QUASI-EXACT SOLVABILITY

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Abstract. This paper surveys recent work on quasi-exactly solvable Schrödinger operators and Lie algebras of differential operators.

1. Introduction.

Lie algebraic and Lie group theoretic methods have played a significant role in the development of quantum mechanics since its inception. In the classical applications, the Lie group appears as a symmetry group of the Hamiltonian operator, and the associated representation theory provides an algebraic means for computing the spectrum. Of particular importance are the exactly solvable problems, such as the harmonic oscillator or the hydrogen atom, whose point spectrum can be completely determined using purely algebraic methods. The fundamental concept of a "spectrum generating algebra" was introduced by Arima and Iachello, [4], [5], to study nuclear physics, and subsequently, by Iachello, Alhassid, Gürsey, Levine, Wu and their collaborators, was also successfully applied to molecular dynamics and spectroscopy, [19], [22], and scattering theory, [1], [2], [3]. The Schrödinger operators amenable to the algebraic approach assume a "Lie algebraic form", meaning that they belong to the universal enveloping algebra of the spectrum generating algebra. Lie algebraic operators reappeared in the discovery of Turbiner, Shifman, Ushveridze, and their collaborators, [26], [28], [29], [33], of a new class of physically significant spectral problems, which they named "quasi-exactly solvable", having the property that a (finite) part of the point spectrum can be determined using purely algebraic methods. This is an immediate consequence of the additional requirement that the hidden symmetry algebra preserve a finite-dimensional representation space consisting of smooth wave functions. In this case, the Hamiltonian restricts to a linear transformation on the representation space, and hence the associated eigenvalues can be computed by purely algebraic methods, meaning matrix eigenvalue calculations. Finally, one must decide the "normalizability" problem of whether the resulting "algebraic" eigenfunctions are square integrable and therefore represent true bound states of the system. Connections with conformal field

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theory, [15], [25], [27], and the theory of orthogonal polynomials, [30], [31], [32], lend additional impetus for the study of such problems.

In this paper we will survey recent work done by the authors in the mathematical study of Lie algebraic and quasi-exactly solvable Schrödinger operators. Our principal focus has been in the classification of quasi-exactly solvable problems, especially those of physical importance. The one-dimensional case is relatively easy for two important reasons. First, there is just a single discrete family of finite-dimensional (quasi-exactly solvable) Lie algebras of first order differential operators on the line, which are well-known realizations of the $\mathfrak{sl}(2, \mathbb{R})$ algebra. Second, every second order differential operator on $\mathbb{R}$ can, up to an overall sign, be (locally) mapped to a Schrödinger operator. The determination of explicit normalizability conditions on the resulting operators and Lie algebraic coefficients relies on an unusual combination of elementary asymptotics and classical invariant theory. Therefore, in one dimension, we can determine a complete list of quasi-exactly solvable potentials and explicit normalizability conditions.

In higher dimensions, much less is known. Only a few special examples of quasi-exactly solvable problems in two dimensions have appeared in the literature to date, [28], all of which are constructed using semi-simple Lie algebras. Complete lists of finite-dimensional Lie algebras of differential operators are known in two (complex) dimensions; there are essentially 24 different classes, some depending on parameters. The quasi-exactly solvable condition imposes a remarkable quantization constraint on the cohomology parameters classifying these Lie algebras. This phenomenon of the "quantization of cohomology" has recently been been given an algebro-geometric interpretation, [9]. Any of the resulting quasi-exactly solvable Lie algebras of differential operators can be used to construct new examples of two-dimensional quasi-exactly solvable spectral problems. An additional complication is that, in higher dimensions, not every elliptic second order differential operator is equivalent to a Schrödinger operator (*i.e.*, minus Laplacian plus potential), so not every Lie algebraic operator can be assigned an immediate physical meaning. The resulting "closure conditions" are quite complicated to solve, and so the problem of completely classifying quasi-exactly solvable Schrödinger operators in two dimensions appears to be too difficult to solve in full generality. A variety of interesting examples are given in [14], and we present a few particular cases of interest here.

2. Quasi-Exactly Solvable Schrödinger Operators.

Let M denote an open subset of Euclidean space $\mathbb{R}^n$ with coordinates $x = (x^1, \ldots, x^n)$. The time-independent Schrödinger equation for a differential operator $\mathcal{H}$ is the eigenvalue problem

$$\mathcal{H}[\psi] = \lambda\psi. \tag{1}$$

In the quantum mechanical interpretation, a (self-adjoint) differential operator $\mathcal{H}$ plays the role of the quantum "Hamiltonian" of the system. A nonzero wave function $\psi(x)$ is called *normalizable* if it is square integrable, *i.e.*, lies in the Hilbert space $\mathrm{L}^2(\mathbb{R}^n)$, and so represents a physical bound state of the quantum mechanical system, the corresponding eigenvalue determining the associated energy level. While it is of great interest to know the bound states and energy levels of a given operator, complete explicit lists of eigenvalues and eigenfunctions are known for only a handful of classical "exactly solvable" operators, such as the harmonic oscillator. For the vast majority of quantum mechanical problems, the spectrum can, at best, only be approximated by numerical computation. The quasi-exactly solvable systems occupy an important intermediate station, in that a finite part of the spectrum can be computed by purely algebraic means.

To describe the general form of a quasi-exactly solvable problem, we begin with a finite-dimensional Lie algebra $\mathfrak{g}$ spanned by r linearly independent first-order differential operators

$$T^a = \sum_{i=1}^{n} \xi^{ai}(x)\,\frac{\partial}{\partial x^i} + \eta^a(x), \qquad a = 1, \ldots, r, \tag{2}$$

whose coefficients ξ^{ai}, η^a are smooth functions of x. The Lie algebra assumption requires that the commutator between two such operators can be written as a linear combination of the operators: $[T^a, T^b] = T^aT^b - T^bT^a = \sum_c C^c_{ab}T^c$, where the C^c_{ab} are the structure constants of the Lie algebra $\mathfrak{g}$. Note that each differential operator is a sum, $T^a = \mathbf{v}^a + \eta^a$, of a *vector field* $\mathbf{v}^a = \sum \xi^{ai}\partial/\partial x^i$ (which may be zero) and a *multiplication operator* η^a.

A differential operator is said to be *Lie algebraic* if it lies in the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$, meaning that it can be expressed as a polynomial in the operators T^a. In particular, a second order differential operator is Lie algebraic if it can be written as a quadratic combination

$$-\mathcal{H} = \sum_{a,b} c_{ab}T^aT^b + \sum_a c_aT^a + c_0, \tag{3}$$

for certain constants c_{ab}, c_a, c_0. (The minus sign in front of the Hamiltonian is taken for later convenience.) If some of the operators T^a generating the Lie algebra are pure multiplication operators, then one could allow higher degree combinations in (3); however, it is not hard to show that such Lie algebraic operators can always be re-expressed in a quadratic form, (3), for some possibly larger Lie algebra $\mathfrak{g}$, and so we are not losing any generality with the form (3). Note that the commutator $[T^c, \mathcal{H}]$ of the Hamiltonian with any generator of $\mathfrak{g}$, while still of the same Lie algebraic form, is not in general a multiple of the Hamiltonian $\mathcal{H}$ (unless $\mathcal{H}$ happens to be a Casimir for $\mathfrak{g}$). Therefore, the "hidden symmetry algebra" $\mathfrak{g}$ is not a symmetry algebra in the traditional sense. Lie algebraic operators appeared in the early work of Iachello, Levine, Alhassid, Gürsey and collaborators in the algebraic approach to scattering theory, [1], [2], [22].

The condition of quasi-exact solvability imposes an additional constraint on the Lie algebra and hence on the type of operators which are allowed. A Lie algebra of first order differential operators $\mathfrak{g}$ will be called *quasi-exactly solvable* if it possesses a finite-dimensional representation space (or module) $\mathcal{N} \subset C^\infty$ consisting of smooth functions; this means that if $\psi \in \mathcal{N}$ and $T^a \in \mathfrak{g}$, then $T^a(\psi) \in \mathcal{N}$. A differential operator $\mathcal{H}$ is called *quasi-exactly solvable* if it lies in the universal enveloping algebra of a quasi-exactly solvable Lie algebra of differential operators. Clearly, the module $\mathcal{N}$ is an invariant space for the Hamiltonian $\mathcal{H}$, *i.e.*, $\mathcal{H}(\mathcal{N}) \subset \mathcal{N}$, and hence $\mathcal{H}$ restricts to a linear matrix operator on $\mathcal{N}$. We will call the eigenvalues and corresponding eigenfunctions for the restriction $\mathcal{H}|\mathcal{N}$ *algebraic* since they can be computed by algebraic methods for matrix eigenvalue problems. (This does not mean that these functions are necessarily "algebraic" in the traditional pure mathematical sense.) Note that the number of such "algebraic" eigenvalues and eigenfunctions equals the dimension of $\mathcal{N}$. So far we have not imposed any normalizability conditions on the algebraic eigenfunctions, but, if they are normalizable, then the corresponding algebraic eigenvalues give part of the point spectrum of the differential operator.

It is of great interest to know when a given differential operator is in Lie algebraic or quasi-exactly solvable form. There is not, as far as we know, any direct test on the operator in question that will answer this in general. Consequently, the best approach to this problem is to effect a complete classification of such operators under an appropriate notion of equivalence. In order to classify Lie algebras of differential operators, and hence Lie algebraic and quasi-exactly solvable Schrödinger operators, we need to precisely specify the allowable changes of variables.

Definition 1. Two differential operators are *equivalent* if they can be mapped into each other by a combination of change of independent variable,

$$\bar{x} = \varphi(x), \tag{4}$$

and "gauge transformation"

$$\overline{\mathcal{H}} = e^{\sigma(x)} \cdot \mathcal{H} \cdot e^{-\sigma(x)}. \tag{5}$$

The transformations (4), (5), have two key properties. First, they respect the commutator between differential operators, and therefore preserve their Lie algebra structure. Second, they preserve the spectral problem (1) associated with the differential operator $\mathcal{H}$, so that if $\psi(x)$ is an eigenfunction of $\mathcal{H}$ with eigenvalue λ, then the transformed (or "gauged") function

$$\overline{\psi}(\bar{x}) = e^{\sigma(x)}\psi(x), \qquad \text{where} \qquad \bar{x} = \xi(x), \tag{6}$$

is the corresponding eigenfunction of $\overline{\mathcal{H}}$ having the same eigenvalue. Therefore this notion of equivalence is completely adapted to the problem of classifying quasi-exactly solvable Schrödinger operators. The gauge factor $\mu(x) = e^{\sigma(x)}$ in (5) is *not* necessarily unitary, *i.e.*, $\sigma(x)$ is not restricted to be purely imaginary, and hence does not necessarily preserve the normalizability properties of the associated eigenfunctions. Therefore, the problem of normalizability of the resulting algebraic wavefunctions must be addressed.

Definition 2. A quasi-exactly solvable Schrödinger operator is called *normalizable* if every algebraic eigenfunction is normalizable. An operator is called *partially normalizable* if some of the algebraic eigenfunctions are normalizable.

Let us summarize the basic steps that are required in order to obtain a complete classification of quasi-exactly solvable operators and their algebraic physical states.
1. Classify finite-dimensional Lie algebras of differential operators.

2. Determine which Lie algebras are quasi-exactly solvable.
3. Solve the equivalence problem for differential operators.
4. Determine normalizability conditions.
5. Solve the associated matrix eigenvalue problem.

The remainder of this survey will discuss what is now known about these problems, except the last which is merely an exercise in linear algebra.

3. Equivalence of Differential Operators.

Consider a second-order linear differential operator

$$-\mathcal{H} = \sum_{i,j=1}^{n} g^{ij}(x)\,\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} h^i(x)\,\frac{\partial}{\partial x^i} + k(x), \tag{7}$$

defined on an open subset $M \subset \mathbb{R}^n$. We are interested in studying the problem of when two such operators are equivalent under the combination of change of variables and gauge transformations (4), (5). Of particular importance is the question of when $\mathcal{H}$ is equivalent to a Schrödinger operator, which we take to mean an operator $\mathcal{S} = -\Delta + V(x)$, where Δ denotes either the flat space Laplacian or, more generally, the Laplace-Beltrami operator over a curved manifold. (Operators on Riemannian manifolds with non-zero curvature can be viewed as constrained quantum mechanical systems, *e.g.*, a particle moving on a sphere, [6].) This definition of Schrödinger operator excludes the introduction of a magnetic field, which, however, can also be handled by these methods. There is an essential difference between one-dimensional and higher dimensional spaces in the solution to the equivalence problem for second order differential operators.

Theorem 3. *Let*

$$-\mathcal{H} = P(x)\,\frac{d^2}{dx^2} + \widehat{Q}(x)\,\frac{d}{dx} + \widehat{R}(x), \qquad x \in \mathbb{R},$$

be a second order differential operator such that $P(x) > 0$. Then the change of variables

$$\bar{x} = \varphi(x) = \int^x \frac{dy}{\sqrt{P(y)}}\,, \tag{8}$$

and gauge factor

$$\mu(x) = \left|P(x)\right|^{-1/4} \exp\left\{\int^x \frac{\widehat{Q}(y)}{2P(y)}\,dy\right\}, \tag{9}$$

will place the operator $\mathcal{H}$ into Schrödinger form

$$\mu(x)\cdot\mathcal{H}\cdot\frac{1}{\mu(x)} = -\,\frac{d^2}{d\bar{x}^2} + V(\bar{x}).$$

The potential is given by

$$V(\bar{x}) = -\frac{3P'^2 - 8\widehat{Q}P' + 4\widehat{Q}^2}{16P} - \widehat{R} + \frac{1}{2}\widehat{Q}' - \frac{1}{4}P'', \tag{10}$$

where the right hand side is evaluated at $x = \varphi^{-1}(\bar{x})$. Moreover, if $\psi(x)$ is a wave function in the original coordinates, then

$$\overline{\psi}(\bar{x}) = \mu(\varphi^{-1}(\bar{x}))\,\psi(\varphi^{-1}(\bar{x})) \tag{11}$$

will be the corresponding wave function in the physical (Schrödinger) coordinates.

In higher space dimensions, it is no longer true that every second order differential operator is locally equivalent to a Schrödinger operator of the form $-\Delta + V(x)$, where $-\Delta$ is the flat space Laplacian — explicit equivalence conditions were first found by É. Cotton, [7]. Since the symbol of a linear differential operator is invariant under coordinate transformations, we begin by assuming that the operator is elliptic, meaning that the symmetric matrix $\widehat{\mathbf{g}}(x) = \big(g^{ij}(x)\big)$ determined by the leading coefficients of $-\mathcal{H}$ is positive definite. Owing to the induced transformation rules under the change of variables (7), we interpret the inverse matrix $\mathbf{g}(x) = \widehat{\mathbf{g}}(x)^{-1} = \big(g_{ij}(x)\big)$ as defining a Riemannian metric

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}(x)dx^i dx^j, \tag{12}$$

on the subset $M \subset \mathbb{R}^n$. We will follow the usual tensor convention of raising and lowering indices with respect to the Riemannian metric (12). We rewrite the differential operator (7) in a more natural coordinate-independent form

$$\mathcal{H} = -\sum_{i,j=1}^{n} g^{ij}(\nabla_i - A_i)(\nabla_j - A_j) + V, \tag{13}$$

where ∇_i denotes covariant differentiation using the associated Levi-Civita connection. Physically, $A(x) = (A_1(x), \dots , A_n(x))$ can be thought of as a (generalized) magnetic vector potential; in view of its transformation properties, we define the associated *magnetic one-form*

$$\omega = \sum_{i=1}^{n} A_i(x)\,dx^i. \tag{14}$$

(Actually, to qualify as a physical vector potential, A must be purely imaginary and satisfy the stationary Maxwell equations, but we will not impose this additional physical constraint in our definition of the mathematical magnetic one-form (14).) The explicit formulas relating the covariant form (13) to the standard form (7) of the differential operator are

$$\begin{aligned} A^i &= \sum_{j=1}^{n} g^{ij} A_j = -\frac{h^i}{2} + \frac{1}{2\sqrt{g}} \sum_{j=1}^{n} \frac{\partial\big(\sqrt{g}\,g^{ij}\big)}{\partial x^j}\,, \\ V &= -k + \sum_{i=1}^{n} \left[A_i\,A^i - \frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x^i}\,\big(\sqrt{g}\,A^i\big) \right], \end{aligned} \tag{15}$$

where $g(x) = \det\big(g_{ij}(x)\big) > 0$. Each second order elliptic operator then is uniquely specified by a metric, a magnetic one-form, and potential function $V(x)$. In particular, if the magnetic form vanishes, so $A = 0$, then $\mathcal{H}$ has the form of a Schrödinger operator $\mathcal{H} = -\Delta + V$, where Δ is the Laplace-Beltrami operator associated with the metric (12).

The application of a gauge transformation (5) does not affect the metric or the potential; however the magnetic one-form is modified by an exact one-form: $\omega \mapsto \omega + d\sigma$. Consequently, the "magnetic two-form" $\Omega = d\omega$, whose coefficients represent the associated magnetic field, *is* unaffected by gauge transformations.

Theorem 4. *Two elliptic second order differential operators $\mathcal{H}$ and $\overline{\mathcal{H}}$ are (locally) equivalent under a change of variables $\bar{x} = \varphi(x)$ and gauge transformation (5) if and only if their metrics, their magnetic two-forms, and their potentials are mapped to each other*

$$\varphi^*\big(d\bar{s}^2\big) = ds^2, \qquad \varphi^*\big(\,\overline{\Omega}\,\big) = \Omega, \qquad \varphi^*\big(\,\overline{V}\,\big) = V. \tag{16}$$

(Here φ^* denotes the standard pull-back action of φ on differential forms; in particular, $\varphi^*(\overline{V}) = \overline{V} \circ \varphi$.)

In particular, an elliptic second order differential operator is equivalent to a Schrödinger operator $-\Delta + V$ if and only if its magnetic one-form is closed: $d\omega = \Omega = 0$. Moreover, since the curvature tensor associated with the metric is invariant, the Laplace-Beltrami operator Δ will be equivalent to the flat space Laplacian if and only if the metric ds^2 is flat, *i.e.*, has vanishing Riemannian curvature tensor.

4. Lie Algebras of Differential Operators.

In this section, we summarize what is known about the classification problem for Lie algebras of first order differential operators. Any finite-dimensional Lie algebra $\mathfrak{g}$ of first order differential operators has a basis of the form

$$T^1 = \mathbf{v}^1 + \eta^1(x), \ldots, T^r = \mathbf{v}^r + \eta^r(x), \quad T^{r+1} = \zeta^1(x), \ldots, T^{r+s} = \zeta^s(x), \tag{17}$$

cf. (2). Here $\mathbf{v}^1, \ldots, \mathbf{v}^r$ are linearly independent vector fields spanning an r-dimensional Lie algebra $\mathfrak{h}$. The functions $\zeta^1(x), \ldots, \zeta^s(x)$ define multiplication operators, and span a finite-dimensional $\mathfrak{h}$-module $\mathcal{M} \subset C^\infty$ of smooth functions, hence $\mathbf{v}^i(\zeta^j) = \sum b^{ij}_k \zeta^k$ for constants b^{ij}_k. The functions $\eta^a(x)$ must satisfy additional constraints in order that the operators (17) span a Lie algebra; these conditions can be conveniently expressed using the basic theory of Lie algebra cohomology, [20]. Define the "one-cochain" on the vector field Lie algebra $\mathfrak{h}$ by the linear map $F\colon \mathfrak{h} \to C^\infty$ which satisfies $\langle F\,; \mathbf{v}^a\rangle = \eta^a$. Since we can add in any constant coefficient linear combination of the ζ^b's to the η^a's without changing the Lie algebra $\mathfrak{g}$, we should interpret the η^a's as lying in the quotient module $C^\infty/\mathcal{M}$, and hence regard F as a $C^\infty/\mathcal{M}$–valued cochain. It is straightforward to see that the collection of differential operators (17) spans a Lie algebra if and only if the cochain F satisfies

$$\mathbf{v}\langle F\,;\mathbf{w}\rangle - \mathbf{w}\langle F\,;\mathbf{v}\rangle - \langle F\,;[\mathbf{v},\mathbf{w}]\rangle \in \mathcal{M} \quad \text{forall} \quad \mathbf{v},\mathbf{w} \in \mathfrak{h}. \tag{18}$$

The left hand side of (18) is just the evaluation $\langle \delta_1 F\,;\mathbf{v},\mathbf{w}\rangle$ of the *coboundary* of the 1-cochain F, hence (18) expresses the fact that the cochain F must be a $C^\infty/\mathcal{M}$–valued *cocycle*. A 1-cocycle is itself a coboundary, written $F = \delta_0\sigma$ for some $\sigma(x) \in C^\infty$ if and only if $\langle F\,;\mathbf{v}\rangle = \mathbf{v}(\sigma)$ for all $\mathbf{v} \in \mathfrak{h}$. It can be shown that two cocycles will differ by a coboundary $\delta_0\sigma$ if and only if the corresponding Lie algebras are equivalent under the gauge transformation (5). Therefore two cocycles lying in the same *cohomology class* in the cohomology space $H^1(\mathfrak{h}, C^\infty/\mathcal{M}) = \operatorname{Ker}\delta_1/\operatorname{Im}\delta_0$, will give rise to equivalent Lie algebras of differential operators. In summary, then, we have the following fundamental characterization of Lie algebras of first order differential operators.

Theorem 5. *There is a one-to-one correspondence between equivalence classes of finite dimensional Lie algebras $\mathfrak{g}$ of first order differential operators on M and equivalence classes of triples $\big[\mathfrak{h}, \mathcal{M}, [F]\big]$, where*

1. *$\mathfrak{h}$ is a finite-dimensional Lie algebra of vector fields,*
2. *$\mathcal{M} \subset C^\infty$ is a finite-dimensional $\mathfrak{h}$-module of functions,*
3. *$[F]$ is a cohomology class in $H^1(\mathfrak{h}, C^\infty/\mathcal{M})$.*

Based on Theorem 5, there are three basic steps required to classify finite dimensional Lie algebras of first order differential operators. First, one needs to classify the finite dimensional Lie algebras of vector fields $\mathfrak{h}$ up to changes of variables; this was done by Lie in one and two dimensions under the assumption that the Lie algebra has no singularities — not every vector field in the Lie algebra vanishes at a common point. (Lie further claimed to have completed the classification in three dimensions, [23], but the complete results were never published.) Secondly, for each of these Lie algebras, one needs to classify all possible finite-dimensional $\mathfrak{h}$-modules $\mathcal{M}$ of C^∞ functions. Trivial modules, valid for any Lie algebra of vector fields are the zero module $\mathcal{M} = 0$, which consists of the zero function alone, and that containing just the constant functions, which we write $\mathcal{M} = \{1\}$. Finally, for each of the modules $\mathcal{M}$, one needs to determine the first cohomology space $H^1(\mathfrak{h}, C^\infty/\mathcal{M})$. As the tables indicate, the cohomology classes are parametrized by one or more continuous parameters or, in a few cases, smooth functions.

It is then a fairly straightforward matter to determine when a given Lie algebra satisfies the quasi-exactly solvable condition that it admit a non-zero finite-dimensional module $\mathcal{N} \subset C^\infty$. A simple lemma says that we can always, without loss of generality, take the Lie algebra $\mathfrak{g}$ to be represented by a triple $\big[\mathfrak{h}, \{1\}, [F]\big]$ with $\mathcal{M} = \{1\}$. (Indeed, $\mathfrak{g}$ admits a finite-dimensional module if and only if $\mathcal{M} = \{1\}$ or $\mathcal{M} = 0$, and, in the latter case, $\mathfrak{g}$ can always be enlarged to include constant functions without destroying its quasi-exact solvability.) Remarkably, *in all known cases*, the cohomology parameters are "quantized", the quasi-exact solvability requirement forcing them to assume at most a discrete set of distinct values. This intriguing phenomenon of "quantization of cohomology" has been geometrically explained in terms of line bundles on complex surfaces in [9].

In one dimension, meaning $M = \mathbb{R}$ or $\mathbb{C}$, the classification of non-singular quasi-exactly solvable Lie algebras is straightforward, and summarized in the following table for the real case. The complex classification is exactly the same, but with $\mathbb{C}$

replacing $\mathbb{R}$. (This, though, is particular to one-dimensional Lie algebras.)

Basis	Structure	Action	Module	Cohomology	Q.E.S.
∂_x	$\mathbb{R}$	$x+\beta$	$\operatorname{Ker}\mathcal{D}$	0	0
$\partial_x, x\partial_x$	$\mathfrak{h}_2$	$\alpha x+\beta$	$\mathcal{P}^{(n)}$	0 or $\mathbb{R}$	0
$\partial_x, x\partial_x, x^2\partial_x$	$\mathfrak{sl}(2,\mathbb{R})$	$\frac{\alpha x+\beta}{\gamma x+\delta}$	$\{1\}$	$\mathbb{R}$	$\mathbb{N}$

Here the first column provides a basis for the Lie algebra of vector fields $\mathfrak{h}$. The second column gives the structure as an abstract Lie algebra; we use $\mathfrak{h}_2 = \mathbb{R} \ltimes \mathbb{R}$ to denote the unique solvable two-dimensional Lie algebra, $\ltimes$ denoting semi-direct product. The third column indicates the associate group action. The fourth column lists the general $\mathfrak{h}$–module $\mathcal{M}$; in particular, $\operatorname{Ker}\mathcal{D} = \operatorname{Span}\{x^i e^{\lambda x} \mid 0 \le i \le n_\lambda\}$ denotes the solution space to a linear, constant coefficient ordinary differential equation $\mathcal{D}\psi = 0$, while $\mathcal{P}^{(n)}$ denotes the space of polynomials of degree at most n. As for the cohomology, in case 2 it is only nontrivial if $\mathcal{M} = 0$, in which case it is represented by the operators $\partial_x, x\partial_x + c$ for $c \in \mathbb{R}$; in Case 3, it is represented by $\partial_x, x\partial_x + c, x^2\partial_x + 2cx$, $c \in \mathbb{R}$. The final column discusses when a Lie algebra satisfies the requirement of quasi-exactly solvability, assuming $\mathcal{M} = \{1\}$, *i.e.*, the algebra $\mathfrak{g}$ contains the constant functions. In Case 3, the cohomology parameter is quantized, $c = -\frac{1}{2}n$, where $n \in \mathbb{N}$ is a non-negative integer (which plays the physical role of spin). The corresponding module is $\mathcal{N} = \mathcal{P}^{(n)}$, the space of polynomials of degree n. Therefore, we have proven the basic classification theorem for quasi-exactly solvable Lie algebras of differential operators in one dimension.

Theorem 6. *Every (non-singular) finite-dimensional quasi-exactly solvable Lie algebra of first order differential operators in one (real or complex) variable is, locally, equivalent to a subalgebra of one of the Lie algebras*

$$\widehat{\mathfrak{g}}_n = \operatorname{Span}\{\ \partial_x, \quad x\partial_x, \quad x^2\partial_x - n\,x, \quad 1\ \}, \tag{19}$$

where $n \in \mathbb{N}$. For $\widehat{\mathfrak{g}}_n$, the associated module $\mathcal{N} = \mathcal{P}^{(n)}$ consists of the polynomials of degree at most n.

Turning to the two-dimensional classification, a number of additional complications present themselves. First, as originally shown by Lie, there are many more equivalence classes of finite-dimensional Lie algebras of vector fields. Moreover, the classification results in $\mathbb{R}^2$ and $\mathbb{C}^2$ are no longer the same — here we just present the complex case since the real classification has yet to be completed. Another complication is that the modules $\mathcal{M}$ for the vector field Lie algebras are no longer necessarily spanned by monomials, a fact that makes the determination of the cohomology considerably more difficult. Tables 1 – 3 at the end of the paper summarize our classification results for finite-dimensional Lie algebras of differential operators in two complex variables, [10], [12]. Lie's classification of nonsingular finite-dimensional Lie algebras of vector fields on $\mathbb{C}^2$ is summarized in Table 1. The first column exhibits a basis of the algebra, and the second indicates its structure as an abstract Lie algebra. The last column indicates where the Lie algebra lies in Lie's "Gruppenregister", [24]. (We have, in a few cases, employed different coordinate systems than Lie.) Table 2 describes the different finite-dimensional modules for each of these Lie algebras. The first column tells whether the module is necessarily spanned by monomials, *i.e.*, single terms of the indicated form. (In cases 5 and 20, we have monomials unless $\alpha \in \mathbb{Q}^+$ or $r < \alpha \in \mathbb{Q}^+$ are positive rational numbers, respectively.) The second column indicates a typical term in a basis element for the module — the non-monomial basis elements will be linear combinations of terms of the indicated type; i, j always denote nonnegative integers. The third column either indicates ranges of indices which must be included, or, in the case of an arrow, indicates other indices which must be included if the given one is. For instance, in Case 19, if the monomial $x^i y^j e^{\mu x}$ belongs to the module, so must the monomials $x^{i-1} y^j e^{\mu x}$ and $x^{i+r} y^{j+1} e^{(\mu+\lambda)x}$ (provided $i > 0$ and/or $j > 0$) for each exponent λ appearing in the Lie algebra. Finally, $R_k^{n,m}(z)$ denotes the polynomial

$$R_k^{n,m}(z) = \frac{d^k}{dz^k}(z-1)^{m+n}(z+1)^n,$$

which, for $m = 0$, is a multiple of the ultraspherical Gegenbauer polynomial $C_{2n-k}^{n+k+(1/2)}(z)$, [8; vol. 2]. Table 3 describes the cohomology spaces $H^1(\mathfrak{h}, C^\infty/\mathcal{M})$ for each of the Lie algebras and corresponding modules. The first column indicates the dimension of the cohomology space, and the second column gives a representative cocycle of each nontrivial cohomology class. Only the vector fields $\mathbf{v}^a$ which are actually modified are indicated, *i.e.*, those for which $\eta^a = \langle F\,; \mathbf{v}^a \rangle \not\equiv 0$, *cf.* (17). In Case 4, $\operatorname{Div} \mathcal{M} = \{ f_x + g_y \mid f, g \in \mathcal{M} \}$. Finally, Table 4 describes the quantization condition resulting from the quasi-exactly solvability assumption that, assuming $\mathcal{M} = \{1\}$, the Lie algebra admit a finite-dimensional module $\mathcal{N}$. If the cohomology is trivial, so $\mathfrak{g}$ is spanned by vector fields and the constant functions, then it automatically satisfies the quasi-exactly solvable condition, with the associated finite-dimensional modules being explicitly described in Table 2. The maximal algebras, namely Case 11, $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, Case 15, $\mathfrak{sl}(3)$, and Case 24, $\mathfrak{gl}(2) \ltimes \mathbb{R}^r$, play an important role in Turbiner's theory of differential equations in two dimensions with orthogonal polynomial solutions, [32].

5. Quasi-Exactly Solvable Operators on the Line.

Let us now specialize to problems in one dimension. In view of Theorem 6, we let $n \in \mathbb{N}$ be a nonnegative integer, and consider the Lie algebra $\mathfrak{g}_n$ spanned by the differential operators

$$T^- = T_n^- = \frac{d}{dz}, \qquad T^0 = T_n^0 = z\,\frac{d}{dz} - \frac{n}{2}, \qquad T^+ = T_n^+ = z^2\,\frac{d}{dz} - nz, \tag{20}$$

which satisfy the standard $\mathfrak{sl}(2)$ commutation relations. In this section, we shall use z instead of x for the "canonical coordinate", retaining x for the physical coordinate in which the operator takes Schrödinger form. The Lie algebra $\widehat{\mathfrak{g}}_n$ in Theorem 6 is merely a central extension (by the constant functions) of the subalgebra $\mathfrak{g}_n$, hence $\widehat{\mathfrak{g}}_n$ is isomorphic to the Lie algebra $\mathfrak{gl}(2)$ of all 2×2 matrices. Note that since $\widehat{\mathfrak{g}}_n$ and $\mathfrak{g}_n$ only differ by inclusion of constant functions, in our analysis of Lie algebraic differential operators we can, without loss of generality, concentrate on the Lie algebra $\mathfrak{g}_n$, since any Lie algebraic operator (3) for the full algebra $\widehat{\mathfrak{g}}_n$ is automatically a Lie algebraic operator for the subalgebra $\mathfrak{g}_n$.

Thus, the most general second order quasi-exactly solvable Hamiltonian in one space dimension can therefore be written in the form

$$\begin{aligned}-\mathcal{H} = c_{++}(T^+)^2 + c_{+0}[T^+T^0 + T^0T^+] + c_{00}(T^0)^2 + c_{+-}[T^+T^- + T^-T^+]+ \\ + c_{0-}[T^0T^- + T^-T^0] + c_{--}(T^-)^2 + c_+T^+ + c_0T^0 + c_-T^- + c_*.\end{aligned} \tag{21}$$

Substituting the explicit formulas (20) for T^+, T^0, T^- into (21), we find that every quasi-exactly solvable operator can be written in the canonical form

$$-\mathcal{H} = P(z)\,\frac{d^2}{dz^2} + \left\{ Q(z) - \frac{n-1}{2}P'(z) \right\} \frac{d}{dz} + \left\{ R - \frac{n}{2}Q'(z) + \frac{n(n-1)}{12}P''(z) \right\}, \tag{22}$$

where

$$\begin{aligned}P(z) &= c_{++}z^4 + 2c_{+0}z^3 + (2c_{+-} + c_{00})z^2 + 2c_{0-}z + c_{--}, \\ Q(z) &= c_+z^2 + c_0z + c_-, \\ R(z) &= \frac{n^2+2n}{12}c_{00} - \frac{n^2+2n}{3}c_{+-} + c_*,\end{aligned} \tag{23}$$

are (general) polynomials of respective degrees $4, 2, 0$. Since the module $\mathcal{N}$ is the space $\mathcal{P}^{(n)}$ of polynomials of degree at most n, the algebraic eigenfunctions (21) will, in the z-coordinates, just be polynomials $\chi_k(z) \in \mathcal{P}^{(n)}$. In terms of the standard basis $\chi_k(z) = z^k$, $k = 0, \ldots, n$, the restricted Hamiltonian $\mathcal{H} \mid \mathcal{P}^{(n)}$ takes the form of a pentadiagonal matrix, or, if $c_{++} = c_{--} = 0$, a tridiagonal matrix. Thus, for a normalizable one-dimensional quasi-exactly solvable operator, there are $n+1$ algebraic eigenfunctions which, in the canonical z coordinates, are polynomials of degree at most n.

Specializing the solution to the equivalence problem given by Theorem 3 to the particular operator (22), we find that the change of variables (16) required to place the operator into physical (Schrödinger) form will, in general, be given by an elliptic integral

$$x = \varphi(z) = \int^z \frac{dy}{\sqrt{P(y)}}\,. \tag{24}$$

The corresponding gauge factor is

$$\mu(z) = \left|P(z)\right|^{-n/4}\, \exp\left\{ \int^z \frac{Q(y)}{2P(y)}\,dy \right\}. \tag{25}$$

The potential is given by

$$V(x) = -\frac{n(n+2)\left(PP'' - \frac{3}{4}P'^2\right) + 3(n+1)(QP' - 2PQ') - 3Q^2}{12P} - R, \tag{26}$$

where the right hand side is evaluated at $z = \varphi^{-1}(x)$. In the physical coordinate, the associated algebraic wave functions will then take the form

$$\psi(x) = \mu(\varphi^{-1}(x))\,\chi(\varphi^{-1}(x)), \tag{27}$$

where $\chi(z)$ is a polynomial of degree at most n.

The canonical form (22) of a quasi-exactly solvable differential operator is not unique, since there is a "residual" symmetry group which preserves the Lie algebra $\mathfrak{g}_n$. Not surprisingly, this group is $\mathrm{GL}(2,\mathbb{R})$, which acts on the (projective) line by linear fractional (Möbius) transformations

$$z \longmapsto w = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \det A = \Delta = \alpha\delta - \beta\gamma \neq 0. \tag{28}$$

To describe the induced action of the transformations (28) on the quasi-exactly solvable operators (22), we first recall the basic construction of the finite-dimensional irreducible rational representations of the general linear group $\mathrm{GL}(2,\mathbb{R})$.

Definition 7. Let $n \geq 0$, i be integers. The irreducible multiplier representation $\rho_{n,i}$ of $\mathrm{GL}(2,\mathbb{R})$ is defined on the space $\mathcal{P}^{(n)}$ of polynomials of degree at most n by the transformation rule $\widehat{P} = \rho_{n,i}(P)$, where

$$\widehat{P}(z) = (\gamma z + \delta)^n (\alpha\delta - \beta\gamma)^i \; P\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right), \qquad P \in \mathcal{P}^{(n)}. \tag{29}$$

The multiplier representation $\rho_{n,i}$ has infinitesimal generators given by the differential operators (5) combined with the operator of multiplication by i representing the diagonal subalgebra (center) of $\mathfrak{gl}(2,\mathbb{R})$. The action (28) induces an automorphism of the Lie algebra $\mathfrak{g}_n$, which is isomorphic to the representation $\rho_{2,-1}$, and, consequently, preserves the class of quasi-exactly solvable operators associated with the algebra $\mathfrak{g}_n$. Moreover, the corresponding gauge action

$$\widehat{\mathcal{H}}(z) = (\gamma z + \delta)^n \cdot \mathcal{H} \cdot (\gamma z + \delta)^{-n}, \tag{30}$$

will preserve the space of quasi-exactly solvable Hamiltonians (22). Identifying the operator $\mathcal{H}$ with the corresponding quartic, quadratic and constant polynomials (23), we find that the action (30) of $\mathrm{GL}(2,\mathbb{R})$ on the space of quasi-exactly solvable operators is isomorphic to the sum of three irreducible representations, $\rho_{4,-2} \oplus \rho_{2,-1} \oplus \rho_{0,0}$; the quartic $P(z)$ transforms according to $\rho_{4,-2}$, the quadratic $Q(z)$ according to $\rho_{2,-1}$, while R is constant. Finally, the associated module, which is just the space of polynomials $\mathcal{P}^{(n)}$, transforms according to the representation $\rho_{n,0}$.

Using the action of $\mathrm{GL}(2,\mathbb{R})$, we can place the gauged operator (22) into a simpler canonical form, based on the invariant theoretic classification of canonical forms for real quartic polynomials, [18].

Theorem 8. *Under the representation $\rho_{4,-2}$ of $\mathrm{GL}(2,\mathbb{R})$, every nonzero real quartic polynomial $P(z)$ is equivalent to one of the following canonical forms:*

$$\nu(1-z^2)(1-k^2z^2), \qquad \nu(1-z^2)(1-k^2+k^2z^2), \qquad \nu(1+z^2)(1+k^2z^2),$$
$$\nu(z^2-1), \quad \nu(z^2+1), \quad \nu z^2, \quad \nu(z^2+1)^2, \quad z, \quad 1,$$

where ν and $0 < k < 1$ are real constants.

The nine cases correspond to the positions of the complex roots of the quartic, which are, respectively, four simple real roots, two simple real roots and two simple complex conjugate roots, four simple complex roots, one double and two simple real roots, one double real and two simple complex roots, two double real roots, two double complex roots, one triple and one simple real roots, and one quadruple real root. Note that we are allowing ∞ to be a root, whose order is defined to be $4 - d$ where d is the degree of the polynomial. (This makes the concept of order of a root of a quartic polynomial invariant under the representation $\rho_{4,-2}$.)

Remark. If P has four simple roots, then the Schrödinger equation (1) for the operator (22), (23), has four regular singularities, and hence, by a *complex* linear fractional transformation, can be mapped to a form of Heun's equation

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\varepsilon}{z-1} + \frac{\delta}{z-a}\right)\frac{dy}{dz} + \frac{\alpha\beta z + b}{z(z-1)(z-a)}\, y = 0, \tag{31}$$

with $\alpha + \beta - \gamma - \delta - \varepsilon + 1 = 0$, which amounts to placing P into the complex canonical form $P(z) = \nu z(z-1)(z-a)$ with the roots at $0, 1, a$ and ∞. Therefore, the algebraic eigenfunctions can be expressed in terms of Heun polynomials, or, in the case of multiple roots, confluent Heun polynomials, [8; vol. 3].

The solution to the normalizability problem begins with a detailed analysis of the elliptic integral (24). Consider an interval $z_0 < z < z_1$ where $P(z) > 0$ is positive and vanishes at the endpoints, $P(z_0) = P(z_1) = 0$. Simple asymptotic analysis of (24) immediately implies that if both z_0 and z_1 are simple roots, then (8) defines z as a periodic function of x; therefore, the potential $V(x)$ and the corresponding algebraic eigenfunctions are periodic functions of x, and cannot be normalizable. (However, if they have no singularities, they do contribute, albeit minutely, to the continuous spectrum of the operator.) If $P(z) > 0$ is everywhere positive and has no real roots including ∞, then the same conclusion of periodicity holds. Therefore, a necessary condition for the quasi-exactly solvable operator (22), (23), to be normalizable is that the quartic polynomial $P(z)$ have at least one multiple real root, which must lie at the end of an interval of positivity. Thus, the class of quasi-exactly solvable potentials naturally splits into two subclasses — the periodic potentials, which are never normalizable, and the non-periodic potentials, which are sometimes normalizable.

Tedious but direct calculations based on (24), (25), (27), produce the explicit change of variables, the potential, and eigenfunctions for the above normal forms in physical coordinates. Each of the classes of potentials is a linear combination of four elementary and/or elliptic functions, plus a constant which we absorb into the eigenvalue. The potentials naturally divide into two classes, which are listed in the following two Tables. In each case, the four coefficients are not arbitrary, but satisfy a single complicated algebraic equation and one or more inequalities; for simplicity, we only exhibit these in the non-periodic case. For the more general class of Lie algebraic potentials see [21].

First, the periodic quasi-exactly solvable potentials correspond to the cases when the roots of P are simple, of which there are five cases given in the following Table.

$$\begin{array}{ll} 1. & \nu(1-z^2)(1-k^2z^2), \\ 2. & \nu(1-z^2)(1-k^2+k^2z^2), \\ 3. & \nu(1+z^2)(1+(1-k^2)z^2), \\ 4. & \nu(1-z^2), \\ 5. & \nu(z^2+1)^2, \end{array} \tag{32}$$

where $\nu > 0$, $0 < k < 1$. The explicit formulas for the corresponding potentials follows. In the first three cases, the corresponding potentials are written in terms of the standard Jacobi elliptic functions of modulus k, [8; vol. 2]. Also, as remarked above, the coefficients A, B, C, D are not arbitrary, although the explicit constraints are too complicated to write here.

Periodic Quasi-Exactly Solvable Potentials

1. $\mathrm{dn}^{-2}\sqrt{\nu}\,x\big(A\,\mathrm{sn}\sqrt{\nu}\,x+B\big)+\mathrm{cn}^{-2}\sqrt{\nu}\,x\big(C\,\mathrm{sn}\sqrt{\nu}\,x+D\big)$
2. $\mathrm{dn}^{-2}\sqrt{\nu}\,x\big(A\,\mathrm{cn}\sqrt{\nu}\,x+B\big)+\mathrm{sn}^{-2}\sqrt{\nu}\,x\big(C\,\mathrm{cn}\sqrt{\nu}\,x+D\big)$
3. $A\,\mathrm{cn}\sqrt{\nu}\,x\,\mathrm{sn}\sqrt{\nu}\,x+B\,\mathrm{cn}^2\sqrt{\nu}\,x+C\,\mathrm{dn}^{-2}\sqrt{\nu}\,x\big(\mathrm{cn}\sqrt{\nu}\,x\,\mathrm{sn}\sqrt{\nu}\,x+D\,\mathrm{cn}^2\sqrt{\nu}\,x\big)$
4. $A\sin^2\sqrt{\nu}\,x+B\sin\sqrt{\nu}\,x+C\tan\sqrt{\nu}\,x\sec\sqrt{\nu}\,x+D\sec^2\sqrt{\nu}\,x$
5. $A\cos 4\sqrt{\nu}\,x+B\cos 2\sqrt{\nu}\,x+C\sin 2\sqrt{\nu}\,x+D\sin 4\sqrt{\nu}\,x$

The elliptic functions $\mathrm{sn}\,y, \mathrm{cn}\,y, \mathrm{dn}\,y$ are periodic of period $4K$ (or $2K$ in the case of $\mathrm{dn}\,y$), where

$$K=\int_0^1\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

is the complete elliptic integral of modulus k. Moreover, $\mathrm{dn}\,y$ is never zero, $\mathrm{cn}\,y$ vanishes at odd multiples of K, and $\mathrm{sn}\,y$ vanishes at even multiples of K; also $\mathrm{cn}\,4mK = 1 = \mathrm{sn}(4m+1)K$, $\mathrm{cn}(4m+2)K = -1 = \mathrm{sn}(4m+3)K$. Therefore, the potentials in cases 1, 2 and 4 have singularities unless $C = D = 0$. In Cases 3 and 5, the potential has no singularities, reflecting the fact that in these cases $P(z)$ has no real roots. Case 3 includes the Lamé equation, [8; vol. 3].

The non-periodic potentials correspond to the cases with one or two multiple roots, which are given in the following Table.

$$\begin{array}{lll} 1. & \nu(z^2+1), & (-\infty,\infty) \\ 2. & \nu(z^2-1), & [1,\infty) \quad \text{or} \quad (-\infty,-1] \\ 3. & \nu z^2, & (0,\infty) \quad \text{or} \quad (-\infty,0) \\ 4. & z, & [0,\infty) \\ 5. & 1. & (-\infty,\infty) \end{array} \tag{33}$$

Here $\nu > 0$, and the second column indicates the interval(s) where P is positive, with square brackets designating simple roots. In cases 2 and 3 there are two possible intervals of positivity, and hence potentially two different physical Hamiltonians, although we can readily switch from one to the other by a discrete reflection $z \mapsto -z$.

The explicit formulas for the corresponding potentials follow.

Non-periodic Quasi-Exactly Solvable Potentials

1. $A\sinh^2\sqrt{\nu}\,x + B\sinh\sqrt{\nu}\,x + C\tanh\sqrt{\nu}\,x\,\mathrm{sech}\,\sqrt{\nu}\,x + D\,\mathrm{sech}^2\sqrt{\nu}\,x$
2. $A\cosh^2\sqrt{\nu}\,x + B\cosh\sqrt{\nu}\,x + C\coth\sqrt{\nu}\,x\,\mathrm{csch}\,\sqrt{\nu}\,x + D\,\mathrm{csch}^2\sqrt{\nu}\,x$
3. $Ae^{2\sqrt{\nu}\,x} + Be^{\sqrt{\nu}\,x} + Ce^{-\sqrt{\nu}\,x} + De^{-2\sqrt{\nu}\,x}$
4. $Ax^6 + Bx^4 + Cx^2 + \dfrac{D}{x^2}$
5. $Ax^4 + Bx^3 + Cx^2 + Dx$

In cases 2 and 4, the potential has a singularity at $x = 0$ unless $C+D = 0$ (Case 2) or $D = 0$ (Case 4). The nonsingular potentials in Case 4 are the anharmonic oscillator potentials discussed in detail in [29], [27]. The algebraic constraints satisfied by the coefficients are given in the following table.

Non-Periodic Constraints

1. $[B \pm 2(n+1)\sqrt{\nu A}]^4 + A(4D-\nu)[B \pm 2(n+1)\sqrt{\nu A}]^2 - 4A^2C^2 = 0, \quad A \geq 0.$
2. $[B \pm 2(n+1)\sqrt{\nu A}]^4 - A(4D+\nu)[B \pm 2(n+1)\sqrt{\nu A}]^2 + 4A^2C^2 = 0, \quad A \geq 0.$
3. $\pm C\sqrt{A} \pm B\sqrt{D} = 2(n+1)\sqrt{\nu AD}, \qquad A, D \geq 0.$
4. $16A^3[(4n+5)(4n+3) - 4D] \pm 32(n+1)A^{3/2}(B^2 - 4AC) + (B^2 - 4AC)^2 = 0,$
   $A \geq 0, \qquad D \geq -\frac{1}{4}.$
5. $8A^2D + B(B^2 - 4AC) = \pm 16(n+1)A^{5/2}, \qquad A \geq 0.$

According to Turbiner, [29], a potential is *exactly solvable* if it does not explicitly depend on the discrete "spin" parameter n, since, in this case, one can find representation spaces of arbitrarily large dimension and thereby (if normalizable) produce infinitely many eigenvalues by algebraic methods. Note that since the gauge transformation (8), (9), can explicitly depend on n, exact solvability cannot be detected in the canonical coordinates, but depends on the final physical form of the operator. The exactly solvable nonperiodic potentials are characterized by the condition $A = B = 0$, or, in case 3, $C = D = 0$. In Case 2, there is an additional inequality, $|C| \leq D + \frac{1}{4}\nu$, to be satisfied. The exactly solvable potentials include the (restricted) Pöschl–Teller potentials (Case 1), the Morse potentials (Case 3), the radial harmonic oscillator (provided $D = l(l+1)$, $l \in \mathbb{N}$) (Case 4), and the harmonic oscillator (Case 5), as well as a number of new and interesting cases not noted before in the literature.

Normalizability Conditions

1. $c_{++} = c_{+0} = c_{0-} = 0, \quad c_{--} = 2c_{+-} + c_{00}, \quad c_{+} = 0, \quad c_0 < -nc_{--}.$
2. $c_{++} = c_{+0} = c_{0-} = 0, \quad c_{--} = -2c_{+-} - c_{00}$
   $\big(c_{+} + c_0 + c_{-} = -nc_{--} \quad \text{or} \quad c_{+} + c_0 + c_{-} = -(n+2)c_{--}\big),$
   $\big(c_{+} < 0 \quad \text{or} \quad c_{+} = 0, \quad c_0 < nc_{--}\big).$
3. $c_{++} = c_{+0} = c_{0-} = c_{--} = 0\big),$
   $\big(c_{+} < 0, \quad \text{or} \quad c_{+} = 0, \quad c_0 < -2nc_{+-} - nc_{00})\big),$
   $\big(c_{-} > 0, \quad \text{or} \quad c_{-} = 0, \quad c_0 > 2nc_{+-} + nc_{00}\big).$
4. $c_{++} = c_{+0} = 2c_{+-} + c_{00} = c_{--} = 0, \qquad c_{0-} = \tfrac{1}{2},$
   $\big(c_{-} = \tfrac{1}{2}n \quad \text{or} \quad c_{-} = \tfrac{1}{2}n + 1\big),$
   $\big(c_{+} < 0, \quad \text{or} \quad c_{+} = 0, \quad c_0 < 0\big).$
5. $c_{++} = c_{+0} = 2c_{+-} + c_{00} = c_{0-} = 0, \quad c_{--} = 1, \quad c_{+} = 0, \quad c_0 < 0.$

Analysis of the explicit formulas for the eigenfunctions based on Theorem 3 yields a complete set of conditions for the normalizability of the non-periodic potentials, which are written in terms of the coefficients of the quadratic polynomial $Q(z)$ as given in (23). Note that this requires that $P(z)$ be in canonical form. Thus, although this table provides a complete list of normalizable potentials, it does not give the most general set of normalizable Lie algebraic coefficients, but rather, describes a single representative set of coefficients for each $\mathrm{GL}(2,\mathbb{R})$ orbit of normalizable quasi-exactly solvable operators. See [13] for partial normalizability conditions.

It is possible to deduce explicit, general normalizability conditions on the Lie algebraic coefficients c_{ab}, by using the fact that such conditions must be invariant under the action of the group $\mathrm{GL}(2,\mathbb{R})$. Therefore, normalizability conditions can be written in terms of the classical joint invariants and covariants of the pair of polynomials P, Q. The simplest of these necessary conditions stems from our earlier observation that, for normalizability, the quartic polynomial P must have a multiple root. This condition can be expressed in an invariant manner by the vanishing of the discriminant of P, which results in a sixth degree algebraic constraint, having the explicit form

$$\begin{aligned}\big[12c_{++}c_{--} - 12c_{+0}c_{0-} + (2c_{+-} + c_{00})^2\big]^3 &= \\ &= \left(\det\begin{vmatrix} 6c_{++} & 3c_{+0} & 2c_{+-} + c_{00} \\ 3c_{+0} & 2c_{+-} + c_{00} & 3c_{0-} \\ 2c_{+-} + c_{00} & 3c_{0-} & 6c_{--}\end{vmatrix}\right)^2 .\end{aligned} \tag{34}$$

Additional normalizability conditions require further joint covariants, which can be explicitly written in terms of the classical transvectants.

Definition 9. Let $F(z) \in \mathcal{P}^{(m)}$ and $G(z) \in \mathcal{P}^{(n)}$ be polynomials, and suppose that $r \le \min\{m,n\}$. The r-th *transvectant* of F and G is the polynomial

$$(F,G)^{(r)} = \sum_{k=0}^{r} (-1)^k \binom{r}{k} \frac{(m-r+k)!}{(m-r)!}\,\frac{(n-k)!}{(n-r)!}\, F^{(r-k)}(z)\, G^{(k)}(z). \tag{35}$$

Theorem 10. *Suppose* $F \in \mathcal{P}^{(m)}$ *and* $G \in \mathcal{P}^{(n)}$ *are polynomials transforming under* $\mathrm{GL}(2, \mathbb{R})$ *according to the representations* $\rho_{m,i}$, $\rho_{n,j}$, *respectively. Then the* r*-th transvectant* $(F,\, G)^{(r)} \in \mathcal{P}^{(m+n-2r)}$ *is a polynomial of degree at most* $m+n-2r$ *and transforms according to the representation* $\rho_{m+n-2r,i+j+r}$.

The fact that $(F,\, G)^{(r)}$ has degree at most $m + n - 2r$ is *not* obvious from the formula (20). The First Fundamental Theorem of classical invariant theory, [16], [18], states that every joint covariant of a system of polynomials is a suitably homogeneous polynomial in the succesive transvectants of the system. The transvectants are not all independent, and, except for a few low order cases (which, fortunately, include the case of a quartic and a quadratic), the determination of a complete list of algebraically independent covariants (a Hilbert Basis) is a difficult, unsolved problem.

Theorem 11. *Let* P *be a quartic polynomial and* Q *a quadratic polynomial. Then a complete system of irreducible covariants for the pair* P, Q *is provided by the polynomials themselves, the discriminant* $\Delta = (Q,\, Q)^{(2)}$ *of the quadratic, the covariants* $H = (P,\, P)^{(2)}$, $T = (H,\, P)^{(1)}$, *and invariants* $i = (P,\, P)^{(4)}$, $j = (H,\, P)^{(4)}$ *of the quartic, and the following 11 joint covariants:*

$$(P,\, Q)^{(1)}, \quad (P,\, Q)^{(2)}, \quad (P,\, Q^2)^{(3)}, \quad (P,\, Q^2)^{(4)}, \quad (H,\, Q)^{(1)}, \quad (H,\, Q)^{(2)}$$
$$(H,\, Q^2)^{(3)}, \quad (H,\, Q^2)^{(4)}, \quad (T,\, Q)^{(2)} \quad (T,\, Q^2)^{(4)}, \quad (T,\, Q^3)^{(6)}.$$

For example, the discriminant of a quartic is given by the combination

$$\frac{1}{9216}\left\{\frac{1}{96}\left[(P,\, P)^{(4)}\right]^3 - \left[(H,\, P)^{(4)}\right]^2\right\}.$$

Using the covariants in Theorem 11, we can now state the explicit necessary and sufficient conditions for normalizability of a quasi-exactly solvable operator on the line. In each case, the first line gives the invariant conditions for a quartic P to be equivalent to one of our five canonical forms with a multiple root; these conditions automatically imply the discriminantal condition (34). The subcases a) and b) are different alternatives; also, in Cases 2 and 4, n^* means either n or $n+2$. Finally, we write $F \succ G$ if $F(z) \geq G(z)$ for all z, and $F \not\equiv G$. Each of these conditions, when written out, gives a (very) complicated, but explicit condition on the Lie algebraic coefficients of our quasi-exactly solvable operator.

Theorem 12. *A one-dimensional quasi-exactly solvable operator* (21) *is normalizable if and only if its Lie algebraic coefficients satisfy one of the following sets of* *constraints.*

I. $i^3 = j^2 \neq 0, \qquad iH - jP \succ 0.$
$(iH - jP,\, Q^2)^{(4)} = 0, \qquad 3(T,\, Q)^{(2)} \succ 10n(iH - jP).$

II. $i^3 = j^2 \neq 0, \qquad iH - jP \prec 0.$

a) $(iH - jP,\, Q^2)^{(4)} \neq 0,$

$$12i^2(T,\, Q^3)^{(6)} + 1800n^* i^2 (iH - jP,\, Q^2)^{(4)} = \\ = 25\sqrt{(3jP - 3iH,\, Q^2)^{(4)}}\Big[(2jH + i^2P,\, Q^2)^{(4)} - 144ij(Q,\, Q)^{(2)}\Big].$$

b) $(iH - jP,\, Q^2)^{(4)} = 0,$

$$25\big[(P,\, Q^2)^{(3)}\big]^2 (iH - jP) = 864(Q,\, Q)^{(2)}\big[3(T,\, Q)^{(2)} + 10n^*(iH - jP)\big]^2,$$

$3(T,\, Q)^{(2)} \prec 10n(iH - jP).$

III. $i^3 = j^2 \neq 0, \qquad T = 0.$

a) $(H,\, Q^2)^{(4)} - 48i(Q,\, Q)^{(2)} > 0.$

b) $(H,\, Q^2)^{(4)} = 48i(Q,\, Q)^{(2)},\ (H,\, Q)^{(2)} - 8iQ \not\equiv 0, \qquad \dfrac{(P,\, Q^2)^{(3)}}{(H,\, Q)^{(2)} - 8iQ} < 12n.$

IV. $i = j = 0, \qquad H \not\equiv 0.$

a) $(T,\, Q^3)^{(6)} > 0,\ 2400(Q,\, Q)^{(2)}(H,\, Q^2)^{(4)} - 25[(P,\, Q^2)^{(4)}]^2 = 64n^*(T,\, Q^3)^{(6)},$

b) $(T,\, Q^3)^{(6)} = 0, \qquad (P,\, Q)^{(2)} \succ 0, \qquad 3P(H,\, Q)^{(1)} - 6H(P,\, Q)^{(1)} = 8n^* H^2.$

V. $i = j = T = H = 0,$
$(P,\, Q^2)^{(4)} = 0, \qquad$ and $\qquad (P,\, Q)^{(1)} \succ 0.$

6. Two Dimensional Problems.

There are a number of additional difficulties in the two dimensional problem which do not appear in the scalar case. First, there are several different classes of quasi-exactly solvable Lie algebras available. Even more important is the fact that, according to Theorem 3, there are nontrivial "closure conditions" which must be satisfied in order that the magnetic one-form associated with a given Hamiltonian operator be closed and hence the operator be equivalent, under a gauge transformation (5) to a Schrödinger operator. Unfortunately, in all but trivial cases, the closure conditions associated with a quasi-exactly solvable Hamiltonian (3) corresponding to the generators of one of the quasi-exactly solvable Lie algebras on our list are *nonlinear algebraic equations* in the coefficients c_{ab}, c_a, c_0, and it appears to be impossible to determine their general solution. Nevertheless, there are useful simplifications of the general closure conditions which can be effectively used to generate large classes of planar quasi-exactly solvable and exactly solvable Schrödinger operators, both for flat space as well as curved metrics.

Suppose that the Lie algebra $\mathfrak{g}$ is spanned by linearly independent first-order differential operators as in (2). Substituting these into the general Lie algebraic form (3), we find that the operator assumes the form (7) with

$$
\begin{aligned}
g^{ij} &= \sum_{a,b=1}^{r} c_{ab}\xi^{ai}\xi^{bj}, \\
h^{i} &= \sum_{a,b=1}^{r} \left[c_{ab}\left(\xi^{aj}\,\frac{\partial \xi^{bi}}{\partial x^j} + 2\eta^a\xi^{bi} \right) + c_a\xi^{ai} \right], \qquad (36)\\
k &= \sum_{a,b=1}^{r} \left[c_{ab}\left(\xi^{aj}\,\frac{\partial \eta^{b}}{\partial x^j} + \eta^a\eta^{b} \right) + c_a\eta^{a} \right].
\end{aligned}
$$

The magnetic form ω, (14), and potential V for the covariant form (13) of the Hamiltonian are then computed using formulas (15). The *closure conditions* $d\omega = 0$ are equivalent to the solvability of the system of partial differential equations

$$
\sum_{a,b=1}^{r} c_{ab}\xi^{ai}\sum_{j=1}^{n}\left(\xi^{bj}\,\frac{\partial \tau}{\partial x^j} + \frac{\partial \xi^{bj}}{\partial x^j} \right) = \sum_{a=1}^{r}\xi^{ai}\left[2\sum_{b=1}^{r} c_{ab}\eta^b + c_a \right], \qquad i = 1,\dots,n, \tag{37}
$$

for a scalar function $\tau(x)$, given by $\tau = 2\sigma + \frac{1}{2}\log\det g$ in terms of the gauge factor e^{σ} required to place the operator in Schrödinger form. The closure conditions (37) are extremely complicated to solve in full generality, but a useful subclass of solutions can be obtained from the *simplified closure conditions*

$$
\sum_{i=1}^{n}\left(\xi^{ai}\,\frac{\partial \tau}{\partial x^i} + \frac{\partial \xi^{ai}}{\partial x^i} \right) - 2\eta^a = k^a, \qquad a = 1,\dots,r. \tag{38}
$$

where $k^1,\dots,k^r$ are constants. Any solution $\tau(x)$ of equations (38) will generate an infinity of solutions to the full closure conditions (37), with c_{ab} arbitrary, and $c_a = \sum_b c_{ab}k^b$. The case $k^a = 0$ and $\mathfrak{g}$ semi-simple was investigated in [25]. Although the simplified closure conditions can be explicitly solved for such Lie algebras, with the exception of $\mathfrak{so}(3)$, their solutions are found to generate quasi-exactly solvable Schrödinger operators that are *not* normalizable, and hence of limited use. Note that even when the simplified closure conditions do not have any acceptable solutions, the *full* closure conditions (37) may be compatible and may give rise to normalizable operators.

Consider, in the first place, the Lie algebra $\mathfrak{g} \simeq \mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$ of type 11 spanned by the first-order differential operators

$$
T^1 = \partial_x, \quad T^2 = \partial_y, \quad T^3 = x\partial_x, \quad T^4 = y\partial_y, \quad T^5 = x^2\partial_x - nx, \quad T^6 = y^2\partial_y - my,
$$

where $n, m \in \mathbb{N}$. The particular choice

$$
(c_{ab}) = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},
$$

$$
(c_a) = \Big(0, 0, -(1+4n), -(1+4m), 0, 0\Big), \qquad c_0 = \tfrac{3}{4} + m^2 + n^2,
$$

of Lie algebraic coefficients lead to a quasi-exactly solvable Hamiltonian with Riemannian metric

$$g^{11} = (1+x^2)(2+x^2), \quad g^{12} = (1+x^2)(1+y^2), \quad g^{22} = (1+y^2)(2+y^2),$$

which has complicated curvature, and potential

$$\begin{aligned} 4V &= -y^2 - \frac{(1+2n)(3+2n)}{1+x^2} - \frac{(1+2m)(3+2m)}{1+y^2} \\ &\quad - \frac{17+12y^2-y^4+2xy(6+5y^2)}{3+x^2+y^2} + \frac{5(3+2xy)(1+y^2)(2+y^2)}{(3+x^2+y^2)^2}. \end{aligned}$$

A second interesting solution is

$$\begin{aligned} (c_{ab}) &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}, \\ (c_a) &= \Big(0, 0, -2n, -8m, 0, 0\Big), \qquad c_0 = -n - 4m. \end{aligned}$$

The Riemannian metric is

$$g^{11} = (1+x^2)^2, \qquad g^{12} = (1+x^2)(1+y^2), \qquad g^{22} = 4(1+y^2)^2,$$

which has zero curvature. The potential $V = 0$ also vanishes. The coordinate transformation

$$\bar{x} = \frac{1}{2\sqrt{3}}\big(4\arctan x - \arctan y\big), \qquad \bar{y} = \frac{1}{2}\arctan y, \tag{39}$$

maps the whole plane onto a bounded rectangle, so this example describes the physical situation of a free particle confined to a bounded rectangle. The algebraic eigenfunctions have the form

$$\psi(\bar{x}, \bar{y}) = \frac{P(x,y)}{(1+x^2)^{\frac{n}{2}}\,(1+y^2)^{\frac{m}{2}}},$$

where P is a polynomial of degree less than or equal to n in x and less than or equal to m in y, and we re-express x, y using (39).

The Lie algebra of type 15 provides a realization of $\mathfrak{sl}(3,\mathbb{R})$ in terms of first-order planar differential operators, given by

$$\begin{gathered} T^1 = \partial_x, \quad T^2 = \partial_y, \quad T^3 = x\partial_x, \quad T^4 = y\partial_x, \quad T^5 = x\partial_y, \quad T^6 = y\partial_y, \\ T^7 = x^2\partial_x + xy\partial_y - nx, \qquad T^8 = xy\partial_x + y^2\partial_y - ny, \end{gathered}$$

with $n \in \mathbb{N}$ admits the finite-dimensional module consisting of polynomials of total degree in x and y less than or equal to n. The quasi-exactly solvable Hamiltonian

$$\begin{aligned}\mathcal{H} = \left(T^1\right)^2 + \left(T^2\right)^2 + 2\left(T^7\right)^2 + 2\left(T^8\right)^2 + \\ + T^1T^7 + T^7T^1 + T^2T^8 + T^8T^2 + (3+2n)\left(T^3 + T^6\right)\end{aligned}$$

has contravariant metric coefficients

$$g^{11} = x^2(1+\rho) + 1, \qquad g^{12} = xy(1+\rho), \qquad g^{22} = y^2(1+\rho) + 1,$$

where $\rho = 1+2(x^2+y^2)$, whose Gaussian curvature $\kappa = -2\rho$ is negative everywhere. The potential is given by

$$4V = -3\rho - (7+16n+8n^2) + \frac{14+24n+8n^2+(22+24n+8n^2)\rho}{\rho^2+1}.$$

If we look for solutions of the Schrödinger equation depending only on the "radial" coordinate ρ, we end up with an effectively one-dimensional Schrödinger operator

$$-\widehat{\mathcal{H}} = 4(\rho-1)(\rho^2+1)\,\frac{d^2}{d\rho^2} + \Big[(6-4n)\rho^2+(8n+4)\rho-4n-2\Big]\,\frac{d}{d\rho} + \big[(n^2-n)\rho-n^2-n\big].$$

which does appear among the list of purely one-dimensional quasi-exactly solvable Hamiltonians, albeit with a different cohomology parameter. The question of whether the class of one-dimensional quasi-exactly solvable Schrödinger operators can be significantly enlarged via looking at reductions of two-dimensional quasi-exactly solvable Hamiltonians remains unanswered.

Next let $\mathfrak{g}$ be the noncompact Lie algebra of type 24 for $r = 1$, spanned by

$$T^1 = \partial_x, \quad T^2 = \partial_y, \quad T^3 = x\partial_x, \quad T^4 = x\partial_y, \quad T^5 = y\partial_y, \tag{40}$$

and

$$T^6 = x^2\partial_x + xy\partial_y - nx, \qquad \text{where} \qquad n \in \mathbb{N}.$$

This Lie algebra admits the finite-dimensional module $\mathcal{N}$ spanned by the monomials x^iy^j with $i+j \le n$. The Lie algebraic coefficients

$$(c_{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(c_a) = \Big(0, 0, -2n, 0, -1-2n, 0\Big), \qquad c_0 = n^2+n+1,$$

give a quasi-exactly solvable operator with flat Riemannian metric

$$g^{11} = (1+x^2)^2, \qquad g^{12} = xy(1+x^2), \qquad g^{22} = (1+x^2)(1+y^2).$$

Flat coordinates are given by

$$\bar{x} = \arctan x, \qquad \bar{y} = \operatorname{arcsinh} \frac{y}{\sqrt{1+x^2}},$$

which maps the plane to an open infinite strip $(-\pi/2, \pi/2) \times \mathbb{R}$. The potential in these coordinates is

$$V(\bar{x}, \bar{y}) = -\frac{(n+1)(n+2)}{2} \operatorname{sech}^2 \bar{y}.$$

which is simply a Pöschl–Teller potential in $\bar{y}$. The algebraic eigenfunctions take the form

$$\psi(\bar{x}, \bar{y}) = \frac{\cos^n \bar{x}}{\cosh^{n+1} \bar{y}}\, P(\tan \bar{x}, \sec \bar{x}\, \sinh \bar{y}),$$

where $P(x, y)$ is a polynomial of total degree at most n in (x, y). Notice that this potential, the preceding zero potential, and, indeed, all other flat potentials that we have found satisfy a conjecture of Turbiner: any quasi-exactly solvable Hamiltonian on a flat manifold in more than one dimension is necessarily separable. However, we do not know whether this conjecture holds in general.

Finally, let $\mathfrak{g}$ be a general Lie algebra of type 24, spanned by the first-order differential operators (40), $T^6 = x^2\partial_x + rxy\partial_y - nx$, and $T^{6+i} = x^{i+1}\partial_y$, $i = 1, \dots, r-1$. The module $\mathcal{N}$ is spanned by the monomials $x^i y^j$ with $i + rj \leq n$, $j \leq l$. For $m \in \mathbb{N}$, $A, B > 0$, the Schrödinger operator with metric

$$g^{11} = Ax^2 + B, \quad g^{12} = (1+m)Axy, \quad g^{22} = (Ax^2+B)^m + A(1+m)^2y^2,$$

and potential

$$V = -\frac{\lambda AB(1+m)^2(Ax^2+B)^m}{(Ax^2+B)^{1+m} + AB(1+m)^2y^2}, \qquad m \leq r \neq 2(m+1),$$

is normalizable and quasi-exactly solvable with respect to $\mathfrak{g}$, provided that the parameter λ is large enough. The metric in this case has constant negative Gaussian curvature $\kappa = -A$. Furthermore, since the potential V does not depend on the cohomology parameter n, the above Hamiltonian is *exactly solvable*. Moreover, the potential is also independent of r, hence we have constructed a single exactly solvable Hamiltonian which is associated to an infinite number of inequivalent Lie algebras of arbitrarily large dimension.

TABLE 1

FINITE-DIMENSIONAL LIE ALGEBRAS OF VECTOR FIELDS IN $\mathbb{C}^2$

	Generators	Structure	Label
1.	$\{\partial_x\}$	$\mathbb{C}$	E
2.	$\{\partial_x, x\partial_x\}$	$\mathfrak{h}_2$	$C1$
3.	$\{\partial_x, x\partial_x, x^2\partial_x\}$	$\mathfrak{sl}(2)$	$C4$
4.	$\{\partial_x, \partial_y\}$	$\mathbb{C}^2$	$D1$
5.	$\{\partial_x, \partial_y, x\partial_x + \alpha y\partial_y\} \quad \alpha \neq 0$	$\mathbb{C} \ltimes \mathbb{C}^2$	$C8, D3$
6.	$\{\partial_x, \partial_y, x\partial_x, y\partial_y\}$	$\mathfrak{h}_2 \oplus \mathfrak{h}_2$	$C3$
7.	$\{\partial_x, \partial_y, x\partial_x - y\partial_y, y\partial_x, x\partial_y\}$	$\mathfrak{sl}(2) \ltimes \mathbb{C}^2$	$A3$
8.	$\{\partial_x, \partial_y, x\partial_x, y\partial_x, x\partial_y, y\partial_y\}$	$\mathfrak{gl}(2) \ltimes \mathbb{C}^2$	$A2$
9.	$\{\partial_x, \partial_y, x\partial_x, x^2\partial_x\}$	$\mathfrak{gl}(2)$	$C5$
10.	$\{\partial_x, \partial_y, x\partial_x, y\partial_y, x^2\partial_x\}$	$\mathfrak{sl}(2) \oplus \mathfrak{h}_2$	$C6$
11.	$\{\partial_x, \partial_y, x\partial_x, y\partial_y, x^2\partial_x, y^2\partial_y\}$	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	$C7$
12.	$\{\partial_x + \partial_y, x\partial_x + y\partial_y, x^2\partial_x + y^2\partial_y\}$	$\mathfrak{sl}(2)$	$C9$
13.	$\{\partial_x, 2x\partial_x - y\partial_y, x^2\partial_x - xy\partial_y\}$	$\mathfrak{sl}(2)$	$B\delta 1$
14.	$\{\partial_x, x\partial_x, y\partial_y, x^2\partial_x - xy\partial_y\}$	$\mathfrak{gl}(2)$	$B\delta 2$
15.	$\{\partial_x, \partial_y, x\partial_x, y\partial_x, x\partial_y, y\partial_y,$ $x^2\partial_x + xy\partial_y, xy\partial_x + y^2\partial_y\}$	$\mathfrak{sl}(3)$	$A1$
16.	$\{\xi_1(x)\partial_y, \dots, \xi_r(x)\partial_y\}$	$\mathbb{C}^r$	$B\alpha 1$
17.	$\{\xi_1(x)\partial_y, \dots, \xi_r(x)\partial_y, y\partial_y\}$	$\mathbb{C} \ltimes \mathbb{C}^r$	$B\alpha 2$
18.	$\{\partial_x, x^i e^{\lambda x}\partial_y \mid 0 \leq i \leq r_\lambda\}$	$\mathbb{C} \ltimes \mathbb{C}^r$	$B\beta 1, D2$
19.	$\{\partial_x, y\partial_y, x^i e^{\lambda x}\partial_y \mid 0 \leq i \leq r_\lambda\}$	$\mathbb{C}^2 \ltimes \mathbb{C}^r$	$B\beta 2, C2$
20.	$\{\partial_x, \partial_y, x\partial_x + \alpha y\partial_y, x\partial_y, \dots, x^r\partial_y\}$	$\mathfrak{h}_2 \ltimes \mathbb{C}^{r+1}$	$B\gamma 1, 2$
21.	$\{\partial_x, \partial_y, x\partial_y, \dots, x^{r-1}\partial_y, x\partial_x + (ry + x^r)\partial_y\}$	$\mathfrak{h}_2 \ltimes \mathbb{C}^r$	$B\gamma 3$
22.	$\{\partial_x, \partial_y, x\partial_x, x\partial_y, y\partial_y, x^2\partial_y, \dots, x^r\partial_y\}$	$(\mathfrak{h}_2 \oplus \mathbb{C}) \ltimes \mathbb{C}^{r+1}$	$B\gamma 4$
23.	$\{\partial_x, \partial_y, 2x\partial_x + ry\partial_y, x\partial_y, x^2\partial_x + rxy\partial_y,$ $x^2\partial_y, \dots, x^r\partial_y\}$	$\mathfrak{sl}(2) \ltimes \mathbb{C}^{r+1}$	$B\delta 3$
24.	$\{\partial_x, \partial_y, x\partial_x, x\partial_y, y\partial_y, x^2\partial_x + rxy\partial_y,$ $x^2\partial_y, \dots, x^r\partial_y\}$	$\mathfrak{gl}(2) \ltimes \mathbb{C}^{r+1}$	$B\delta 4$

In Cases 16 and 17 we assume $r > 1$. In Cases 18 and 19, $r = \sum r_\lambda$, and the exponents λ belong to some finite set Λ. In cases 20–24, we assume $r \geq 1$.

TABLE 2

FINITE-DIMENSIONAL MODULES FOR LIE ALGEBRAS OF VECTOR FIELDS

	Monomials?	Generators	Rules
1.	No	$x^i e^{\lambda x} g(y)$	$(i,\lambda,g) \longrightarrow (i-1,\lambda,g)$,
2.	Yes	$x^i g(y)$	$(i,g) \longrightarrow (i-1,g)$
3.	Yes	$g(y)$	
4.	No	$x^i y^j e^{\lambda x+\mu y}$	$(i,j,\lambda,\mu) \longrightarrow (i-1,j,\lambda,\mu),(i,j-1,\lambda,\mu)$
5.	$\alpha \in \mathbb{Q}^+$	$x^i y^j$	$(i,j) \longrightarrow (i-1,j),(i,j-1)$
6.	Yes	$x^i y^j$	$(i,j) \longrightarrow (i-1,j),(i,j-1)$
7.	Yes	$x^i y^j$	$0 \le i+j \le n$
8.	Yes	$x^i y^j$	$0 \le i+j \le n$
9.	Yes	$y^j e^{\mu y}$	$0 \le j \le n_\mu$
10.	Yes	y^j	$0 \le j \le n$
11.	Yes	1	
12.	No	$(x-y)^{k-n} R^{n,0}_{2n-k}\left(\dfrac{x+y}{x-y}\right)$	$0 \le k \le 2n$
13.	Yes	$x^i y^n$	$0 \le i \le n,\ n \in S$
14.	Yes	$x^i y^n$	$0 \le i \le n,\ n \in S$
15.	Yes	1	
16.	No	$y^j g(x)$	$(j,g) \longrightarrow (j-1,g\cdot\xi_k)$
17.	Yes	$y^j g(x)$	$(j,g) \longrightarrow (j-1,g\cdot\xi_k)$
18.	No	$x^i y^j e^{\mu x}$	$(i,j,\mu) \longrightarrow (i-1,j,\mu),(i+r_\lambda,j-1,\mu+\lambda)$
19.	Yes	$x^i y^j e^{\mu x}$	$(i,j,\mu) \longrightarrow (i-1,j,\mu),(i+r_\lambda,j-1,\mu+\lambda)$
20.	$r < \alpha \in \mathbb{Q}^+$	$x^i y^j$	$(i,j) \longrightarrow (i-1,j),(i+r,j-1)$
21.	Yes	$x^i y^j$	$(i,j) \longrightarrow (i-1,j),(i+r,j-1)$
22.	Yes	$x^i y^j$	$(i,j) \longrightarrow (i-1,j),(i+r,j-1)$
23.	Yes	1	
24.	Yes	1	

TABLE 3

COHOMOLOGIES FOR LIE ALGEBRAS OF VECTOR FIELDS

	Dimension	Representatives
1.	0	
2.	∞	$x\partial_x + h(y), h \notin \mathcal{M}$
3.	∞	$x\partial_x + h(y), \quad x^2\partial_x + 2xh(y)$
4.	$\dim(\mathcal{M}/\operatorname{Div}\mathcal{M}) < \infty$	$\partial_y + h(x,y),\ h_x \in \mathcal{M}, h \neq \psi_y$ with $\psi_x \in \mathcal{M}$
5.	$0\ (\alpha \notin \mathbb{Q}^-), 1$ or $0\ (\alpha \in \mathbb{Q}^-)$	$x\partial_x + \alpha y\partial_y + c_1 x^i y^j$ or $\partial_y + c_1 x^i y^j$
6.	$0\ (\mathcal{M} \neq 0),\ 2\ (\mathcal{M} = 0)$	$x\partial_x + c_1, \quad y\partial_y + c_2$
7.	$0\ (\mathcal{M} \neq \{1\}),\ 1\ (\mathcal{M} = \{1\})$	$\partial_y + 2c_1 x, \quad y\partial_x + c_1 y^2, \quad x\partial_y + c_1 x^2$
8.	$0\ (\mathcal{M} \neq 0),\ 1\ (\mathcal{M} = 0)$	$x\partial_x + c_1, \quad y\partial_y + c_1$
9.	1	$x\partial_x + c_1, x^2\partial_x + 2c_1 x$
10.	$1\ (\mathcal{M} \neq 0),\ 2\ (\mathcal{M} = 0)$	$x\partial_x + c_1, \quad x^2\partial_x + 2c_1 x, \quad y\partial_y + c_2$
11.	2	$x\partial_x + c_1,\ x^2\partial_x + 2c_1 x,\ y\partial_y + c_2, y^2\partial_y + 2c_2 y$
12.	1	$x^2\partial_x + y^2\partial_y + c_1(x-y)$
13.	1	$x^2\partial_x - xy\partial_y + c_1 y^{-2}$
14.	$0\ (1 \in \mathcal{M}),\ 1\ (1 \notin \mathcal{M})$	$x\partial_x + c_1, \quad y\partial_y + 2c_1$
15.	1	$x\partial_x + c_1, x^2\partial_x + xy\partial_y + 3c_1 x, y\partial_y + c_1, xy\partial_x + y^2\partial_y + 3c_1 y$
16.	$\infty^r + k, \quad k < \infty$	$\xi_i(x)\partial_y + f_i(x)y^j$
17.	∞	$y\partial_y + f(x)$
18.	$< \infty$	$x^k e^{\lambda x}\partial_y + c_{i,j}^{\lambda,k} x^{i+k} y^j e^{\lambda x}$
19.	1	$y\partial_y + c_1 x^n$
20.	$0\ (\alpha \notin \mathbb{Q}),\ 1$ or $0\ (\alpha \in \mathbb{Q})$	$x\partial_x + \alpha y\partial_y + c_1 x^i y^j$, or $\xi_k\partial_y + c_k x^{i+k} y^j,\ k \geq l \geq 0$
21.	$0\ (\mathcal{M} \neq 0),\ 1\ (\mathcal{M} = 0)$	$x\partial_x + (ry + x^r y)\partial_y + c_1$
22.	$0\ (\mathcal{M} \neq 0),\ 2\ (\mathcal{M} = 0)$	$x\partial_x + c_1, \quad y\partial_y + c_2$
23.	$1\ (r > 2)$	$2x\partial_x + ry\partial_y + c_1, \quad x^2\partial_x + rxy\partial_y + c_1 x$
	$2\ (r = 2)$	$x\partial_x + y\partial_y + c_1, x\partial_y + c_2, x^2\partial_x + 2xy\partial_y + 2c_1 x + 2c_2 y, x^2\partial_y + 2c_2 x$
24.	$1\ (\mathcal{M} \neq 0),\ 2\ (\mathcal{M} = 0)$	$x\partial_x + c_1, y\partial_y + c_2, x^2\partial_x + rxy\partial_y + (2c_1 + rc_2)x$

TABLE 4

QUASI-EXACTLY SOLVABLE LIE ALGEBRAS OF DIFFERENTIAL OPERATORS

	Quantization condition	Module
1.	0	
2.	0	
3.	$h = -\frac{n}{2}, \quad n \geq 0,$	$\{x^i g(y) \mid i \leq n, g \in S\}$
4.	0	
5.	0	
6.	0	
7.	0	
8.	0	
9.	$c_1 = -\frac{n}{2}, \quad n \geq 0,$	$\{x^i y^j e^{\mu y} \mid i \leq n, j \leq m_\mu\}$
10.	$c_1 = -\frac{n}{2}, \quad n \geq 0,$	$\{x^i y^j \mid i \leq n, j \leq m\}$
11.	$c_1 = -\frac{n}{2},\ c_2 = -\frac{m}{2}, \quad n, m \geq 0,$	$\{x^i y^j \mid i \leq n, j \leq m\}$
12.	$c_1 = \frac{n}{2},$ †	$\left\{(x-y)^{m+\frac{n}{2}-k} R_k^{m,n}\left(\frac{x+y}{x-y}\right) \;\middle|\; 0 \leq k \leq 2m+n, m \in S\right\}$
13.	0	
14.	0	
15.	$c_1 = -\frac{n}{3}, \quad n \geq 0,$	$\{x^i y^j \mid i + j \leq n\}$
16.	0	
17.	0	
18.	0	
19.	0	
20.	0	
21.	0	
22.	0	
23.	$c_1 = -n, \quad n \geq 0,$	$\{x^i y^j \mid i + rj \leq n, j \leq l\}$
24.	$c_1 = -\frac{n}{2},\ c_2 = 0, \quad n \geq 0,$	$\{x^i y^j \mid i + rj \leq n, j \leq l\}$

†In case 12, there is no positivity restriction on n, and $S = \{m \mid m \geq max(0, -n)\}$ is a finite set of integers.

References

1. Y. Alhassid, J. Engel and J. Wu, *Algebraic approach to the scattering matrix*, Phys. Rev. Lett. **53** (1984), 17–20.
2. Y. Alhassid, F. Gürsey and F. Iachello, *Group theory approach to scattering*, Ann. Phys. **148** (1983), 346–380.
3. Y. Alhassid, F. Gürsey and F. Iachello, *Group theory approach to scattering. II. The euclidean connection*, Ann. Phys. **167** (1986), 181–200.
4. A. Arima and F. Iachello, *Interacting boson model of collective nuclear states. I – The vibrational limit*, Ann. Phys. **99** (1976), 253–317.
5. A. Arima and F. Iachello, *Interacting boson model of collective nuclear states. I – The rotational limit*, Ann. Phys. **111** (1978), 201–238.
6. N.L. Balazs and A. Voros, *Chaos on the pseudosphere*, Physics Reports **143** (1986), 109–240.
7. É. Cotton, *Sur les invariants différentiels de quelques équations linéaires aux dérivées partielles du second ordre*, Ann. École Norm. **17** (1900), 211–244.
8. A. Erdélyi and H. Bateman, *Higher Transcendental Functions*, McGraw–Hill, New York, 1953.
9. A. González–López, J. Hurtubise, N. Kamran and P.J. Olver, *Quantification de la cohomologie des algèbres de Lie de champs de vecteurs et fibrés en droites sur des surfaces complexes compactes*, preprint, McGill University, 1992.
10. A. González–López, N. Kamran and P.J. Olver, *Lie algebras of first order differential operators in two complex variables*, Differential Geometry, Global Analysis, and Topology, A. Nicas and W.F. Shadwick, eds., Canadian Math. Soc. Conference Proceedings, vol. 12, Amer. Math. Soc., Providence, R.I., 1991, pp. 51–84.
11. A. González–López, N. Kamran and P.J. Olver, *Quasi–exactly solvable Lie algebras of first order differential operators in two complex variables*, J. Phys. A **24** (1991), 3995–4008.
12. A. González–López, N. Kamran and P.J. Olver, *Lie algebras of differential operators in two complex variables*, American J. Math. (to appear).
13. A. González–López, N. Kamran and P.J. Olver, *Normalizability of one-dimensional quasi–exactly solvable Schrödinger operators*, Commun. Math. Phys. (to appear).
14. A. González–López, N. Kamran and P.J. Olver, *New quasi–exactly solvable Hamiltonians in two dimensions*, preprint, Univ. of Maryland, 1992.
15. A. Gorsky, *Relationship between exactly solvable and quasi-exactly solvable versions of quantum mechanics with conformal block equations in 2D theories*, JETP Lett. **54** (1991), 289–292.
16. J.H. Grace and A. Young, *The Algebra of Invariants*, Cambridge Univ. Press, Cambridge, 1903.
17. I.S. Gradshteyn and I.W. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York, 1965.
18. G.B. Gurevich, *Foundations of the Theory of Algebraic Invariants*, P. Noordhoff Ltd., Groningen, Holland, 1964.
19. F. Iachello and R. Levine, *Algebraic approach to rotation–vibration spectra. I – Diatomic molecules*, J. Chem. Phys. **77** (1982), 3046–3055.
20. N. Jacobson, *Lie Algebras*, Interscience Publ. Inc., New York, 1962.
21. N. Kamran and P.J. Olver, *Lie algebras of differential operators and Lie-algebraic potentials*, J. Math. Anal. Appl. **145** (1990), 342–356.
22. R.D. Levine, *Lie algebraic approach to molecular structure and dynamics*, Mathematical Frontiers in Computational Chemical Physics, D.G. Truhlar, ed., IMA Volumes in Mathematics and its Applications, Vol. 15, Springer-Verlag, New York, 1988, pp. 245–261.
23. S. Lie, *Theorie der Transformationsgruppen*, vol. 3, B.G. Teubner, Leipzig, 1893.
24. S. Lie, *Gruppenregister*, Gesammelte Abhandlungen, vol. 5, B.G. Teubner, Leipzig, 1924, pp. 767–773.
25. A.Y. Morozov, A.M. Perelomov, A.A. Rosly, M.A. Shifman and A.V. Turbiner, *Quasi-exactly solvable quantal problems: one-dimensional analogue of rational conformal field theories*, Int. J. Mod. Phys. **5** (1990), 803–832.

26. M.A. Shifman, *New findings in quantum mechanics (partial algebraization of the spectral problem)*, Int. J. Mod.. Phys. A **126** (1989), 2897–2952.
27. M.A. Shifman, *Quasi-exactly solvable spectral problems and conformal field theory*, preprint, Theoretical Physics Inst., Univ. of Minnesota, 1992.
28. M.A. Shifman and A.V. Turbiner, *Quantal problems with partial algebraization of the spectrum*, Commun. Math. Phys. **126** (1989), 347–365.
29. A.V. Turbiner, *Quasi-exactly solvable problems and* $\mathfrak{sl}(2)$ *algebra*, Commun. Math. Phys. **118** (1988), 467–474.
30. A.V. Turbiner, *Lie algebras and polynomials in one variable*, J. Phys. A **25** (1992), L1087–L1093.
31. A.V. Turbiner, *On polynomial solutions of differential equations*, J. Math. Phys. **33** (1992), 3989–3993.
32. A.V. Turbiner, *Lie algebraic approach to the theory of polynomial solutions. III. Differential equations in two real variables and general outlook*, preprint, ETH-TH/92-34, 1992.
33. A. G. Ushveridze, *Quasi-exactly solvable models in quantum mechanics*, Sov. J. Part. Nucl. **20** (1989), 504–528.

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