Divergence Invariant Variational Problems

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In [**3**], Harvey Brown states:

> One of the most remarkable methodological trends in modern physics has been the *a priori* use of symmetry principles to constrain the action principles of the non-gravitational interactions in quantum electrodynamics (QED) and particle physics.

In other words, the starting point of a modern physical theory is to postulate the relevant physical symmetries, and then construct an invariant variational principle (Lagrangian) whose Euler–Lagrange equations form the field equations whose solutions determine the associated physics. However, the theory of differential invariants, [**10**, **11**], implies that there are infinitely many inequivalent invariant Lagrangians that could be employed for this purpose. What is not addressed is the question I pose in [**12**]:

> How does this affect the resulting physics governed by the ostensibly different field equations? ... So, either the underlying physics is, in some rather vague and as yet undefined sense, "the same", in which case choosing the simplest invariant Lagrangian makes sense on purely practical grounds, or, more worryingly, the two physical theories are different, which then begs the question as to which invariant variational problem describes the correct physics — how does one decide among an infinite range of possibilities?

The following note begins with a very preliminary investigation into this issue in the simplest possible case: the one-dimensional classical free particle. It ends with a new result characterizing divergence invariant Lagrangians and some speculation as to their role in fundamental physics.

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Let us consider the variational principle for the one-dimensional free particle[†]

$$I[u] = \int \tfrac{1}{2}\, u_t^2 \, dt, \tag{1}$$

where the independent and dependent variables $t, u \in \mathbb{R}$ are scalars, and we use subscripts to denote derivatives. The Euler–Lagrange equation is simply

$$u_{tt} = 0. \tag{2}$$

Consider the one-parameter group of Galilean symmetries

$$(t,u) \longmapsto (t, u + \varepsilon\, t), \qquad \varepsilon \in \mathbb{R}. \tag{3}$$

The prolonged action is

$$u_t \longmapsto u_t + \varepsilon, \qquad u_{tt} \longmapsto u_{tt}, \qquad \ldots\ . \tag{4}$$

The Lagrangian is not strictly Galilean invariant since

$$\tfrac{1}{2}\, u_t^2 \longmapsto \tfrac{1}{2}\,(u_t + \varepsilon)^2 = \tfrac{1}{2}\, u_t^2 + \varepsilon\, u_t + \tfrac{1}{2}\,\varepsilon^2.$$

However since the additional terms are a total t derivative,

$$\varepsilon\, u_t + \tfrac{1}{2}\,\varepsilon^2 = D_t\big(\,\varepsilon\, u + \tfrac{1}{2}\,\varepsilon^2 t\,\big),$$

the Lagrangian is divergence invariant, [**10**]. The corresponding Noether conservation law is

$$D_t(t\, u_t - u) = t\, u_{tt} = 0. \tag{5}$$

The fundamental differential invariants, [**10**, **11**], of the Galilean group (3) are[‡]

$$t, \quad v = u_t - \frac{u}{t}, \quad u_{tt}, \quad u_{ttt}, \quad \ldots\ , \tag{6}$$

where, because t is fixed, the invariant differential operator is simply D_t. Note in particular that

$$D_t v = u_{tt} - \frac{u_t}{t} + \frac{u}{t^2} = u_{tt} - \frac{v}{t}.$$

Let us "invariantize" the Lagrangian in (1), and consider the variational principle

$$J[u] = \int \tfrac{1}{2}\, v^2 \, dt = \int \tfrac{1}{2}\,(u_t - u/t)^2 \, dt, \tag{7}$$

[†] The mass of the particle plays no role for now, and will be ignored for simplicity.

[‡] These are easy to find directly, but the method of moving frames, [**4**, **7**], can be effectively used in more complicated situations.

which, because it is expressed in terms of the differential invariants times the invariant "volume form" dt, is strictly invariant under the Galilean group (3), as can be easily checked directly. Since

$$\frac{1}{2}\left(u_t - \frac{u}{t}\right)^2 = \tfrac{1}{2}\,u_t^2 - \frac{u\,u_t}{t} + \frac{u^2}{2\,t^2} = \tfrac{1}{2}\,u_t^2 - D_t\left(\frac{u^2}{2\,t}\right),$$

the variational problems $(1,7)$ are equivalent and lead to the same free particle Euler–Lagrange equation (2). This alternative strictly invariant variational principle can, in fact, be found in Noether's original paper [**9**; p. 245]; see also [**3**; eq. (9)].

Now, according to Lie's Theorem, [**8**] — see also [**11**; Theorem 7.27] — the most general strictly Galilean invariant variational problem has the form

$$K[\,u\,] = \int F(t, v, u_{tt}, u_{ttt}, \ \ldots\ , u_{n\,t})\, dt = \int F(t, u_t - u/t, u_{tt}, u_{ttt}, \ \ldots\ , u_{n\,t})\, dt \qquad (8)$$

where $u_{n\,t}$ denotes the n^{th} derivative of u with respect to t, and F is an arbitrary function, assumed sufficiently smooth for later purposes. Let us study the case when $F = F(v)$ depends only on v. The Euler–Lagrange equation is

$$E = -\,D_t F'(v) - \frac{F'(v)}{t} = -\,F''(v)\,v_t - \frac{F'(v)}{t} = -\,F''(u_t - u/t)\,u_{tt} + \frac{B(u_t - u/t)}{t} = 0, \quad (9)$$

where

$$B(v) = v\,F''(v) - F'(v).$$

Since, when written in terms of v, the Euler–Lagrange equation (9) is separable, we can integrate once:

$$\int \frac{F''(v)}{F'(v)}\, dv = -\int \frac{dt}{t}, \qquad \text{and hence} \qquad F'(v) = \frac{1}{c\,t},$$

for some integration constant $c \in \mathbb{R}$. On any interval where $F''(v) \neq 0$, we can use the Inverse Function Theorem to solve the latter equation for

$$v = u_t - \frac{u}{t} = G(ct), \quad \text{where} \quad G(x) = F'^{-1}(1/x).$$

The resulting first order ordinary differential equation for u is linear, and its general solution is

$$u(t) = t\big(\,a + H(ct)\,\big), \qquad \text{where} \qquad H'(x) = \frac{G(x)}{x}, \qquad (10)$$

and where a is a second integration constant. In particular, for Noether's variational principle (7), $F(v) = \frac{1}{2}\,v^2$, so that $G(x) = 1/x$, $H(x) = -1/x$, and so

$$u(t) = a\,t - 1/c = a\,t + b, \quad \text{where} \quad b = -1/c,$$

recovering the standard linear motion of a free particle. The nonlinear motion (10) induced by the alternative Galilean-invariant Lagrangian is mathematically quite different, and there is no obvious relationship between the two, neither mathematical nor physical.

Of course, (3) is just a one-parameter subgroup of the full Galilean group, which also includes the translations in both t and u:

$$(t, u) \longmapsto (t + a, u + \varepsilon\, t + b), \qquad a, b, \varepsilon \in \mathbb{R}. \tag{11}$$

The translations do not affect the prolonged action, which coincides with (4), and the above calculation shows that the free particle Lagrangian is strictly invariant under the translations, but divergence invariant under the Galilean boost. On the other hand, the alternative Lagrangian (7) is strictly invariant under the Galilean boost, but only divergence invariant under the translations. (The latter can be checked either infinitesimally — see below — or directly using the group transformations (11).) The conservation laws corresponding to the two independent translation symmetries are

$$D_t(u_t) = u_{tt} = 0, \qquad D_t\big(\tfrac{1}{2}u_t^2\big) = u_t u_{tt} = 0, \tag{12}$$

the first representing conservation of momentum, and the second conservation of energy[†]. Since $u_t = a$ is constant, the Galilean conservation law (5) implies that the motion is linear in time: $u = a\,t + b$.

For the full Galilean group (11), the differential invariants are $u_{tt}, u_{ttt}, \ \ldots\ ,$ and hence Lie's Theorem says that the most general strictly Galilean-invariant variational problem takes the form

$$\widetilde{K}[u] = \int F(u_{tt}, u_{ttt}, \ \ldots\ , u_{nt})\, dt. \tag{13}$$

In particular, *there are no non-constant strictly Galilean-invariant first order Lagrangians* and the preceding "invariantization" trick does not work!

The latter result can be easily proved directly without invoking Lie's Theorem. As in [**10**, **11**], a Lagrangian $L(t, u, u_t, \ \ldots\ )$ is strictly invariant under a connected transformation group G if and only if

$$\mathrm{pr}\, \mathbf{v}(L) + L\, D_t \xi = 0 \tag{14}$$

for all infinitesimal generators $\mathbf{v} = \xi\, \partial_x + \varphi\, \partial_t$ of G, where pr $\mathbf{v}$ denotes its prolongation to jet space. In the case of the Galilean group (11), a basis for the prolonged infinitesimal generators is provided by

$$\mathrm{pr}\, \mathbf{v}_1 = \partial_t, \qquad \mathrm{pr}\, \mathbf{v}_2 = \partial_u, \qquad \mathrm{pr}\, \mathbf{v}_3 = t\,\partial_u + \partial_{u_t}. \tag{15}$$

Thus, a first order Lagrangian $L(t, u, u_t)$ satisfies the infinitesimal invariance condition (14) if and only if

$$\frac{\partial L}{\partial t} = \frac{\partial L}{\partial u} = \frac{\partial L}{\partial u_t} = 0,$$

and hence L must be constant.

† Recall that we are ignoring the mass parameter here.

On the other hand, it is still possible to modify the free particle Lagrangian to obtain a family of higher order Galilean divergence-invariant Lagrangians of the following form:

$$\widehat{K}[u] = \int \left[\tfrac{1}{2} m u_t^2 + f u + F(u_{tt}, u_{ttt}, \ \ldots \ , u_{nt}) \right] dt, \tag{16}$$

where we can regard the constant $m \in \mathbb{R}$ as the mass parameter and, in the case of Newtonian mechanics, the constant $f \in \mathbb{R}$ as a uniform external force. (More generally, f/m represents the constant acceleration a force-free body experiences in a Galilean invariant theory that violates Newton's first law of motion.) The Lagrangian is strictly Galilean invariant if and only if $m = f = 0$. If F depends nonlinearly on the n^{th} order derivative u_{nt}, the corresponding Euler–Lagrange equation has order $2n$, and its solutions appear to have very little to do with physical free particle motion.

In particular, there is a two parameter family of divergence-invariant first order Lagrangians that are not strictly invariant, obtained by setting $F = 0$:

$$\widehat{K}_1[u] = \int \left[\tfrac{1}{2} m u_t^2 + f u \right] dt. \tag{17}$$

One can replace the free particle term by Noether's Lagrangian (7), producing the alternative family

$$\widetilde{K}_1[u] = \int \left[\tfrac{1}{2} m (u_t - u/t)^2 + f u \right] dt, \tag{18}$$

with the same properties. In both cases, the corresponding Galilean-invariant Euler–Lagrange equations are

$$m u_{tt} = f, \qquad \text{with quadratic solutions} \qquad u(t) = \frac{f t^2}{2m} + a t + b, \tag{19}$$

provided $m \neq 0$. The three associated Noether conservation laws are

$$\begin{aligned} D_t(m u_t - f t) &= m u_{tt} - f = 0, \\ D_t\big(\tfrac{1}{2} m u_t^2 - f u \big) &= u_t (m u_{tt} - f) = 0, \\ D_t\big(m (t u_t - u) - \tfrac{1}{2} f t^2 \big) &= t (m u_{tt} - f) = 0, \end{aligned} \tag{20}$$

representing, as before, conservation of momentum and energy, along with the quadratic dependence of the motion on time.

In view of these considerations, an interesting question arises. Given a variational symmetry group, how can one characterize the most general divergence-invariant Lagrangian, and under what conditions does an equivalent strictly invariant Lagrangian exist? These questions are answered using the machinery of the (invariant) variational bicomplex, [**1**, **2**, **7**, **14**], as follows.

We shall work at the infinitesimal level, using results in [**10**]. Thus, let $\mathbf{v}$ be a generalized vector field and $\mathbf{v}_Q$ its evolutionary representative. According to Proposition 5.52 and the identity (5.86) in [**10**], a Lagrangian L is (infinitesimally) divergence invariant under $\mathbf{v}$ — in the language of [**10**], $\mathbf{v}$ is a variational symmetry — if and only if $\mathbf{v}_Q$ is also a variational symmetry and hence satisfies

$$E\big[\,\mathrm{pr}\;\mathbf{v}_Q(L)\,\big] = \mathrm{pr}\;\mathbf{v}_Q\big[\,E(L)\,\big] + D_Q^* E(L) = 0, \tag{21}$$

where D_Q^* denotes the adjoint of the Fréchet derivative of Q. As in [**2**], we will refer to vector fields satisfying (21) as *distinguished symmetries* of the Euler–Lagrange equations. The identity (21) is used to prove Theorem 5.53 in [**10**], which states that every variational symmetry is a symmetry of the Euler–Lagrange equations. (The converse is not valid, the most common counterexamples of non-distinguished symmetries being the generators of groups of scaling transformations.)

We have thereby established a basic lemma:

Lemma 1. *A Lagrangian L is divergence invariant under a generalized vector field $\mathbf{v}$ if and only if $\mathbf{v}$ is a distinguished symmetry of the Euler–Lagrange equations $E(L) = 0$.*

The next step is to reformulate this result using the invariant Euler–Lagrange complex, [**1**, **2**, **7**, **14**]. Let $\mathfrak{g}$ be a Lie algebra of prolonged (generalized) vector fields acting on jet space and hence on the differential forms thereon. A differential form ω is $\mathfrak{g}$-invariant if and only if all its Lie derivatives vanish:

$$\mathbf{v}(\omega) = 0 \qquad \text{for all} \qquad \mathbf{v} \in \mathfrak{g}. \tag{22}$$

In particular, $\mathfrak{g}$-invariance of a Lagrangian form λ is equivalent to strict invariance of the associated variational problem, while $\mathfrak{g}$-invariance of the associated Euler–Lagrange source form $E(\lambda)$ encodes the fact that the generators of $\mathfrak{g}$ are distinguished symmetries of the Euler–Lagrange equations, and hence, by Lemma 1, the Lagrangian form is divergence invariant. The difference between strict and divergence invariance of the Lagrangian is thus prescribed by the cohomology class of the Euler–Lagrange source form.

More precisely, let $H^*(\mathcal{E}_\mathfrak{g})$ denote the cohomology of the $\mathfrak{g}$-invariant Euler–Lagrange complex, denoted by $\mathcal{E}_\mathfrak{g}$. By definition, a $\mathfrak{g}$-invariant form is *closed* if its differential vanishes; it is *exact* if it is locally the differential of a $\mathfrak{g}$-invariant form. In particular, a source form is closed if it is (locally) an Euler–Lagrange source form for some variational problem, and it is exact if it comes from a $\mathfrak{g}$-invariant Lagrangian form. Thus, as a consequence of Lemma 1, we deduce our main result.

Theorem 2. *Let p denote the number of independent variables. A divergence $\mathfrak{g}$-invariant Lagrangian form λ is equivalent to a strictly $\mathfrak{g}$-invariant Lagrangian form if and only if the cohomology class of its Euler–Lagrange source form vanishes*:

$$0 = \big[\,E(\lambda)\,\big] \in H^{p+1}(\mathcal{E}_\mathfrak{g}). \tag{23}$$

A precursor of this Theorem can be found in [**1**]. Methods for computing the invariant cohomology spaces when the Lie algebra $\mathfrak{g}$ consists of the infinitesimal generators of either a finite-dimensional Lie group G or an infinite-dimensional Lie pseudo-group of projectable[†] transformations can be found in [**2**]. In the former case, it is proved that $H^{p+1}_{\mathfrak{g}}(\mathcal{E})$ is isomorphic to the Lie algebra cohomology space $H^{p+1}(\mathfrak{g})$, cf. [**6**]. This result was generalized to Lie groups of point transformations in [**14**], where algorithmic moving frame methods are applied to implement the isomorphism. This, establishes the following result.

Corollary 3. *Let G be a connected Lie group of point transformations. Then the condition $H^{p+1}(\mathfrak{g}) = \{0\}$ is necessary and sufficient for every divergence-invariant Lagrangian to be strictly invariant.*

In particular, if $\dim \mathfrak{g} \leq p$, then $H^{p+1}(\mathfrak{g}) = \{0\}$ automatically, and hence Corollary 3 applies. In the preceding example, there is one independent variable t, and hence $p = 1$. Thus, in this situation, Noether's "trick" of replacing the free particle Lagrangian with an equivalent strictly invariant Lagrangian can be applied to *any* one-parameter symmetry group, and hence divergence inequivalence is a multi-parameter phenomenon.

On the other hand, if $\mathfrak{g}_3$ denotes the three-dimensional Galilean Lie algebra spanned by (15), then it can be shown that $H^2(\mathfrak{g}_3) = \mathbb{R}^2$, and hence there is a two parameter family of non-zero cohomology classes, which correspond to the divergence invariant Lagrangians (17), or, alternatively, (18). Theorem 2 implies that every divergence invariant Lagrangian under the full Galilean group (11) is locally equivalent to one of the above form (16).

Example 4. An even simpler example is the abelian Lie algebra $\mathfrak{g}_2$ spanned by the first two vector fields in (15), generating the translations $(x,u) \longmapsto (x+a, u+b)$. In this case, the differential invariants are $u_t, u_{tt}, u_{ttt}, \ldots$, while $H^2(\mathfrak{g}_2) = \mathbb{R}$, and hence there is a one-dimensional space of divergence invariant Lagrangians that are not strictly translation invariant. Indeed, it is not hard to see that every divergence invariant variational problem has the form

$$I[u] = \int \left[\, f\,u + F(u_t, u_{tt}, \ \ldots \ , u_{nt}) \,\right] dt, \tag{24}$$

where $f \in \mathbb{R}$ and the second term is strictly invariant.

Example 5. A more substantial example is provided by combining the Galilean group with the scalings of time and space, producing the five-parameter transformation group

$$(t,u) \ \longmapsto \ (\lambda t + a, \mu u + \varepsilon t + b), \qquad a, b, \varepsilon, \in \mathbb{R}, \quad \lambda, \mu \in \mathbb{R} \setminus \{0\}. \tag{25}$$

It is easily checked that this forms a symmetry group of the (unforced) free particle equation (2). In addition, the particular scaling symmetries

$$(t,u) \ \longmapsto \ (\mu^2 t, \mu u), \qquad \mu \in \mathbb{R} \setminus \{0\} \tag{26}$$

[†] Projectable means that the transformations of the independent variables do not depend on the dependent variables; in bundle-theoretic terms they are fiber preserving maps. All the group actions considered here, and most of those arising in physics, are projectable.

form a strict one-parameter variational symmetry group of both the standard Lagrangian (1) and the Noether alternative (7). The corresponding conservation law is

$$D_t(u\,u_t - t\,u_t^2) = (u - 2\,t\,u_t)\,u_{tt} = 0. \tag{27}$$

On the other hand, the other scaling symmetries of the Euler–Lagrange equation are not distinguished symmetries of the corresponding source form, and thus are *not* symmetries of either Lagrangian, neither strict nor divergence. They act by rescaling the Lagrangian by a non-unit factor. Thus, they fail to produce a conservation law, but instead according to [**13**] — see also [**10**; Exercise 5.35] — lead to the following divergence identity:

$$D_t(uu_t) = u\,u_{tt} + u_t^2 = u_t^2, \tag{28}$$

which is valid on solutions to (2). Note that the right hand side is twice the Lagrangian, i.e., twice the kinetic energy. Such scaling identities are used to establish Clausius' Virial Theorem that governs the behavior of the average kinetic energy of a system, [**5**].

Let us finish by determining the structure of variational problems that admit the full group (25). The Lie algebra $\mathfrak{g}_5$ is spanned by the three Galilean infinitesimal generators (15) along with the two prolonged scaling generators

$$\begin{aligned} \operatorname{pr} \mathbf{v}_4 &= t\,\partial_t - u_t \partial_{u_t} - 2\,u_{tt}\partial_{u_{tt}} - 3\,u_{ttt}\partial_{u_{ttt}} - \;\cdots\;, \\ \operatorname{pr} \mathbf{v}_5 &= u\,\partial_u + u_t\partial_{u_t} + u_{tt}\partial_{u_{tt}} + u_{ttt}\partial_{u_{ttt}} + \;\cdots\;. \end{aligned} \tag{29}$$

Using either the method of moving frames [**4**], or a direct computation, one finds a complete system of differential invariants,

$$I_4 = \frac{u_{tt}u_{tttt}}{u_{ttt}^2}, \qquad I_5 = \frac{u_{tt}^2 u_{ttttt}}{u_{ttt}^3}, \qquad \ldots \qquad I_n = \frac{u_{tt}^{n-3} u_{n\,t}}{u_{ttt}^{n-2}}, \tag{30}$$

for $n \geq 4$, along with the fundamental (contact) invariant one-form and dual invariant differentiation operator

$$\omega = \frac{u_{ttt}}{u_{tt}}\,dt, \qquad \mathcal{D} = \frac{u_{tt}}{u_{ttt}}\,D_t, \tag{31}$$

the latter mapping differential invariants to higher order differential invariants. Thus, by Lie's Theorem, the most general strictly invariant variational problem has the form

$$I[u] = \int F(I_4, I_5, \;\ldots\;, I_n)\,\omega,$$

where F is an arbitrary function of the basic differential invariants (30).

To check for divergence-invariant Lagrangians, we compute the Lie algebra cohomology, finding $H^2(\mathfrak{g}_5) = \mathbb{R}$. This implies that there is a unique, up to constant multiple, divergence invariant Lagrangian that is not equivalent to a strictly invariant Lagrangian. A moving frame-based computation using the methods in [**14**] is the easiest way to find the following third order representative†:

$$J[u] = \int \frac{u_{ttt} \log u_{ttt} - u_{ttt}}{u_{tt}} \, dt, \tag{32}$$

with invariant sixth order Euler–Lagrange equation

$$D_t^2 \left(\frac{u_{tt} u_{tttt} - u_{ttt}^2}{u_{tt}^2 u_{ttt}} \right) = 0. \tag{33}$$

One can verify directly that (32) is divergence invariant under (25), but is not equivalent to a strictly invariant Lagrangian. Of course, this final example is a mere mathematical curiosity, as such higher order variational principles presumably have very little relevance to physics.

Let us close with a wild speculation that Theorem 2 may provide an answer (or at least a partial answer) to the original question. Namely, we propose that variational principles of physical relevance are distinguished by arising from nonzero cohomology classes of the underlying physical symmetry group, or, equivalently, are based on divergence invariant Lagrangians that are not equivalent to any strictly invariant Lagrangian. Of course, this requires using the full physical symmetry group, as restriction to a subgroup, especially if it is just a one-parameter subgroup, may lead to an equivalent strictly invariant variational principle of physical relevance, as we saw in our key example. According to Theorem 2 and the identification with Lie algebra cohomology, $H^{p+1}(\mathcal{E}_{\mathfrak{g}}) \simeq H^{p+1}(\mathfrak{g})$, when $\mathfrak{g}$ is a finite-dimensional Lie algebra of infinitesimal point transformations, such variational problems form a finite-dimensional space, whereas the strictly invariant variational problems depend upon an arbitrary function of the infinite family of independent differential invariants which are of arbitrarily high order. In particular, if the relevant cohomology space is one-dimensional, such a variational principle is unique up to constant multiple. Furthermore, according to computations in [**2**], the relevant cohomology spaces continue to be finite (and low) dimensional for a variety of nontrivial infinite-dimensional pseudo-group actions of physical and mathematical importance. (On the other hand, if the cohomology vanishes, there are no distinguished variational principles, and one must use other criteria to characterize physically relevant invariant variational principles.) Thus, divergence invariance or, equivalently, cohomological considerations may be fundamental to the symmetry-driven formulation of physical theories. In this direction, it would be of great interest to determine the cohomology for the infinite-dimensional group underlying the standard model, although this will be a very challenging computation.

† The Lagrangian is only defined on the subset of jet space where $u_{tt} \neq 0, \ \ u_{ttt} > 0$, and hence should be restricted to functions $u = f(t)$ whose jet lies in this subset. Interestingly, the Euler–Lagrange equation is less restricted, especially if one clears denominators after evaluating the total derivatives.

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