariants* associated with a moving frame $\rho^{(n)}$ of order n (or less) are given by

$$I^{(n)}(z^{(n)}) = w^{(n)} \circ \sigma^{(n)}(z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot z^{(n)}. \tag{11.2}$$

Theorem 11.4. *If $J(x, u^{(n)})$ is any n^{th} order differential invariant, then, locally, J is a function of the normalized n^{th} order differential invariants, i.e., $J = H \circ I^{(n)}$.*

Remark: The fundamental normalized differential invariants are not necessarily functionally independent. Indeed, typically we normalize some of the components of the fundamental lifted invariant $w^{(n)}$ by setting them equal to constants; the corresponding normalized differential invariants will then, of course, also be constant. However, Theorem 4.5 does imply that the n^{th} order normalized differential invariants contain *all* of the n^{th} order differential invariants. In particular, any lower order differential invariants, including those on jet bundles where G does not yet act freely, will appear as functional combinations of the $I^{(n)}$.

As in the order zero case, given an arbitrary differential function $F\colon \mathrm{J}^n \to \mathbb{R}$, then $L = F \circ w^{(n)}\colon \mathcal{B}^{(n)} \to \mathbb{R}$ defines a lifted differential invariant, and hence $J = L \circ \sigma^{(n)} = F \circ I^{(n)}$ defines a differential invariant, called the *invariantization* of F with respect to the given moving frame. Thus a moving frame provides a natural way to construct a differential invariant from any differential function! Theorem 10.3 just says that if F itself is a differential invariant, then $F \circ w^{(n)}$ is independent of the group parameters, and hence $J = F$. Thus, invariantization defines a projection, depending on the moving frame, from the space of differential functions to the space of differential invariants. One case of interest is the j^{th} total derivative $D_j F(x, u^{(n+1)})$ of a differential function $F(x, u^{(n)})$. The corresponding lifted invariant coincides with the j^{th} invariant derivative of the lifted invariant $L = F \circ w^{(n)}$, so that $D_j F \circ w^{(n+1)} = \mathcal{E}_j L$. Consequently, the invariantization of $D_j F$ is given by

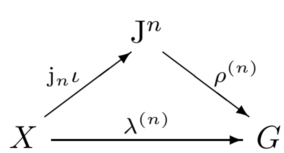
$$\mathcal{E}_j L \circ \sigma^{(n)} = D_j F \circ I^{(n+1)}. \tag{11.3}$$

As in the order zero case, in applications to equivalence problems, one restricts the moving frame to a submanifold.

Definition 11.5. An n^{th} *order moving frame restricted* to a p-dimensional submanifold $\iota\colon X \to S \subset M$ whose n-jet $\mathrm{j}_n S$ lies in the domain of definition of $\rho^{(n)}$ is defined as the composition

$$\lambda^{(n)} = \rho^{(n)} \circ \mathrm{j}_n \iota : X \longrightarrow G. \tag{11.4}$$

Equivalently, a n^{th} order moving frame is a smooth map $\lambda^{(n)}\colon X \to G$ which factors through an equivariant map from J^n to G:



(11.5)

generalizing the order zero construction (4.17). Theorems 4.4 and 9.8 serve to characterize the submanifolds which admit moving frames.

Theorem 11.6. *A submanifold $S \subset M$ admits a (locally defined) n^{th} order moving frame if and only if S is regular of order n, i.e., $\mathrm{j}_n S \subset \mathcal{V}^n$.*

Thus, in the analytic category, a submanifold S admits a moving frame (of some sufficiently high order) if and only if its isotropy subgroup G_S acts freely on S.

The practical implementation of the higher order moving frame construction relies on the higher order version of the normalization method. Consider a point $z^{(n)} \in \mathcal{V}^n$ contained in the regular subset of the n^{th} jet space. According to Proposition 4.10, in a neighborhood of $z^{(n)}$, we can choose a regular system of r lifted differential invariants $L(g, x, u^{(n)})$ having maximal rank $r = \operatorname{rank} d_G L = \dim G$. The Implicit Function Theorem allows us to solve the normalization equations

$$L_1(g, z^{(n)}) = c_1, \qquad \ldots \qquad L_r(g, z^{(n)}) = c_r, \tag{11.6}$$

for the group parameters g in terms of $z^{(n)}$ provided the normalization constants $c = (c_1, \ldots, c_r)$ belong to the image of L. Typically, one chooses L to be r suitable components of the fundamental lifted differential invariant $w^{(n)} = g^{(n)} \cdot z^{(n)}$ that have as low an order as possible, subject to the maximal rank condition, although this is by no means essential to the implementation of the method. The solution to the normalization equations (11.6) determines an n^{th} order moving frame $g = \rho^{(n)}(x, u^{(n)})$. Substituting the formula for the moving frame into the remaining lifted invariants produces a complete system of differential invariants on the open neighborhood of $z^{(n)} \in \mathcal{V}^n$ where $\rho^{(n)}$ is defined.

In terms of the invariant local coordinates $w^{(n)} = (y, v^{(n)})$ on $\mathcal{B}^{(n)}$, the fundamental normalized differential invariants $I^{(n)} = (\sigma^{(n)})^* w^{(n)}$ associated with the given moving frame $g = \rho^{(n)}(x, u^{(n)})$ are

$$\begin{aligned} &J^i(x, u^{(n)}) = y^i(\rho^{(n)}(x, u^{(n)}), x, u), && i = 1, \ldots, p, \\ &I^\alpha_K(x, u^{(l)}) = v^\alpha_K(\rho^{(n)}(x, u^{(n)}), x, u^{(k)}), && \alpha = 1, \ldots, q, \quad k = \#K \geq 0. \end{aligned} \tag{11.7}$$

In the second formula, $l = \max\{n, k\}$. As above, some of these may be constant and/or functionally dependent due to normalizations. However, (11.7) do include a complete system of differential invariants, meaning that, provided $k \geq n$, any other k^{th} order differential invariant can be locally expressed as a function of the J^i and I^α_K for $\#K \leq k$.

Example 11.7. Consider the elementary similarity group $G = \mathbb{R}^+ \ltimes \mathbb{R}^2$ acting on $M = \mathbb{R}^2$ via

$$(x, u) \longmapsto (\alpha x + a, \alpha^3 u + b). \tag{11.8}$$

If we choose $y = \alpha x + a$ as the lifted independent variable, then $\eta = d_H y = \alpha\, dx$ is the corresponding horizontal invariant form, with invariant differential operator $\mathcal{E} = D_y = \alpha^{-1} D_x$. Successively applying $\mathcal{E}$ to the dependent order zero lifted invariant v produces the complete system of higher order lifted invariants: $v_n = \mathcal{E}^n v = \alpha^{3-n} u_n$, where $u_n = D_x^n u$, and $n \geq 1$. Therefore, on $\mathcal{B}^{(4)}$, say, the lifted invariants $w^{(4)}$ are

$$y = \alpha x + a,\ v = \alpha^3 u + b,\ v_y = \alpha^2 u_x,\ v_{yy} = \alpha u_{xx},\ v_{yyy} = u_{xxx},\ v_{yyyy} = \alpha^{-1} u_{xxxx}. \tag{11.9}$$

The simplest first order moving frame is found by normalizing $y = v = 0$, $u_x = 1$, whereby

$$a = -\frac{x}{\sqrt{u_x}}, \qquad b = -\frac{u}{u_x^{3/2}}, \qquad \alpha = \frac{1}{\sqrt{u_x}}, \tag{11.10}$$

which is well-defined on the subset $\widetilde{\mathcal{V}}^1 = \{u_x > 0\}$. The resulting normalized fourth order differential invariants $I^{(4)}$ are obtained by substituting (11.10) into the lifted invariants:

$$J^1 = 0, \qquad I_0 = 0, \qquad I_1 = 1, \qquad I_2 = \frac{u_{xx}}{\sqrt{u_x}}, \qquad I_3 = u_{xxx}, \qquad I_4 = \sqrt{u_x}\, u_{xxxx}.$$

The moving frame (11.10) applies to curves $u = f(x)$ provided the tangent is not horizontal, so $u_x \neq 0$. If the curve has a horizontal tangent, then one can construct a second order moving frame by using the alternative normalization $v_{yy} = 1$, with

$$a = -\frac{x}{u_{xx}}, \qquad b = -\frac{u}{u_{xx}^2}, \qquad \alpha = \frac{1}{u_{xx}}, \tag{11.11}$$

which is well-defined on the subdomain $\mathcal{V}^2 = \{u_{xx} \neq 0\}$, and hence applies to curves with horizontal tangent at a point, but not those with inflection points. (Curves with horizontal inflection points can be handled by a yet higher order normalization.) The moving frame (11.11) leads to a slightly different normalized fourth order differential invariant $I^{(4)}$:

$$\widetilde{J} = 0, \qquad \widetilde{I}_0 = 0, \qquad \widetilde{I}_1 = \frac{u_x}{u_{xx}^2}, \qquad \widetilde{I}_2 = 1, \qquad \widetilde{I}_3 = u_{xxx}, \qquad \widetilde{I}_4 = u_{xx} u_{xxxx},$$

all of which are, naturally, functions of the previous normalized differential invariants on their common domain of definition. Note that the two moving frames correspond to different choices of cross-section of J^2, namely $\{(0,0,1,k)\}$ for (11.10) and $\{(0,0,\widetilde{k},1)\}$ for (11.11).

Remark: In his thesis, I. Lisle, [**15**; Ex. 4.4.21], introduces a "naïve elimination method" for determining differential invariants that is essentially the same as the normalization method used here. Our theory of normalization demonstrates how Lisle's method can be formalized into a practical and elegant alternative to the more traditional methods for computing differential invariants.

12. Higher Order Moving Coframes.

The final ingredients in our general theory are the jet space counterparts of the moving coframe forms. These will produce the normalized invariant differential operators that can be used to recursively construct complete systems of higher order differential invariants, and will govern the equivalence and symmetry properties of submanifolds.

Definition 12.1. The *moving coframe* of order n associated with an order n moving frame $\rho^{(n)}\colon \mathrm{J}^n \to G$ is the extended differential system $\Sigma^{(n)} = \{\boldsymbol{\zeta}^{(n)}, dI^{(n)}, I^{(n)}\}$ consisting of the pull-back $\boldsymbol{\zeta}^{(n)} = (\rho^{(n)})^* \boldsymbol{\mu}$ of the Maurer–Cartan forms to J^n, along with the n^{th} order normalized differential invariants $I^{(n)}$ and their differentials.

Lemma 12.2. *The n^{th} order moving coframe $\Sigma^{(n)}$ forms a $G^{(n)}$–coframe on $\mathcal{V}^n$.*

In other words, $\Sigma^{(n)}$ is involutive and its symmetry group coincides with the n^{th} prolongation of G acting on J^n. Lemma 12.2 is a direct consequence of Lemma 6.4. Any other $G^{(n)}$–coframe on J^n is invariantly related to the moving coframe, meaning that its functions are combinations of the differential invariants, and the one-forms are linear combinations of the moving coframe forms, with differential invariant coefficients. A particularly useful $G^{(n)}$–coframe can be constructed using the method in Theorem 6.6.

Theorem 12.3. *Let $\rho^{(n)}\colon \mathcal{V}^n \to G$ be a right moving frame on the n^{th} jet bundle over M. The extended coframe $\Gamma^{(n)} = \{\boldsymbol{\gamma}^{(n)}, I^{(n)}\}$ consisting of the normalized differential invariants*

$$I^{(n)}(z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot z^{(n)} = (\sigma^{(n)})^* w^{(n)}, \tag{12.1}$$

and the one-forms

$$\boldsymbol{\gamma}^{(n)} = \rho^{(n)}(z^{(n)}) \cdot dz^{(n)} = (\sigma^{(n)})^* d_J w^{(n)}, \tag{12.2}$$

forms a $G^{(n)}$–coframe on J^n, and hence is invariantly related to the moving coframe $\Sigma^{(n)}$.

Example 12.4. Consider the elementary similarity group (11.8). The moving coframe $\Sigma^{(2)}$ is obtained by applying the moving frame map (11.10) to the right invariant Maurer-Cartan forms

$$\mu^1 = \frac{d\alpha}{\alpha}, \qquad \mu^2 = da - \frac{a}{\alpha} d\alpha, \qquad \mu^3 = db - \frac{3b}{\alpha} d\alpha. \tag{12.3}$$

The resulting moving coframe forms are

$$\zeta^1 = -\frac{du_x}{2u_x}, \qquad \zeta^3 = -\frac{du}{u_x^{3/2}}, \qquad \zeta^2 = -\frac{dx}{\sqrt{u_x}}, \qquad \zeta^4 = d\left(\frac{u_{xx}}{\sqrt{u_x}}\right),$$

where the final form is the differential of the fundamental second order differential invariant $I_2 = u_x^{-1/2} u_{xx}$. The second order extended coframe $\Gamma^{(2)}$ is obtained by applying the moving frame map (11.10) to the jet differentials of the lifted invariants

$$d_J y = \alpha\, dx, \qquad d_J v = \alpha^3\, du, \qquad d_J v_y = \alpha^2\, du_x, \qquad d_J v_{yy} = \alpha\, du_{xx},$$

leading to

$$\gamma^1 = \frac{dx}{\sqrt{u_x}}, \qquad \gamma^2 = \frac{du}{u_x^{3/2}}, \qquad \gamma^3 = \frac{du_x}{u_x}, \qquad \gamma^4 = \frac{du_{xx}}{\sqrt{u_x}}.$$

The invariant relation

$$\gamma^1 = -\zeta^2, \qquad \gamma^2 = -\zeta^3, \qquad \gamma^3 = -2\zeta^1, \qquad \gamma^4 = \zeta^4 - I_2 \zeta^1,$$

between the two $G^{(2)}$–coframes follows from (6.4), using the coefficients of the prolonged infinitesimal generators for the given transformation group.

As in Section 10, we use the local coordinates $(x, u^{(\infty)})$ on J^∞ and lifted coordinates $(g, y, v^{(\infty)})$ on $\mathcal{B}^{(\infty)}$. The pull-back of the lifted contact-invariant coframe $\boldsymbol{\eta} = d_H y$ under the moving frame section will produce a contact-invariant coframe, from which we can construct the required invariant differential operators.

Definition 12.5. The *normalized contact-invariant coframe* is the pull-back of the lifted contact-invariant coframe:

$$\boldsymbol{\omega} = (\sigma^{(n)})^* \boldsymbol{\eta} = (\sigma^{(n)})^* d_H y. \tag{12.4}$$

Lemma 12.6. *The horizontal one-forms $\boldsymbol{\omega} = (\sigma^{(n)})^* d_H y$ are linearly independent at a point $z^{(n)}$ in the domain of definition of the moving frame map if and only if $z^{(n)} = \mathrm{j}_n S|_z$ is the n-jet of a transverse submanifold $S \subset M$.*

Proof: In terms of our bundle coordinates, the transversality of S implies $z^{(n)} \in \mathrm{J}^n E$. According to Proposition 10.7, the one-forms $\boldsymbol{\eta}$ will be linearly independent at a point $(g, z^{(n)}) \in G \times \mathrm{J}^n E \subset \mathcal{B}^{(n)}$ if and only if $g^{(1)} \cdot \pi^n_1(z^{(n)}) \in \mathrm{J}^1 E$, which automatically implies $g^{(n)} \cdot z^{(n)} \in \mathrm{J}^n E$. Therefore, $\boldsymbol{\omega}$ will be linearly independent if and only if $k^{(n)} = \rho^{(n)}(z^{(n)}) \cdot z^{(n)} \in \mathrm{J}^n E$. But, by construction, $k^{(n)} \in K^{(n)}$ is the cross-section representative of the orbit of $G^{(n)}$ through $z^{(n)}$, and hence lies in the coordinate chart $\mathrm{J}^n E$ used to construct the moving frame. *Q.E.D.*

In local coordinates, the normalized one-forms $\boldsymbol{\omega}$ are therefore obtained by using the moving frame to replace the group parameters in (10.7), so

$$\omega^i = \sum_{j=1}^{p} D_j y^i(\rho^{(n)}(x, u^{(n)}), x, u^{(n)})\, dx^j = \sum_{j=1}^{p} P^i_j(x, u^{(n)})\, dx^j, \tag{12.5}$$

whose coefficient matrix $P = (\sigma^{(n)})^* \mathbf{D} y$ is the pull-back of the total Jacobian matrix of the independent lifted invariants.

Remark: The coefficients P^i_j *cannot* be obtained by invariantly differentiating the normalized invariants $J^i = (\sigma^{(n)})^* y^i$; in other words, $\omega^i \neq d_H J^i$. Indeed, in many cases, the y's are normalized to be constant, whereas the ω's are clearly not zero. This is because the operations of total differentiation and normalization *do not commute*.

The invariant differential operators associated with the horizontal coframe (12.5) are obtained by normalizing the lifted invariant differential operators (10.10), so that the $\mathcal{E}_i$ on $\mathcal{B}^{(\infty)}$ project, by $\sigma^{(n)}$, to G-invariant total differential operators on J^∞. In coordinates,

$$\mathcal{D}_i = \sum_{j=1}^{p} Q^j_i(x, u^{(n)})\, D_j = \sum_{j=1}^{p} Z^j_i(\rho^{(n)}(x, u^{(n)}), x, u^{(1)})\, D_j, \tag{12.6}$$

where $Q = P^{-1} = (\sigma^{(n)})^* Z$ can be constructed directly from (10.10).

Example 12.7. Consider the similarity group (11.8). Under the first order moving frame map (11.10), the lifted horizontal form $\eta = d_H y = \alpha\, dx$ reduces to $\omega = u_x^{-1/2}\, dx$. Similarly the lifted invariant differential operator $\mathcal{E} = \alpha^{-1} D_x$ reduces to $\mathcal{D} = \sqrt{u_x}\, D_x$, which maps differential invariants to higher order differential invariants. However, $\mathcal{D}$ does not directly produce the normalized invariants. For example, v_{yy} normalizes to $I_2 = u_x^{-1/2} u_{xx}$, but $v_{yyy} = \mathcal{E}(v_{yy})$ normalizes to $I_3 = u_{xxx}$, which is not the same as

$\mathcal{D} I_2 = u_{xxx} + u_x^{-1} u_{xx}^2$. The second order moving frame (11.11) produces a different horizontal one-form, namely $\widetilde{\omega} = u_{xx}^{-1}\, dx$, whose invariant differential operator $\widetilde{\mathcal{D}} = u_{xx} D_x$ produces yet another hierarchy of differential invariants, which, naturally, are functions of the normalized differential invariants. The explicit formulae relating these different hierarchies of differential invariants will be found in the next section.

13. Recurrence Formulae, Commutation Relations, and Syzygies.

We have now introduced the basic ingredients in the regularized theory of moving frames. In this section, we discuss several important consequences of our constructions. These include recurrence formulae and general classification results for differential invariants, commutation formulae for the associated invariant differential operators, and, finally, a general syzygy classification. The results are all illustrated at the end of the section by a particular example arising in classical invariant theory.

An important point, encountered in Example 12.7, is that the normalized invariant differential operators, unlike their lifted counterparts, do *not* directly produce the normalized differential invariants. For example, consider the normalized differential invariant $I^\alpha = (\sigma^{(n)})^* v^\alpha$ corresponding to the lifted zero$^{\text{th}}$ order invariant v^α as in (11.7). Applying an invariant differential operator to I^α produces a higher order differential invariant $\mathcal{D}_K I^\alpha$, but this is *not*, in general, equal to its normalized counterpart $I^\alpha_K = (\sigma^{(n)})^* v^\alpha_K = (\sigma^{(n)})^* [\mathcal{E}_K v^\alpha]$. For example, if we normalize $v^\alpha = c^\alpha$, then $I^\alpha = c^\alpha$ is constant, and so its derivatives are all zero, but the higher order I^α_K are generally *not* trivial. The goal is to determine a recursive formula for constructing the I^α_K directly without having to appeal to the lifted invariants. Our starting point is formula (10.14), to which we apply the moving frame pull-back $(\sigma^{(n)})^*$. A difficulty is that, while $(\sigma^{(n)})^*$ trivially commutes with the differential d, it does *not* commute with the operations d_H and d_V. Therefore, we rewrite

$$d_H v^\alpha_K = d v^\alpha_K - d_V v^\alpha_K - d_G v^\alpha_K \tag{13.1}$$

before applying $(\sigma^{(n)})^*$. We find

$$\begin{aligned} \sum_{i=1}^p I^\alpha_{K,i}\, \omega^i = (\sigma^{(n)})^* (d_H v^\alpha_K) &= d I^\alpha_K - (\sigma^{(n)})^* (d_V v^\alpha_K) - (\sigma^{(n)})^* (d_G v^\alpha_K) \\ &= d_H I^\alpha_K - \pi_H \big[(\sigma^{(n)})^* (d_G v^\alpha_K) \big] \\ &= \sum_{i=1}^p (\mathcal{D}_i I^\alpha_K)\, \omega^i - \pi_H \big[(\sigma^{(n)})^* (d_G v^\alpha_K) \big]. \end{aligned} \tag{13.2}$$

The next to last equality is obtained by applying the horizontal projection π_H, noting that the left hand side is a horizontal form. Moreover, the pull-back of any lifted contact form, such as $d_V v^\alpha_K$, remains a contact form on J^∞. The second summand in the final line of (13.2) provides the correction terms that relate the differential invariants $I^\alpha_{K,i}$ and $\mathcal{D}_i I^\alpha_K$.

To find the explicit formula for these correction terms, we adapt Theorem 3.10 to the case of the n^{th} order regularized action of G on $\mathcal{B}^{(n)}$. Since v^α_K is a component of the lifted invariant $w^{(n)} = g^{(n)} \cdot z^{(n)}$, equation (3.8) implies that, at a point $w^{(n)} \in \mathcal{B}^{(n)}$, we can write the group differential in terms of the Maurer–Cartan forms on $\mathcal{B}^{(n)}$:

$$
\begin{aligned}
d_G y^i &= \sum_{\kappa=1}^{r} \xi^i_\kappa(w)\, \mu^\kappa, && i = 1, \ldots, p, \\
d_G v^\alpha_K &= \sum_{\kappa=1}^{r} \varphi^\alpha_{K,\kappa}(w^{(k)})\, \mu^\kappa, && \alpha = 1, \ldots, r, \quad k = \#K.
\end{aligned}
\tag{13.3}
$$

The coefficients in (13.3) are the invariant counterparts of the coefficients $\xi^i_\kappa(z)$, $\varphi^\alpha_{K,\kappa}(z^{(k)})$ of the prolonged infinitesimal generator pr $\mathbf{v}_\kappa$, as given in (9.1). Substituting (13.3) into (13.2) and its counterpart for $d_H y^i$ using (11.7) leads to the key system of identities

$$
\begin{aligned}
\omega^i &= d_H J^i - \sum_{\kappa=1}^{r} \xi^i_\kappa(I^{(0)})\, \zeta^\kappa_H, && i = 1, \ldots, p, \\
\sum_{i=1}^{p} I^\alpha_{K,i}\, \omega^i &= d_H I^\alpha_K - \sum_{\kappa=1}^{r} \varphi^\alpha_{K,\kappa}(I^{(k)})\, \zeta^\kappa_H, && \alpha = 1, \ldots, r, \quad k = \#K.
\end{aligned}
\tag{13.4}
$$

Here $\boldsymbol{\zeta}^{(n)}_H = \{\zeta^1_H, \ldots, \zeta^r_H\} = \pi_H(\boldsymbol{\zeta}^{(n)}) = \pi_H((\rho^{(n)})^* \boldsymbol{\mu})$. The coefficients in (13.4) are obtained by invariantizing the coefficients of the prolonged infinitesimal generators of the group action (9.1), meaning that we replace the jet coordinates $z^{(k)}$ by the fundamental normalized differential invariants $I^{(k)}$. Note that if G acts transitively on J^k, then there are no nonconstant k^{th} order differential invariants, and hence in such cases the coefficients of order k or less will be automatically constant. The first terms on the right hand side of (13.4) can be re-expressed in terms of the contact-invariant coframe $\boldsymbol{\omega}$ using the associated invariant differential operators, as in (9.7), so

$$
d_H J^i = \sum_{j=1}^{p} (\mathcal{D}_j J^i)\, \omega^j, \qquad d_H I^\alpha_K = \sum_{j=1}^{p} (\mathcal{D}_j I^\alpha_K)\, \omega^j. \tag{13.5}
$$

On the other hand, the horizontal components of the Maurer–Cartan forms can themselves be written in terms of our contact-invariant coframe,

$$
\zeta^\kappa_H = \sum_{j=1}^{p} K^\kappa_j [I^{(n+1)}(x, u^{(n+1)})]\, \omega^j, \qquad \kappa = 1, \ldots, r, \tag{13.6}
$$

where the coefficients are certain differential invariants of order $n+1$. Substituting (13.5), (13.6), into (13.2) produces the fundamental *recurrence formulae* for the differential invariants:

$$
\mathcal{D}_j J^i = \delta^i_j + M^i_j, \qquad \mathcal{D}_j I^\alpha_K = I^\alpha_{K,j} + M^\alpha_{K,j}. \tag{13.7}
$$

The "correction terms", that account for the non-commuting of the processes of normalization and "horizontalization", are explicitly given by

$$
\begin{aligned}
M^i_j &= \sum_{\kappa=1}^{r} \xi^i_\kappa(I^{(0)}) K^\kappa_j(I^{(n+1)}), \\
M^\alpha_{K,j} &= \sum_{\kappa=1}^{r} \varphi^\alpha_{K,\kappa}(I^{(k)}) K^\kappa_j(I^{(n+1)}),
\end{aligned}
\qquad
\begin{aligned}
& i,j = 1,\dots,p, \\
& \alpha = 1,\dots,q, \quad \#K = k.
\end{aligned}
\tag{13.8}
$$

There are similar recurrence formulae for higher order differentiated invariants,

$$
\mathcal{D}_J I^\alpha_K = I^\alpha_{J,K} + M^\alpha_{K,J}, \tag{13.9}
$$

where the higher order correction terms can be determined by iterating the basic recurrence formulae (13.7).

The coefficients K^κ_j in (13.6) can, in fact, be explicitly determined from a subset of the identities (13.7). Suppose, for simplicity, that we are normalizing r components of $w^{(n)}$ to be constant. The corresponding invariants, J^i and I^α_K will then also be constant, and hence the horizontal derivative term on the right hand side of (13.4) will vanish. For these particular forms, (13.4) reduces to a system of r linear equations relating the horizontal moving coframe forms $\zeta^1_H, \dots, \zeta^r_H$ to the contact-invariant coframe forms $\omega^1, \dots, \omega^p$. The coefficients of these linear equations are differential invariants of order $\leq n+1$. (On the right hand side, the coefficients are of order $\leq n$, while $(n+1)^{\text{st}}$ order differential invariants can appear on the left.) Since $G^{(n)}$ acts freely, its infinitesimal generators are linearly independent on the domain of definition of $\rho^{(n)}$, and hence transversality of the cross-section used to normalize the differential invariants implies that the coefficient matrix for this linear system is invertible. Solving for one-forms $\boldsymbol{\zeta}_H$ produces the required system of coefficients in (13.6).

Remark: In the method of moving coframes, [**9**], one normalizes the lifted differential invariants arising from the linear dependencies among the horizontal components of the moving coframe forms. In this case, the coefficients in (13.6) will be the chosen normalization constants and/or differential invariants. Typically, one is able to normalize all the coefficients to be constant up until the final step, at which point the fundamental differential invariants appear as coefficients.

The key observation is that the correction term (13.8) is a (typically nonlinear) function of the differential invariants of order $\leq k$, *provided* $k \geq n+1$, where n is the order of the chosen moving frame. This immediately implies provides a new proof, and a refined version of, Theorem 9.13.

Theorem 13.1. *Suppose G acts freely on $\mathcal{V}^n \subset \mathrm{J}^n$. Then, locally, every differential invariant on $\mathcal{V}^\infty = (\pi^\infty_n)^{-1}\mathcal{V}^n$ can be found by successively applying the invariant differential operators $\mathcal{D}_1, \dots, \mathcal{D}_p$ to a generating set of differential invariants of order at most $n+1$, namely the independent components of $I^{(n+1)}$.*

The commutation formulae (9.11) for the invariant differential operators (12.6) can now be explicitly determined from the moving frame formulae. In view of (10.6), (12.4), we can compute

$$\begin{aligned} d_H\boldsymbol{\omega} &= \pi_H(d\boldsymbol{\omega}) = \pi_H[(\sigma^{(n)})^*(d\, d_H y)] = \\ &= \pi_H[(\sigma^{(n)})^*(d_G d_H y)] = -\pi_H[(\sigma^{(n)})^*(d_H d_G y)]. \end{aligned} \tag{13.10}$$

Here we used the fact that d_V produces (lifted) contact forms, which do not contribute to the horizontal two-form $d_H\boldsymbol{\omega}$. Applying d_H to to (13.3), and noting that $d_H\boldsymbol{\mu} = 0$, we find

$$d_H d_G y^k = \sum_{\kappa=1}^{r} [d_H \xi_\kappa^k(w)] \wedge \mu^\kappa = \sum_{j=1}^{p}\sum_{\kappa=1}^{r} \mathcal{E}_j(\xi_\kappa^k(w))\, \eta^j \wedge \mu^\kappa, \tag{13.11}$$

where $\xi_\kappa^k(z)$ is the coefficient of $\partial/\partial x^k$ in the infinitesimal generator $\mathbf{v}_\kappa$. Combining (11.3), (13.6), (13.10) and (13.11), proves that

$$d_H \omega^k = \sum_{i,j=1}^{p}\sum_{\kappa=1}^{r} K_i^\kappa[I^{(n+1)}]\, (D_j\xi_\kappa^k)[I^{(1)}]\, \omega^i \wedge \omega^j,$$

where $(D_i\xi_\kappa^k)[I^{(1)}]$ is obtained by substituting the first order normalized differential invariant into the total derivative $D_i\xi_\kappa^k(z^{(1)})$ of the coefficient ξ_κ^k of the infinitesimal generator $\mathbf{v}_\kappa$. Therefore, by (9.10), the commutation coefficients in (9.11) are explicitly given by

$$A_{ij}^k = \sum_{\kappa=1}^{r} K_j^\kappa[I^{(n+1)}]\, (D_i\xi_\kappa^k)[I^{(1)}] - K_i^\kappa[I^{(n+1)}]\, (D_j\xi_\kappa^k)[I^{(1)}]\,. \tag{13.12}$$

Our fundamental recurrence formulae (13.7) also provide a resolution of the syzygy problem for differential invariants. First, in the normalization context, the solution is now trivial. According to our general construction, given a moving frame of order n, the normalized differential invariants (11.7) provide a complete system of differential invariants of order $k \geq n$. Assume, for simplicity, that the normalization consists of setting $r = \dim G$ components† of the n^{th} order lifted invariants $w^{(n)}$ to be constant. Then the remaining components will pull-back to functionally independent differential invariants. Therefore, all syzygies among the normalized differential invariants (11.7) occur through the normalization equations, and hence are of order at most n, the order of the moving frame.

The more subtle question is to understand the syzygies among the differentiated invariants $\mathcal{D}_J I_\nu$, arising from a generating system of differential invariants. If we choose the generating system to be the nonconstant normalized differential invariants of order $\leq n+1$, then the resulting syzygies will be of two kinds. Those involving differential invariants of order $\leq n$ will depend on the precise structure of the normalizations and the invariants themselves. Once these are understood, the higher order syzygies are more regular. Before attempting to formulate a general theorem, let us consider a simple example. Suppose

† In the more general situation where we normalize certain functional combinations of the components of $w^{(n)}$, one must modify the subsequent constructions accordingly.

our moving frame has order n and that the normalized differential invariant I^α_K of order $n = \#K$ is constant. Suppose that the normalized differential invariants $I^\alpha_{K,i}$ and $I^\alpha_{K,j}$ of order $n+1$ are not constant, and can be taken as part of the generating set of differential invariants. Since the correction terms in (13.7) have order k for $k \geq n+1$, we have

$$\mathcal{D}_j I^\alpha_{K,i} = I^\alpha_{K,i,j} + M^\alpha_{K,i,j}, \qquad \mathcal{D}_i I^\alpha_{K,j} = I^\alpha_{K,i,j} + M^\alpha_{K,j,i},$$

where the correction terms $M^\alpha_{K,i,j}$ and $M^\alpha_{K,j,i}$ are differential invariants of order $\leq n+1$ that are not necessarily equal. Therefore, we deduce a syzygy between the differentiated invariants

$$\mathcal{D}_j I^\alpha_{K,i} - \mathcal{D}_i I^\alpha_{K,j} = M^\alpha_{K,i,j} - M^\alpha_{K,j,i},$$

where the right hand side is a differential invariant of order $n+1$. The constant normalized differential invariant I^α_K is a "phantom differential invariant" that provides the seed for the syzygy. A general syzygy theorem for differential invariants can now be straightforwardly proved using these basic observations.

Definition 13.2. A *phantom differential invariant* is a constant normalized differential invariant.

Warning: Theorems 13.3 and 13.4 below, taken from the original version of the paper, are not correct as stated. Theorem 13.3 requires minimality of the moving frame, and a corrected theorem and discussion can be found in Olver, P.J., "Generating differential invariants", *J. Math. Anal. Appl.* **333** (2007) 450–471. See also Hubert, E., "Differential invariants of a Lie group action: syzygies on a generating set", *J. Symb. Comp.* **44** (2009) 382–416. A more accurate version of the Syzygy Theorem 13.4, also valid for Lie pseudo-groups, can be found in Olver, P.J., and Pohjanpelto, J., "Differential invariant algebras of Lie pseudo-groups", *Adv. Math.* **222** (2009) 1746–1792.

Theorem 13.3. *A generating system of differential invariants consists of* a) *all non-phantom differential invariants J^i and I^α coming from the non-normalized zero*$^{\text{th}}$ *order lifted invariants y^i, v^α, and* b) *all non-phantom differential invariants of the form $I^\alpha_{J,i}$ where I^α_J is a phantom differential invariant.*

Theorem 13.4. *All syzygies among the differentiated invariants arising from the generating system constructed in Theorem 13.3 are differential consequences of the following three fundamental types:*

(i) $\mathcal{D}_j J^i = \delta^i_j + M^i_j$, *when J^i is non-phantom,*

(ii) $\mathcal{D}_J I^\alpha_K = c + M^\alpha_{K,J}$, *when I^α_K is a generating differential invariant, while $I^\alpha_{J,K} = c$ is a phantom differential invariant, and*

(iii) $\mathcal{D}_J I^\alpha_{LK} - \mathcal{D}_K I^\alpha_{LJ} = M^\alpha_{LK,J} - M^\alpha_{LJ,K}$, *where I^α_{LK} and I^α_{LJ} are generating differential invariants the multi-indices $K \cap J = \varnothing$ are disjoint and non-zero, while L is an arbitrary multi-index.*

Remark: One can often use the syzygies to substantially reduce the generating system of differential invariants. In such cases, one must accordingly modify the remaining syzygies.

Example 13.5. We now illustrate the preceding formulae with a nontrivial example. Let $M = \mathbb{R}^3$, with coordinates x^1, x^2, u. Consider the action of GL(2) defined by the order zero invariants

$$y^1 = \alpha x^1 + \beta x^2, \qquad y^2 = \gamma x^1 + \delta x^2, \qquad v = (\alpha\delta - \beta\gamma)u = \lambda u, \tag{13.13}$$

where $\lambda = \alpha\delta - \beta\gamma$. This action plays a key role in the classical invariant theory of binary forms, when u is a homogeneous polynomial, cf. [**18**]. The lifted contact-invariant coframe and associated invariant differential operators are

$$\begin{aligned} \eta^1 &= d_H y^1 = \alpha\, dx^1 + \beta\, dx^2, \qquad & \mathcal{E}_1 &= \lambda^{-1}(\delta D_1 - \gamma D_2), \\ \eta^2 &= d_H y^2 = \gamma\, dx^1 + \delta\, dx^2, \qquad & \mathcal{E}_2 &= \lambda^{-1}(-\beta D_1 + \alpha D_2), \end{aligned} \tag{13.14}$$

where D_i is the total derivative with respect to x^i. The lifted differential invariants are thus $v_{jk} = (\mathcal{E}_1)^j (\mathcal{E}_2)^k v$; in particular

$$v_1 = \delta u_1 - \gamma u_2, \qquad v_2 = -\beta u_1 + \alpha u_2, \qquad v_{11} = \frac{\delta^2 u_{11} - 2\gamma\delta u_{12} + \gamma^2 u_{22}}{\lambda},$$
$$v_{12} = \frac{-\beta\delta u_{11} + (\alpha\delta + \beta\gamma)u_{12} - \alpha\gamma u_{22}}{\lambda}, \qquad v_{22} = \frac{\beta^2 u_{11} - 2\alpha\beta u_{12} + \alpha^2 u_{22}}{\lambda}.$$

If we normalize using the cross-section

$$y^1 = 1, \qquad y^2 = 0, \qquad v_1 = 1, \qquad v_2 = 0, \tag{13.15}$$

we are led to the first order moving frame

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{x^1 u_1 + x^2 u_2} \begin{pmatrix} u_1 & u_2 \\ -x^2 & x^1 \end{pmatrix}. \tag{13.16}$$

This moving frame is well-defined on surfaces $u = f(x, y)$ provided the relative invariant $x^1 u_1 + x^2 u_2 \neq 0$. (Different choices of cross-section lead to other types of constraints. For example, if $u \neq 0$, then we could normalize $v = 1$ instead of, say, $v_2 = 0$.) The resulting normalized differential invariants are $I^{(2)} = (J^1, J^2, I, I_1, I_2, I_{11}, I_{12}, I_{22}) = (\sigma^{(2)})^* w^{(2)}$, where

$$J^1 = 1, \qquad J^2 = 0, \qquad I = \frac{u}{x^1 u_1 + x^2 u_2}, \qquad I_1 = 1, \qquad I_2 = 0,$$
$$I_{11} = \frac{(x^1)^2 u_{11} + 2x^1 x^2 u_{12} + (x^2)^2 u_{22}}{x^1 u_1 + x^2 u_2}, \quad I_{12} = \frac{-x^1 u_2 u_{11} + (x^1 u_1 - x^2 u_2)u_{12} + x^2 u_1 u_{22}}{x^1 u_1 + x^2 u_2},$$
$$I_{22} = \frac{(u_2)^2 u_{11} - 2u_1 u_2 u_{12} + (u_1)^2 u_{22}}{x^1 u_1 + x^2 u_2}. \tag{13.17}$$

The normalized coframe and associated invariant differential operators are

$$\begin{aligned} \omega^1 &= \frac{u_1\, dx^1 + u_2\, dx^2}{x^1 u_1 + x^2 u_2} = \frac{d_H u}{x^1 u_1 + x^2 u_2}, \qquad & \mathcal{D}_1 &= x^1 D_1 + x^2 D_2, \\ \omega^2 &= \frac{-x^2\, dx^1 + x^1\, dx^2}{x^1 u_1 + x^2 u_2}, & \mathcal{D}_2 &= -u_2 D_1 + u_1 D_2. \end{aligned} \tag{13.18}$$

The invariant differential operators are well-known: $\mathcal{D}_1$ is the scaling process and $\mathcal{D}_2$ the Jacobian process in classical invariant theory. The prolonged infinitesimal generator coefficient matrix and its invariantized counterpart are, up to second order,

$$\begin{pmatrix} x^1 & x^2 & 0 & 0 \\ 0 & 0 & x^1 & x^2 \\ u & 0 & 0 & u \\ 0 & 0 & -u_2 & u_1 \\ u_2 & -u_1 & 0 & 0 \\ -u_{11} & 0 & -2u_{12} & u_{11} \\ 0 & -u_{11} & -u_{22} & 0 \\ u_{22} & -2u_{12} & 0 & -u_{22} \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ I & 0 & 0 & I \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ -I_{11} & 0 & -2I_{12} & I_{11} \\ 0 & -I_{11} & -I_{22} & 0 \\ I_{22} & -2I_{12} & 0 & -I_{22} \end{pmatrix}. \tag{13.19}$$

The invariant linear relations

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ I_{11} & I_{12} \\ I_{12} & I_{22} \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta^1_H \\ \zeta^2_H \\ \zeta^3_H \\ \zeta^4_H \end{pmatrix} = \begin{pmatrix} -\zeta^1_H \\ -\zeta^3_H \\ -\zeta^4_H \\ \zeta^2_H \end{pmatrix}, \tag{13.20}$$

follow from (13.6) and the subsequent remarks. The left hand side in (13.20) is obtained by pulling back the lifted contact-invariant one-forms

$$d_H y^1 = \eta^1, \qquad d_H y^2 = \eta^2, \qquad d_H v_1 = v_{11}\eta^1 + v_{12}\eta^2, \qquad d_H v_2 = v_{12}\eta^1 + v_{22}\eta^2,$$

corresponding to our choice (13.15) of normalizations; the matrix on the right hand side is the minor consisting of first, second, fourth and fifth rows of the invariantized matrix (13.19), again governed by the normalizations. We rewrite (13.20) in the matrix form

$$\begin{pmatrix} \zeta^1_H \\ \zeta^2_H \\ \zeta^3_H \\ \zeta^4_H \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ I_{12} & I_{22} \\ 0 & -1 \\ -I_{11} & -I_{12} \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}. \tag{13.21}$$

The coefficients K^κ_j in (13.6) are the entries of the coefficient matrix in (13.21). The commutator between the two invariant differential operators,

$$[\mathcal{D}_1, \mathcal{D}_2] = -I_{12}\mathcal{D}_1 + (I_{11} - 1)\mathcal{D}_2, \tag{13.22}$$

now follows from our general formula (9.11), (13.12). Indeed, the $(D_i \xi^k_\kappa)[I^{(1)}]$ are obtained by first computing the total derivatives of the independent variable coefficient matrix (which consists of the first two rows of (13.19))

$$\begin{pmatrix} x^1 & x^2 & 0 & 0 \\ 0 & 0 & x^1 & x^2 \end{pmatrix}$$

and then invariantizing by substituting the normalized differential invariants (13.17) for the jet coordinates. In this particular case, the latter process is trivial since the total derivatives are all either 1 or 0.

The correction terms to the recurrence formula can be easily obtained by multiplying the invariantized matrix (13.19) by the coefficient matrix (13.21); the resulting matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ I & 0 & 0 & I \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ -I_{11} & 0 & -2I_{12} & I_{11} \\ 0 & -I_{11} & -I_{22} & 0 \\ I_{22} & -2I_{12} & 0 & -I_{22} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ I_{12} & I_{22} \\ 0 & -1 \\ -I_{11} & -I_{12} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -I(1+I_{11}) & -I I_{12} \\ -I_{11} & -I_{12} \\ -I_{12} & -I_{22} \\ (1-I_{11})I_{11} & (2-I_{11})I_{12} \\ -I_{11}I_{12} & (1-I_{11})I_{22} \\ (I_{11}-1)I_{22} - \\ \quad -2I_{12}^2 & -I_{12}I_{22} \end{pmatrix}. \tag{13.23}$$

contains the correction terms in (13.7) — the columns correspond to normalized differential invariants and the rows to invariant differential operators. Specifically, we have

$$\begin{aligned} &\mathcal{D}_1 J^1 = \delta_1^1 - 1 = 0, && \mathcal{D}_2 J^1 = \delta_2^1 - 0 = 0, \\ &\mathcal{D}_1 J^2 = \delta_1^2 - 0 = 0, && \mathcal{D}_2 J^2 = \delta_2^2 - 1 = 0, \\ &\mathcal{D}_1 I = I_1 - I(1+I_{11}) = 1 - I(1+I_{11}), && \mathcal{D}_2 I = I_2 - I\, I_{12} = -I\, I_{12}, \\ &\mathcal{D}_1 I_1 = I_{11} - I_{11} = 0, && \mathcal{D}_2 I_1 = I_{12} - I_{12} = 0, \\ &\mathcal{D}_1 I_2 = I_{12} - I_{12} = 0, && \mathcal{D}_2 I_2 = I_{22} - I_{22} = 0, \\ &\mathcal{D}_1 I_{11} = I_{111} + (1-I_{11})I_{11}, && \mathcal{D}_2 I_{11} = I_{112} + (2-I_{11})I_{12}, \\ &\mathcal{D}_1 I_{12} = I_{112} - I_{11}I_{12}, && \mathcal{D}_2 I_{12} = I_{122} + (1-I_{11})I_{22}, \\ &\mathcal{D}_1 I_{22} = I_{122} + (I_{11}-1)I_{22} - 2I_{12}^2, && \mathcal{D}_2 I_{22} = I_{222} - I_{12}I_{22}. \end{aligned}$$

Here $I_{ijk} = (\sigma^{(1)})^* v_{ijk}$ are the third order normalized differential invariants. An alternative method for computing the correction matrix (13.23) that avoids the intermediate system (13.21) is to first perform a Gauss–Jordan *column* reduction on the invariantized coefficient matrix (13.19) making the chosen normalization rows — in the present case rows 1,2,4,5 — into an identity matrix, and then multiply by the pulled-back coefficient matrix corresponding to the horizontal derivatives of the normalized lifted invariants, as given on the left hand side of (13.20); the result will be minus the correction matrix. In the present case, (13.23) is *minus* the matrix product

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ I & 0 & I & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -I_{11} & -2I_{12} & I_{11} & 0 \\ 0 & -I_{22} & 0 & I_{11} \\ I_{22} & 0 & -I_{22} & 2I_{12} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ I_{11} & I_{12} \\ I_{12} & I_{22} \end{pmatrix}. \tag{13.24}$$

According to Theorem 13.3, we can take $I, I_{11}, I_{12}, I_{22}$ as our generating system of differential invariants. The third row of this system of identities produces the syzygies of the second type. Actually, this means that we can use I to generate I_{11}, I_{12}, leaving only I_{22} as an additional fundamental invariant. There are three fundamental syzygies of the third type:

$$\begin{aligned}
&\mathcal{D}_1 I_{12} - \mathcal{D}_2 I_{11} = -2 I_{12}, \\
&\mathcal{D}_1 I_{22} - \mathcal{D}_2 I_{12} = 2(I_{11}-1) I_{22} - 2 I_{12}^2, \\
&\mathcal{D}_1^2 I_{22} - \mathcal{D}_2^2 I_{11} = 2 I_{22} \mathcal{D}_1 I_{11} + (5 I_{12} - 2) \mathcal{D}_1 I_{12} + (3 I_{11} - 5) \mathcal{D}_1 I_{22} - \\
&\qquad\qquad\qquad\qquad - (2 I_{11} - 5)(I_{11} - 1) I_{12} + 4 (I_{11} - 1) I_{12}^2.
\end{aligned}$$

The final syzygy comes from extending our recurrence formulae on to the next order, by appending the appropriate columns to the prolonged vector field coefficient matrix (13.19). Using I to generate I_{11} and I_{12} will modify the syzygies accordingly.

14. Equivalence, Symmetry, and Rigidity.

We now reach the culmination of the paper. The fundamental problems that have motivated the development of the theory of moving frames are equivalence and symmetry of submanifolds under a Lie transformation group G, as introduced in Section 7. If G acts freely on M, then, as we saw, the basic order zero theory, as described in Theorems 7.7 and 7.8, provides the solution. However, in the non-free case, we need to prolong in order to make the group act (locally) freely. Since two submanifolds are equivalent under the action of G on M if and only if their n-jets are equivalent under the prolonged action of $G^{(n)}$ on J^n, we can then readily adapt our earlier results.

When we restrict the $G^{(n)}$–coframe on J^n to a submanifold, the resulting linear dependencies among the restricted one-forms lead to additional invariants. In the order zero context, these invariants are not directly predicted by the moving coframe, but appear to depend on the submanifold itself. An important fact is that, in the jet bundle context, they are merely the restrictions of *higher order differential invariants*! Thus, even in the order zero case, the jet bundle constructions lead to significant new information.

We start with the $G^{(n)}$–coframe $\Gamma^{(n)} = \{\boldsymbol{\gamma}^{(n)}, I^{(n)}\}$ on J^n constructed in Theorem 12.3. Let $\iota: X \to M$ parametrize a submanifold $S = \iota(X)$, so that $\mathrm{j}_n \iota: X \to \mathrm{J}^n$ parametrizes the corresponding n-jet $\mathrm{j}_n S$. We assume that $\mathrm{j}_n S$ lies in the domain of definition of our chosen order n moving frame $\rho^{(n)}$, which implies that S is order n regular. Let $\Xi^{(n)} = (\mathrm{j}_n \iota)^* \Gamma^{(n)}$ denote the restriction of the n^{th} order coframe to S. As in the order zero case, the one-form system $\Xi^{(n)}$ is overdetermined on X, and we need to reduce it to an extended coframe. Now since $\mathrm{j}_n \iota$ annihilates the contact forms, only the horizontal components of the forms in $\Gamma^{(n)}$ will contribute to the one-forms in $\Xi^{(n)}$. Therefore, the linear dependencies among these one-forms will arise from the linear dependencies among the horizontal components of the one-forms in the $G^{(n)}$–coframe. The one-forms $\boldsymbol{\gamma}^{(n)} = (\sigma^{(n)})^* d_J w^{(n)}$ are, by definition, the pull-backs of the jet differentials of the lifted invariants. We have already used the horizontal components $d_H y$ of the "independent variable" lifted invariants to construct the contact-invariant coframe $\boldsymbol{\omega} = (\sigma^{(n)})^* d_H y$. The remaining "dependent variable" lifted invariants will lead to additional contact-invariant horizontal forms $\boldsymbol{\delta}^{(n)} = (\sigma^{(n)})^* d_H v^{(n)}$, which must be invariant linear combinations of the contact-invariant coframe $\boldsymbol{\omega}$. According to (13.2),

$$\delta_K^\alpha = (\sigma^{(n)})^* d_H v_K^\alpha = \sum_{i=1}^{p} I_{K,i}^\alpha \, \omega^i, \qquad \alpha = 1, \ldots, q, \quad \#K \geq 0. \tag{14.1}$$

The coefficient $I_{K,i}^\alpha$ is the normalized differential invariant of order $\#K+1$. Therefore, *the linear dependencies among the horizontal forms* $\boldsymbol{\gamma}_H^{(n)} = \{\boldsymbol{\omega}, \boldsymbol{\delta}^{(n)}\}$ *are the differential invariants of order* $n+1$. With this in mind, we make the following definition.

Definition 14.1. The n^{th} order *differential invariant coframe* on J^n is the extended horizontal coframe

$$\Delta^{(n)} = \{\boldsymbol{\omega}, I^{(n)}\} \tag{14.2}$$

consisting of the contact-invariant coframe and the n^{th} order normalized differential invariants.

Proposition 14.2. *The horizontal components of the* n^{th} *order moving coframe* $\Sigma_H^{(n)} = \{\boldsymbol{\zeta}_H^{(n)}, d_H I^{(n)}, I^{(n)}\}$ *or its normalized counterpart* $\Gamma_H^{(n)} = \{\boldsymbol{\gamma}_H^{(n)}, I^{(n)}\}$ *are invariantly related to the differential invariant coframe* $\Delta^{(n+1)} = \{\boldsymbol{\omega}, I^{(n+1)}\}$ *of order* $n+1$.

Proof: Formula (14.1) shows that $\Gamma_H^{(n)}$ is invariantly related to $\Delta^{(n+1)}$. Moreover, since $\Sigma^{(n)}$ is invariantly related to $\Gamma^{(n)}$, the same is true for $\Sigma_H^{(n)}$. In particular, the fact that $d_H I^{(n)}$ can be written as a linear combination of $\boldsymbol{\omega}$ with $(n+1)^{\text{st}}$ order differential invariant coefficients is immediate from (9.7). *Q.E.D.*

We now restrict the coframes to a regular submanifold $S = \iota(X)$. Let $\Upsilon^{(n)} = \{\boldsymbol{\varpi}, J^{(n)}\} = (\mathrm{j}_n\iota)^*\Delta^{(n)}$ denote the restriction of the differential invariant coframe to S. Transversality implies that $\boldsymbol{\varpi} = (\mathrm{j}_n\iota)^*\boldsymbol{\omega}$ will form a coframe on the parameter space X, while $J^{(n)} = (\mathrm{j}_n\iota)^* I^{(n)}$ corresponds to the pull-back of the n^{th} order normalized differential invariants to X.

Proposition 14.3. *Let* $S = \iota(X)$ *be a submanifold whose* n *jet lies in the domain of definition of the given moving frame. Then* $\Xi^{(n)} = (\mathrm{j}_n\iota)^*\Gamma^{(n)}$ *is invariantly related to the restricted* $(n+1)^{\text{st}}$ *order differential invariant coframe* $\Upsilon^{(n+1)} = (\mathrm{j}_{n+1}\iota)^*\Delta^{(n+1)}$.

Remark: A key point is that, by construction, the invariant relation does not depend on the particular submanifold S and hence we can replace $\Xi^{(n)}$ by $\Upsilon^{(n+1)}$ without altering the equivalence relations between different submanifolds.

If $\Upsilon^{(n+1)}$ is not involutive, then we need to extend it by appending additional derived invariants. A second key fact is that the derived invariants are merely the differential invariants of the next higher order restricted to S. This is an immediate consequence of (13.4) and (13.6).

Proposition 14.4. *The* k^{th} *order derived coframe* $(\Upsilon^{(n+1)})^{(k)}$ *for the restricted differential invariant coframe* $\Upsilon^{(n+1)}$ *is invariantly related to the coframe* $\Upsilon^{(n+k+1)}$.

Remark: We can now interpret the additional invariants that arose in the order zero construction — they are the differential invariants associated with the freely acting transformation group on M.

Definition 14.5. The k^{th} order *differential invariant classifying manifold* $\mathcal{C}^{(k)}(S)$ associated with a submanifold $\iota: X \to M$ is the manifold parametrized by the normalized differential invariants of order k, namely $J^{(k)} = I^{(k)} \circ \mathrm{j}_k \iota$. The submanifold S is *order k regular* if $\mathcal{C}^{(k)}(S)$ is an embedded submanifold of its classifying space $Z^{(k)}$ (which can, in fact, be identified with $\mathrm{J}^k E$).

Definition 14.6. The *differential invariant order* of S with respect to an n^{th} order moving frame $\rho^{(n)}$ is the minimal integer $s \geq n$ such that the extended coframe $\Upsilon^{(s)}$ is involutive. The *differential invariant rank* of S is $t = \operatorname{rank} \Upsilon^{(s)} = \dim \mathcal{C}^{(s)}(S)$.

Remark: The differential invariant order defined here is slightly different from the order defined earlier. For instance, if the $(n+1)^{\text{st}}$ order differential invariants $I^{(n+1)}$ provide a complete system of invariants on S, then S will have differential invariant order $n+1$, but will be an order zero submanifold with respect to the restricted coframe $\Upsilon^{(n+1)}$.

Theorem 14.7. *Let $S \subset M$ be a regular p-dimensional submanifold of differential invariant rank t with respect to the transformation group G. Then its isotropy group G_S is a $(p-t)$-dimensional subgroup of G acting locally freely on S.*

In particular, the maximally symmetric submanifolds are those of rank 0, where all the differential invariants are constants. See [**5**, **14**], for a general characterization of such submanifolds as group orbits in the case when $M = G/H$ is a homogeneous space.

In the fully regular case, the ranks $t_k = \operatorname{rank} dJ^{(k)} = \dim \mathcal{C}^{(k)}(S)$ of the k^{th} order fundamental differential invariants on S are all constant for[†] $k \geq n$, and satisfy

$$t_n < t_{n+1} < t_{n+2} < \cdots < t_s = t_{s+1} = \cdots = t \leq p, \tag{14.3}$$

where t is the differential invariant rank and s the differential invariant order. Generically, a p-dimensional submanifold will have differential invariant order n equal to the stabilization order of the group, provided there are at least p functionally independent differential invariants of order $\leq n$; if G admits less than p independent n^{th} order differential invariants, then the generic differential invariant order will be $n+1$. According to (9.6), the latter situation occurs only when

$$q\binom{p+n}{n} < r = \dim G \leq p + q\binom{p+n}{n} = \dim \mathrm{J}^n. \tag{14.4}$$

If S is fully regular, then its differential invariant order is always bounded by either $n+p-1$ or, possibly, $n+p$; the latter case only occurs if all n^{th} order differential invariants are constant, and there is but one independent differential invariant appearing at each order $n+1 \leq k \leq n+p$. In this context, it is instructive to reconsider the higher order submanifold discussed in Example 7.9.

† The differential invariant ranks for $k < n$ will not play any significant role.

Example 14.8. Consider the Lie group $G = \mathbb{R}^3$ acting by translations on $M = \mathbb{R}^3$. For a moving frame of order zero, the generating differential invariants for surfaces $u = f(x, y)$ are just the derivatives u_x, u_y. Any nonplanar solution to the nonlinear partial differential equation‡ $u_y = \frac{1}{2}u_x^2$ will define a surface of rank 2 and differential invariant order 2. (The function $u(x, y) = -x^2/2y$ discussed in Example 7.9 above is a particular case.) Indeed, the second order differential invariants are u_{xx}, $u_{xy} = u_x u_{xx}$, and $u_{yy} = u_x^2 u_{xx}$. The single independent invariant u_{xx} is, however, not a function of the first order invariant u_x, since their Jacobian matrix is

$$\frac{\partial(u_x, u_{xx})}{\partial(x, y)} = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xxx} & u_{xxy} \end{pmatrix} = \begin{pmatrix} u_{xx} & u_x u_{xx} \\ u_{xxx} & u_x u_{xxx} + u_{xx}^2 \end{pmatrix} = u_{xx}^3 \neq 0,$$

since u is nonplanar. Thus one must use the third order differential invariant classifying manifold to characterize such solutions.

Remark: The nonplanar solutions to the differential equation in Example 14.8 provide examples of non-reducible partially invariant submanifolds, where we are using Ovsiannikov's terminology, [**22**]. Ondich, [**21**], discusses conditions that a partially invariant solution be "non-reducible", meaning that it is not invariant under a (continuous) subgroup of the symmetry group G, and hence has maximal rank p. In the moving frame approach, then, one can completely characterize non-reducible partially invariant solutions to partial differential equations as those whose graphs are submanifolds of higher order and maximal rank.

The fundamental equivalence theorem for submanifolds under general transformation group actions is a direct consequence of the corresponding Equivalence Theorem 7.2 for submanifolds under free actions.

Theorem 14.9. *Let $S, \overline{S} \subset M$ be regular p-dimensional submanifolds whose n-jets lie in the domain of definition of a moving frame map $\rho^{(n)}$. Then S and $\overline{S}$ are (locally) congruent, $\overline{S} = g \cdot S$ if and only if they have the same differential invariant order s and their classifying manifolds of order $s + 1$ are identical*: $\mathcal{C}^{(s+1)}(\overline{S}) = \mathcal{C}^{(s+1)}(S)$.

Finally, we discuss rigidity theorems for submanifolds under transformation groups. These come in two varieties. Roughly speaking, a rigidity result says that, under certain conditions, a submanifold is uniquely determined by its k-jet for some finite order k.

Definition 14.10. A submanifold S is *order k congruent* to a submanifold $\overline{S}$ at a point $z \in S$ if there is a group transformation $g \in G$ such that S and $g \cdot \overline{S}$ have order k contact at the point z.

We shall call S order k congruent to $\overline{S}$ if this occurs for every $z \in S$. Note that the group transformation $g = g(z)$ may vary from point to point. If $G^{(k)}$ acts freely on J^k, then the group transformation $g(z)$ determining the contact is uniquely determined. The

‡ Any other first order nonlinear equation $u_y = F(u_x)$ relating the two differential invariants will also work.

first rigidity theorem, which generalizes results in Griffiths, [**12**], Green, [**11**], and Jensen, [**14**], states that order k congruence implies congruence provided k is sufficiently large. The *rigidity order* of S is the minimal k for which this applies. For example, the rigidity order of a circle under the Euclidean group is two, since the only curves that are second order congruent to a circle are translates of it. On the other hand, a generic curve in the plane has rigidity order 3 under the Euclidean group.

Theorem 14.11. *Let $S \subset M$ be a regular p-dimensional submanifold which has differential invariant order s with respect to a given moving frame. Then S has rigidity order at most $s+1$. In other words, a submanifold $\overline{S}$ is order $s+1$ congruent to S at every point $z \in S$ if and only if $S = g \cdot \overline{S}$ for a fixed $g \in G$.*

Proof: Note first that $\overline{S}$ and $\overline{S}_g = g \cdot \overline{S}$ have identical classifying manifolds. Moreover, if S and $\overline{S}_g$ have order $s+1$ contact at a common point z, then their $(s+1)$–jets coincide, and hence their order $s+1$ differential invariant classifying manifolds agree at the point z. Therefore, the two submanifolds are order $s+1$ congruent at every point if and only if their order $s+1$ differential invariant classifying manifolds are identical: $\mathcal{C}^{(s+1)}(\overline{S}) = \mathcal{C}^{(s+1)}(S)$. Therefore, the result is an immediate consequence of Theorem 14.9. *Q.E.D.*

The simplest case is when the order of the moving frame equals the stabilization order of the group G. Generically, the rigidity order of a regular submanifold will be either $n+1$ or $n+2$, depending on whether (14.4) holds. Barring higher order singularities, the maximal rigidity order will be $n+p+1$. Jensen, [**14**], appears to assert that the rigidity order is at most $n+1$, but does not consider (non-generic) submanifolds of higher order, as in Example 14.8, or having other types of singularities.

A second type of rigidity theorem shows that one can uniquely characterize the group transformation mapping congruent submanifolds by knowing their order of contact.

Definition 14.12. A p-dimensional submanifold $S \subset M$ is said to be order k *rigid* if the only congruent submanifold $\overline{S} = g \cdot S$ which has k^{th} order contact with S at a point is S itself.

In other words, if $\overline{S} = g \cdot S$, then the condition $\mathrm{j}_k \overline{S}|_{z_0} = \mathrm{j}_k S|_{z_0}$ at $z_0 \in S \cap \overline{S}$ implies $g \in G_S$ and so $\overline{S} = S$. The second rigidity theorem can now be stated.

Theorem 14.13. *Let G act freely on $\mathcal{V}^n \subset \mathrm{J}^n$. Let S be an order n regular p-dimensional submanifold which has differential invariant order $s \geq n$. Then S is rigid of order $s+1$.*

Proof: We let $\rho^{(n)}$ be a moving frame defined in a neighborhood of S. Let $\overline{S} = g \cdot S$ have contact at order $s+1$ at $z_0 \in S \cap \overline{S}$. Let $z_0^{(s+1)} = \mathrm{j}_{s+1}\overline{S}|_{z_0} = \mathrm{j}_{s+1}S|_{z_0} \in \mathcal{V}^{s+1}$. Congruence implies that S and $\overline{S}$ have identical differential invariant classifying manifolds $\mathcal{C}^{(s+1)}(\overline{S}) = \mathcal{C}^{(s+1)}(S)$, which are parametrized by their $(s+1)$–jets. Theorem 5.16 implies uniqueness of the group transformation g defining the congruence map once we specify that it fix the common point $z_0^{(s+1)}$. Finally, freeness of the action of G on $\mathcal{V}^{s+1}$ implies that $g = e$, which proves rigidity. *Q.E.D.*

Remark: If G only acts locally freely on $\mathcal{V}^{s+1}$, then Theorem 14.13 reduces to a local rigidity result, i.e., $(s+1)^{\rm st}$ order contact of $\overline{S} = g \cdot S$ and S implies that the congruence transformation g must lie in a discrete subgroup of G. However, since the higher order differential invariants completely determine the higher order jets of the submanifolds, one can eliminate the discrete ambiguity provided $G^{(k)}$ acts freely on the appropriate subset of J^k for k sufficiently large.

Example 14.14. Consider the translation action $z \mapsto z + a$ of $G = \mathbb{R}^2$ on $M = \mathbb{R}^2$. The derivative coordinates u_x, u_{xx}, u_{xxx}, ... provide a complete system of differential invariants. The classifying curve of a generic curve $u = f(x)$ is parametrized by (u_x, u_{xx}). However, singularities may require us to prolong to higher order in order to assure rigidity. For example, the curve C given by $u = x^4 - 2x^2$ has second order contact at $z_0 = (1, 0)$ with its translate by $a = (2, 0)$. Moreover, the first two differential invariants $\{u_x, u_{xx}\}$ have rank 1 on C. However, C is not regular of differential invariant order 2 because its second order classifying curve intersects itself at the point $u_x = 0$, $u_{xx} = 8$, which permits second order non-rigidity. The curve C is locally rigid at order 1, and completely rigid at order 3.

15. Examples.

We now demonstrate the preceding theory with several additional examples. Only space precludes discussing a more extensive range of examples in this paper. However, all of the classical examples, including Euclidean, affine and projective geometry, as well as an extensive variety of new transformation group actions (e.g., conformal geometry) not previously treated by the classical moving frame techniques, can be directly handled by our regularized techniques.

Example 15.1. We return to the multiplier representation

$$(x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}, \frac{u}{\gamma x + \delta} \right), \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2), \tag{15.1}$$

of the general linear group $\mathrm{GL}(2)$ on $\mathbb{R}^2$ that was studied in depth in part I, [**9**], and plays a fundamental role in classical invariant theory and the calculus of variations. The right lifted invariants of order zero are just

$$y = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad v = \frac{u}{\gamma x + \delta}. \tag{15.2}$$

Choosing y as the lifted independent variable, its jet differential

$$\eta = d_J y = \frac{\alpha\delta - \beta\gamma}{(\gamma x + \delta)^2}\, dx \tag{15.3}$$

determines the lifted horizontal invariant form. The corresponding invariant differential operator is

$$\mathcal{E} = D_y = \frac{(\gamma x + \delta)^2}{\alpha\delta - \beta\gamma} D_x. \tag{15.4}$$

Applying $\mathcal{E}$ recursively to the dependent lifted invariant v leads to the lifted differential invariants $v_k = \mathcal{E}^k v$, the first few of which are

$$v_y = \frac{(\gamma x+\delta)u_x - \gamma u}{\alpha\delta-\beta\gamma}\,, \qquad v_{yy} = \frac{(\gamma x+\delta)^3 u_{xx}}{(\alpha\delta-\beta\gamma)^2}\,,$$
$$v_{yyy} = \frac{(\gamma x+\delta)^5 u_{xxx} + 3\gamma(\gamma x+\delta)^4 u_{xx}}{(\alpha\delta-\beta\gamma)^3}\,, \tag{15.5}$$
$$v_{yyyy} = \frac{(\gamma x+\delta)^7 u_{xxxx} + 8\gamma(\gamma x+\delta)^6 u_{xxx} + 12\gamma^2(\gamma x+\delta)^5 u_{xx}}{(\alpha\delta-\beta\gamma)^4}\,.$$

These formulae coincide with the transformation laws for the prolonged group action. On the regular subdomain $V = \{u u_{xx} > 0\} \subset \mathrm{J}^2$, we can choose the cross-section defined by the normalizations

$$y = 0, \qquad v = 1, \qquad v_y = 0, \qquad v_{yy} = 1. \tag{15.6}$$

Solving for the group parameters gives

$$\alpha = \sqrt{u u_{xx}}, \qquad \beta = -x\sqrt{u u_{xx}}, \qquad \gamma = u_x, \qquad \delta = u - x u_x. \tag{15.7}$$

These serve to parametrize a right $\mathrm{GL}(2)$ moving frame of order two:

$$\rho^{(2)}(x,u,u_x,u_{xx}) = \begin{pmatrix} \sqrt{u u_{xx}} & -x\sqrt{u u_{xx}} \\ u_x & u - x u_x \end{pmatrix}. \tag{15.8}$$

The left moving frame computed in [**9**] is obtained by inverting:

$$\widetilde{\rho}^{(2)}(x,u,u_x,u_{xx}) = \rho^{(2)}(x,u,u_x,u_{xx})^{-1} = \frac{1}{\sqrt{u^3 u_{xx}}} \begin{pmatrix} u - x u_x & x\sqrt{u u_{xx}} \\ -u_x & \sqrt{u u_{xx}} \end{pmatrix}. \tag{15.9}$$

Substituting the moving frame normalizations (15.7) into the higher order lifted differential invariants leads to the normalized differential invariants; the first nonconstant ones are obtained by normalizing v_{yyy} and v_{yyyy}:

$$I = \frac{u u_{xxx} + 3 u_x u_{xx}}{\sqrt{u u_{xx}^3}}\,, \qquad J = \frac{u^2 u_{xxxx} + 8 u u_x u_{xxx} + 12 u_x^2 u_{xx}}{u u_{xx}^2}. \tag{15.10}$$

Incidentally, the Replacement Theorem 10.3 implies that we can also write I and J using the *same* formulae in the lifted invariants, e.g., $I = v^{-1/2} v_{yy}^{-3/2}(v v_{yyy} + 3 v_y v_{yy})$. Applying the normalizations (15.7) to the lifted horizontal form (15.3) leads to the contact-invariant one-form and the associated invariant differential operator:

$$\omega = \sqrt{\frac{u_{xx}}{u}}\, dx, \qquad \mathcal{D} = \sqrt{\frac{u}{u_{xx}}}\, D_x. \tag{15.11}$$

The jet differentials of the second order lifted invariants are

$$\begin{aligned} d_J y &= \frac{\alpha\delta-\beta\gamma}{(\gamma x+\delta)^2}\, dx\,, & d_J v_y &= \frac{(\gamma x+\delta)\, du_x - \gamma\, du + \gamma u_x\, dx}{\alpha\delta-\beta\gamma}\,, \\ d_J v &= \frac{du}{\gamma x+\delta} - \frac{\gamma u\, dx}{(\gamma x+\delta)^2}\,, & d_J v_{yy} &= \frac{(\gamma x+\delta)^3\, du_{xx} + 3\gamma(\gamma x+\delta)^2 u_{xx}\, dx}{(\alpha\delta-\beta\gamma)^2}\,. \end{aligned} \tag{15.12}$$

The right-invariant Maurer–Cartan forms on GL(2) are the entries of the matrix product $dA \cdot A^{-1}$, namely

$$\begin{aligned} \mu^1 &= \frac{\delta\, d\alpha - \gamma\, d\beta}{\alpha\delta - \beta\gamma}, & \mu^2 &= \frac{-\beta\, d\alpha + \alpha\, d\beta}{\alpha\delta - \beta\gamma}, \\ \mu^3 &= \frac{\delta\, d\gamma - \gamma\, d\delta}{\alpha\delta - \beta\gamma}, & \mu^4 &= \frac{-\beta\, d\gamma + \alpha\, d\delta}{\alpha\delta - \beta\gamma}. \end{aligned} \tag{15.13}$$

The eight one-forms (15.12), (15.13) form a coframe on $\mathcal{B}^{(2)} = \mathrm{GL}(2) \times \mathrm{J}^2$ whose symmetry group coincides with the right lifted action of GL(2). The group differentials can be written as invariant linear combinations of the Maurer–Cartan forms:

$$\begin{aligned} d_G y &= y\,\mu^1 + \mu^2 - y^2\,\mu^3 - y\,\mu^4, & d_G v &= -yv\,\mu^3 - v\,\mu^4, \\ d_G v_y &= -v_y\,\mu^1 + (y v_y - v)\,\mu^3, & d_G v_{yy} &= -2v_{yy}\,\mu^1 + 3y v_{yy}\,\mu^3 + v_{yy}\,\mu^4, \end{aligned} \tag{15.14}$$

and can replace the Maurer–Cartan forms in the lifted coframe. The coefficients in (15.14) are given directly by formula (3.8). As in Example 6.7, we write down the coefficient matrix corresponding to the prolonged infinitesimal generators of GL(2); we find, to order 4,

$$\begin{pmatrix} x & 1 & -x^2 & -x \\ 0 & 0 & -xu & -u \\ -u_x & 0 & xu_x - u & 0 \\ -2u_{xx} & 0 & 3xu_{xx} & u_{xx} \\ -3u_{xxx} & 0 & 5xu_{xxx} + 3u_{xx} & 2u_{xxx} \\ -4u_{xxxx} & 0 & 7xu_{xxxx} + 8u_{xxx} & 3u_{xxxx} \end{pmatrix}. \tag{15.15}$$

The lifted version is obtained by replacing x and u by y and v:

$$\begin{pmatrix} y & 1 & -y^2 & -y \\ 0 & 0 & -yv & -v \\ -v_y & 0 & yv_y - v & 0 \\ -2v_{yy} & 0 & 3yv_{yy} & v_{yy} \\ -3v_{yyy} & 0 & 5yv_{yyy} + 3v_{yy} & 2v_{yyy} \\ -4v_{yyyy} & 0 & 7yv_{yyyy} + 8v_{yyy} & 3v_{yyyy} \end{pmatrix}. \tag{15.16}$$

The first four rows of (15.16) then give the coefficients in (15.14). The normalized matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ -2 & 0 & 0 & 1 \\ -3I & 0 & 3 & 2I \\ -4J & 0 & 8I & 3J \end{pmatrix} \tag{15.17}$$

is obtained by substituting (15.6), (15.10) into (15.16); in other words, (15.17) is the invariantization of the infinitesimal generator coefficient matrix (15.15).

Since GL(2) acts transitively on the open subset of J^2 under consideration, we can find the moving coframe on J^2 either by normalizing the jet differentials (15.12) or the

Maurer–Cartan forms (15.13). The former become $\boldsymbol{\gamma} = (\rho^{(2)})^* d_J w$, so that

$$\begin{aligned}
\gamma^1 &= (\sigma^{(2)})^* d_J y = \sqrt{\frac{u_{xx}}{u}}\, dx, \\
\gamma^2 &= (\sigma^{(2)})^* d_J v = \frac{du - u_x\, dx}{u}\,, \\
\gamma^3 &= (\sigma^{(2)})^* d_J v_y = \frac{du_x - u_{xx}\, dx}{\sqrt{u u_{xx}}} + \frac{u_x (du - u_x\, dx)}{\sqrt{u^3 u_{xx}}} + \sqrt{\frac{u_{xx}}{u}}\, dx, \\
\gamma^4 &= (\sigma^{(2)})^* d_J v_y = \frac{du_{xx} - u_{xxx}\, dx}{u_{xx}} + \frac{u u_{xxx} + 3 u_x u_{xx}}{u_{xx}}\, dx,
\end{aligned} \tag{15.18}$$

where we have explicitly written out the contact and horizontal components, the latter being invariant linear combinations of the invariant one-form ω. Indeed, in view of (15.6), (15.10),

$$\begin{aligned}
\gamma^1_H &= (\sigma^{(2)})^* d_H y = \omega, \\
\gamma^2_H &= (\sigma^{(2)})^* d_H v = (\sigma^{(2)})^* (v_y d_H y) = 0, \\
\gamma^3_H &= (\sigma^{(2)})^* d_H v_y = (\sigma^{(2)})^* (v_{yy} d_H y) = \omega, \\
\gamma^4_H &= (\sigma^{(2)})^* d_H v_{yy} = (\sigma^{(2)})^* (v_{yyy} d_H y) = I\, \omega.
\end{aligned} \tag{15.19}$$

On the other hand, substituting (15.17) in the general identity (6.4) produces the explicit linear dependencies:

$$\gamma^1 = -\zeta^2, \qquad \gamma^2 = \zeta^4, \qquad \gamma^3 = \zeta^3, \qquad \gamma^4 = 2\zeta^1 - \zeta^4, \tag{15.20}$$

Combining (15.19), (15.20) yields the corresponding formulae for the horizontal components of the moving coframe:

$$\zeta^1_H = \tfrac{1}{2} I\, \omega, \qquad \zeta^2_H = -\omega, \qquad \zeta^3_H = \omega, \qquad \zeta^4_H = 0, \tag{15.21}$$

reconfirming our moving coframe computation in part I.

Substituting (15.17), (15.21) into the general formula (13.7), (13.8), produces the explicit formula connecting the normalized and derived differential invariants. The easiest way to compute the correction terms is to multiply the matrix (15.17) by the column vector $\left(\frac{1}{2} I, -1, 1, 0\right)^T$ whose entries are given in (15.21); the result is a column vector

$$\left(1, 0, 1, -1 - I, 3 - \tfrac{3}{2} I^2, 8I - 2IJ\right)^T$$

whose entries are the correction terms. (Alternatively, one can use column operations as in Example 13.5.) For example, the last two entries imply

$$\mathcal{D} I = J - \tfrac{3}{2} I^2 + 3, \qquad \mathcal{D} J = K - 2IJ + 8, \tag{15.22}$$

where $K = (\sigma^{(2)})^* v_{yyyyy}$ is the fifth order normalized differential invariant. Note that we can iterate to find higher order correction terms, e.g.,

$$\mathcal{D}^2 I = \mathcal{D} J - 3I\, \mathcal{D} I = K - 5IJ + \tfrac{9}{2} I^3 - 9I + 8.$$

Example 15.2. Consider the intransitive action of the orthogonal group O(3) on surfaces in three-dimensional space $M = \mathbb{R}^3$. Assume that the surface is given as the graph of a function $u = f(x^1, x^2)$. The order zero invariants are

$$\begin{pmatrix} y^1 \\ y^2 \\ v \end{pmatrix} = R \begin{pmatrix} x^1 \\ x^2 \\ u \end{pmatrix}, \qquad R = (R^i_j) \in \mathrm{O}(3). \tag{15.23}$$

The lifted contact-invariant coframe and associated invariant differential operators are

$$\begin{aligned} \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} &= \begin{pmatrix} R^1_1 + R^1_3 u_1 & R^1_2 + R^1_3 u_2 \\ R^2_1 + R^2_3 u_1 & R^2_2 + R^2_3 u_2 \end{pmatrix} \begin{pmatrix} dx^1 \\ dx^2 \end{pmatrix}, \\ \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{pmatrix} &= \begin{pmatrix} R^1_1 + R^1_3 u_1 & R^1_2 + R^1_3 u_2 \\ R^2_1 + R^2_3 u_1 & R^2_2 + R^2_3 u_2 \end{pmatrix}^{-T} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}. \end{aligned} \tag{15.24}$$

The lifted invariants are $v_{jk} = (\mathcal{E}_1)^j (\mathcal{E}_2)^k v$; in particular

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} R^1_1 + R^1_3 u_1 & R^1_2 + R^1_3 u_2 \\ R^2_1 + R^2_3 u_1 & R^2_2 + R^2_3 u_2 \end{pmatrix}^{-T} \begin{pmatrix} R^3_1 + R^3_3 u_1 \\ R^3_2 + R^3_3 u_2 \end{pmatrix}.$$

The normalization

$$y^1 = 0, \qquad v_1 = 0, \qquad v_2 = 0, \tag{15.25}$$

leads to a particularly simple first order moving frame. Introduce the column vectors

$$z = (x_1, x_2, u)^T, \qquad n = \frac{N}{|\,N\,|} = \frac{(-u_1, -u_2, 1)^T}{\sqrt{1 + u_1^2 + u_2^2}},$$

which respectively define the point on the surface, and the corresponding unit normal. Then

$$R = (t \;\; \widehat{t} \;\; n)^T, \qquad \text{where} \qquad t = \frac{z \wedge n}{|\, z \wedge n \,|}, \qquad \widehat{t} = n \wedge t = \frac{z - (z \cdot n) n}{|\, z \wedge n \,|}, \tag{15.26}$$

define distinguished, orthogonally equivariant, unit tangent vectors. The moving frame (15.26) applies to surfaces provided that the unit normal is not parallel to the point z. Pulling back the remaining lifted invariants leads to the first order differential invariants

$$\begin{aligned} J &= (\sigma^{(1)})^* y^2 = \frac{(z \cdot n)^2 - |\, z \,|^2}{|\, z \wedge n \,|} = -|\, z \wedge n \,| = -\sqrt{|\, z \,|^2 - (z \cdot n)^2}, \\ I &= (\sigma^{(1)})^* v = z \cdot n. \end{aligned} \tag{15.27}$$

(It's interesting that we don't obtain the invariant $|z|$ directly; it is of course a function of the fundamental invariants (15.27).)

The contact-invariant coframe and invariant differential operators are obtained by pulling back the horizontal differentials of the y^i, so

$$\begin{aligned} \boldsymbol{\omega} &= A\, dx, \\ \mathcal{D} &= A^{-T}\, \mathbf{D}, \end{aligned} \qquad \text{where} \qquad A = \begin{pmatrix} t \cdot t_1 & t \cdot t_2 \\ \widehat{t} \cdot t_1 & \widehat{t} \cdot t_2 \end{pmatrix}. \tag{15.28}$$

Here $t_1 = (1, 0, u_1)$, $t_2 = (0, 1, u_2)$ are the coordinate tangent vectors to the surface, and A is the transpose of their coefficient matrix with respect to the moving frame tangent vectors t, $\widehat{t}$. A generating system of differential invariants requires the corresponding normalized second order invariants:

$$\begin{pmatrix} I_{11} & I_{12} \\ I_{12} & I_{22} \end{pmatrix} = \frac{1}{|\,N\,|}\,\mathcal{D}^2 u = \frac{A^{-T}(\nabla^2 u)A^{-1}}{\sqrt{1+u_1^2+u_2^2}}\,. \tag{15.29}$$

Here $\nabla^2 u$ is the usual Hessian matrix of u, so $\mathcal{D}^2 u$ represents an "equivariant Hessian". However, using (13.7), the recurrence relations (or syzygies)

$$\mathcal{D}_1 J = I\, I_{12}, \qquad \mathcal{D}_2 J = 1 + I\, I_{22}, \qquad \mathcal{D}_1 I = -J\, I_{12}, \qquad \mathcal{D}_2 I = -J\, I_{22},$$

show that only I, J, and I_{11} are required to form a generating system of differential invariants.

Example 15.3. Consider the action of the rotation group SO(3) on $M = \mathbb{R}^4$ corresponding to the lifted zero$^{\text{th}}$ order invariants

$$y = R\,x, \qquad v = u \qquad \text{with} \qquad R \in \mathrm{SO}(3), \tag{15.30}$$

where $y = (y^1, y^2, y^3)$, $x = (x^1, x^2, x^3)$. In this case, the differential invariants were found in [**18**; Chapter 5] by an ad hoc approach; the moving frame method allows us to be systematic. The lifted invariant one-forms and corresponding invariant differential operators are

$$\eta^i = d_H y^i = \sum_{j=1}^{3} R^i_j\, dx^j, \qquad \mathcal{E}_i = \sum_{j=1}^{3} R^i_j\, D_j \qquad i = 1, 2, 3.$$

The lifted invariants are then

$$y^i = \sum_{j=1}^{3} R^i_j x^j, \qquad v = u, \qquad v_i = \sum_{j=1}^{3} R^i_j u_j, \qquad v_{ij} = \sum_{k,l=1}^{3} R^i_k R^j_l u_{kl}, \qquad \cdots\,.$$

To determine a first order moving frame, we consider the cross-section

$$y^2 = 0, \qquad y^3 = 0, \qquad v_3 = 0. \tag{15.31}$$

The normalization equations (15.31) can be solved provided $x \wedge \nabla u \neq 0$, where $\nabla u = (u_1, u_2, u_3)$. The solution is $R = (a\,b\,c)^T$, where the column vectors

$$a = \frac{x}{|\,x\,|}, \qquad b = a \wedge c = \frac{(x\cdot\nabla u)x - |\,x\,|^2\nabla u}{|\,x\,|\,|\,x\wedge\nabla u\,|}, \qquad c = \frac{x\wedge\nabla u}{|\,x\wedge\nabla u\,|}, \tag{15.32}$$

define a rotationally equivariant orthonormal frame. The resulting first order invariants are

$$J^1 = |\,x\,|, \qquad J^2 = J^3 = 0, \qquad I = u, \qquad I_1 = \frac{x\cdot\nabla u}{|\,x\,|}, \qquad I_2 = -\frac{|\,x\wedge\nabla u\,|}{|\,x\,|}, \qquad I_3 = 0. \tag{15.33}$$

Of course, one can eliminate the denominators since they are invariant themselves. The corresponding contact-invariant coframe and invariant differential operators are

$$\widetilde{\omega}^1 = x \cdot dx, \qquad \widetilde{\omega}^2 = \big[(x \cdot \nabla u)x - |\,x\,|^2 \nabla u\big] \cdot dx, \qquad \widetilde{\omega}^3 = (x \wedge \nabla u) \cdot dx,$$
$$\widetilde{\mathcal{D}}_1 = x \cdot \mathbf{D}, \qquad \widetilde{\mathcal{D}}_2 = \nabla u \cdot \mathbf{D}, \qquad \widetilde{\mathcal{D}}_3 = (x \wedge \nabla u) \cdot \mathbf{D}, \tag{15.34}$$

where the tildes indicate that we have dropped the invariant denominators arising from a direct pull-back via (15.32). We leave it to the reader to deduce the commutator formulae. A complete generating system of differential invariants requires second order invariants:

$$\begin{aligned} I_{11} &= x^T\,(\nabla^2 u)\,x, & I_{12} &= x^T\,(\nabla^2 u)\,\nabla u, & I_{13} &= x^T\,(\nabla^2 u)\,(x \wedge \nabla u), \\ I_{22} &= \nabla u^T\,(\nabla^2 u)\,\nabla u, & I_{23} &= \nabla u^T\,(\nabla^2 u)\,(x \wedge \nabla u), & I_{33} &= (x \wedge \nabla u)^T\,(\nabla^2 u)\,(x \wedge \nabla u). \end{aligned} \tag{15.35}$$

However, using either the recurrence relations (keeping in mind that we modified the invariant differential operators and second order invariants from their normalized versions) or directly computing, we see that only three differential invariants,

$$J^1 = |\,x\,|, \qquad I = u, \qquad I_{33} = (x \wedge \nabla u)^T\,(\nabla^2 u)\,(x \wedge \nabla u),$$

are required to generate all the rest.

16. Partial Regularization.

The one draw-back to the regularized method as presented so far is that one needs to compute a sufficient number of higher order lifted differential invariants before commencing the normalization procedure. This can be quite computationally intensive — for instance, in the case of projective geometry of curves in the plane, cf. [**6**], one needs to prolong to sixth order derivatives in order to specify a complete set of normalizations. In the classical Cartan approach, as well as our earlier method of moving coframes, cf. [**9**], one avoids having to to perform a complete prolongation before starting to normalize. A similar option exists in the regularized method; one can, provided some care is taken, normalize lower order lifted invariants by solving for some of the group parameters, and then using these simplified expressions to compute higher order, *partially regularized* lifted invariants. The optimal strategy is to normalize globally defined lifted invariants, but regularize locally defined ones. This allows one to construct, with a minimal amount of computation, a partially regularized moving frame that applies to all submanifolds. The full normalization can then be accomplished for particular classes of submanifolds satisfying appropriate regularity conditions.

An essential complication is that the lifted invariant differential operators that are used to construct the higher order invariants *cannot be directly normalized*! Indeed, unlike their fully lifted or their fully normalized counterparts, partially normalized invariant differential operators will often contain additional terms involving derivatives with respect to the remaining group parameters. As pointed out by I. Anderson (personal communication), the additional terms can be interpreted as coming from the reduction of the flat connection on the regularized bundle to the appropriate partially normalized principal subbundle. These terms are correctly predicted by the moving coframe approach, but are less transparent when using a direct approach based on the lifted invariants. The resulting theory has yet to be fully developed, and lack of space precludes a detailed treatment in the present paper.

We shall content ourselves with treating one final illustrative example, that of curves in the plane under the special affine group; see [**9**] for details. We shall demonstrate how a regularized version of our moving coframe method can be used to perform a globally defined partial regularization that includes nonconvex curves. Also, for variety, and since the classical results are in terms of the left moving frame, we will use the left regularized action in this example. Let $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$ act on $M = \mathbb{R}^2$ according to

$$g \cdot (x, u) = (\alpha x + \beta u + a, \gamma x + \delta u + b), \qquad \alpha\delta - \beta\gamma = 1. \tag{16.1}$$

The zero$^{\text{th}}$ order left lifted invariants are the components of $g^{-1} \cdot (x, u)$, i.e.,

$$y = \delta(x - a) - \beta(u - b), \qquad v = -\gamma(x - a) + \alpha(u - b). \tag{16.2}$$

In the fully regularized approach, we compute the higher order lifted invariants by successively differentiating v with respect to y using the lifted invariant differential operator

$$\mathcal{E} = \frac{1}{\delta - u_x \beta} D_x \tag{16.3}$$

associated with the invariant horizontal form $\eta = d_H y = (\delta - \beta u_x)dx$. The first few are

$$v_y = \mathcal{E} v = -\frac{\gamma - \alpha u_x}{\delta - \beta u_x}, \qquad v_{yy} = \mathcal{E} v_y = -\frac{u_{xx}}{(\delta - \beta u_x)^3},$$
$$v_{yyy} = \mathcal{E} v_{yy} = -\frac{(\delta - \beta u_x)u_{xxx} + 3\beta u_{xx}^2}{(\delta - \beta u_x)^5},$$
$$v_{yyyy} = \mathcal{E} v_{yyy} = -\frac{u_{xxxx}(\delta - \beta u_x)^2 + 10 u_{xx} u_{xxx} \beta(\delta - \beta u_x) + 15 u_{xx}^3 \beta^2}{(\alpha + \beta u_x)^7}$$

By choosing the cross-section $\{(0,0,0,1,0)\} \subset \mathrm{J}^3$ we obtain the classical equi-affine moving frame

$$\beta = -\frac{1}{3} u_{xx}^{-5/3} u_{xxx}, \qquad \alpha = u_{xx}^{-1/3}, \qquad \gamma = -u_x u_{xx}^{-1/3}, \qquad a = x, \qquad b = u. \tag{16.4}$$

The first differential invariant is found by applying the moving frame normalizations to the next lifted invariant v_{yyyy}, leading to the equi-affine curvature

$$\kappa = \frac{3 u_{xx} u_{xxxx} - 5 u_{xxx}^2}{3 u_{xx}^{8/3}}. \tag{16.5}$$

In the partial normalization approach, we try to normalize lifted invariants as they appear, and thereby avoid the long computations required to initially produce the general higher order lifted invariants. For example, we can normalize the zero$^{\text{th}}$ order lifted invariants $y = v = 0$ by setting $a = x$, $b = u$. In the moving coframe method, we substitute these normalizations into the independent left invariant Maurer–Cartan forms

$$\begin{gathered} \mu^1 = \delta\, d\alpha - \beta\, d\gamma, \qquad \mu^2 = \delta\, d\beta - \beta\, d\delta, \qquad \mu^3 = \alpha\, d\gamma - \gamma\, d\alpha, \\ \nu^1 = \delta\, da - \beta\, db, \qquad \nu^2 = -\gamma\, da + \alpha\, db. \end{gathered} \tag{16.6}$$

The linear dependency between the horizontal components

$$\nu_H^1 = (\delta - \beta u_x)\, dx, \qquad \nu_H^2 = (-\gamma + \alpha u_x)\, dx,$$

produces the first order lifted invariant v_y, which can, of course, be constructed directly. Normalizing $\nu_H^2 = 0$ produces the partial normalizations

$$a = x, \qquad b = u, \qquad \gamma = \alpha u_x, \qquad \delta = \beta u_x + \frac{1}{\alpha}, \tag{16.7}$$

the final formula being a consequence of unimodularity. The partially normalized Maurer–Cartan forms

$$\nu^1 = \alpha^{-1}\, dx - \beta(du - u_x\, dx), \qquad \nu^2 = \alpha(du - u_x\, dx),$$

include the basic invariant contact form, while

$$\nu_H^1 = \omega = \alpha^{-1}\, dx \tag{16.8}$$

is a contact-invariant horizontal form. Now, the key complication is that even though one might be tempted to directly normalize the invariant differential operator (16.3), the resulting total differential operator $\widehat{\mathcal{E}} = \alpha\, D_x$, which is dual to the horizontal form (16.8), is *not an invariant differential operator*! In other words, applying $\widehat{\mathcal{E}}$ to the higher order partially normalized differential invariants *does not produce lifted differential invariants*. For example, the linear dependency between the horizontal component of $\mu^3 = \alpha^2\, du_x$ and ω leads to the second order partially normalized differential invariant

$$J = \alpha^3 u_{xx},$$

which agrees with the reduction of the lifted invariant v_{yy} under the partial normalizations (16.7). However, $\alpha D_x J = \alpha^4 u_{xxx}$ does not agree with the reduction of v_{yyy} under (16.7), which is

$$K = \alpha^4 u_{xxx} + 3\alpha^5 \beta u_{xx}^2. \tag{16.9}$$

Indeed, $\alpha^4 u_{xxx}$ is not even a lifted invariant! Thus we cannot use the directly normalized total differential operator to compute higher order partially normalized invariants. One resolution of this difficulty relies on adapting the moving coframe method, [**9**]. The remaining partially normalized Maurer–Cartan forms are

$$\mu^1 = \alpha^{-1}\, d\alpha - \alpha\beta\, du_x, \qquad \mu^2 = -\alpha^{-2}\beta\, d\alpha + \alpha^{-1}\, d\beta - \beta^2\, du_x, \qquad \mu^3 = \alpha^2 du_x. \tag{16.10}$$

If $L(\alpha, \beta, x, u^{(n)})$ is any function, then

$$dL \equiv (D_x L)\, dx + L_\alpha\, d\alpha + L_\beta\, d\beta \equiv (\alpha D_x L + \beta J L_\alpha)\omega + (\alpha L_\alpha - \beta L_\beta)\mu^1 + \alpha L_\beta \mu^2,$$

where $\equiv$ indicates that we have omitted the unimportant vertical (contact) components. We conclude that if L is any lifted invariant, then so are

$$\mathcal{D}L = \alpha D_x L + \beta J L_\alpha = \alpha D_x L + \alpha^3 \beta u_{xx} L_\alpha,$$
$$\mathcal{F}_1(L) = \alpha L_\alpha - \beta L_\beta, \qquad \mathcal{F}_2(L) = \alpha L_\beta.$$

For example, $\mathcal{D}J = K$, $\mathcal{F}_1 J = 3J$, $\mathcal{F}_2 J = 0$, while

$$\mathcal{D}K = L = \alpha^5 u_{xxxx} + 10\alpha^6 \beta u_{xx} u_{xxx} + 15\alpha^7 \beta^2 u_{xx}^3, \qquad \mathcal{F}_1 K = 5K, \qquad \mathcal{F}_2 K = 10JK,$$

where L is obtained by substituting (16.7) into v_{yyyy}. Therefore, higher order partially normalized differential invariants are given by successively applying the invariant differential operator

$$\mathcal{D} = \alpha D_x + \beta J \partial_\alpha = \alpha D_x + \alpha^3 \beta \, u_{xx} \partial_\alpha \tag{16.11}$$

to the fundamental invariant $J = \alpha^3 u_{xx}$. (Since the operators $\mathcal{F}_1 = \alpha\partial_\alpha - \beta\partial_\beta$ and $\mathcal{F}_2 = \alpha\partial_\beta$ preserve the order of differential invariants, they will not produce anything new.) Note the appearance of additional "connection terms" involving derivatives with respect to the remaining group parameters in (16.11); these have no counterpart in either the fully lifted theory or the fully normalized version. They can be interpreted as arising from the total derivative component of the reduction of the flat connection on $\mathcal{B}^{(\infty)}$, to the subbundle specified by the normalizations (16.7). As usual, further reductions rely on imposing genericity assumptions on the curve. In the standard case, one assumes that $u_{xx} \neq 0$, which allows us to perform the non-global normalization $J = 1$, $K = 0$, leading to the standard moving frame (16.4). See [**9**] for further details.

17. Conclusions.

In this paper we have provided a general theoretical foundation for the method of moving frames for finite-dimensional Lie transformation groups. The regularization procedure is also of great practical applicability, and gives a powerful tool for investigating the differential invariants, equivalence and symmetry properties of submanifolds under quite general transformation groups. Further applications that warrant further research and development include:

(1) An immediate application of the moving frame method would be to the classification of the differential invariants associated with many of the transformation groups arising in physics. As remarked in [**18**], to date such classifications have not been completed, even for some of the most fundamental groups of physical importance.

(2) In [**11**], M. Green gives various intriguing numerical formulae for the number of differential invariants for curves in a homogeneous space. These formulae were generalized in [**18**], but the extension to surfaces and higher dimensional submanifolds remains open. The resolution of the syzygy problem given here should provide insight into resolving such generalizations.

(3) The completion of the theory of partial regularization of Section 16 and the determination of explicit connection formulae would greatly aid in the practical application of the method to concrete problems.

(4) The variational tricomplex given by the operators d_H, d_V and d_G on the regularized bundle could have important applications to the study of differential equations, variational problems, and conservation laws under the action of symmetry groups, and thus deserves a detailed investigation.

(5) Applications to Ovsiannikov's method of partially invariant solutions using the remarks after Example 14.8 appear to be quite promising.

(6) The commutation formulae and syzygy classifications will have important applications to Lisle's "frame method" for symmetry classification of partial differential equations, [**15**].

(7) An inductive approach to complicated equivalence problems was described in [**18**], and is based on the solution to a simpler problem based on a subgroup of the full group. Lisle, [**15**], successfully uses an inductive approach to determining the invariant differential operators, which indicates that a general implementation of inductive methods for moving frames would not be difficult. Inductive formulae have the advantage of expressing invariant quantities for the larger group in terms of those associated with the subgroup.

(8) Finally, a theoretical justification of the moving frame method for infinite pseudo-groups, as illustrated in [**9**], corresponding to the finite-dimensional theory described here, would be of great significance. Such a theory would, we believe, be an important aid in further developing the general theory and applications of Lie pseudo-groups.

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