Moving Frames

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1. Introduction.

First introduced by Gaston Darboux, the theory of moving frames ("repères mobiles") is most closely associated with the name of Élie Cartan, [**5**], who molded it into a powerful and algorithmic tool for studying the geometric properties of submanifolds and their invariants under the action of a transformation group.In the 1970's, several researchers, cf. [**11**, **10**, **6**, **14**], began the attempt to place Cartan's intuitive constructions on a firm theoretical foundation. A significant step was to begin the process of disassociating the theory of moving frames from reliance on frame bundles and connections. More recently, [**7**, **8**], Mark Fels and I formulated a new approach to the basic moving frame theory that can be systematically applied to general transformation groups. These notes provide a quick survey of the basic ideas underlying our constructions.

New and significant applications of these results have been developed in a wide variety of directions. In [**21**, **1**], the theory was applied to produce new algorithms for solving the basic symmetry and equivalence problems of polynomials that form the foundation of classical invariant theory. In [**18**], the differential invariants of projective surfaces were classified and applied to generate integrable Poisson flows arising in soliton theory. In [**7**], the moving frame algorithm was extended to include infinite-dimensional pseudo-group actions. In [**4**], the characterization of submanifolds via their differential invariant signatures

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was applied to the problem of object recognition and symmetry detection. The moving frame method provides a direct route to the classification of joint invariants and joint differential invariants, [**8**, **23**], establishing a geometric counterpart of what Weyl, [**26**], in the algebraic framework, calls the first main theorem for the transformation group. The approximation of higher order differential invariants by joint differential invariants and, generally, ordinary joint invariants leads to fully invariant finite difference numerical schemes, [**24**], first proposed in [**3**, **4**]. Applications to the construction of invariant numerical algorithms and the theory of geometric integration, [**2**, **19**], are under development.

Throughout this paper, G will denote an r-dimensional Lie group acting smoothly on an m-dimensional manifold M. Let $G_S = \{ g \in G \mid g \cdot S = S \}$ denote the *isotropy subgroup* of a subset $S \subset M$, and $G_S^* = \bigcap_{z \in S} G_z$ its *global isotropy subgroup*, which consists of those group elements which fix all points in S. The group G acts *freely* if $G_z = \{e\}$ for all $z \in M$, *effectively* if $G_M^* = \{e\}$, and *effectively on subsets* if $G_U^* = \{e\}$ for every open $U \subset M$. Local versions of these concepts are defined by replacing $\{e\}$ by a discrete subgroup of G. A non-effective group action can be replaced by an equivalent effective action of the quotient group G/G_M^*, and so we shall always assume that G acts locally effectively on subsets. A group acts *semi-regularly* if all its orbits have the same dimension; in particular, an action is locally free if and only if it is semi-regular with r-dimensional orbits. The action is *regular* if, in addition, each point $x \in M$ has arbitrarily small neighborhoods whose intersection with each orbit is connected.

Definition 1.1. A *moving frame* is a smooth, G-equivariant map $\rho : M \to G$.

The group G acts on itself by left or right multiplication. If $\rho(z)$ is any right-equivariant moving frame then $\widetilde{\rho}(z) = \rho(z)^{-1}$ is left-equivariant and conversely. All classical moving frames are left equivariant, but, in many cases, the right versions are easier to compute.

Theorem 1.2. *A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.*

Of course, most interesting group actions are *not* free, and therefore do not admit moving frames in the sense of Definition 1.1. There are two basic methods for converting a non-free (but effective) action into a free action. The first is to look at the product action of G on several copies of M, leading to joint invariants. The second is to prolong the group action to jet space, which is the natural setting for the traditional moving frame theory, and leads to differential invariants. Combining the two methods of prolongation and product will lead to joint differential invariants. In applications of symmetry constructions to numerical approximations of derivatives and differential invariants, one requires a unification of these different actions into a common framework, called "multispace", [**24**]; the simplest version is the blow-up construction of algebraic geometry, [**12**].

The practical construction of a moving frame is based on Cartan's method of *normalization*, [**15**, **5**], which requires the choice of a (local) *cross-section* to the group orbits.

Theorem 1.3. *Let G act freely, regularly on M, and let K be a cross-section. Given $z \in M$, let $g = \rho(z)$ be the unique group element that maps z to the cross-section: $g \cdot z = \rho(z) \cdot z \in K$. Then $\rho : M \to G$ is a right moving frame for the group action.*

Given local coordinates $z = (z_1, \ldots, z_m)$ on M, let $w(g, z) = g \cdot z$ be the explicit formulae for the group transformations. The right moving frame $g = \rho(z)$ associated with a *coordinate cross-section* $K = \{ z_1 = c_1, \ldots, z_r = c_r \}$ is obtained by solving the *normalization equations*

$$w_1(g, z) = c_1, \qquad \ldots \qquad w_r(g, z) = c_r, \tag{1.1}$$

for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the coordinates $z = (z_1, \ldots, z_m)$.

Theorem 1.4. *If $g = \rho(z)$ is the moving frame solution to the normalization equations* (1.1), *then the functions*

$$I_1(z) = w_{r+1}(\rho(z), z), \qquad \ldots \qquad I_{m-r}(z) = w_m(\rho(z), z), \tag{1.2}$$

form a complete system of functionally independent invariants.

Definition 1.5. The *invariantization* of a scalar function $F: M \to \mathbb{R}$ with respect to a right moving frame ρ is the the invariant function $I = \iota(F)$ defined by $I(z) = F(\rho(z) \cdot z)$.

In particular, if $I(z)$ is an invariant, then $\iota(I) = I$, so invariantization defines a projection, depending on the moving frame, from functions to invariants.

Traditional moving frames are obtained by prolonging the group action to the n^{th} order (extended) jet bundle $\mathrm{J}^n = \mathrm{J}^n(M, p)$ consisting of equivalence classes of p-dimensional submanifolds $S \subset M$ modulo n^{th} order contact; see [**20**; Chapter 3] for details. The n^{th} order *prolonged* action of G on J^n is denoted by $G^{(n)}$.

An n^{th} *order moving frame* $\rho^{(n)}: \mathrm{J}^n \to G$ is an equivariant map defined on an open subset of the jet space. In practical examples, for n sufficiently large, the prolonged action $G^{(n)}$ becomes regular and free on a dense open subset $\mathcal{V}^n \subset \mathrm{J}^n$, the set of *regular jets*.

Theorem 1.6. *An n^{th} order moving frame exists in a neighborhood of a point $z^{(n)} \in \mathrm{J}^n$ if and only if $z^{(n)} \in \mathcal{V}^n$ is a regular jet.*

Although there are no known counterexamples, for general (even analytic) group actions only a local theorem, [**25, 22**], has been established to date.

Theorem 1.7. *A Lie group G acts locally effectively on subsets of M if and only if for $n \gg 0$ sufficiently large, $G^{(n)}$ acts locally freely on an open subset $\mathcal{V}^n \subset \mathrm{J}^n$.*

We can now apply our normalization construction to produce a moving frame and a complete system of differential invariants in the neighborhood of any regular jet. Choosing local coordinates $z = (x, u)$ on M — considering the first p components $x = (x^1, \ldots, x^p)$ as independent variables, and the latter $q = m - p$ components $u = (u^1, \ldots, u^q)$ as dependent variables — induces local coordinates $z^{(n)} = (x, u^{(n)})$ on J^n with components u^α_J representing the partial derivatives of the dependent variables with respect to the independent variables. We compute the prolonged transformation formulae

$$w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}, \qquad \text{or} \qquad (y, v^{(n)}) = g^{(n)} \cdot (x, u^{(n)})$$

by implicit differentiation of the v's with respect to the y's. For simplicity, we restrict to a coordinate cross-section by choosing $r = \dim G$ components of $w^{(n)}$ to normalize to constants:

$$w_1(g, z^{(n)}) = c_1, \qquad \ldots \qquad w_r(g, z^{(n)}) = c_r. \tag{1.3}$$

Solving the normalization equations (1.3) for the group transformations leads to the explicit formulae $g = \rho^{(n)}(z^{(n)})$ for the right moving frame. Moreover, substituting the moving frame formulae into the unnormalized components of $w^{(n)}$ leads to the *fundamental* n^{th} *order differential invariants*

$$I^{(n)}(z^{(n)}) = w^{(n)}(\rho^{(n)}(z^{(n)}), z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot z^{(n)}. \tag{1.4}$$

In terms of the local coordinates, the fundamental differential invariants will be denoted

$$H^i(x, u^{(n)}) = y^i(\rho^{(n)}(x, u^{(n)}), x, u), \qquad I^\alpha_K(x, u^{(k)}) = v^\alpha_K(\rho^{(n)}(x, u^{(n)}), x, u^{(k)}). \tag{1.5}$$

In particular, those corresponding to the normalization components (1.3) of $w^{(n)}$ will be constant, and are known as the *phantom differential invariants*.

Theorem 1.8. *Let $\rho^{(n)}\colon \mathrm{J}^n \to G$ be a moving frame of order $\leq n$. Every n^{th} order differential invariant can be locally written as a function $J = \Phi(I^{(n)})$ of the fundamental n^{th} order differential invariants. The function Φ is unique provided it does not depend on the phantom invariants.*

The *invariantization* of a differential function $F\colon \mathrm{J}^n \to \mathbb{R}$ with respect to the given moving frame is the differential invariant $J = \iota(F) = F \circ I^{(n)}$. As before, invariantization defines a projection, depending on the moving frame, from the space of differential functions to the space of differential invariants.

Example 1.9. Let us illustrate the theory with a very simple, well-known example: curves in the Euclidean plane. The orientation-preserving Euclidean group SE(2) acts on $M = \mathbb{R}^2$, mapping a point $z = (x, u)$ to

$$y = x\cos\theta - u\sin\theta + a, \qquad v = x\sin\theta + u\cos\theta + b. \tag{1.6}$$

For a parametrized curve $z(t) = (x(t), u(t))$, the prolonged group transformations

$$v_y = \frac{dv}{dy} = \frac{x_t\sin\theta + u_t\cos\theta}{x_t\cos\theta - u_t\sin\theta}, \qquad v_{yy} = \frac{d^2v}{dy^2} = \frac{x_t u_{tt} - x_{tt} u_t}{(x_t\cos\theta - u_t\sin\theta)^3}, \tag{1.7}$$

and so on, are found by successively applying implicit differentiation operator

$$D_y = \frac{1}{x_t\cos\theta - u_t\sin\theta}\, D_t \tag{1.8}$$

to v. The classical Euclidean moving frame for planar curves, [**13**], follows from the cross-section normalizations

$$y = 0, \qquad v = 0, \qquad v_y = 0. \tag{1.9}$$

Solving for the group parameters $g = (\theta, a, b)$ leads to the right-equivariant moving frame

$$\theta = -\tan^{-1}\frac{u_t}{x_t}\,, \qquad a = -\frac{x x_t + u u_t}{\sqrt{x_t^2+u_t^2}}\,, \qquad b = \frac{x u_t - u x_t}{\sqrt{x_t^2+u_t^2}}\,. \tag{1.10}$$

The inverse group transformation $g^{-1} = (\widetilde{\theta}, \widetilde{a}, \widetilde{b})$ is the classical left moving frame, [**5**, **13**]: one identifies the translation component $(\widetilde{a}, \widetilde{b}) = (x, u) = z$ as the point on the curve, while the columns of the rotation matrix $\widetilde{R} = (\mathbf{t}, \mathbf{n})$ are the unit tangent and unit normal vectors. Substituting the moving frame normalizations (1.10) into the prolonged transformation formulae (1.7), results in the fundamental differential invariants

$$v_{yy} \longmapsto \kappa = \frac{x_t u_{tt} - x_{tt} u_t}{(x_t^2+u_t^2)^{3/2}}\,, \qquad v_{yyy} \longmapsto \frac{d\kappa}{ds}\,, \qquad v_{yyyy} \longmapsto \frac{d^2\kappa}{ds^2} + 3\kappa^3, \tag{1.11}$$

where $D_s = (x_t^2+u_t^2)^{-1/2} D_t$ is the arc length derivative — which is itself found by substituting the moving frame formulae (1.10) into the implicit differentiation operator (1.8). A complete system of differential invariants for the planar Euclidean group is provided by the curvature and its successive derivatives with respect to arc length: $\kappa, \kappa_s, \kappa_{ss}, \ldots$.

The one caveat is that the first prolongation of SE(2) is only locally free on J^1 since a 180° rotation has trivial first prolongation. The even derivatives of κ with respect to s change sign under a 180° rotation, and so only their absolute values are fully invariant. The ambiguity can be removed by including the second order constraint $v_{yy} > 0$ in the derivation of the moving frame. Extending the analysis to the full Euclidean group E(2) adds in a second sign ambiguity which can only be resolved at third order. See [**23**] for complete details.

As we noted in the preceding example, substituting the moving frame normalizations into the implicit differentiation operators $D_{y^1}, \ldots, D_{y^p}$ associated with the transformed independent variables gives the fundamental invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ that map differential invariants to differential invariants.

Theorem 1.10. *If $\rho^{(n)}\colon \mathrm{J}^n \to G$ is an n^{th} order moving frame, then, for any $k \geq n+1$, a complete system of k^{th} order differential invariants can be found by successively applying the invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ to the non-constant* (*non-phantom*) *fundamental differential invariants $I^{(n+1)}$ of order at most $n+1$.*

Thus, the moving frame provides two methods for computing higher order differential invariants. The first is by normalization — plugging the moving frame formulae into the higher order prolonged group transformation formulae. The second is by invariant differentiation of the lower order invariants. These two processes lead to different differential invariants; for instance, see the last formula in (1.11). The fundamental *recurrence formulae*

$$\mathcal{D}_j H^i = \delta^i_j - L^i_j, \qquad \mathcal{D}_j I^\alpha_K = I^\alpha_{K,j} - M^\alpha_{K,j}, \tag{1.12}$$

connecting the normalized and the differentiated invariants (1.5) are of critical importance for the development of the theory, and in applications too.

A remarkable fact, [**8**, **9**], is that the *correction terms* L^i_j, $M^\alpha_{K,j}$ can be effectively computed, *without knowledge of the explicit formulae for the moving frame or the normalized differential invariants.* Let

$$\operatorname{pr} \mathbf{v}_\kappa = \sum_{i=1}^{p} \xi^i_\kappa(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{k=\#J\geq 0} \varphi^\alpha_{J,\kappa}(x,u^{(k)}) \frac{\partial}{\partial u^\alpha_J}\,, \qquad \kappa = 1,\ldots,r,$$

be a basis for the Lie algebra $\mathfrak{g}^{(n)}$ of infinitesimal generators of $G^{(n)}$. The coefficients $\varphi^\alpha_{J,\kappa}(x,u^{(k)})$ are given by the standard *prolongation formula* for vector fields, cf. [**20**], and are assembled as the entries of the n^{th} order *Lie matrix*

$$\mathbf{L}_n(z^{(n)}) = \begin{pmatrix} \xi^1_1 & \ldots & \xi^p_1 & \varphi^1_1 & \ldots & \varphi^q_1 & \ldots & \varphi^\alpha_{J,1} & \ldots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ \xi^1_r & \ldots & \xi^p_r & \varphi^1_r & \ldots & \varphi^q_r & \ldots & \varphi^\alpha_{J,r} & \ldots \end{pmatrix}. \tag{1.13}$$

The rank of $\mathbf{L}_n(z^{(n)})$ equals the dimension of the orbit through $z^{(n)}$. The *invariantized Lie matrix* is obtained by $\mathbf{I}_n = \iota(\mathbf{L}_n) = \mathbf{L}_n(I^{(n)})$, replacing the jet coordinates $z^{(n)} = (x,u^{(n)})$ by the corresponding fundamental differential invariants (1.4). We perform a Gauss–Jordan row reduction on the matrix $\mathbf{I}_n$ so as to reduce the $r\times r$ minor whose columns correspond to the normalization variables $z_1,\ldots,z_r$ in (1.3) to an $r\times r$ identity matrix — let $\mathbf{K}_n$ denote the resulting matrix of differential invariants. Further, let $\mathbf{Z}(x,u^{(n)}) = (D_i z_\kappa)$ denote the $p\times r$ matrix whose entries are the total derivatives of the normalization coordinates $z_1,\ldots,z_r$, and $\mathbf{W} = \iota(\mathbf{Z}) = \mathbf{Z}(I^{(n)})$ its invariantization. The main result is that the correction terms in (1.12) are the entries of the matrix product

$$\mathbf{W}\cdot\mathbf{K}_n = \mathbf{M}_n = \begin{pmatrix} L^1_1 & \ldots & L^p_1 & M^1_1 & \ldots & M^q_1 & \ldots & M^\alpha_{K,1} & \ldots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ L^1_r & \ldots & L^p_p & M^1_r & \ldots & M^q_r & \ldots & M^\alpha_{K,r} & \ldots \end{pmatrix}. \tag{1.14}$$

Example 1.11. The infinitesimal generators of the planar Euclidean group SE(2) are

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_u, \qquad \mathbf{v}_3 = -u\partial_x + x\partial_u.$$

Prolonging these vector fields to J^5, we find the fifth order Lie matrix

$$\mathbf{L}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -u & x & 1+u_x^2 & 3u_xu_{xx} & M_3 & M_4 & M_5 \end{pmatrix}, \tag{1.15}$$

where

$$M_3 = 4u_xu_{xxx} + 3u_{xx}^2, \qquad M_4 = 5u_xu_{xxxxx} + 10u_{xx}u_{xxx}, \\ M_5 = 6u_xu_{xxxxx} + 15u_{xx}u_{xxxx} + 10u_{xxx}^2.$$

Under the normalizations (1.9), the fundamental differential invariants are

$$y \longmapsto J = 0, \qquad v \longmapsto I = 0, \qquad v_y \longmapsto I_1 = 0, \qquad v_{yy} \longmapsto I_2 = \kappa, \tag{1.16}$$

and, in general, $v_k = D_y^k v \longmapsto I_k$; see (1.11). The recurrence formulae will express each normalized differential invariant I_k in terms of arc length derivatives of $\kappa = I_2$. Using (1.16), the invariantized Lie matrix takes the form

$$\iota(\mathbf{L}_5) = \mathbf{I}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3\,\kappa^2 & 10\,\kappa\, I_3 & 15\,\kappa\, I_4 + 10 I_3^2 \end{pmatrix}.$$

Since our chosen cross-section (1.9) is based on the jet coordinates x, u, u_x that index the first three columns of $\mathbf{I}_5$ is already in the appropriate row-reduced form, and so $\mathbf{K}_5 = \mathbf{I}_5$. Differentiating the normalization variables and then invariantizing produces the matrices

$$\mathbf{Z} = (\, 1 \;\; u_x \;\; u_{xx} \,), \qquad \iota(\mathbf{Z}) = \mathbf{W} = (\, 1 \;\; 0 \;\; I_2 \,) = (\, 1 \;\; 0 \;\; \kappa \,).$$

Therefore, the fifth order correction matrix is

$$\mathbf{M}_5 = \mathbf{W} \cdot \mathbf{K}_5 = \left(\, 1 \;\; 0 \;\; 0 \;\; 0 \;\; 3\kappa^3 \;\; 10\kappa^2 I_3 \;\; 15\kappa^2 I_4 + 10\kappa I_3^2 \,\right),$$

whose entries are the required the correction terms. The recurrence formulae (1.12) can then be read off in order:

$$\begin{aligned} &D_s J = D_s(0) = 1 - 1, && D_s I = D_s(0) = 0 - 0, && \\ &D_s I_1 = D_s(0) = 0 - 0, && D_s I_2 = D_s \kappa = I_3 - 0, && \\ &D_s I_3 = I_4 - 3\kappa^3, && D_s I_4 = I_5 - 10\kappa^2 I_3, && D_s I_5 = I_6 - 15\kappa^2 I_4 - 10\kappa\, I_3^2, \end{aligned}$$

We conclude that the higher order normalized differential invariants are given in terms of arc length derivatives of the curvature κ by

$$\begin{aligned} I_2 &= \kappa, \qquad I_3 = \kappa_s, \qquad I_4 = \kappa_{ss} + 3\,\kappa^3, \\ I_5 &= \kappa_{sss} + 19\,\kappa^2 \kappa_s, \qquad I_6 = \kappa_{ssss} + 34\,\kappa^2 \kappa_{ss} + 48\,\kappa\kappa_s^2 + 45\,\kappa^4 \kappa_s, \end{aligned}$$

and so on. The direct derivation of these and similar formulae is, needless to say, considerably more tedious. Even sophisticated computer algebra systems have difficulty owing to the appearance of rational algebraic functions in many of the expressions.

A *syzygy* is a functional dependency $H(\,\ldots\, \mathcal{D}_J I_\nu \,\ldots\,) \equiv 0$ among the fundamental differentiated invariants. In Weyl's algebraic formulation of the "Second Main Theorem" for the group action, [**26**], syzygies are defined as algebraic relations among the joint invariants. Here, since we are classifying invariants up to functional independence, there are no algebraic syzygies, and so the classification of differential syzygies is the proper setting for the Second Main Theorem in the geometric/analytic context. See [**8**, **23**] for examples and applications.

Theorem 1.12. *A generating system of differential invariants consists of* a) *all non-phantom differential invariants H^i and I^α coming from the un-normalized zero*th *order jet coordinates y^i, v^α, and* b) *all non-phantom differential invariants of the form $I^\alpha_{J,i}$ where I^α_J is a phantom differential invariant. The fundamental syzygies among the differentiated invariants are*

(*i*) $\mathcal{D}_j H^i = \delta^i_j - L^i_j$, *when H^i is non-phantom,* (*ii*) $\mathcal{D}_J I^\alpha_K = c - M^\alpha_{K,J}$, *when* I^α_K *is a generating differential invariant, while* $I^\alpha_{J,K} = c$ *is a phantom differential invariant, and*

(*iii*) $\mathcal{D}_J I^\alpha_{LK} - \mathcal{D}_K I^\alpha_{LJ} = M^\alpha_{LJ,K} - M^\alpha_{LK,J}$, *where* I^α_{LK} *and* I^α_{LJ} *are generating differential invariants and* $K \cap J = \varnothing$ *are disjoint and non-zero.*

All other syzygies are all differential consequences of these generating syzygies.

Two submanifolds $S, \overline{S} \subset M$ are said to be *equivalent* if $\overline{S} = g \cdot S$ for some $g \in G$. A *symmetry* of a submanifold is a group transformation that maps S to itself, and so is an element $g \in G_S$. As emphasized by Cartan, [**5**], the solution to the equivalence and symmetry problems for submanifolds is based on the functional interrelationships among the fundamental differential invariants restricted to the submanifold.

A submanifold $S \subset M$ is called *regular* of order n at a point $z_0 \in S$ if its n^{th} order jet $\mathrm{j}_n S|_{z_0} \in \mathcal{V}^n$ is regular. Any order n regular submanifold admits a (locally defined) moving frame of that order — one merely restricts a moving frame defined in a neighborhood of z_0 to it: $\rho^{(n)} \circ \mathrm{j}_n S$. Thus, only those submanifolds having singular jets at arbitrarily high order fail to admit any moving frame whatsoever. The complete classification of such *totally singular submanifolds* appears in [**22**]; an analytic version of this result is:

Theorem 1.13. *Let G act effectively, analytically. An analytic submanifold $S \subset M$ is totally singular if and only if G_S does not act locally freely on S itself.*

Given a regular submanifold S, let $J^{(k)} = I^{(k)} \mid S = I^{(k)} \circ \mathrm{j}_k S$ denote the k^{th} order *restricted differential invariants*. The k^{th} order *signature* $\mathcal{S}^{(k)} = \mathcal{S}^{(k)}(S)$ is the set parametrized by the restricted differential invariants; S is called *fully regular* if $J^{(k)}$ has constant rank $0 \leq t_k \leq p = \dim S$. In this case, $\mathcal{S}^{(k)}$ forms a submanifold of dimension t_k — perhaps with self-intersections. In the fully regular case,

$$t_n < t_{n+1} < t_{n+2} < \cdots < t_s = t_{s+1} = \cdots = t \leq p,$$

where t is the *differential invariant rank* and s the *differential invariant order* of S.

Theorem 1.14. *Let $S, \overline{S} \subset M$ be regular p-dimensional submanifolds with respect to a moving frame $\rho^{(n)}$. Then S and $\overline{S}$ are (locally) equivalent, $\overline{S} = g \cdot S$, if and only if they have the same differential invariant order s and their signature manifolds of order $s+1$ are identical: $\mathcal{S}^{(s+1)}(\overline{S}) = \mathcal{S}^{(s+1)}(S)$.*

Example 1.15. A curve in the Euclidean plane is uniquely determined, modulo translation and rotation, from its curvature invariant κ and its first derivative with respect to arc length κ_s. Thus, the curve is uniquely prescribed by its *Euclidean signature curve* $\mathcal{S} = \mathcal{S}(C)$, which is parametrized by the two differential invariants (κ, κ_s). The Euclidean (and equi-affine) signature curves have been applied to the problems of object recognition and symmetry detection in digital images in [**4**].

Theorem 1.16. *If $S \subset M$ is a fully regular p-dimensional submanifold of differential invariant rank t, then its symmetry group G_S is an $(r-t)$–dimensional subgroup of G that acts locally freely on S.*

A submanifold with maximal differential invariant rank $t = p$ is called *nonsingular*. Theorem 1.16 says that these are the submanifolds with only discrete symmetry groups. The *index* of such a submanifold is defined as the number of points in S map to a single generic point of its signature, i.e., $\operatorname{ind} S = \min \{ \# \, \sigma^{-1}\{\zeta\} \mid \zeta \in \mathcal{S}^{(s+1)} \}$, where $\sigma(z) = J^{(s+1)}(z)$ denotes the *signature map* from S to its order $s+1$ signature $\mathcal{S}^{(s+1)}$. Incidentally, a point on the signature is non-generic if and only if it is a point of self-intersection of $\mathcal{S}^{(s+1)}$. The index is equal to the number of symmetries of the submanifold, a fact that has important implications for the computation of discrete symmetries in computer vision, [**4**], and in classical invariant theory, [**1**, **21**].

Theorem 1.17. *If S is a nonsingular submanifold, then its symmetry group is a discrete subgroup of cardinality $\# G_S = \operatorname{ind} S$.*

At the other extreme, a rank 0 or *maximally symmetric* submanifold has all constant differential invariants, and so its signature degenerates to a single point.

Theorem 1.18. *A regular p-dimensional submanifold S has differential invariant rank 0 if and only if it is an orbit, $S = H \cdot z_0$, of a p-dimensional subgroup $H = G_S \subset G$.*

For example, in planar Euclidean geometry, the maximally symmetric curves have constant Euclidean curvature, and are the circles and straight lines. Each is the orbit of a one-parameter subgroup of SE(2), which also forms the symmetry group of the orbit.

In equi-affine planar geometry, when $G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$ acts on planar curves, the maximally symmetric curves are the conic sections, which admit a one-parameter group of equi-affine symmetries. The straight lines are totally singular, and admit a three-parameter equi-affine symmetry group, which, in accordance with Theorem 1.13, does not act freely thereon. In planar projective geometry, with $G = \mathrm{SL}(3, \mathbb{R})$ acting on $M = \mathbb{RP}^2$, the maximally symmetric curves, having constant projective curvature, are the "W–curves" studied by Lie and Klein, [**16**, **17**].

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