A Higher Order Moving Frame for Equi-Affine Plane Curves

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This is a supplement to Example 3.3 of [**3**], correcting the original statement about the computation of a higher order moving frame for plane curves $C \subset \mathbb{R}^2$ under the action of the special affine (or equi-affine) group SA(2) at an inflection point. Also note further corrections to that Example in the updated version posted on the author's website:

http://www.math.umn.edu/∼olver/mf_/smf.pdf

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As in [**3**; Example 3.3], the equi-affine geometry of curves in the plane is governed by the standard action

$$g\colon (x,u) \longmapsto (\alpha\, x + \beta\, u + a, \gamma\, x + \delta\, u + b), \qquad \alpha\,\delta - \beta\,\gamma = 1, \tag{1}$$

of the special affine group, $g \in \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$, acting on $M = \mathbb{R}^2$. To obtain a left equivariant moving frame, we begin by inverting the group transformations. The components of $w = (y,v) = g^{-1} \cdot (x,u)$ are

$$y = \delta\,(x-a) - \beta\,(u-b), \qquad v = -\,\gamma\,(x-a) + \alpha\,(u-b). \tag{2}$$

Let

$$dy = \sigma\, dx \qquad \text{where} \qquad \sigma = \delta - \beta\, u_x, \tag{3}$$

so that the dual implicit differentiation operator is

$$D_y = (D_x y)^{-1} D_x = \sigma^{-1} D_x. \tag{4}$$

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The explicit formulae for the fifth prolongation are obtained by repeatedly applying D_y to v, yielding

$$\begin{aligned}
v_y &= \frac{\alpha\, u_x - \gamma}{\sigma}, \qquad v_{yy} = \frac{u_{xx}}{\sigma^3}, \qquad v_{yyy} = \frac{\sigma\, u_{xxx} + 3\,\beta\, u_{xx}^2}{\sigma^5},\\
v_{yyyy} &= \frac{\sigma^2 u_{xxxx} + 10\,\beta\,\sigma u_{xx} u_{xxx} + 15\,\beta^2 u_{xx}^3}{\sigma^7},\\
v_{yyyyy} &= \frac{\sigma^3 u_{xxxxx} + \beta\,\sigma^2\bigl(15\, u_{xx} u_{xxxx} + 10\, u_{xxx}^2\bigr) + 105\,\beta^2 \sigma u_{xx}^2 u_{xxx} + 105\,\beta^3 u_{xx}^4}{\sigma^9}.
\end{aligned} \tag{5}$$

The standard moving frame, which has order 3, requires that the curve not have an inflection point, i.e., that $u_{xx} \neq 0$. See the corrected version of [**3**; Example 3.3] for that computation using the equivariant moving frame calculus, and [**2**] for the classical Cartan approach.

To obtain a moving frame that allows inflection points, we note that the prolonged action of SA(2) is free on $\mathcal{V}^5 = \mathrm{J}^5 \setminus \mathcal{S}^5$, where $\mathcal{S}^5 = \{u_{xx} = u_{xxx} = 0\}$ is the singular subvariety. We can thus use the cross-section†

$$y = 0, \qquad v = 0, \qquad v_y = 0, \qquad v_{yyy} = 1, \qquad v_{yyyyy} = 0, \tag{6}$$

to compute the (left) equivariant moving frame. Substituting the prolongation formulae (2–5), the first two normalization equations produce the translation components: $a = x$, $b = u$. The third implies

$$\gamma = \alpha\, u_x. \tag{7}$$

We then skip to the last equation: $v_{yyyyy} = 0$. Since the numerator is a homogeneous cubic polynomial in β, σ, we can solve for

$$\beta = Q\, \sigma, \tag{8}$$

where Q is a complicated rational algebraic function of $u_x, \ldots, u_{xxxxx}$, which can be explicitly written down using the Cardano formula for the roots of the cubic, [**4**].

Remark: When $u_{xx} = 0$, the cubic degenerates into the product of σ^2 and a homogeneous linear polynomial in β, σ; in other words, as $u_{xx} \to 0$, two of its three roots go off to ∞ on the projective line. To keep the moving frame well-defined at such an inflection point, one thus needs to choose the branch in the Cardano formula that remains finite.

Substituting (8) into the remaining normalization equation $v_{yyy} = 1$ yields

$$\sigma^5 = \bigl(u_{xxx} + 3\, u_{xx}^2 Q\bigr)\, \sigma, \qquad \text{and therefore} \qquad \sigma = S = \sqrt[4]{u_{xxx} + 3\, u_{xx}^2\, Q}\,. \tag{9}$$

The resulting moving frame will be valid on the subset

$$\mathcal{V}^5_+ = \bigl\{\, u_{xxx} + 3\, u_{xx}^2 Q > 0 \,\bigr\} \subset \mathcal{V}^5.$$

† Setting $v_{yyyy} = 0$ requires $u_{xx} = 0$, and so is not of use for this purpose.

If $u_{xxx} + 3u_{xx}^2 Q < 0$, one uses the alternative cross-section normalization $v_{yyy} = -1$ with analogous results. Also, as with almost all moving frame calculations in the literature, we ignore the discrete ambiguity in the branch of the fourth root caused by the fact that the prolonged action of SA(2) is locally but not globally free on $\mathcal{V}^5$, [**1**].

Finally, substituting formulas $(3, 7, 8, 9)$ into the unimodularity constraint (1) will produce the formulae for the fifth order left-equivariant moving frame $\rho: \mathcal{V}^5_+ \to \mathrm{SA}(2)$ corresponding to the cross-section (6):

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1/S & Q\,S \\ u_x/S & S(1+u_x Q) \end{pmatrix}, \qquad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix}. \tag{10}$$

The first two fundamental differential invariants are

$$\begin{aligned} v_{yy} &\longmapsto I_2 = \iota(u_{xx}) = \frac{u_{xx}}{S^3}, \\ v_{yyyy} &\longmapsto I_4 = \iota(u_{xxxx}) = \frac{u_{xxxx} + 10\,u_{xx}u_{xxx}\,Q + 15\,u_{xx}^3\,Q^2}{S^5}, \end{aligned} \tag{11}$$

where ι denotes the invariantization map associated with the moving frame, [**1**]. Both I_2 and I_4 are fifth order differential invariants, valid at nondegenerate inflection points. (Their expressions in terms of the equi-affine curvature and its arc length derivatives can be straightforwardly obtained using the recurrence formulae and the Replacement Rule, [**1**].) The (contact-)invariant one-form $\omega = \iota(dx)$ is obtained by normalizing $dy = \sigma\,dx$, whence

$$\omega = S\,dx \quad \text{with dual invariant differential operator} \quad \mathcal{D} = (1/S)D_x, \tag{12}$$

where S, as given in (9), has order 5, which is again valid at suitably nondegenerate inflection points.

Since the moving frame has order 5, according to [**1**; Theorem 13.1], all higher order differential invariants can be obtained by invariantly differentiating those of order ≤ 6, namely I_2, I_4, and $I_6 = \iota(u_{xxxxxx})$. Moreover, according to the first two recurrence formulae[†]

$$\mathcal{D}I_2 = \frac{9\,I_2^3 I_6 - 45\,I_2^2 I_4^2 + 30\,I_2 I_4 + 40}{20(3\,I_2 I_4 + 2)}, \qquad \mathcal{D}I_4 = \frac{3\,I_2^2 I_4 I_6 - 8\,I_2 I_6 - 15\,I_2 I_4^3 - 10\,I_4^2}{4(3\,I_2 I_4 + 2)}, \tag{13}$$

knowing I_2 and I_4 and differentiating either one of them produces I_6, and hence all higher order differential invariants.

At an inflection point, $u_{xx} = 0$ with $u_{xxx} \neq 0$, the formulas simplify dramatically:

$$Q = -\tfrac{1}{10}\,u_{xxx}^{-2} u_{xxxxx}, \qquad S = u_{xxx}^{1/4},$$

and hence the moving frame map at such points is given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} u_{xxx}^{-1/4} & -\tfrac{1}{10} u_{xxx}^{-7/4} u_{xxxxx} \\ u_x u_{xxx}^{-1/4} & u_{xxx}^{1/4} - \tfrac{1}{10} u_x u_{xxx}^{-7/4} u_{xxxxx} \end{pmatrix}, \qquad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix}. \tag{14}$$

[†] See [**1**; Theorem 13.4] for the symbolic calculus used to determine them.

The lowest order restricted differential invariants are

$$I_2 = 0, \qquad I_4 = u_{xxx}^{-5/4} u_{xxxx}. \tag{15}$$

However, one cannot directly use invariant differentiation to generate the higher order restricted differential invariants since setting $u_{xx} = 0$ does not commute with differentiation.

References

[**1**] Fels, M., and Olver, P.J., Moving coframes. II. Regularization and theoretical foundations, *Acta Appl. Math.* **55** (1999), 127–208.

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[**3**] Olver, P.J., Moving frames and singularities of prolonged group actions, *Selecta Math.* **6** (2000), 41–77.

[**4**] van der Waerden, B.L., *Algebra*, vols. 1 & 2, Springer, New York, 1991.