New Quasi-Exactly Solvable Hamiltonians in Two Dimensions

Artemio González-López§
Departamento de Física Teórica II
Universidad Complutense
28040 Madrid
SPAIN

Niky Kamran¶
Department of Mathematics
McGill University
Montréal, Québec
CANADA H3A 2K6

Peter J. Olver†‡
Department of Mathematics
University of Maryland
College Park, MD
U.S.A. 20742

Abstract

Quasi-exactly solvable Schrödinger operators have the remarkable property that a part of their spectrum can be computed by algebraic methods. Such operators lie in the enveloping algebra of a finite-dimensional Lie algebra of first order differential operators — the "hidden symmetry algebra". In this paper we develop some general techniques for constructing quasi-exactly solvable operators. Our methods are applied to provide a wide variety of new explicit two-dimensional examples (on both flat and curved spaces) of quasi-exactly solvable Hamiltonians, corresponding to both semi-simple and more general classes of Lie algebras.

§ Supported in Part by DGICYT Grant PS 89–0011.
¶ Supported in Part by an NSERC Grant.
† Supported in Part by NSF Grant DMS 92–04192.
‡ On leave from School of Mathematics, University of Minnesota, Minneapolis, Minnesota, U.S.A. 55455.

1 Introduction

The spectral problems of non-relativistic quantum mechanics fall within two general categories. In the first category, we have the small number of so-called exactly solvable problems, that is Schrödinger operators whose entire spectrum can be determined by algebraic methods. The simplest example of such a problem is given by the harmonic oscillator. In the second category, we have the Schrödinger operators whose complete spectrum cannot be computed exactly, but only approximated numerically at the very best.

Over the past decade, there has been a fair amount of interest in trying to construct physically significant systems which may not be exactly solvable, but for which part of the spectrum can be computed exactly by algebraic methods. In the early 1980's, Alhassid, Gürsey, Iachello, Levine and collaborators, [3], [1], [12], introduced the concept of a "spectrum generating algebra" to construct models for complicated molecules whose point spectrum could be analyzed algebraically. Independently, Turbiner, Ushveridze, Shifman and their collaborators were led to define a class of spectral problems which they called "quasi-exactly solvable", [18], [21], [15]. This class is characterized by the the property that the Schrödinger operator is Lie algebraic, *i.e.* expressible as a bilinear combination of first-order differential operators spanning a finite-dimensional Lie algebra $\mathfrak{g}$, and such that $\mathfrak{g}$ admits a finite-dimensional module $\mathcal{N}$ of smooth functions. Thus if $\mathcal{N}$ is k-dimensional, then one can obtain k eigenvalues (counting multiplicities) of the Schrödinger operator H by computing the spectrum of the Hermitian operator (with respect to an appropriate inner product) obtained by restriction of H to the k-dimensional vector space $\mathcal{N}$. The Lie algebra $\mathfrak{g}$ is to be thought of as a "hidden" symmetry algebra for the quasi-exactly solvable problem, whose presence underlies the partial solvability of the spectral problem. In the context of applications to molecular dynamics, Levine [12] posed the problem of classifying the Lie-algebraic operators under the equivalence relation defined by smooth changes of the independent variables and rescalings of the wave function.

The general procedure to be followed in order to solve Levine's classification problem is quite clear in principle, although the difficulties involved in implementing this procedure in practice are enormous. Indeed, one first needs a classification up to local diffeomorphisms of the finite-dimensional Lie algebras of first-order differential operators. As we discuss in detail in Section 3 (*cf.* Theorem 3.3), this can be shown to amount to the classification of triples $(\mathfrak{h}, \mathfrak{m}, [F])$, where $\mathfrak{h}$ is a finite-dimensional Lie algebra of vector fields, $\mathfrak{m}$ is a finite-dimensional $\mathfrak{h}$-module of smooth functions, and $[F]$ is a cohomology class in $H^1(\mathfrak{h}; C^\infty(\mathbb{R}^n; \mathbb{R})/\mathfrak{m})$, [6]. Next, one has to determine amongst these Lie algebras of first-order differential operators those admitting a finite-dimensional module of smooth functions. Finally, one has to solve the equivalence problem for second-order differential operators under smooth changes of the independent variables and rescalings of the dependent variable, thereby determining when a given Schrödinger operator can be written in the required bilinear form using one of the Lie algebras obtained above. To our knowledge, it is only on the line and, very recently, in the plane that a complete classification for the finite-dimensional Lie algebras of first-order differential operators is known. In [13], [10], the Lie algebras of differential operators are classified on the line. In [6], they are classified in the plane, and in [7] the Lie algebras on the line or in the plane which admit a finite-dimensional module of smooth functions are obtained. Remarkably, the cohomology spaces $H^1\left(\mathfrak{h}; C^\infty\left(\mathbb{R}^n;\mathbb{R}\right)/\mathfrak{m}\right)$ for all the Lie algebras obtained in [7] depend on a finite number of cohomology parameters, and these cohomology parameters are all restricted to take on discrete values in order for the $\mathfrak{g}$-module $\mathcal{N}$ to be finite-dimensional. On the other hand, the solution of the equivalence problem for second-order differential operators is relatively straightforward (see [9] for ordinary differential operators and [5] for partial differential operators).

At the present time, a complete classification and a list of normal forms is available for the one-dimensional quasi-exactly solvable spectral problems, [18], [21], and there are a few known classes of quasi-exactly solvable problems in two dimensions, [17]. The "hidden" symmetry algebra in these two-dimensional examples is always given by a representation of either of the compact Lie algebras $\mathfrak{su}(3)$, $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ or $\mathfrak{so}(3)$ in terms of differential operators. Yet we know from the classification performed in [6] that there is a wide array of equivalence classes of Lie algebras of differential operators in the plane (admitting a finite-dimensional module of smooth functions) which extends considerably beyond the few equivalence classes considered by Shifman and Turbiner in [17]. It is therefore natural to use the classification of the Lie algebras of first-order differential operators given in [7] to construct new example of quasi-exactly solvable spectral problems for Schrödinger operators in two dimensions. This is precisely what we do in this paper, where particular emphasis is put on using non-compact "hidden" symmetry algebras to obtain Schrödinger operators defined globally on $\mathbb{R}^2$ with a square integrable invariant module $\mathcal{N}$. Recently uncovered connections with conformal field theory, [14], [8], [16], quantum chaos, [4], and the theory of orthogonal polynomials, [19], lend an added impetus to this study.

Our paper is organized as follows. In Section 2, we recall the solution of the equivalence problem for differential operators in the form needed for the purposes of our discussion. Section 3 is devoted to a description of the method we shall use to construct new quasi-exactly solvable Schrödinger operators, based on the classification of Lie algebras of differential operators in the plane. In Section 4, we present our new quasi-exactly solvable potentials and discuss some of their properties.

2 Equivalence of Differential Operators

Let M and $\overline{M}$ be open subsets of $\mathbb{R}^n$ with local coordinates given respectively by $(x^1,\ldots,x^n)$ and $(\bar{x}^1,\ldots,\bar{x}^n)$. Consider a second-order linear differential operator

$T : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$ given by

$$T = \sum_{i,j=1}^{n} g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i \frac{\partial}{\partial x^i} + c, \tag{2.1}$$

where g^{ij}, b^i, $1 \le i, j \le n$, and c are real-valued functions of class C^∞ on M. Likewise, consider an operator $\overline{T} : C^\infty(\overline{M};\mathbb{R}) \to C^\infty(\overline{M};\mathbb{R})$ of the form

$$\overline{T} = \sum_{i,j=1}^{n} \bar{g}^{ij} \frac{\partial^2}{\partial \bar{x}^i \partial \bar{x}^j} + \sum_{i=1}^{n} \bar{b}^i \frac{\partial}{\partial \bar{x}^i} + \bar{c}, \tag{2.2}$$

with coefficients $\bar{g}^{ij}$, $\bar{b}^i$, $1 \le i, j \le n$, and $\bar{c}$ in $C^\infty(\overline{M};\mathbb{R})$.

We say that the operators T and $\overline{T}$ are *equivalent* if there exists a diffeomorphism $\varphi : M \to \overline{M}$ and a nowhere vanishing function $\mu \in C^\infty(M;\mathbb{R})$ such that

$$\left(\overline{T}\psi\right) \circ \varphi = \mu \cdot T\chi, \tag{2.3}$$

for every $\chi \in C^\infty(M;\mathbb{R})$, where

$$\psi = (\mu \cdot \chi) \circ \varphi^{-1}. \tag{2.4}$$

Defining the linear operator (depending on μ and φ) $\mathcal{F} : C^\infty(M;\mathbb{R}) \to C^\infty(\overline{M};\mathbb{R})$ by

$$\mathcal{F}(\chi) = (\mu \cdot \chi) \circ \varphi^{-1}, \tag{2.5}$$

the equivalence between T and $\overline{T}$ is simply expressed by the operator equality

$$\overline{T} = \mathcal{F} \cdot T \cdot \mathcal{F}^{-1}. \tag{2.6}$$

This notion of equivalence is well adapted to the study of spectral problems. Indeed, if T and $\overline{T}$ are equivalent in the sense of equations (2.3) and (2.4), then they will have the same eigenvalues and their eigenfunctions will be related by equation (2.4).

Another important property of the equivalence relation (2.5) that is crucial for our purposes is the following. Consider the Lie algebra $\mathcal{D}(M)$ of linear first-order differential operators D on M,

$$D = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i} + \eta, \tag{2.7}$$

where ξ^i, η, $1 \le i \le n$, are in $C^\infty(M;\mathbb{R})$, and where the Lie bracket between differential operators is given by the usual commutator

$$[D, E] = D\,E - E\,D. \tag{2.8}$$

If we define the equivalence of two first-order differential operators $D \in \mathcal{D}(M)$ and $\overline{D} \in \mathcal{D}(\overline{M})$ as before, *cf.* equations (2.3) and (2.4), then this equivalence relation preserves the Lie bracket, or, more precisely, defines a Lie algebra isomorphism $\mathcal{D}(M) \to \mathcal{D}(\overline{M})$. In other words, if $D_i \in \mathcal{D}(M)$ is equivalent to $\overline{D}_i \in \mathcal{D}(\overline{M})$ for $i = 1, 2$, then $[D_1, D_2]$ will be equivalent to $\left[\overline{D}_1, \overline{D}_2\right]$.

It is a straightforward matter to obtain a workable set of necessary and sufficient conditions for two second-order differential operators T and $\overline{T}$ to be equivalent. We now briefly recall these conditions, which were first given in a paper of É. Cotton [5]. We first observe that if there exist functions φ and μ satisfying the equivalence conditions (2.3) and (2.4), then for every $p \in M$ the quadratic forms associated to the symmetric matrices $\left(g^{ij}(p)\right)_{1\le i,j\le n}$ and $\left(\bar{g}^{ij}(\varphi(p))\right)_{1\le i,j\le n}$ should have same rank and same index. We shall assume throughout this paper that these quadratic forms have rank equal to n and index equal to zero, *i.e.* that they are *positive definite.* We may thus interpret the functions g^{ij}, $1 \le i, j \le n$, as defining the contravariant components of a Riemannian metric

$$(g_{ij}) = (g^{ij})^{-1} \tag{2.9}$$

on M, and likewise for $\bar{g}^{ij}$ on $\overline{M}$. In what follows, we shall use these metrics to raise and lower the indices of tensor fields on M and $\overline{M}$ in the usual way.

It is convenient to rewrite rewrite T in the coordinate-independent form

$$T = \sum_{i,j=1}^{n} g^{ij}(\nabla_i - A_i)(\nabla_j - A_j) + U, \tag{2.10}$$

where ∇_i denotes the covariant differentiation operator with respect to the Levi-Civita connection of (g_{ij}),

$$\begin{aligned} A^i &= \sum_{i,j=1}^{n} g^{ij} A_j \\ &= -\frac{b^i}{2} + \frac{1}{2\sqrt{g}} \sum_{j=1}^{n} \frac{\partial(\sqrt{g}\, g^{ij})}{\partial x^j}, \qquad (2.11) \\ U &= c + \sum_{i=1}^{n} \left[-A_i\, A^i + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g}\, A^i) \right] \qquad (2.12) \end{aligned}$$

and

$$g = \det(g_{ij}) \,. \tag{2.13}$$

Similarly, we may express $\overline{T}$ as

$$\overline{T} = \sum_{i,j=1}^{n} \bar{g}^{ij}(\overline{\nabla}_i - \bar{A}_i)(\overline{\nabla}_j - \bar{A}_j) + \overline{U}. \tag{2.14}$$

Let us also rewrite the non-vanishing scaling function μ in the form

$$\mu = e^{\sigma}; \tag{2.15}$$

defining the metric forms

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j, \qquad d\bar{s}^2 = \sum_{i,j=1}^{n} \bar{g}_{ij} d\bar{x}^i d\bar{x}^j, \tag{2.16}$$

and the *magnetic 1-forms*

$$\omega = \sum_{i=1}^{n} A_i \, dx^i, \qquad \bar{\omega} = \sum_{i=1}^{n} \bar{A}_i \, d\bar{x}^i, \tag{2.17}$$

we have the following result:

Theorem 2.1 *The necessary and sufficient conditions for the differential operators T and $\overline{T}$ given by equations (2.10) and (2.14) to be equivalent under a diffeomorphism $\varphi : M \to \overline{M}$ and rescaling by $\mu = e^\sigma$ are that*

$$\begin{aligned} \varphi^*(d\bar{s}^2) &= ds^2, & (2.18)\\ \varphi^*(\bar{\omega}) &= \omega + d\sigma, & (2.19)\\ \overline{U} \circ \varphi &= U. & (2.20) \end{aligned}$$

In particular, from (2.18) we see that if T and $\overline{T}$ are equivalent then the metrics ds^2 and $d\bar{s}^2$ are necessarily isometric, as already observed.

Definition 2.2 A *Schrödinger operator* on M is a second-order differential operator $H : C^\infty(M; \mathbb{R}) \to C^\infty(M; \mathbb{R})$ of the form

$$H = -\frac{1}{2} \sum_{i,j=1}^{n} g^{ij} \nabla_i \nabla_j + V. \tag{2.21}$$

In the previous formula, $V \in C^\infty(M; \mathbb{R})$ is the *potential function* for the physical system under consideration, and a system of physical units has been chosen so that $\hbar = m = 1$. The Riemannian metric $(g_{ij}) \equiv (g^{ij})^{-1}$ associated to H may have non-zero curvature if the system is constrained, *e.g.* a particle moving on a sphere, but for most of the applications we have in mind the metric will be flat, so that the Schrödinger operator (2.21) will be locally expressible in the form

$$H_{\mathrm{flat}} = -\frac{1}{2} \sum_{i=1}^{n} \left(\frac{\partial}{\partial \bar{x}^i} \right)^2 + V(\bar{x}^1, \ldots, \bar{x}^n) \tag{2.22}$$

in appropriate coordinates $(\bar{x}^1, \ldots, \bar{x}^n)$. Finally, notice that the differential operator

$$\triangle = \sum_{i,j=1}^{n} g^{ij} \nabla_i \nabla_j \tag{2.23}$$

is the well-known *Laplace–Beltrami* operator associated to the metric (g_{ij}) on M.

As special cases of Theorem 2.1, we obtain the following necessary and sufficient conditions for a linear differential operator $-\frac{1}{2}T$ to be equivalent to a flat Schrödinger operator H_{flat} or, more generally, to a Schrödinger operator H on a curved Riemannian manifold:

Corollary 2.3 *The differential operator $-\frac{1}{2}T$, where T is given by equation (2.10), will be equivalent to a Schrödinger operator H_{flat} of the form given by equation (2.22) for some potential function V if and only if:*

i) The metric ds^2 is Riemannian, i.e. *positive-definite.*

ii) The Riemann–Christoffel curvature tensor of ds^2 is identically zero, that is $R^i{}_{jkl} = 0$.

iii) The magnetic 1-form ω is closed: $d\omega = 0$.

If only conditions i) and iii) are satisfied, then $-\frac{1}{2}T$ will be equivalent to a Schrödinger operator H of the form (2.21), and conversely.

It should be noted that in the case $n = 2$, the condition ii) is equivalent to the vanishing of the Gaussian curvature of ds^2. In the case $n = 1$, the above results results simplify considerably. Indeed, in this case condition i) is always locally satisfied up to an overall sign, whereas conditions ii) and iii) are automatically fulfilled. Hence any second-order linear differential operator on the line is locally equivalent to a Schrödinger operator, up to an overall sign.

3 Quasi-Exactly Solvable Schrödinger Operators in $\mathbb{R}^n$

¿From now on, we shall restrict our attention to the case $M = \mathbb{R}^n$. Let, as before, $\mathcal{D}(\mathbb{R}^n)$ denote the set of all first-order differential operators of the form (2.7) globally defined on $\mathbb{R}^n$. Note that the elements of $\mathcal{D}(\mathbb{R}^n)$ have a natural action on smooth functions, so that $C^\infty(\mathbb{R}^n;\mathbb{R})$ has a natural $\mathcal{D}(\mathbb{R}^n)$-module structure.

A linear differential operator T in $\mathbb{R}^n$ is said to be *Lie-algebraic* if it is an element of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of a finite-dimensional Lie subalgebra $\mathfrak{g}$ of $\mathcal{D}(\mathbb{R}^n)$. In particular, a second-order linear differential operator T of the form (2.1) is Lie algebraic if there exist r linearly independent first-order differential operators

$$T^a = \sum_{i=1}^{n} \xi^{ai} \frac{\partial}{\partial x^i} + \eta^a, \quad 1 \le a \le r, \tag{3.1}$$

where ξ^{ai}, η^a, $1 \le a \le r$, $1 \le i \le n$, are in $C^\infty(\mathbb{R}^n;\mathbb{R})$, which span a finite-dimensional Lie algebra under the commutator, and which are such that

$$T = \sum_{a,b=1}^{r} C_{ab} T^a T^b + \sum_{a=1}^{r} C_a T^a + C_0, \tag{3.2}$$

for some real constants C_{ab}, C_a, $1 \leq a, b \leq r$, and C_0. Since the addition of the constant C_0 merely shifts the spectrum of T as a whole, we shall often find convenient to fix the origin of the latter spectrum by setting $C_0 = 0$.

Remark: If $\mathfrak{g}$ contains functions, there might be Lie-algebraic second-order differential operators $T \in \mathfrak{U}(\mathfrak{g})$ expressible as polynomials of degree higher than two in the generators T^a of $\mathfrak{g}$. However, this generalization of (3.2) is only apparent, since it is easy to show that, given a second-order differential operator $T \in \mathfrak{U}(\mathfrak{g})$, it is always possible to extend $\mathfrak{g}$ to another finite-dimensional Lie algebra $\bar{\mathfrak{g}}$ by including suitable functions, in such a way that T is a polynomial of degree no higher than two in the generators of $\bar{\mathfrak{g}}$.

We now define the class of Schrödinger operators which will be the main focus of our study.

Definition 3.1 A Schrödinger operator

$$H = -\frac{1}{2} \sum_{i,j=1}^{n} \bar{g}^{ij}(\bar{x}) \overline{\nabla}_i \overline{\nabla}_j + V(\bar{x}) \tag{3.3}$$

(or, more generally, an arbitrary second-order differential operator) in $\mathbb{R}^n$ is said to be *quasi-exactly solvable* if:

i) H is Lie-algebraic, *i.e.* there exists a finite-dimensional Lie subalgebra $\mathfrak{g}$ of $\mathcal{D}(\mathbb{R}^n)$ such that $H \in \mathfrak{U}(\mathfrak{g})$.

ii) The Lie algebra $\mathfrak{g}$ leaves a finite-dimensional subspace $\overline{\mathcal{N}}$ of $C^\infty(\mathbb{R}^n; \mathbb{R})$ invariant, that is $\mathfrak{g}$ admits $\overline{\mathcal{N}}$ as a finite-dimensional module of smooth functions. In other words, we have a finite-dimensional representation of $\mathfrak{g}$ on a subspace $\overline{\mathcal{N}}$ of $C^\infty(\mathbb{R}^n; \mathbb{R})$.

Let $L^2(M; \rho)$ denote the space of square integrable functions on a manifold M with respect to the measure ρ. If, in addition to i) and ii),

iii) The finite-dimensional module $\overline{\mathcal{N}}$ is a subspace of $L^2\left(\mathbb{R}^n; \sqrt{\bar{g}(\bar{x})}\, d\bar{x}^1 \cdots d\bar{x}^n\right)$,

then H is said to be a *normalizable quasi-exactly solvable* Schrödinger operator. A potential $V \in C^\infty(\mathbb{R}^n; \mathbb{R})$ is called a (*normalizable*) *quasi-exactly solvable potential* if it arises from a (normalizable) quasi-exactly solvable Schrödinger operator.

¿From this definition, we see that the eigenvalue problem for a normalizable quasi-exactly solvable Schrödinger operator H is partly solvable by strictly algebraic constructions. Indeed, from the self-adjointness of H with respect to the inner product associated to the standard measure

$$\sqrt{\bar{g}}\, d\bar{x}^1 \cdots d\bar{x}^n, \tag{3.4}$$

it follows that the restriction of H to the finite-dimensional subspace $\overline{\mathcal{N}}$ is a Hermitian finite-dimensional linear operator (with respect to an appropriate inner product). Hence we can in principle exactly compute $s = \dim \overline{\mathcal{N}}$ eigenvalues of H (counting multiplicities) by diagonalizing the $s \times s$ matrix representing H in any basis of $\overline{\mathcal{N}}$. For this reason, we shall be almost exclusively concerned in this paper with normalizable quasi-exactly solvable Schrödinger operators and potentials.

Another important remark, which we shall systematically exploit in what follows, is that the property of being quasi-exactly solvable is invariant under the equivalence relation defined in the previous section, *cf.* equations (2.3) and (2.4). Indeed, suppose that a Schrödinger operator H is equivalent to a quasi-exactly solvable second-order differential operator $-\frac{1}{2}T \in \mathfrak{U}(\mathfrak{g})$ under a diffeomorphism φ and a rescaling μ, *i.e.*

$$-2H = \mathcal{F} \cdot T \cdot \mathcal{F}^{-1} \tag{3.5}$$

with $\mathcal{F}$ defined by (2.5). Then it is immediate to show that H is quasi-exactly solvable with respect to the finite-dimensional Lie algebra

$$\bar{\mathfrak{g}} = \mathcal{F} \cdot \mathfrak{g} \cdot \mathcal{F}^{-1} \equiv \left\{ \mathcal{F} \cdot X \cdot \mathcal{F}^{-1} \mid X \in \mathfrak{g} \right\}, \tag{3.6}$$

which is isomorphic to $\mathfrak{g}$ and possesses the finite-dimensional $\bar{\mathfrak{g}}$-module

$$\overline{\mathcal{N}} = \mathcal{F} \cdot \mathcal{N} \equiv \{ \mathcal{F} \cdot h \mid h \in \mathcal{N} \}, \tag{3.7}$$

$\mathcal{N}$ being the finite-dimensional $\mathfrak{g}$-module whose existence is guaranteed by the quasi-exact solvability of T. Notice, by contrast, that the class of *normalizable* quasi-exactly solvable second-order differential operators is *not* invariant under the equivalence relation defined by (3.5), *i.e.* H may fail to be normalizable (with respect to the measure (3.4)) even if T is.

In practice, to find examples of normalizable quasi-exactly solvable Schrödinger operators one usually starts with a Lie algebra of first-order differential operators $\mathfrak{g}$ admitting a finite-dimensional $\mathfrak{g}$-module of smooth functions $\mathcal{N}$, expressed in suitable "canonical" coordinates (*cf.* [7]). Next, one tries to construct a Lie-algebraic second-order differential operator $T \in \mathfrak{U}(\mathfrak{g})$ satisfying conditions ii) and iii) of Corollary 2.3. This guarantees that $-\frac{1}{2}\,T$ is equivalent to a Schrödinger operator $H = -\frac{1}{2}\,\mathcal{F}^{-1} \cdot T \cdot \mathcal{F}$, which by the previous remark will be quasi-exactly solvable. Lastly, one has to check whether or not H is normalizable, *i.e.* whether or not $\overline{\mathcal{N}} \subset L^2\,(\mathbb{R}^n; \sqrt{\bar{g}}\,d\bar{x}^1 \cdots d\bar{x}^n)$. Since

$$\mathcal{F} : L^2\,(M; \mu^2 \sqrt{g}\,dx^1 \cdots dx^n) \to L^2\,(\overline{M}; \sqrt{\bar{g}}\,d\bar{x}^1 \cdots d\bar{x}^n) \tag{3.8}$$

is clearly a linear isometry, this is equivalent to checking whether or not

$$\mathcal{N} \subset L^2\,(\mathbb{R}^n; \mu^2 \sqrt{g}\,dx^1 \cdots dx^n). \tag{3.9}$$

If that is the case then, as remarked before, $s = \dim \overline{\mathcal{N}} = \dim \mathcal{N}$ eigenvalues (counting multiplicities) and linearly-independent square-integrable eigenfunctions of H can be computed algebraically. One often computes these "algebraic" eigenvalues of H by diagonalizing the restriction of $-\frac{1}{2}T$ to $\mathcal{N}$, using (2.4) to obtain the eigenfunctions of H from those of $-\frac{1}{2}T$. Notice, finally, that although T need not be (formally) self-adjoint with respect to the inner product associated to the standard measure $\sqrt{g}\,dx^1\cdots dx^n$, it is automatically self-adjoint with respect to the inner product defined by the measure $\mu^2\sqrt{g}\,dx^1\cdots dx^n$. (This is easily proved from the self-adjointness of H with respect to the standard inner product, using (2.6) and the fact that (3.8) is a linear isometry.) In particular, this guarantees that the restriction of T to $\mathcal{N}$ is Hermitian with respect to an appropriate inner product, and therefore has exactly $s = \dim\mathcal{N}$ *real* eigenvalues counting multiplicities, as expected.

Example 3.2 Consider the Lie algebra $\mathfrak{g} \cong \mathfrak{sl}(3) \subset \mathcal{D}(\mathbb{R}^2)$ spanned by the first-order differential operators

$$\begin{gathered} T^1 = p, \quad T^2 = q, \quad T^3 = xp, \quad T^4 = yp, \quad T^5 = xq, \\ T^6 = yq, \quad T^7 = x^2p + xyq - 2x, \quad T^8 = xyp + y^2q - 2y, \end{gathered} \tag{3.10}$$

where we have used the classical notation

$$p = \frac{\partial}{\partial x}, \quad q = \frac{\partial}{\partial y}. \tag{3.11}$$

Let H be the Schrödinger operator (2.21) defined by the contravariant metric components

$$\begin{aligned} g^{11} &= \alpha + x^2(\beta + s), \\ g^{12} &= xy(\beta + s), \\ g^{22} &= \lambda\alpha + y^2(\beta + s), \end{aligned} \tag{3.12}$$

and by the potential

$$V = \frac{-3s^3 - 71\beta s^2 + (195\beta^2 - 48\lambda\alpha\gamma)\,s - \beta\,(121\beta^2 - 72\lambda\alpha\gamma)}{8\,(s^2 + \lambda\alpha\gamma - \beta^2)}, \tag{3.13}$$

where

$$s = \beta + \gamma(\lambda x^2 + y^2), \tag{3.14}$$

the parameters α, γ and λ are positive, and β is nonnegative.

We shall show in Section 4.2.1 that $-2H$ is equivalent under rescaling by

$$\mu = \alpha^{1/4}\gamma^{5/8}\left(s^2 + \lambda\alpha\gamma - \beta^2\right)^{-5/8} \tag{3.15}$$

to the Lie-algebraic differential operator $T \in \mathfrak{U}(\mathfrak{g})$ given by

$$\begin{aligned} T \;=\; & \alpha\left((T^1)^2 + \lambda(T^2)^2\right) + \gamma\left(\lambda(T^7)^2 + (T^8)^2\right) + \\ & \beta\left(T^1T^8 + T^8T^1 + T^2T^8 + T^8T^2 + 7T^3 + 7T^6\right). \end{aligned} \tag{3.16}$$

The Lie algebra $\mathfrak{g}$ leaves invariant the subspace $\mathcal{N} \subset C^\infty(\mathbb{R}^2;\mathbb{R})$ of all polynomials of degree less than or equal to two, whose image $\overline{\mathcal{N}}$ under the rescaling (3.15) is spanned by the functions

$$x^i y^j \left(s^2 + \lambda\alpha\gamma - \beta^2\right)^{-5/8}, \quad 0 \le i+j \le 2. \tag{3.17}$$

Since these functions are obviously square integrable with respect to the volume element

$$(\gamma/\alpha)^{1/2} \left(s^2 + \lambda\alpha\gamma - \beta^2\right)^{-1/2} dx\, dy \tag{3.18}$$

associated to the contravariant metric (3.12), H is a normalizable quasi-exactly solvable Schrödinger operator. Hence we can exactly compute six eigenvalues of H (counting multiplicities) and its corresponding eigenfunctions by diagonalizing the matrix of the restriction of T to $\mathcal{N}$. In the standard basis $\{1, x, y, x^2, xy, y^2\}$ of $\mathcal{N}$, the latter matrix is easily computed from equation (3.16), obtaining

$$\begin{pmatrix} -4\beta & 0 & 0 & 2\alpha & 0 & 2\lambda\alpha \\ 0 & 2\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\beta & 0 & 0 & 0 \\ 2\lambda\gamma & 0 & 0 & 12\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 12\beta & 0 \\ 2\gamma & 0 & 0 & 0 & 0 & 12\beta \end{pmatrix}. \tag{3.19}$$

The eigenvalues of this matrix arranged in order of increasing magnitude are:

$$\begin{aligned} \lambda_1 &= 4\beta - 2\sqrt{16\beta^2 + 2\lambda\alpha\gamma}\,, \\ \lambda_2 &= 2\beta, \\ \lambda_3 &= 12\beta, \\ \lambda_4 &= 4\beta + 2\sqrt{16\beta^2 + 2\lambda\alpha\gamma}\,, \end{aligned} \tag{3.20}$$

with multiplicities 1, 2, 2, and 1, respectively. Its associated eigenvectors are easily found to be

$$\begin{aligned} \chi_1 &= \begin{pmatrix} -\frac{4\beta+\sqrt{16\beta^2+2\lambda\alpha\gamma}}{\gamma} & 0 & 0 & \lambda & 0 & 1 \end{pmatrix}^t; \\ \chi_{21} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^t, \quad \chi_{22} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^t; \\ \chi_{31} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^t, \quad \chi_{32} = \begin{pmatrix} 0 & 0 & 0 & \lambda & 0 & -1 \end{pmatrix}^t; \\ \chi_4 &= \begin{pmatrix} \frac{-4\beta+\sqrt{16\beta^2+2\lambda\alpha\gamma}}{\gamma} & 0 & 0 & \lambda & 0 & 1 \end{pmatrix}^t. \end{aligned} \tag{3.21}$$

Hence H has the eigenvalues (arranged in order of increasing magnitude)

$$\begin{aligned} E_1 &= -2\beta - \sqrt{16\beta^2 + 2\lambda\alpha\gamma}\,, \\ E_2 &= -6\beta, \\ E_3 &= -\beta, \\ E_4 &= -2\beta + \sqrt{16\beta^2 + 2\lambda\alpha\gamma}\,. \end{aligned} \tag{3.22}$$

Taking into account the rescaling (3.15), the eigenfunctions of H associated to its eigenvalues (3.22) are given respectively by

$$
\begin{aligned}
\psi_1 &= \left(s^2+\lambda\alpha\gamma-\beta^2\right)^{-5/8}\left(\frac{-4\beta+\sqrt{16\beta^2+2\lambda\alpha\gamma}}{\gamma}+\lambda x^2+y^2\right);\\
\psi_{21} &= xy\left(s^2+\lambda\alpha\gamma-\beta^2\right)^{-5/8}, \quad \psi_{22}=(\lambda x^2-y^2)\left(s^2+\lambda\alpha\gamma-\beta^2\right)^{-5/8};\\
\psi_{31} &= x\left(s^2+\lambda\alpha\gamma-\beta^2\right)^{-5/8}, \quad \psi_{32}=y\left(s^2+\lambda\alpha\gamma-\beta^2\right)^{-5/8};\\
\psi_4 &= \left(s^2+\lambda\alpha\gamma-\beta^2\right)^{-5/8}\left(-\frac{4\beta+\sqrt{16\beta^2+2\lambda\alpha\gamma}}{\gamma}+\lambda x^2+y^2\right).
\end{aligned}
\tag{3.23}
$$

Notice, however, that we don't have any information as to where in the point spectrum of H lie the eigenvalues (3.22) just found. ■

In the remainder of this section, we shall present a general procedure for classifying the normalizable quasi-exactly solvable potentials in $n \geq 2$ dimensions. Section 4 will be devoted to the implementation of this procedure in the case $n = 2$, allowing us to obtain a wide array of new normalizable quasi-exactly solvable planar potentials.

The first step in the classification of quasi-exactly solvable Schrödinger operators in $\mathbb{R}^n$ is to classify the finite-dimensional Lie subalgebras of $\mathcal{D}(\mathbb{R}^n)$ under diffeomorphisms $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ and rescalings $\mu : \mathbb{R}^n \to \mathbb{R}^*$, acting on $\mathcal{D}(\mathbb{R}^n)$ according to equations (2.3) and (2.4). This first step has a natural formulation in terms of Lie algebra cohomology, [6]. Indeed, any finite-dimensional Lie subalgebra $\mathfrak{g}$ of $\mathcal{D}(\mathbb{R}^n)$ has a basis of the form

$$X_1 + f_1, \ldots, X_l + f_l, h_1, \ldots, h_m, \tag{3.24}$$

where $X_1, \ldots, X_l$ are linearly independent vector fields spanning an l-dimensional Lie subalgebra $\mathfrak{h}$ of the Lie algebra of vector fields on $\mathbb{R}^n$, and where the functions $h_1, \ldots, h_m$ act as multiplication operators on $C^\infty(\mathbb{R}^n; \mathbb{R})$.

Since $\mathfrak{g}$ is a Lie algebra, the functions $h_1, \ldots, h_m$ must necessarily span an $\mathfrak{h}$-module $\mathfrak{m} \subset C^\infty(\mathbb{R}^n; \mathbb{R})$. In other words, $\mathfrak{m}$ must be invariant under the action of $\mathfrak{h}$ (by derivation). Moreover, the functions $f_1, \ldots, f_l$ define a 1-cochain $F : \mathfrak{h} \to C^\infty(\mathbb{R}^n; \mathbb{R})$, according to $\langle F; X_i \rangle = f_i$, $1 \leq i \leq l$. Actually, F ought to be regarded as a $C^\infty(\mathbb{R}^n; \mathbb{R})/\mathfrak{m}$-valued 1-cochain, since the f_i, $1 \leq i \leq l$, are only defined up to linear combinations of the h_j, $1 \leq j \leq m$. Recall now that the coboundary of a $C^\infty(\mathbb{R}^n; \mathbb{R})/\mathfrak{m}$-valued 1-cochain is the alternating bilinear mapping $\delta_1 F : \mathfrak{h} \times \mathfrak{h} \to C^\infty(\mathbb{R}^n; \mathbb{R})/\mathfrak{m}$ defined by the formula

$$\langle \delta_1 F; X, Y \rangle = X(\langle F; Y \rangle) - Y(\langle F; X \rangle) - \langle F; [X, Y] \rangle. \tag{3.25}$$

In order that the differential operators (3.24) span a Lie algebra, it is necessary and sufficient that the right-hand side of equation (3.25) lie in the $\mathfrak{h}$-module $\mathfrak{m}$ for all choices of X and Y in $\mathfrak{h}$, or in other words that F be a $C^\infty(\mathbb{R}^n;\mathbb{R})/\mathfrak{m}$-valued cocycle. On the other hand, two Lie subalgebras of $\mathcal{D}(\mathbb{R}^n)$ will be equivalent under a rescaling given by equations (2.3) and (2.4) with $\varphi = \mathrm{id}_{\mathbb{R}^n}$ if and only if their corresponding 1-cocycles differ by a 1-cocycle G of the form $\langle G; X_i\rangle = X_i(-\log\mu) = X_i(-\sigma)$, $1 \le i \le l$, that is by a 1-coboundary $G = \delta_0(-\sigma)$. Thus we have:

Theorem 3.3 *There is a one-to-one correspondence between equivalence classes of finite-dimensional subalgebras of $\mathcal{D}(\mathbb{R}^n)$ and equivalence classes of triples $(\mathfrak{h}, \mathfrak{m}, [F])$, where:*

i) $\mathfrak{h}$ *is a finite-dimensional Lie algebra of vector fields.*

ii) $\mathfrak{m} \subset C^\infty(\mathbb{R}^n;\mathbb{R})$ *is a finite-dimensional* $\mathfrak{h}$*-module.*

iii) $[F]$ *is a cohomology class in* $H^1(\mathfrak{h}; C^\infty(\mathbb{R}^n;\mathbb{R})/\mathfrak{m}) \simeq \ker\delta_1/\mathrm{im}\,\delta_0$.

If one assumes that the triples $(\mathfrak{h}, \mathfrak{m}, [F])$ have been classified, then one has a complete list of the Lie-algebraic second-order differential operators by taking quadratic combinations of the form (3.2). The classification of triples $(\mathfrak{h}, \mathfrak{m}, [F])$ is however a highly non-trivial problem. The complete answer is only known in the case of Lie algebras of differential operators in two complex variables, that is finite-dimensional subalgebras $\mathfrak{g} \subset \mathcal{D}(\mathbb{C}^2)$, [6].

The next step towards the classification of the quasi-exactly solvable potentials is to determine which of the (equivalence classes of) Lie subalgebras $\mathfrak{g} \subset \mathcal{D}(\mathbb{R}^n)$ obtained in Theorem 3.3 admit a finite-dimensional module $\mathcal{N}$ of functions in $C^\infty(\mathbb{R}^n;\mathbb{R})$. Again, this is a fairly complicated problem which has been solved completely in the case of finite-dimensional subalgebras $\mathfrak{g}$ of $\mathcal{D}(\mathbb{C}^2)$, [7]. Remarkably, for all the Lie algebras obtained in Reference [7], the condition that there exist a finite-dimensional module of functions forces the cohomology parameters arising in Theorem 3.3 to take only a discrete set of values. This phenomenon of the "quantization of cohomology" is a fascinating problem that deserves further investigation.

The next step in the classification of quasi-exactly solvable Schrödinger operators is to determine which of the Lie-algebraic second-order operators obtained as a result of the first two steps are equivalent to a Schrödinger operator. To this effect we must apply Corollary 2.3, the content of which we now formulate a bit more explicitly.

Let us first rewrite equation (3.2) a bit differently as

$$T = -2H_{\mathrm{gauge}} = \sum_{a,b=1}^{r} C_{ab}T^aT^b + \sum_{a=1}^{r} C_aT^a, \tag{3.26}$$

where the Lie-algebraic operator H_{gauge} will be referred to as the "gauge Hamiltonian", since it is to be equivalent to a Schrödinger operator under the combined effect of a rescaling or "gauge transformation" on the dependent variable and a change of independent variables, as per equations (2.3) and (2.4). (Notice that the rescaling (2.4) with $\varphi = \mathrm{id}_{\mathbb{R}^n}$, though it is not a unitary transformation of $L^2(\mathbb{R}^n; \sqrt{g}\,dx^1\dots dx^n)$, is nevertheless a genuine unitary operator from $L^2(\mathbb{R}^n; \mu^2\sqrt{g}\,dx^1\ldots dx^n)$ onto $L^2(\mathbb{R}^n; \sqrt{g}\,dx^1\ldots dx^n)$.) By expressing T in the forms (2.1) and (3.26), and using (3.1), we obtain

$$
\begin{aligned}
g^{ij} &= \sum_{a,b=1}^{r} C_{ab}\xi^{ai}\xi^{bj}, && (3.27)\\
b^i &= \sum_{a,b=1}^{r}\left[C_{ab}\left(\xi^{aj}\frac{\partial\xi^{bi}}{\partial x^j}+2\eta^a\xi^{bi}\right)+C_a\xi^{ai}\right], && (3.28)\\
c &= \sum_{a,b=1}^{r}\left[C_{ab}\left(\xi^{aj}\frac{\partial\eta^{b}}{\partial x^j}+\eta^a\eta^{b}\right)+C_a\eta^{a}\right], && (3.29)
\end{aligned}
$$

The functions A_i, $1\le i\le n$, and U are computed from the above equations using (2.11) and (2.12). According to Corollary 2.3, the necessary and sufficient conditions for the existence of a "gauge transformation"

$$\psi=\mu\chi=e^{\sigma}\chi, \tag{3.30}$$

transforming the eigenvalue problem

$$H_{\text{gauge}}\chi=E\chi \tag{3.31}$$

into the "physical" eigenvalue problem

$$H\psi=E\psi, \tag{3.32}$$

where H is the Schrödinger operator (2.21), is that

$$\sum_{a,b=1}^{r}\xi^{ai}\left[C_{ab}\left(\sum_{j=1}^{n}\left(\xi^{bj}\frac{\partial\alpha}{\partial x^j}+\frac{\partial\xi^{bj}}{\partial x^j}\right)-2\eta^b\right)-C_a\right]=0 \tag{3.33}$$

for $1\le i\le n$, where

$$\alpha=2\sigma+\frac{1}{2}\log g. \tag{3.34}$$

The conditions (3.33) will be called the *closure conditions*, since they simply express that the magnetic 1-form $\omega=\sum_{i=1}^n A_i dx^i$ should be closed. If the closure conditions are satisfied, then the potential V in the Schrödinger operator is given by

$$
\begin{aligned}
V &= -\frac{U}{2}\\
&= -\frac{c}{2}+\frac{1}{2}\sum_{i=1}^{n}\left[-A_i\,A^i+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,A^i\right)\right]. && (3.35)
\end{aligned}
$$

The fourth and final step required to classify the normalizable quasi-exactly solvable Schrödinger operators is to ensure that the image $\overline{\mathcal{N}}$ of the $\mathfrak{g}$-module $\mathcal{N}$ under the transformations (2.3) and (2.4) which map the eigenvalue problem (3.31) for the gauge Hamiltonian to the eigenvalue problem (3.32) for the physical Hamiltonian (3.3) is such that $\overline{\mathcal{N}} \subset L^2\left(\mathbb{R}^n; \sqrt{\bar{g}}\, d\bar{x}^1 \ldots d\bar{x}^n\right)$. Explicitly, this means that if $\{\bar{u}_1(\bar{x}), \ldots, \bar{u}_s(\bar{x})\}$ is any basis of $\overline{\mathcal{N}}$, then

$$\int_{\mathbb{R}^n} |\bar{u}_a|^2 \sqrt{\bar{g}}\, d\bar{x}^1 \ldots d\bar{x}^n < \infty, \quad 1 \le a \le s, \tag{3.36}$$

or equivalently

$$\int_{\mathbb{R}^n} |u_a(x)|^2 \, e^{\alpha(x)}\, dx^1 \ldots dx^n < \infty, \quad 1 \le a \le s \tag{3.37}$$

for a basis $\{u_1(x), \ldots, u_s(x)\}$ of $\mathcal{N}$. Note that the square-integrability conditions (3.36) or (3.37) are invariant under global diffeomorphisms $\varphi : \mathbb{R}^n \to \mathbb{R}^n$.

We conclude this section with an important remark. Since the closure conditions are extremely complicated to solve in full generality, it is highly desirable to simplify them whenever possible. A natural way of obtaining a subset of the set of solutions to the closure conditions (3.33) is to consider the *simplified closure conditions*

$$\sum_{i=1}^{n} \left(\xi^{ai} \frac{\partial \alpha}{\partial x^i} + \frac{\partial \xi^{ai}}{\partial x^i} \right) - 2\eta^a = k^a, \quad 1 \le a \le r, \tag{3.38}$$

where $(k^1, \ldots, k^r)$ is a constant vector and

$$C_a = \sum_{b=1}^{r} C_{ab}\, k^b. \tag{3.39}$$

It is important to observe that the simplified closure conditions are independent of the C_{ab}'s: therefore, a particular solution $\alpha(x^1, \ldots, x^n)$ of equations (3.38) will generate an infinity of solutions to the full closure conditions (3.33), with C_{ab} completely arbitrary and C_a given by (3.39). Introducing the volume form

$$\Omega = e^{\alpha}\, dx^1 \cdots dx^n, \tag{3.40}$$

equations (3.38) can be rewritten as

$$\mathcal{L}_{X^a} \Omega = (k^a + 2\,\eta^a)\, \Omega, \quad 1 \le a \le r, \tag{3.41}$$

where

$$X^a = \sum_{i=1}^{n} \xi^{ai} \frac{\partial}{\partial x^i} \tag{3.42}$$

denotes the projection of the generator T^a of $\mathfrak{g}$ into the set $\mathfrak{X}(\mathbb{R}^n)$ of vector fields on $\mathbb{R}^n$. In particular, if the cohomology is trivial (*i.e.* $\eta^a = 0$ for all $a = 1, \ldots, r$, or equivalently $\mathfrak{g} = \mathfrak{h}$) a solution of the simplified closure conditions with $k^a = 0$ for all a will yield a $\mathfrak{g}$-invariant volume form on $\mathbb{R}^n$. The advantage of formulating the simplified closure conditions in this way lies in the fact that very general conditions ensuring the existence of such an invariant volume form are known. For instance, if the action of $\mathfrak{g} = \mathfrak{h}$ on (an open subset of) $\mathbb{R}^n$ is transitive, it suffices that the isotropy subgroup of a generic point of $\mathbb{R}^n$—and hence of any point—be compact. In particular, since the isotropy subgroup of a point is always closed, it is automatically compact if $\mathfrak{g}$ itself is compact. Therefore, if $\mathfrak{g}$ is compact and acts transitively then the simplified closure conditions are automatically compatible. In the two-dimensional case, these two requirements are clearly met by the $\mathfrak{so}(3)$ algebra spanned by the vector fields

$$T^1 = y\,q - x\,p, \quad T^2 = x\,y\,p + (1+y^2)\,q, \quad T^3 = (1+x^2)\,p + x\,y\,q, \tag{3.43}$$

cf. [14]. Thus, without any explicit computations, we are assured of the existence of an infinite number of solutions to the full closure conditions for this algebra, with C_{ab} arbitrary and $C_a = 0$ for all $a = 1, \ldots, r$ (since $k^a = 0$ for all a), [17].

In the previous example, one can explicitly solve the simplified closure conditions and compute the gauge factor $\mu = e^\sigma$ from equation (3.34). When this is done, it is found that the finite-dimensional $\mathfrak{so}(3)$-module of smooth functions $\mathcal{N}$ is a subspace of $L^2\left(\mathbb{R}^n; \mu^2\sqrt{g}\,dx^1\cdots dx^n\right)$, and therefore all the solutions of the closure conditions obtained in this way give rise to *normalizable* quasi-exactly solvable Schrödinger operators. In fact, the simplified closure conditions can be explicitly solved for *all* the two-dimensional (complex) Lie algebras of first-order differential operators listed in reference [7]. However—with the only exception of $\mathfrak{so}(3)$ noted above—their solutions are found to generate quasi-exactly solvable Schrödinger operators that are *not* normalizable. Hence in the two-dimensional case the simplified closure conditions are of very limited use. It is however important to keep in mind that, even when the simplified closure conditions don't have any acceptable solutions, the *full* closure conditions (3.33) may be compatible and may give rise to normalizable quasi-exactly solvable Schrödinger operators, as we shall see in the next example.

Example 3.4 For each $n \in \mathbb{N}$, let $\mathfrak{g}_n \cong \mathfrak{sl}(3)$ be the Lie subalgebra of $\mathcal{D}(\mathbb{R}^2)$ generated by the first-order differential operators

$$\begin{aligned} &T^1 = p, \quad T^2 = q, \quad T^3 = xp, \quad T^4 = yp, \quad T^5 = xq, \\ &T^6 = yq, \quad T^7 = x^2p + xyq - nx, \quad T^8 = xyp + y^2q - ny. \end{aligned} \tag{3.44}$$

We shall show in Section 4 that the closure conditions (3.33) for the Lie algebra $\mathfrak{g}_n$ have a solution for every $n \in \mathbb{N}$. (For $n = 2$, this fact has been implicitly used in the previous example.) On the other hand, it is easy to show that the simplified closure conditions (3.38) don't have any solutions in this case.

Indeed, from the first two equations in (3.38) we obtain (dropping an irrelevant additive constant)

$$\alpha(x,y) = k^1x + k^2y. \tag{3.45}$$

Substituting this into the following four equations in (3.38) we get the identities

$$k^1x + 1 \;=\; k^3,$$

$$
\begin{aligned}
k^1 y &= k^4, \\
k^2 x &= k^5, \\
k^2 y + 1 &= k^6,
\end{aligned}
\tag{3.46}
$$

from which it follows that

$$
k^1 = k^2 = k^4 = k^5 = 0; \quad k^3 = k^6 = 1, \tag{3.47}
$$

whence

$$
\alpha = 0. \tag{3.48}
$$

But substituting the latter equation into the last two equations (3.38) we obtain

$$
\begin{aligned}
(3 - 2n)\, x &= k^7, \\
(3 - 2n)\, y &= k^8,
\end{aligned}
\tag{3.49}
$$

which can only be satisfied if $n = 3/2 \notin \mathbb{N}$. ■

4 New Quasi-Exactly Solvable Potentials in $\mathbb{R}^2$

In the preceding section, we presented the general procedure to be followed in order to determine and classify the normalizable quasi-exactly solvable potentials in $n \geq 2$ dimensions. We shall now implement this procedure in the case $n = 2$. This will enable us to obtain a wide array of new normalizable quasi-exactly solvable planar potentials besides the examples already known through the work of Shifman and Turbiner, [17]. Our choice to work in two dimensions is motivated by the fact that it is only for $n = 2$ that a complete classification is available of the Lie algebras of first-order differential operators admitting a finite-dimensional module of smooth functions [7].

Before we proceed to describe our new potentials, it is important to summarize the basic requirements on a Lie-algebraic operator T given by equation (3.26) in order for it to give rise to a normalizable quasi-exactly solvable potential in $\mathbb{R}^2$. The significance of these requirements will be illustrated with concrete examples.

The first requirement, which stems from Corollary 2.3, is that the metric (g_{ij}) defined in terms of the generators of the Lie algebra $\mathfrak{g}$ and the constants C_{ab} by (3.27) be globally Riemannian, that is we should have

$$
g^{11} > 0, \quad g^{11}\, g^{22} - (g^{12})^2 > 0, \tag{4.1}
$$

everywhere in $\mathbb{R}^2$. Indeed, if these conditions were violated, then the Schrödinger operator (2.21) would fail to have a principal symbol of the required negative signature, which is necessary for the physical interpretation of the eigenvalue problem of H as the non-relativistic Schrödinger equation of a particle on a two-dimensional

Riemannian manifold. A case in point is the following example considered by Shifman and Turbiner [17, eqn. (25)]:

$$T = x^2\,(1+x^2)\,\frac{\partial^2}{\partial x^2} - 2\,(1+x^2)\,(1+y^2)\,\frac{\partial^2}{\partial x\,\partial y} + y^2\,(1+y^2)\,\frac{\partial^2}{\partial y^2} + \cdots. \tag{4.2}$$

This differential operator is hyperbolic everywhere, instead of being elliptic.

The second requirement, which we now discuss at length, has to do with the issues of square integrability and boundary conditions. The fundamental requirement is of course that the eigenfunctions arising from the application of the algebraic method be normalizable, so that they can be physically interpreted in standard quantum-mechanical fashion as the particle's probability amplitude. Suppose that we have a Schrödinger operator (2.21) such that (g_{ij}) is positive-definite and has zero Gaussian curvature everywhere in $\mathbb{R}^2$. Thus there exist local coordinates $(\bar{x},\bar{y})$ in which H takes the form H_{flat} given by equation (2.22) with $n=2$:

$$H_{\text{flat}} = -\frac{1}{2}\left(\frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2}\right) + V(\bar{x},\bar{y}). \tag{4.3}$$

It may happen that the change of variables which puts the operator (2.21) in its normal "physical" form (4.3) will map the whole plane $\mathbb{R}^2$ onto an open proper subset (say a bounded open rectangle), as would be the case with a covering map. We will thus have to deal with the issue of boundary conditions for the wave functions in the $(\bar{x},\bar{y})$ coordinates at the boundary of this set. These boundary conditions will play a crucial role for the existence of admissible wave functions, since in general they will not be automatically satisfied by the eigenfunctions found by applying the "algebraic" method. Let us illustrate these ideas with two informative examples.

Example 4.1 Consider the Schrödinger operator

$$\begin{aligned} T \;=\; & A\,(1+x^2)^2\frac{\partial^2}{\partial x^2} + 2\,(1+x^2)\,(1+y^2)\frac{\partial^2}{\partial x\,\partial y} + B\,(1+y^2)^2\frac{\partial^2}{\partial y^2} \\ & + 2\,(1+x^2)[A\,(1-n)\,x - m\,y]\frac{\partial}{\partial x} \\ & + 2\,(1+y^2)[-n\,x + B\,(1-m)\,y]\frac{\partial}{\partial y} \\ & + A\,(n-1)n\,x^2 + 2\,m\,n\,x\,y + B\,(m-1)m\,y^2, \end{aligned} \tag{4.4}$$

where $A>0$ and $B>0$ are real parameters, m and n are nonnegative integers, and

$$AB > 1. \tag{4.5}$$

This operator will be shown below (see Section 4.1.4) to be quasi-exactly solvable, with constant potential and hidden symmetry algebra $\mathfrak{g} = \mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$. The contravariant metric components are given by

$$g^{11} = A\,(1+x^2)^2, \quad g^{12} = (1+x^2)\,(1+y^2), \quad g^{22} = B\,(1+y^2)^2, \tag{4.6}$$

so that the metric is globally Riemannian and has zero Gaussian curvature everywhere in $\mathbb{R}^2$. The coordinate transformation and rescaling of the wave function mapping the operator (4.4) to the normal form (4.3) are given by

$$\bar{x} = \frac{B \arctan x - \arctan y}{\sqrt{B\,(AB-1)}}, \quad \bar{y} = \frac{1}{\sqrt{B}} \arctan y; \tag{4.7}$$

$$\mu = \left(1+x^2\right)^{-\frac{n}{2}} \left(1+y^2\right)^{-\frac{m}{2}}. \tag{4.8}$$

The coordinate transformation (4.7) maps the whole plane onto a bounded open rectangle; therefore this example describes the physical situation of a free particle confined to a bounded rectangle. The natural boundary conditions in the $(\bar{x}, \bar{y})$ coordinates are that the eigenfunctions should vanish at the boundary of the rectangle, so that the particle has zero probability of getting to the boundary and escaping. From the form of the gauge factor μ as given by (4.8), and the fact that any finite-dimensional module of smooth functions for the Lie algebra $\mathfrak{g}$ is spanned by polynomials, we see that the eigenfunctions arising from the application of the algebraic method for quasi-exactly solvable problems are of the form

$$\psi(\bar{x}, \bar{y}) = \frac{P(x,y)}{(1+x^2)^{\frac{n}{2}}\,(1+y^2)^{\frac{m}{2}}}, \tag{4.9}$$

where P is a polynomial of degree less than or equal to n in x and less than or equal to m in y, and $(x,y) \in \mathbb{R}^2$ is mapped to $(\bar{x}, \bar{y})$ in the rectangle according to the local inverse of the diffeomorphism (4.7). The condition that $\psi\,(\bar{x}, \bar{y})$ vanish at the boundary of the rectangle, that is when x and y tend to infinity, should now be imposed as an additional requirement on the eigenfunctions of H, since it won't automatically hold in general. ∎

Example 4.2 Consider the Schrödinger operator given by

$$\begin{aligned} T \;=\; & \left(1+x^2\right)^2 \frac{\partial^2}{\partial x^2} + 2\,x\,y\,\left(1+x^2\right) \frac{\partial^2}{\partial x\,\partial y} + \left(1+x^2\right)\left(1+y^2\right) \frac{\partial^2}{\partial y^2} \\ & + 2\,(1-n)\,x\,\left(1+x^2\right) \frac{\partial}{\partial x} + 2\left(-n + (1-n)\,x^2\right) y \frac{\partial}{\partial y} \\ & + (n-1)\,n\,x^2 + n\,(n+1) + 1. \end{aligned} \tag{4.10}$$

We shall see in Section 4.3.2 that this differential operator is quasi-exactly solvable, with hidden symmetry algebra isomorphic to $\mathfrak{gl}(2) \ltimes \mathbb{R}^2$. The metric, whose contravariant components are given by

$$g^{11} = (1+x^2)^2, \quad g^{12} = x\,y\,(1+x^2), \quad g^{22} = (1+x^2)\,(1+y^2), \tag{4.11}$$

is again globally Riemannian with zero Gaussian curvature everywhere. The change of coordinates and rescaling which map the operator (4.10) to the normal form (4.3) are given by

$$\bar{x} = \arctan x, \quad \bar{y} = \operatorname{arcsinh} \frac{y}{\sqrt{1+x^2}}; \tag{4.12}$$

$$\mu = \frac{\sqrt{1+x^2}}{(1+x^2+y^2)^{\frac{1+n}{2}}}. \tag{4.13}$$

The coordinate transformation (4.12) maps the plane to an open infinite strip $(-\pi/2, \pi/2) \times \mathbb{R}$ in a locally diffeomorphic way. The eigenfunctions obtained by the algebraic method are of the form

$$\psi\,(\bar{x}, \bar{y}) = \frac{\cos^n \bar{x}}{\cosh^{n+1} \bar{y}}\, P(\tan \bar{x}, \sec \bar{x}\, \sinh \bar{y}), \tag{4.14}$$

where $P(x, y)$ is a polynomial of total degree n in (x, y). Again, the natural boundary conditions are that the eigenfunctions vanish at the boundary of the strip, that is

$$\psi\left(-\frac{\pi}{2}, \bar{y}\right) = \psi\left(\frac{\pi}{2}, \bar{y}\right) = 0, \quad \forall \bar{y} \in \mathbb{R}. \tag{4.15}$$

As in the previous example, only some of the eigenfunctions (4.14) will fulfill the boundary conditions (4.14). The potential $V(\bar{x}, \bar{y})$ is remarkably simple:

$$V(\bar{x}, \bar{y}) = -\frac{1}{2}\,(1+n)\,(2+n)\,\operatorname{sech}^2 \bar{y}. \tag{4.16}$$

It is independent of $\bar{x}$, and is a restricted Pöschl–Teller potential in $\bar{y}$.

It should be pointed out at this stage that the natural boundary conditions chosen above are by no means the only ones one could consider from a geometrical point of view. For instance, one could identify the sides of the strip in Example 4.2 and consider the motion of a particle on an infinite cylinder. The boundary conditions would then be periodic in $\bar{x}$, *i.e.*

$$\psi\left(-\frac{\pi}{2}, \bar{y}\right) = \psi\left(\frac{\pi}{2}, \bar{y}\right), \quad \forall \bar{y} \in \mathbb{R}. \tag{4.17}$$

■

The above examples show that in many cases the range of the "physical" coordinates is not all of $\mathbb{R}^2$, but an open subset $M \in \mathbb{R}^2$ with nonempty boundary ∂M. This happens typically in the case of metrics of zero Gaussian curvature—where there is a natural set of Cartesian coordinates—if the coordinate transformation mapping (x, y) to $(\bar{x}, \bar{y})$ is a covering map. In general, if there is a natural system of "physical" coordinates suggested by the geometry associated to the underlying Riemannian metric (say a metric of constant curvature on a hyperboloid), then one has to go through the additional steps of determining the range of these coordinates, choosing natural boundary conditions for the eigenfunctions, and determining which of the eigenfunctions obtained by the algebraic method fulfill these conditions along with square integrability. As shown by Examples 4.1 and 4.2, such a procedure is by no means unique (*e.g.* a cylinder *vs.* an infinite strip). In any case, we shall always require the gauge Hamiltonian (3.26) to be defined over all of $\mathbb{R}^2$ in the (x,y) coordinates, and to have a principal symbol of positive-definite signature everywhere.

The third requirement is that of irreducibility, namely that the gauge Hamiltonian (3.26) should always explicitly involve elements T^a of the hidden symmetry algebra $\mathfrak{g}$ which, together with their commutators $\left[T^a, T^b\right]$, span the entire Lie algebra $\mathfrak{g}$. Indeed, if the T^a's generated a proper subalgebra $\mathfrak{g}_0$, then we could have a situation where non-trivial cohomology classes for $\mathfrak{g}$ become trivial for $\mathfrak{g}_0$, and thus give rise to terms in the gauge Hamiltonian (3.26) which could be eliminated by a gauge transformation. Let us illustrate this by an example.

Example 4.3 Consider the Lie algebra spanned by the differential operators

$$T^1 = p + q, \quad T^2 = x\,p + y\,q, \quad T^3 = x^2\,p + y^2\,q + \frac{n}{2}\,(x - y), \tag{4.18}$$

where as before

$$p = \frac{\partial}{\partial x}, \quad q = \frac{\partial}{\partial y}. \tag{4.19}$$

This Lie algebra, which is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, corresponds to Case 12 in the classification [7] of Lie algebras of planar first-order differential operators admitting a finite-dimensional module of smooth functions. It admits a finite-dimensional module of smooth functions if and only if the cohomology parameter n is an integer constant. But the two-dimensional subalgebra $\mathfrak{g}_0$ spanned by T^2 and T^3 is isomorphic to the Lie algebra $\mathfrak{h}_2$—the unique non-abelian (solvable) two-dimensional Lie algebra—which has trivial cohomology under the assumption of the existence of a finite-dimensional module of smooth functions (see [7, Case 2]). ■

Before proceeding to present our list of new normalizable quasi-exactly solvable planar potentials, we should make some general comments on these potentials.

First of all, our list is by no means complete. We restricted ourselves to three Lie algebras out of the list of 24 obtained in [7], namely realizations of the Lie algebras $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(3,\mathbb{R})$, already considered by Shifman and Turbiner [17], and $\mathfrak{gl}(2,\mathbb{R}) \ltimes \mathbb{R}^{r+1}$, corresponding to Case 24 in the classification of reference [7]. For each Lie algebra $\mathfrak{g}$, we shall give below its realization in terms of first-order differential operators T^a, $1 \le a \le \dim \mathfrak{g}$, and then present the various parametrized families of normalizable quasi-exactly solvable potentials with a brief discussion of their salient features. As before, we shall use the notation

$$T = \sum_{a,b=1}^{r} C_{ab} T^a T^b + \sum_{a=1}^{r} C_a T^a + C_0, \quad r = \dim \mathfrak{g}. \tag{4.20}$$

Our choice was made after much trial and error with the other Lie algebras in our list, leading almost invariably to metrics which were not globally Riemannian, or to eigenfunctions which were not square integrable.

Secondly, the normalizable quasi-exactly solvable potentials we obtained initially were much more complicated than the ones we present below. We chose to specialize the values of some parameters in order to make the expressions more compact and tractable while retaining the essential properties of the general potentials. We should also point out that the new potentials we have obtained for the Lie algebras $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ all arise from solving the full closure conditions (3.33).

Thirdly, the proportionality between the potential and the Gaussian curvature according to

$$V = \frac{3}{16} K, \tag{4.21}$$

proved by Losev and Turbiner [11] for quasi-exactly solvable Hamiltonians associated to the hidden symmetry algebra $\mathfrak{su}(2)$ with zero cohomology, is in general not satisfied (even asymptotically). Specific counterexamples are given by many of the potentials in the list below.

However, we have found further evidence in favor of a conjecture of Turbiner [20] to the effect that the quasi-exactly solvable potentials on a flat background metric are separable in the "physical" coordinates, *i.e.* are of the form

$$V(\bar{x}, \bar{y}) = f(\bar{x}) + g(\bar{y}) \tag{4.22}$$

in suitable Cartesian coordinates $(\bar{x}, \bar{y})$.

4.1 $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$

There is, up to local diffeomorphisms and rescalings (2.3) and (2.4), a unique representation of this Lie algebra in terms of first-order planar differential operators admitting a finite module of smooth functions $\mathcal{N}$. This representation corresponds to Case 11 in the classification of reference [7], and is given explicitly by

$$\begin{gathered} T^1 = p, \quad T^2 = q, \quad T^3 = x\,p, \quad T^4 = y\,q, \\ T^5 = x^2\,p - n\,x, \quad T^6 = y^2\,q - m\,y, \end{gathered} \tag{4.23}$$

where the quantized cohomology parameters n and m are nonnegative integers. Any finite-dimensional $\mathfrak{g}$-module of smooth functions admits a basis of polynomials $P(x, y)$ whose degree in x is less than or equal to n and whose degree in y is less than or equal to m. The eigenfunctions arising from the application of the algebraic method to Hamiltonians admitting $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ as a hidden symmetry algebra are therefore of the form

$$\psi(x, y) = \mu(x, y)\, P(x, y), \tag{4.24}$$

where μ is the gauge factor and P is as above.

We have found the following parametrized families of solutions to the closure conditions (3.33):

4.1.1

The first example is the differential operator (4.20) defined by the following values of the coefficients C_{ab}, C_a, $1 \le a, b \le 6$:

$$(C_{ab}) = \begin{pmatrix} A & 1 & 0 & 0 & 0 & 1 \\ 1 & B & 0 & 0 & 1 & 0 \\ 0 & 0 & 1+A & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+B & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \tag{4.25}$$

$$(C_a) = \Big(0, 0, 1-(1+2\,n)\;A, 1-(1+2\,m)\;B, 0, 0\Big), \tag{4.26}$$

with C_0, A and B arbitrary parameters.

The contravariant metric coefficients are given by

$$g^{11} \;=\; \left(1+x^2\right)\,\left(A+x^2\right), \tag{4.27}$$

$$g^{12} \;=\; \left(1+x^2\right)\,\left(1+y^2\right), \tag{4.28}$$

$$g^{22} \;=\; \left(1+y^2\right)\,\left(B+y^2\right). \tag{4.29}$$

The metric will be positive definite provided that

$$A \ge 1, \quad B \ge 1, \quad AB > 1. \tag{4.30}$$

The gauge factor is

$$\mu = \frac{\left[(B-1)\,x^2 + (A-1)\,y^2 + AB - 1\right]^{\frac{1}{4}}}{(1+x^2)^{\frac{1+2\,n}{4}}\,(1+y^2)^{\frac{1+2\,m}{4}}}. \tag{4.31}$$

The Gaussian curvature is given by

$$\begin{aligned} (B-1)\,K \;=\;& A - B + (1+A-2\,B)\,y^2 \\ &+ 3\,(A-1)^2 D^{-2}\,(y^2+1)\,(y^2+B)\,\Big[AB-1 \\ &+ 2\,(B-1)\,x\,y \;+ (A-B)\,y^2\Big] \\ &+ (A-1)D^{-1}\,\Big[2 + B - 4\,A\,B + B^2 + (5\,B - 4\,A - 1)\,y^4 \\ &+ 2\,\Big(1 + 2\,B - 2\,A\,(1+B) + B^2\Big)\;y^2 \\ &+ (1-B)\,x\,y\,(6\,y^2 + B + 5)\Big], \quad B \ne 1, \end{aligned} \tag{4.32}$$

and

$$K = -(1+2\,x^2), \quad B = 1, \tag{4.33}$$

where

$$D = (B-1)\,x^2 + (A-1)\,y^2 + AB - 1. \tag{4.34}$$

The expression for the potential is even more cumbersome, so we shall not display it here. Instead, we present two particular instances of the above multiparameter family of quasi-exactly solvable Hamiltonians:

4.1.2

Substituting

$$A = B = 2 \tag{4.35}$$

in the previous example we obtain

$$(C_{ab}) = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \tag{4.36}$$

$$(C_a) = \Big(0, 0, -(1+4\,n), -(1+4\,m), 0, 0\Big). \tag{4.37}$$

We shall also take

$$C_0 = \tfrac{3}{4} + m^2 + n^2. \tag{4.38}$$

The contravariant metric coefficients reduce to

$$g^{11} = \left(1+x^2\right)\,\left(2+x^2\right), \tag{4.39}$$

$$g^{12} = \left(1+x^2\right)\,\left(1+y^2\right), \tag{4.40}$$

$$g^{22} = \left(1+y^2\right)\,\left(2+y^2\right). \tag{4.41}$$

The gauge factor and Gaussian curvature from the previous example simplify to the following expressions:

$$\mu = \frac{(3+x^2+y^2)^{\frac{1}{4}}}{(1+x^2)^{\frac{1+2\,n}{4}}\,(1+y^2)^{\frac{1+2\,m}{4}}}, \tag{4.42}$$

$$K = -y^2 - \frac{8+6\,y^2-y^4+x\,y\,\left(7+6\,y^2\right)}{3+x^2+y^2} + \frac{3\,\left(3+2\,x\,y\right)\,\left(1+y^2\right)\,\left(2+y^2\right)}{\left(3+x^2+y^2\right)^2}. \tag{4.43}$$

Finally, the potential is given by

$$\begin{aligned} 8\,V \;=\; & -y^2 - \frac{(1+2\,n)\,(3+2\,n)}{1+x^2} - \frac{(1+2\,m)\,(3+2\,m)}{1+y^2} \\ & - \frac{17+12\,y^2-y^4+2\,x\,y\,\left(6+5\,y^2\right)}{3+x^2+y^2} + \frac{5\,\left(3+2\,x\,y\right)\,\left(1+y^2\right)\,\left(2+y^2\right)}{\left(3+x^2+y^2\right)^2}. \end{aligned} \tag{4.44}$$

4.1.3

A quasi-exactly solvable potential similar to the previous one is obtained when we set

$$A = \frac{3}{2}, \quad B = 2 \tag{4.45}$$

in the general potential 4.1.1. We quote without further comment the expressions for the coefficients C_{ab}, C_a $(1 \le a, b \le 6)$ and C_0, the contravariant metric coefficients, the gauge factor and the Gaussian curvature:

$$(C_{ab}) = \begin{pmatrix} \frac{3}{2} & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{5}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \tag{4.46}$$

$$(C_a) = \left(0, 0, -\left(\tfrac{1}{2} + 3\,n\right), -(1 + 4\,m), 0, 0\right), \tag{4.47}$$

$$C_0 = \frac{1 + 2\,(2\,m^2 - \,n + n^2)}{4}; \tag{4.48}$$

$$g^{11} \;=\; \left(1 + x^2\right)\left(\frac{3}{2} + x^2\right), \tag{4.49}$$

$$g^{12} \;=\; \left(1 + x^2\right)\left(1 + y^2\right), \tag{4.50}$$

$$g^{22} \;=\; \left(1 + y^2\right)\left(2 + y^2\right); \tag{4.51}$$

$$\mu = \frac{(4 + 2\,x^2 + y^2)^{\frac{1}{4}}}{(1 + x^2)^{\frac{1+2\,n}{4}}\,(1 + y^2)^{\frac{1+2\,m}{4}}}. \tag{4.52}$$

$$\begin{aligned} K \;=\; & 1 - \frac{25 + 30\,x^2 + 6\,x^4 + 13\,x\,y + 12\,x^3\,y}{4 + 2\,x^2 + y^2} \\ & + \frac{6\,(1 + x^2)\,(3 + 2\,x^2)\,(4 + x^2 + 2\,x\,y)}{(4 + 2\,x^2 + y^2)^2}. \end{aligned} \tag{4.53}$$

The potential for this example is given by

$$\begin{aligned} 16\,V \;=\; & -4\,y^2 - \frac{(1 + 2\,n)\,(3 + 2\,n)}{1 + x^2} - \frac{2\,(1 + 2\,m)\,(3 + 2\,m)}{1 + y^2} \\ & - \frac{18 + 24\,x\,y + 2\,y^2 + 20\,x\,y^3 - 9\,y^4}{4 + 2\,x^2 + y^2} \\ & + \frac{5\,(4 + 4\,x\,y - y^2)\,(1 + y^2)\,(2 + y^2)}{(4 + 2\,x^2 + y^2)^2}. \end{aligned} \tag{4.54}$$

4.1.4

The following solution corresponds to the Schrödinger operator discussed in Example 4.1. We take

$$(C_{ab}) = \begin{pmatrix} A & 1 & 0 & 0 & 0 & 1 \\ 1 & B & 0 & 0 & 1 & 0 \\ 0 & 0 & 2\,A & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\,B & 0 & 0 \\ 0 & 1 & 0 & 0 & A & 1 \\ 1 & 0 & 0 & 0 & 1 & B \end{pmatrix}, \tag{4.55}$$

$$(C_a) = \Big(0, 0, -2\,A\,n, -2\,B\,m, 0, 0\Big), \tag{4.56}$$

and

$$C_0 = -\,(A\,n + B\,m). \tag{4.57}$$

The contravariant metric coefficients are given by

$$g^{11} = A\,(1+x^2)^2, \tag{4.58}$$
$$g^{12} = (1+x^2)\,(1+y^2), \tag{4.59}$$
$$g^{22} = B\,(1+y^2)^2. \tag{4.60}$$

The gauge factor is given by

$$\mu = \Big(1+x^2\Big)^{-\frac{n}{2}}\,\Big(1+y^2\Big)^{-\frac{m}{2}}. \tag{4.61}$$

The Gaussian curvature and the potential both vanish:

$$K = V = 0. \tag{4.62}$$

The Cartesian coordinates $(\bar{x}, \bar{y})$ are given by (4.7).

4.1.5

For the following solution, we take

$$(C_{ab}) = \begin{pmatrix} A & 1 & 0 & 0 & 0 & \lambda \\ 1 & B & 0 & 0 & 1 & 0 \\ 0 & 0 & 2\,A & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda\,B + C & 0 & 0 \\ 0 & 1 & 0 & 0 & A & \lambda \\ \lambda & 0 & 0 & 0 & \lambda & \lambda\,C \end{pmatrix}, \tag{4.63}$$

$$(C_a) = \Big(0, 0, -2\,A\,n, -\lambda\,B\,\,(1+2\,m) + C, 0, 0\Big), \tag{4.64}$$

and

$$
\begin{aligned}
C_0 \;=\; & \frac{1}{2}\,(\lambda - A\,C)^{-1}\,\Big[2\,\lambda^2\,m\,(1+m)\,B - 2\,\lambda\,m\,(2+m)\,C + 2\,n\,A^2\,C \\
& + A\,\Big(-2\,\lambda\,n - \lambda\,(1+2\,m+2\,m^2)\,B\,C + (1+4\,m+2\,m^2)\,C^2\Big)\Big],
\end{aligned}
\tag{4.65}
$$

where A, B, C and λ are nonnegative real parameters, and $\lambda \neq A\,C$.

The contravariant metric coefficients are given by

$$
g^{11} = A\,(1+x^2)^2, \tag{4.66}
$$

$$
g^{12} = (1+x^2)\,(1+\lambda\,y^2), \tag{4.67}
$$

$$
g^{22} = (1+\lambda\,y^2)\,(B+C\,y^2). \tag{4.68}
$$

The metric will be positive definite everywhere provided that

$$
A>0, \quad A\,B>1, \quad A\,C>\lambda. \tag{4.69}
$$

The gauge factor is given by

$$
\mu = \frac{[A\,B-1+(A\,C-\lambda)y^2]^{\frac{1}{4}}}{(1+x^2)^{\frac{n}{2}}\,(1+\lambda\,y^2)^{\frac{1+2\,m}{4}}}. \tag{4.70}
$$

The Gaussian curvature is given by

$$
K = \frac{C-\lambda B}{AC-\lambda}\left[-\lambda + \frac{2\,(2\lambda AB - AC - \lambda)}{AB-1+(AC-\lambda)y^2} + \frac{3A\,(1-AB)(\lambda B - C)}{(AB-1+(AC-\lambda)y^2)^2}\right]. \tag{4.71}
$$

It is a function of the second coordinate only, and is asymptotically constant.

The potential is given by

$$
\begin{aligned}
\frac{8\,V}{\lambda\,B-C} \;=\; & \frac{-(1+2\,m)\,(3+2\,m)}{1+\lambda\,y^2} + \frac{3\,\lambda - 6\,\lambda\,A\,B + 4\,A\,C - A^2\,B\,C}{(A\,C-\lambda)\,\left(A\,B-1+(A\,C-\lambda)\,y^2\right)} \\
& + \frac{5\,A\,(A\,B-1)\,(\lambda\,B-C)}{(A\,C-\lambda)\,\left(A\,B-1+(A\,C-\lambda)\,y^2\right)^2}.
\end{aligned}
\tag{4.72}
$$

The limiting case $\lambda\,B = C$ leads to a constant potential in a flat background, slightly more general than the one in Example 4.1.4.

4.2 $\mathfrak{sl}(3,\mathbb{R})$

This Lie algebra has a unique realization in terms of first-order planar differential operators admitting a finite-dimensional module of smooth functions, up to local diffeomorphisms and rescalings (2.3) and (2.4). This realization, given by

$$
\begin{gathered}
T^1 = p, \quad T^2 = q, \quad T^3 = x\,p, \quad T^4 = y\,p, \quad T^5 = x\,q, \quad T^6 = y\,q, \\
T^7 = x^2\,p + x\,y\,q - n\,x, \quad T^8 = x\,y\,p + y^2\,q - n\,y,
\end{gathered}
\tag{4.73}
$$

with the parameter n restricted to take nonnegative integer values, corresponds to Case 15 in the classification of reference [7]. The finite-dimensional module $\mathcal{N}$ consists of polynomials $P(x,y)$ of total degree in x and y less than or equal to n. The eigenfunctions obtained by applying the algebraic method to any Hamiltonian possessing $\mathfrak{sl}(3)$ as a hidden symmetry algebra must therefore be of the form (4.24), where μ is the gauge factor corresponding to each particular Hamiltonian, and P is as before.

We have found the following multiparameter family of Hamiltonians with $\mathfrak{sl}(3)$ hidden symmetry:

4.2.1

The principal symbol of the multiparameter family of Hamiltonians we shall present is generated only by the basis elements T^1, T^2, T^7 and T^8. However, since these operators form a generating set for the Lie algebra (4.73), this example meets the irreducibility requirement defined earlier in this Section.

The gauge Hamiltonian for this example is given by

$$\begin{aligned} -2\,H_{\mathrm{gauge}} \;&=\; A\left[(T^1)^2+\lambda\,(T^2)^2\right]+C\left[(T^7)^2+\lambda\,(T^8)^2\right] \\ &\quad +B\left[\left\{T^1,T^7\right\}+\left\{T^2,T^8\right\}+(3+2\,n)\,(T^3+T^6)\right], \end{aligned} \tag{4.74}$$

where $\left\{T^a,T^b\right\}=T^a\,T^b+T^b\,T^a$ denotes the anticommutator of the differential operators T^a and T^b, and the parameters A, B, C and λ satisfy the inequality

$$\delta\equiv\lambda\,A\,C-B^2\neq 0. \tag{4.75}$$

The contravariant metric coefficients are given by

$$g^{11} \;=\; A+x^2\,(B+\rho), \tag{4.76}$$
$$g^{12} \;=\; x\,y\,(B+\rho), \tag{4.77}$$
$$g^{22} \;=\; \lambda\,A+y^2\,(B+\rho), \tag{4.78}$$

where

$$\rho=B+C\left(\lambda\,x^2+y^2\right). \tag{4.79}$$

The metric will be positive definite everywhere if the parameters satisfy

$$A>0,\quad B\geq 0,\quad C>0,\quad \lambda>0. \tag{4.80}$$

The gauge factor is

$$\mu=\exp\left(\frac{3+2n}{4\sqrt{\delta}}\,\arctan\frac{B\,\rho}{\sqrt{\delta}}\right)\,(\rho^2+\delta)^{-\frac{1+2n}{8}}. \tag{4.81}$$

Hence the eigenfunctions (4.24) obtained applying the algebraic method will automatically be square integrable with respect to the measure $\sqrt{g}\,dx\,dy$.

The Gaussian curvature, which is particularly simple for this example,

$$K = -2\rho = -2\,\Big[B + C\,\Big(\lambda\,x^2 + y^2\Big)\Big]\,, \tag{4.82}$$

is negative everywhere.

Finally, the potential is given by

$$\begin{aligned} 8\,V \;\; = \;\; & -3\,\rho - (7 + 16\,n + 8\,n^2)\,B \\ & + (\rho^2 + \lambda\,A\,C - B^2)^{-1}\,\Big[-16(1+n)(2+n)B^3 \\ & + \lambda\,(23 + 36\,n + 12\,n^2)\,A\,B\,C \\ & + \rho\,\Big(16\,(1+n)\,(2+n)\,B^2 - \lambda\,(1 + 2\,n)\,(5 + 2\,n)\,A\,C\Big)\Big]\,. \end{aligned} \tag{4.83}$$

If we look for solutions of the Schrödinger equation with potential (4.83) depending only on the "radial" coordinate ρ, we end up with an effective one-dimensional Schrödinger operator H_ρ given by

$$-2\,H_\rho = f(\rho)\,\frac{\partial^2}{\partial\rho^2} + g(\rho)\,\frac{\partial}{\partial\rho} + h(\rho), \tag{4.84}$$

where

$$\begin{aligned} f(\rho) &= 4\,(B - \rho)\,\Big(-\rho^2 + B^2 - \lambda\,A\,C\Big)\,, \\ g(\rho) &= 2\,\Big[(3 - 2\,n)\,\rho^2 + 2\,(1 + 2\,n)\,\rho\,B - (5 + 2\,n)\,B^2 + 2\,\lambda\,A\,C\Big]\,, \\ h(\rho) &= n\,[(n-1)\,\rho - (n+1)\,B]\,. \end{aligned} \tag{4.85}$$

Comparing these formulas with [9, equation (20)], it is straightforward to show that the one-dimensional Schrödinger operator (4.84) does not admit $\mathfrak{sl}(2,\mathbb{R})$ as a hidden symmetry algebra. However, a subset of the eigenfunctions of (4.84) could in principle be computed algebraically by looking at the eigenfunctions of the two-dimensional quasi-exactly solvable Hamiltonian (4.74) arising from the algebraic method which are functions of ρ only (*cf.* Example 3.2). This suggests that the class of one-dimensional quasi-exactly solvable Schrödinger operators could be significantly enlarged via suitable reduction from two-dimensional quasi-exactly solvable Hamiltonians.

4.3 $\mathfrak{gl}(2,\mathbb{R}) \ltimes \mathbb{R}^{r+1}$

The last Lie algebra we shall deal with, which from the algebraic point of view is the semidirect sum of $\mathfrak{gl}(2,\mathbb{R})$ with a $(r+1)$-dimensional abelian ideal, corresponds to Case 24 in the classification of reference [7]. This class of Lie algebras, depending on a positive integer r, is uniquely realized, up to diffeomorphisms and rescalings (2.3) and (2.4), in terms of planar first-order differential operators possessing a finite module of smooth functions. Explicitly, this realization is given, for $r = 1$, by

$$T^1 = p, \quad T^2 = q, \quad T^3 = x\,p, \quad T^4 = x\,q, \quad T^5 = y\,q, \qquad T^6 = x^2\,p + r\,x\,y\,q - n\,x, \tag{4.86}$$

where n is a nonnegative integer. For $r > 1$ we must also include the differential operators

$$T^{6+i} = x^{i+1}\,q, \qquad 1 \le i \le r-1. \tag{4.87}$$

Any $\mathfrak{g}$-invariant finite-dimensional module of smooth functions $\mathcal{N}$ consists of polynomials $P(x,y)$ of the form

$$P(x,y) = \sum_{\substack{i,j\ge 0\\ i+r\,j\le n}} a_{ij}\,x^i\,y^j. \tag{4.88}$$

We have found the following multiparameter families of solutions for this Lie algebra:

4.3.1

For the first solution, we shall take

$$r = 1, \tag{4.89}$$

so that $\mathfrak{g}$ is 6-dimensional. The coefficients C_{ab} and C_a, $1 \le a, b \le 6$, have the following values:

$$(C_{ab}) = \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & C & 0 \\ 0 & 0 & 0 & \frac{B\,C}{A\,D}\,(E - 2\,C + D) & 0 & 0 \\ 0 & 0 & C & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C}{A}\,(E-C) \end{pmatrix}, \tag{4.90}$$

$$(C_a) = \Big(0, 0, E - 2\,(1+n)C, 0, C - 2\,(1+n)D, 0\Big). \tag{4.91}$$

For the moment, the parameter C_0 is arbitrary, and the other parameters in (4.90) and (4.91) will only be assumed to be nonnegative.

The contravariant metric coefficients are then

$$g^{11} = \frac{1}{A}\,\Big[A + (E-C)\,x^2\Big]\,(A + Cx^2), \tag{4.92}$$

$$g^{12} = \frac{C}{A}\,x\,y\,\Big[A + (E-C)\,x^2\Big], \tag{4.93}$$

$$g^{22} = \frac{C\,x^2}{A\,D}\,\Big[(E-C)D\,y^2 + B\,(E - 2\,C + D)\Big] + D\,y^2 + B. \tag{4.94}$$

This metric is positive definite everywhere provided that the parameters satisfy the additional conditions

$$A > 0, \quad B > 0, \quad E \geq C, \quad E + D \geq 2\,C. \tag{4.95}$$

The gauge factor is

$$\mu = \frac{[A + (E - C)x^2]^{\frac{1}{4}}\,[A\,D + C\,(E - 2\,C + D)x^2]^{\frac{1}{4}}}{(A\,B + B\,C\,x^2 + A\,D\,y^2)^{\frac{1+n}{2}}}. \tag{4.96}$$

Since $r = 1$, the eigenfunctions arising from the application of the algebraic method to this example are of the form (4.24), with P a polynomial of total degree in x and y less than or equal to n. Therefore, to ensure the automatic square integrability of the eigenfunctions we must impose the additional restrictions

$$C > 0, \quad D > 0. \tag{4.97}$$

The Gaussian curvature for this solution is:

$$\begin{aligned} K &= (E - 2\,C)\,C\left[A\,D + C\,(E - 2\,C + D)\,x^2\right]^{-2} \\ &\quad \times \left[A^2 D + 2A\,(E - 2\,C)\,(D - C)\,x^2 + C\,(C - E)\,(E - 2C + D)\,x^4\right]. \end{aligned} \tag{4.98}$$

The potential is too complicated to display in all generality. We shall limit ourselves to presenting two particular cases, instead:

4.3.2

If we set

$$E = 2\,C \tag{4.99}$$

in the previous solution, then (C_{ab}) and (C_a) reduce to

$$(C_{ab}) = \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\,C & 0 & C & 0 \\ 0 & 0 & 0 & \frac{B\,C}{A} & 0 & 0 \\ 0 & 0 & C & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C^2}{A} \end{pmatrix}, \tag{4.100}$$

and

$$(C_a) = \Big(0, 0, -2\,n\,C, 0, C - 2\,(1 + n)\,D, 0\Big). \tag{4.101}$$

We shall also set

$$C_0 = -n\,C + (n + 1)^2\,D. \tag{4.102}$$

The metric, whose contravariant components are given by

$$\begin{aligned} g^{11} &= \frac{1}{A}\,(A+C\,x^2)^2, && (4.103)\\ g^{12} &= \frac{C}{A}\,x\,y\,(A+C\,x^2), && (4.104)\\ g^{22} &= B+\frac{C}{A}\,x^2\,(B+C\,y^2)+D\,y^2, && (4.105) \end{aligned}$$

is everywhere positive definite and flat. In fact, this solution is a slight generalization of Example 4.2.

The gauge factor is

$$\mu = \frac{\sqrt{A+C\,x^2}}{(A\,B+B\,C\,x^2+A\,D\,y^2)^{\frac{1+n}{2}}}. \tag{4.106}$$

The Cartesian coordinates $(\bar{x},\bar{y})$ in which the Schrödinger operator defined by (4.100)–(4.102) adopts its normal form (2.22) are

$$\begin{aligned} \bar{x} &= \frac{1}{\sqrt{A\,C}}\,\arctan\sqrt{C/A}\,x, && (4.107)\\ \bar{y} &= \frac{1}{\sqrt{D}}\,\operatorname{arcsinh}\frac{\sqrt{A\,D}\,y}{\sqrt{B\,(A+C\,x^2)}}. && (4.108) \end{aligned}$$

Therefore, as in Example 4.2, the range of the "physical" coordinates $(\bar{x},\bar{y})$ is a strip $(\bar{x}_{\min},\bar{x}_{\max})\times\mathbb{R}$. Hence, as discussed in Example 4.2, additional boundary conditions will have to be imposed on the eigenfunctions yielded by the algebraic method.

The potential in the physical coordinates is strikingly simple:

$$V = -\frac{1}{2}\,(1+n)\,(2+n)\,D\,\operatorname{sech}^2\left(\sqrt{D}\,\bar{y}\right). \tag{4.109}$$

It is independent of $\bar{x}$, and (as in Example 4.2) is simply a Pöschl–Teller potential in $\bar{y}$. Notice that this potential satisfies the conjecture of Turbiner, according to which a quasi-exactly solvable Hamiltonian on a flat manifold in more than one dimension should be separable.

Taking into account equations (4.106) and (4.108), the algebraically computable eigenfunctions expressed in terms of the physical coordinates are of the form

$$\psi(\bar{x},\bar{y}) = \frac{\cos^n\left(\sqrt{AC}\,\bar{x}\right)}{\cosh^{n+1}\left(\sqrt{D}\,\bar{y}\right)}\,P(x,y), \tag{4.110}$$

with

$$\begin{aligned} x &= \sqrt{A/C}\,\tan\left(\sqrt{AC}\,\bar{x}\right), && (4.111)\\ y &= \sqrt{B/D}\,\sinh\left(\sqrt{D}\,\bar{y}\right)\sec\left(\sqrt{AC}\,\bar{x}\right), && (4.112) \end{aligned}$$

and $P(x,y)$ a polynomial of total degree in x and y not exceeding n.

4.3.3

In this example, which is again based on 4.3.1, we set

$$E = 2\,C - D, \tag{4.113}$$

from which it follows that

$$(C_{ab}) = \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\,C - D & 0 & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C}{A}\,(C-D) \end{pmatrix}, \tag{4.114}$$

$$(C_a) = \Big(0, 0, -2\,n\,C - D, 0, C - 2\,(1+n)D, 0\Big). \tag{4.115}$$

We shall also take

$$C_0 = \frac{1}{4}\,\Big[(1+2\,n)\,(3+2\,n)\,D - 2\,(1+2\,n)\,C\Big]. \tag{4.116}$$

Notice that the matrix (4.114) is degenerate, reflecting the fact that the gauge Hamiltonian (4.20) does not depend on the fourth generator of $\mathfrak{g}$ in this example. The irreducibility condition introduced earlier in this Section is however satisfied, since the remaining generators and their commutators are easily seen to span $\mathfrak{g}$.

The contravariant components of the metric are:

$$g^{11} = \frac{1}{A}\,(A + Cx^2)\,\Big[A + (C-D)\,x^2\Big], \tag{4.117}$$

$$g^{12} = \frac{C}{A}\,x\,y\,\Big[A + (C-D)\,x^2\Big], \tag{4.118}$$

$$g^{22} = B + \frac{C}{A}\,(C-D)\,x^2\,y^2 + D\,y^2. \tag{4.119}$$

The Gaussian curvature simplifies to

$$K = -\frac{C}{A}\,\Big[A + 2\,(C-D)\,x^2\Big], \tag{4.120}$$

and is therefore negative everywhere, by (4.95), (4.97) and (4.113).

The gauge factor reduces to

$$\mu = \frac{[A + (C-D)\,x^2]^{\frac{1}{4}}}{(A\,B + B\,C\,x^2 + A\,D\,y^2)^{\frac{1+n}{2}}}. \tag{4.121}$$

Finally, the potential is given by

$$-8\,V = \frac{4\,(1+n)\,(2+n)\,A\,B\,D}{A\,B + B\,C\,x^2 + A\,D\,y^2} + \frac{A\,D}{(C-D)\,x^2 + A} + \frac{3\,C}{A}(C-D)\,x^2. \tag{4.122}$$

Notice that for this example the potential and the curvature are indeed asymptotically proportional, according to (4.21).

4.3.4

A qualitatively similar solution to the previous one, also based on 4.3.1, is obtained by setting

$$E = D. \tag{4.123}$$

We then have

$$(C_{ab}) = \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & D & 0 & C & 0 \\ 0 & 0 & 0 & 2\,\frac{B\,C}{A\,D}\,(D-C) & 0 & 0 \\ 0 & 0 & C & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C}{A}(D-C) \end{pmatrix}, \tag{4.124}$$

$$(C_a) = \Big(0, 0, -2\,(1+n)\,C + D, 0, C - 2\,(1+n)D, 0\Big), \tag{4.125}$$

and we shall also set

$$C_0 = \frac{1}{4}\Big[2\,(7 + 10\,n + 4\,n^2)C - (11 + 16\,n + 4\,n^2)D\Big] \tag{4.126}$$

The contravariant metric coefficients are then given by

$$\begin{aligned} g^{11} &= \frac{1}{A}\,(A + C\,x^2)\,\Big[A + (D-C)\,x^2\Big], \\ g^{12} &= \frac{C}{A}\,x\,y\,\Big[A + (D-C)\,x^2\Big], \\ g^{22} &= B + D\,y^2 + \frac{C}{A\,D}\,x^2\,(D-C)\,(2\,B + D\,y^2). \end{aligned} \tag{4.127}$$

The gauge factor is

$$\mu = \frac{[A + (D-C)\,x^2]^{\frac{1}{4}}\;[A\,D + 2\,C\,(D-C)\,x^2]^{\frac{1}{4}}}{(A\,B + B\,C\,x^2 + A\,D\,y^2)^{\frac{1+n}{2}}}. \tag{4.128}$$

The Gaussian curvature, which is given by

$$K = C - \frac{D}{2} + \frac{2\,A\,(D-2\,C)\,(D-C)}{A\,D + 2\,C\,(D-C)\,x^2} - \frac{3\,A^2\,D\,(D-2\,C)^2}{2\,[A\,D + 2\,C\,(D-C)\,x^2]^2}, \tag{4.129}$$

is asymptotically constant (positive or negative, but not zero, since in that case it's easy to see that the irreducibility condition would be violated).

The explicit expression for the potential is

$$\begin{aligned} \frac{16\,V}{A} &= \frac{2\,(D-2\,C)}{A + (D-C)\,x^2} + \frac{2\,(D-2\,C)\,(5\,D - 6\,C)}{A\,D + 2\,C\,(D-C)\,x^2} + \frac{5\,A\,D\,(2\,C - D)^2}{[A\,D + 2\,C\,(D-C)\,x^2]^2} \\ &\quad + \frac{8\,(1+n)\,(2+n)\;[B\,(D-2\,C) + 2\,D(D-C)\,y^2]}{A\,B + B\,C\,x^2 + A\,D\,y^2}. \end{aligned} \tag{4.130}$$

4.3.5

In this solution, in which we no longer restrict ourselves to taking $r = 1$, the gauge Hamiltonian is given by

$$\begin{aligned}
-2\,H_{\text{gauge}} \;=\;& B(T^1)^2 + \frac{A}{r}\,(1+m)\,\left\{T^1, T^6\right\} + \frac{A}{r}\,(r-2m-2)\,(T^3)^2 \\
&+ A\,(1+m)^2\,(T^5)^2 + \sum_{a,b\in\{2,4,7,\ldots,m+5\}} C_{ab}\,T^a\,T^b \\
&+ A\left[1 + 2\,(1+m)\,\left(\frac{n}{r} - \lambda\right)\right] T^3 - 2\,A\,\lambda\,(1+m)^2\,T^5 \\
&+ A(1+m)\,\left[\frac{n}{r} - \lambda + (1+m)\,\lambda^2\right]. \qquad (4.131)
\end{aligned}$$

The coefficients C_{ab} with $a, b \in \{2, 4, 7, \ldots, m+5\}$ (*i.e.* those associated to the generators of the form $x^k\,q$, $0 \le k \le m$) are chosen in such a way that

$$\sum_{a,b\in\{2,4,7,\ldots,m+5\}} C_{ab}\,T^a\,T^b = \nu\,(A\,x^2 + B)^m\,\frac{\partial^2}{\partial y^2}, \qquad (4.132)$$

where ν is an arbitrary real constant, and the positive integer $m \le r$ satisfies the inequality

$$r \neq 2\,(m+1). \qquad (4.133)$$

The metric, whose contravariant components are given by

$$\begin{aligned}
g^{11} &= A\,x^2 + B, & (4.134)\\
g^{12} &= (1+m)\,A\,x\,y, & (4.135)\\
g^{22} &= \nu\,(A\,x^2 + B)^m + (1+m)^2\,A\,y^2, & (4.136)
\end{aligned}$$

is positive definite everywhere provided that the parameters A, B and ν are positive. The gauge factor is

$$\mu = \left[\nu\,\left(A\,x^2 + B\right)^{1+m} + A\,B\,(1+m)^2\,y^2\right]^{-\lambda/2}. \qquad (4.137)$$

Hence the algebraic eigenfunctions are square integrable with respect to the measure $\sqrt{g}\,dx\,dy$ for large enough values of the parameter λ. The curvature is negative and constant,

$$K = -A < 0. \qquad (4.138)$$

The potential is given by

$$V = -\frac{1}{2}\,\frac{A\,B\,\nu\,(1+m)^2\,\lambda\,(1+\lambda)\,\left(A\,x^2 + B\right)^m}{\nu\,(A\,x^2 + B)^{1+m} + A\,B\,(1+m)^2\,y^2}. \qquad (4.139)$$

Since the potential does not depend on the cohomology parameter n, the above Hamiltonian is exactly solvable (in the sense of reference [18]). Notice, furthermore, that the potential is also independent of r. We have thus constructed an exactly solvable Hamiltonian associated to an infinite number of inequivalent Lie algebras, *e.g.* the Lie algebras spanned by (4.86) with $r = m, m+1, \ldots$ (and $r \neq 2\,(m+1)$). This underscores the essential nonuniqueness of the correspondence between exactly or quasi-exactly solvable Hamiltonians and hidden symmetry algebras.

Since the metric in this instance has constant negative curvature, it of interest to pass to a system of coordinates in which the metric adopts one of its well known normal forms. For example, in the coordinate system (X, Y) defined by

$$\begin{aligned} x &= \sqrt{\frac{B}{A}}\,\frac{X}{Y}, \quad y = \frac{1}{2}\sqrt{\frac{\nu B^m}{A\,(1+m)^2}}\,Y^{-m-1}\,(1 - R^{2\,m+2}), \\ R &= \sqrt{X^2 + Y^2}; \qquad -\infty < X < \infty, \quad 0 < Y < \infty \end{aligned} \tag{4.140}$$

the metric (4.134)–(4.136) becomes

$$ds^2 = \frac{dX^2 + dY^2}{A\,Y^2}. \tag{4.141}$$

This is Poincaré's famous half-plane model for a manifold of constant negative curvature $-A$. In the new (X, Y) coordinates, the gauge factor and potential are respectively given by

$$\mu = \frac{Y^{(m+1)\,\lambda}}{(1 + R^{2\,m+2})^{\lambda}}, \tag{4.142}$$

and

$$V = -2\,A\,\lambda\,(1+\lambda)\,(1+m)^2\,\frac{Y^2\,R^{2\,m}}{(1 + R^{2\,m+2})^2}. \tag{4.143}$$

One can find, in a totally analogous fashion, a change of coordinates mapping the metric with contravariant components (4.134)–(4.136) to the standard constant negative curvature metric $A^{-1}\,(d\tau^2 + \sinh^2\tau\,d\varphi^2)$ on the two-sheeted hyperboloid. However, the formulas in this case are a little more complicated, so we won't display them here.

Perhaps the most natural coordinate system for this potential is the one defined on the infinite strip $(-\pi/2, \pi/2) \times \mathbb{R}$ by

$$\begin{gathered} x = \sqrt{B/A}\,\tan u, \quad y = \sqrt{\frac{\nu B^m}{A\,(1+m)^2}}\,\sec^{m+1} u\,\sinh(m+1)\,v; \\ -\frac{\pi}{2} < u < \frac{\pi}{2}, \quad -\infty < v < +\infty. \end{gathered} \tag{4.144}$$

The expression of the metric in the (u, v) coordinates is then

$$ds^2 = A^{-1}\sec^2 u\,(du^2 + dv^2). \tag{4.145}$$

This conformally flat version of a constant negative curvature metric is briefly discussed in reference [4]. In the new coordinates, the gauge factor becomes

$$\mu = \cos^{(m+1)\,\lambda} u\,\operatorname{sech}^{\lambda}(m+1)\,v \tag{4.146}$$

and the potential is

$$V = -\frac{1}{2}\, A\,\lambda\,(1+\lambda)\,(1+m)^2\,\cos^2 u\ \mathrm{sech}^2(m+1)\,v. \tag{4.147}$$

The Schrödinger equation is separable in the (u,v) coordinate system. Indeed, from (4.145) it follows that

$$\triangle = A\,\cos^2 u\,\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right), \tag{4.148}$$

and therefore the Schrödinger equation $H\,\psi = E\,\psi$ is easily seen to separate into the two one-dimensional boundary value problems

$$\begin{gathered} -\frac{1}{2}\,\mathcal{U}''(u) - \frac{E}{A}\,\sec^2 u\ \mathcal{U}(u) = -\beta\,\mathcal{U}(u), \\ \int_{-\pi/2}^{\pi/2} \sec^2 u\ |\mathcal{U}(u)|^2\ du < \infty \end{gathered} \tag{4.149}$$

and

$$\begin{gathered} -\frac{1}{2}\,\mathcal{W}''(w) - \frac{1}{2}\,\lambda\,(\lambda+1)\ \mathrm{sech}^2\,w\,\mathcal{W}(w) = \beta\,\mathcal{W}(w), \\ \int_{-\infty}^{\infty} |\mathcal{W}(w)|^2\ dw < \infty, \end{gathered} \tag{4.150}$$

where β is the separation constant and we have set $w = (m+1)\,v$. The potential in (4.150) is a restricted Pöschl–Teller potential; in particular, when $\lambda = N$ is a positive integer we obtain a N-soliton reflectionless potential.

Finally, notice that in the coordinate system (4.144) the "algebraic" square-integrable eigenfunctions of H automatically satisfy the natural boundary condition of vanishing on the boundary $|u| = \pi/2$. Indeed, since the metric volume form is simply $A^{-1}\,\sec^2\,du\,dv$, square integrability of $\psi(u,v)$ implies that that

$$\int_{-\pi/2}^{\pi/2} \sec^2 u\,|\psi|^2\,du < \infty. \tag{4.151}$$

Since, by (4.88), (4.146), and (4.144), the algebraic eigenfunctions are easily seen to be of the form

$$\psi(u,v) = \cos^\beta u\,\phi(u,v) \tag{4.152}$$

with $\phi(u,v)$ smooth, (4.151) immediately yields $\beta > 1/2$, and therefore ψ vanishes at $u = \pm\pi/2$.

References

[1] Y. Alhassid, J. Engel, and J. Wu, Algebraic approach to the scattering matrix, *Phys. Rev. Lett.* **53** (1984), 17–20.

[2] Y. Alhassid, F. Gürsey, and F. Iachello, Group theory approach to scattering, *Ann. Phys.* **148** (1983), 346–380.

[3] Y. Alhassid, F. Gürsey, and F. Iachello, Group theory approach to scattering. II. The euclidean connection, *Ann. Phys.* **167** (1986), 181–200.

[4] N.L. Balazs and A. Voros, Chaos on the pseudosphere, *Physics Reports* **143** (1986), 109–240.

[5] É. Cotton, Sur les invariants différentiels de quelques équations linéaires aux dérivées partielles du second ordre, *Ann. École Norm.* **17** (1900), 211–244.

[6] A. González–López, N. Kamran, and P.J. Olver, Lie algebras of differential operators in two complex variables, *American J. Math.*, to appear.

[7] A. González–López, N. Kamran, and P.J. Olver, Quasi-exactly solvable Lie algebras of first order differential operators in two complex variables, *J. Phys.* **A 24** (1991), 3995–4008.

[8] A. Gorsky, Relationship between exactly solvable and quasi-exactly solvable versions of quantum mechanics with conformal block equations in 2D theories, *JETP Lett.* **54** (1991), 289–292.

[9] N. Kamran and P.J. Olver, Equivalence of differential operators, *SIAM J. Math. Anal.* **20** (1989), 1172–1185.

[10] N. Kamran and P.J. Olver, Lie algebras of differential operators and Lie-algebraic potentials, *J. Math. Anal. Appl.* **145** (1990), 342–356.

[11] A. Losev and A. Turbiner, Multidimensional exactly solvable problems in quantum mechanics and pullbacks of affine co-ordinates on the Grassmannian, preprint, CERN-TH.5837/90, 1990.

[12] R.D. Levine, Lie-algebraic approach to molecular structure and dynamics, *in* "Mathematical Frontiers in Computational Chemical Physics" (D.G. Truhlar, Ed.), IMA Volumes in Mathematics and its Applications, Vol. 15, Springer-Verlag, New York, 1988, pp. 245–261.

[13] W. Miller, Jr., "Lie Theory and Special Functions," Academic Press, New York, 1968.

[14] A.Y. Morozov, A.M. Perelomov, A.A. Rosly, M.A. Shifman, and A.V. Turbiner, Quasi-exactly solvable quantal problems: one-dimensional analogue of rational conformal field theories, *Int. J. Mod. Phys.* **5** (1990), 803–832.

[15] M.A. Shifman, New findings in quantum mechanics (partial algebraization of the spectral problem), *Int. J. Mod. Phys.* **A 126** (1989), 2897–2952.

[16] M.A. Shifman, Quasi-exactly solvable spectral problems and conformal field theory, preprint, Theoretical Physics Inst., Univ. of Minnesota, 1992.

[17] M.A. Shifman and A.V. Turbiner, Quantal problems with partial algebraization of the spectrum, *Commun. Math. Phys.*, **126** (1989), 347–365.

[18] A.V. Turbiner, Quasi-exactly solvable problems and $\mathfrak{sl}(2)$ algebra, *Commun. Math. Phys.* **118** (1988), 467–474.

[19] A.V. Turbiner, Lie algebraic approach to the theory of polynomial solutions, preprint, CERN, CPT-92/P.2679, 1992.

[20] A.V. Turbiner, personal communication.

[21] A.G. Ushveridze, Quasi-exactly solvable models in quantum mechanics, *Sov. J. Part. Nucl.* **20** (1989), 504–528.