Evolution equations possessing infinitely many symmetries

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A general method for finding evolution equations having infinitely many symmetries or flows which preserve them is described. This is applied to the Korteweg–de Vries, modified Korteweg–de Vries, Burgers', and sine–Gordon equations.

The intense research activity of the past ten years surrounding the Korteweg–de Vries (KdV) equation was initially motivated by the discovery of an infinite series of conservation laws.[1] Noether's theorem shows that for a partial differential equation in Lagrangian form (which the KdV equation can be put into) there is an intimate connection between one-parameter symmetry groups of the equation[2] and conservation laws.[3] This was mysterious since the KdV equation possessed only a four-parameter symmetry group. However, it was noticed that the higher order analogs of the KdV equation discovered by Gardner[4] could be reinterpreted as "higher order symmetries" of the equation,[5] shedding some light on the mystery. Thus the more immediate object of interest becomes the symmetry groups, or, in more traditional terminology, the evolution equations whose flows preserve the KdV equation. The advantage of this point of view is that the symmetry groups can be systematically found, as in Theorem 1, in contrast to the ad hoc methods used to discover conservation laws. A recursion formula due to Lenard[4] for the higher order KdV equations is generalized here to provide a method for the construction of infinite series of higher order symmetries of more general evolution equations. In particular, we derive in Example 4 an infinite series of flows, all of which preserve Burgers' equation. (These symmetries however do not give rise to conservation laws since Burgers' equation cannot be placed in Lagrangian form. The precise relationship between symmetries and conservation laws shall be discussed in a future paper.) The methods employed here are differential algebraic in nature, in the same spirit as the recent work of Gelfand and Dikiĭ.[6] The calculations presented in this note will be formal; rigorous mathematical statements and proofs shall appear elsewhere in a more complete exposition.

Let $R\{u\}$ denote the algebra of polynomials in the variable u and its derivatives with respect to a single independent variable x. Given a polynomial $P\in R\{u\}$, let $\{P\}$ denote the differential ideal generated by P; it consists of all polynomials of the form $\sum Q_k D^k P$, where $D=d/dx$, the total derivative. For convenience we abbreviate

$$u_i=\frac{d^i u}{dx^i}, \quad \partial_i=\frac{\partial}{\partial u_i}.$$

Let T be the vector space of all formal polynomial partial differential operators acting on $R\{u\}$; in other words, T consists of all operators of the form

$$\mathcal{D}=\sum_I P_I \partial_I,$$

where $P_I\in R\{u\}$. Here the sum is over all multi-indices $I=(i_0, i_1, \ldots, i_l)$ with $l=0, 1, 2, \cdots$, and $\partial_I=\partial_0^{i_0}\partial_1^{i_1}\cdots\partial_l^{i_l}$. T contains the total derivative operator

$$\frac{d}{dx}=D=\sum_{i=0}^{\infty}u_{i+1}\partial_i. \tag{1}$$

Since T acts on $R\{u\}$, we make T into an algebra by defining the product of $\mathcal{D}$, $\mathcal{D}'\in T$ to be

$$\mathcal{D}\cdot\mathcal{D}'(P)=\mathcal{D}[\mathcal{D}'(P)], \quad P\in R\{u\}.$$

Using Leibnitz' formula, if $\mathcal{D}=\sum P_I\partial_I$ and $\mathcal{D}'=\sum Q_J\partial_J$, then

$$\mathcal{D}\cdot\mathcal{D}'=\sum_{I,J}P_I\sum_{0\le M\le I}\binom{I}{M}\partial_M Q_J\cdot\partial_{I+J-M}. \tag{2}$$

Moreover, T is a Lie algebra with bracket

$$[\mathcal{D},\mathcal{D}']=\mathcal{D}\mathcal{D}'-\mathcal{D}'\mathcal{D}.$$

Now let

$$\mathcal{V}: R\{u\}\to T$$

be the map defined by

$$\mathcal{V}(P)=\sum_{j=0}^{\infty}D^j P\cdot\partial_j. \tag{3}$$

Note that in Ovsjannikov's terminology,[2] $\mathcal{V}(P)$ is the ∞-prolongation of the vector field $P\cdot\partial/\partial u$, in the case P is a polynomial in u alone.

Given a differential polynomial $K\in R\{u\}$, consider the evolution equation

$$u_t=K(u). \tag{4}$$

If we make the assumption that (4) is locally uniquely solvable for arbitrary smooth initial data $u(x,0)=f(x)$, then there is an induced flow on $C^\infty(\mathbb{R})$,

$$\hat{K}_t[f(x)]=u(x,t), \quad f\in C^\infty(\mathbb{R}),$$

where $u(x,t)$ is the solution of (4) with initial data f. If $P\in R\{u\}$ is another polynomial, then we say that the flow generated by P preserves the flow generated by K if

$$\hat{P}_s[\hat{K}_t(f)]=\hat{K}_t[\hat{P}_s(f)]$$

for all $f\in C^\infty(\mathbb{R})$ and all $s,t\in\mathbb{R}$ where the equation is defined.

Theorem 1. Let $P,K\in R\{u\}$. Then the flow generated by P preserves the flow generated by K if and only if

$$\mathcal{V}(P)K-D_tK\in\{u_t-K\}. \tag{5}$$

Condition (5) refers to the partial differential algebra $\hat{R}\{u\}$ consisting of polynomials in u and its partial derivatives with respect to both x and t; $D_t=d/dt$ is the total derivative with respect to t. To verify condition (5) it suffices to replace the variables $\partial^{j+1}u/\partial^j x\partial t$ in the left-hand side by D^jK and equate the resulting expression to 0. In the special case that P is a polynomial in u alone this result is well known; it is just the infinitesimal criterion of invariance of (4) under the one-parameter group with infinitesimal generator $P\cdot\partial/\partial u$.[2] Thus if P, K satisfy criterion (5) we shall say that P is an *infinitesimal higher-order symmetry* of K. Note that $P=K$ trivially satisfies (5).

Next, define the map

$$\mathcal{A}:R\{u\}\to T$$

by

$$\mathcal{A}(P)=\sum_{j=0}^{\infty}\partial_j P\cdot D^j. \tag{6}$$

Note that

$$\mathcal{V}(P)K=\mathcal{A}(K)P \tag{7}$$

for $P, K\in R\{u\}$. Let T_0 be the subspace of T generated by the operator D, i.e., the elements of T_0 are operators of the form $\sum_{i=0}^{\infty}P_iD^i$ with $P_i\in R\{u\}$. Note that T_0 preserves ideals in $R\{u\}$.

Theorem 2: Suppose $\mathcal{D}\in T_0$ satisfies

$$[\mathcal{A}(K)-D_t, \mathcal{D}]P\in\{u_t-K\} \tag{8}$$

for all $P\in R\{u\}$, then K possesses an infinite series of infinitesimal symmetries

$$K^{(j)}=\mathcal{D}^jK, \quad j=0,\ 1,\ 2,\ \cdots. \tag{9}$$

Proof: By induction on j and Eq. (7)

$$[\mathcal{A}(K)-D_t]K^{(j-1)}\in\{u_t-K\}.$$

Using $\equiv$ to denote congruence modulo the ideal $\{u_t-K\}$, condition (8) implies that

$$\begin{aligned}[\mathcal{A}(K)-D_t]K^{(j)}&=[\mathcal{A}(K)-D_t]\mathcal{D}K^{(j-1)}\\ &\equiv\mathcal{D}[\mathcal{A}(K)-D_t]K^{(j-1)}\\ &\equiv 0,\end{aligned}$$

thereby proving the result.

An operator $\mathcal{D}\in T_0$ that satisfies condition (8) will be called a *recursion operator* for K. Practically, to verify the condition that $\mathcal{D}$ be a recursion operator for K it suffices to (a) compute the Lie bracket in $\hat{R}\{u\}$, (b) substitute D^jK whenever the variable $\partial^{j+1}u/\partial x^j\partial t$ appears, and (c) equate the resulting coefficients of D^k for each power k to 0. Thus (8) gives a useful criterion for determining when an evolution equation possesses an infinite number of symmetries.

Actually, to apply Theorem 2 to any interesting equations, it is necessary to enlarge the class T_0 to include more general recursion operators. In particular, we wish to allow recursion operators that involve the inverse total derivative D^{-1}.[7] The problem is that D^{-1} is not well defined on all of $R\{u\}$, so more care in the assumptions is needed to ensure that the conclusions of Theorem 2 still hold for these more general operators. A precise statement of the generalization of this theorem will be deferred to the later rigorous exposition. In this note we shall be content to use condition (8) formally to find a few specific recursion operators.

Example 3: Consider the KdV equation

$$u_t=K(u)=u_{xxx}+uu_x. \tag{10}$$

Here we reprove the result of Lenard that the operator

$$\mathcal{D}=D^2+\tfrac{2}{3}u+\tfrac{1}{3}u_xD^{-1} \tag{11}$$

is (formally) a recursion operator for K. Now

$$\mathcal{A}(K)=A=D^3+uD+u_x.$$

Hence

$$\begin{aligned}A\cdot\mathcal{D}=&D^5+\tfrac{5}{3}uD^3+\tfrac{10}{3}u_xD^2+(3u_{xx}+\tfrac{2}{3}u^2)D\\ &+\tfrac{5}{3}(u_{xxx}+uu_x)+\tfrac{1}{3}(u_{xxxx}+uu_{xx}+u_x^2)D^{-1}\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}\cdot A=&D^5+\tfrac{5}{3}uD^3+\tfrac{10}{3}u_xD^2\\ &+(3u_{xx}+\tfrac{2}{3}u^2)D+(u_{xxx}+uu_x).\end{aligned}$$

Therefore,

$$[A,\mathcal{D}]=\tfrac{2}{3}(u_{xxx}+uu_x)+\tfrac{1}{3}(u_{xxxx}+uu_{xx}+u_x^2)D^{-1}.$$

Furthermore

$$[D_t,\mathcal{D}]=\tfrac{2}{3}u_t+\tfrac{1}{3}u_{xt}D^{-1},$$

so that condition (8) is verified. The infinite series of symmetries

$$K^{(j)}=\mathcal{D}^jK,$$

when put into evolution form

$$u_t=K^{(j)}(u),$$

are just the higher-order analogs of the KdV equation.[4]

Example 4: Consider Burgers' equation

$$u_t=B(u)=u_{xx}+uu_x. \tag{12}$$

We show that B possesses the recursion operator

$$\mathcal{D}=D+\tfrac{1}{2}u+\tfrac{1}{2}u_xD^{-1}. \tag{13}$$

Here

$$\mathcal{A}(B)=A=D^2+uD+u_x.$$

Hence

$$\begin{aligned}A\cdot\mathcal{D}=&D^3+\tfrac{3}{2}uD^2+(\tfrac{5}{2}u_x+\tfrac{1}{2}u^2)D+\tfrac{3}{2}(u_{xx}+uu_x)\\ &+\tfrac{1}{2}(u_{xxx}+uu_{xx}+u_x^2)D^{-1}\end{aligned}$$

and

$$\mathcal{D}\cdot A=D^3+\tfrac{3}{2}uD^2+(\tfrac{5}{2}u_x+\tfrac{1}{2}u^2)D+(u_{xx}+uu_x).$$

Therefore,

$$[A,\mathcal{D}]=\tfrac{1}{2}(u_{xx}+uu_x)+\tfrac{1}{2}(u_{xxx}+uu_{xx}+u_x^2)D^{-1}.$$

Furthermore

$$[D_t,\mathcal{D}]=\tfrac{1}{2}u_t+\tfrac{1}{2}u_{xt}D^{-1},$$

which proves condition (8) formally. Therefore, we have an infinite sequence of flows

$$u_t=B^{(j)}(u)=\mathcal{D}^jB(u)$$

all of which preserve the flow given by Burgers' equation. The first few of these flows are

$$
\begin{aligned}
u_t &= B^{(0)}(u) = u_{xx} + uu_x, \\
u_t &= B^{(1)}(u) = u_{xxx} + \tfrac{3}{2}uu_{xx} + \tfrac{3}{2}u_x^2 + \tfrac{3}{4}u^2u_x, \\
u_t &= B^{(2)}(u) = u_{xxxx} + 2uu_{xxx} + 5u_xu_{xx} \\
&\quad + \tfrac{3}{2}u^2u_{xx} + 3uu_x^2 + \tfrac{1}{2}u^3u_x, \\
u_t &= B^{(3)}(u) = u_{xxxxx} + \tfrac{5}{2}uu_{xxxx} + \tfrac{15}{2}u_xu_{xxx} + 5u_{xx}^2 \\
&\quad + \tfrac{5}{2}u^2u_{xxx} + \tfrac{25}{2}uu_xu_{xx} + \tfrac{15}{4}u_x^3 \\
&\quad + \tfrac{5}{4}u^3u_{xx} + \tfrac{15}{4}u^2u_x^2 + \tfrac{5}{16}u^4u_x.
\end{aligned}
\tag{14}
$$

Example 5: Finally, we consider the modified KdV equation

$$u_t = \tilde{K}(u) = u_{xxx} + u^2u_x, \tag{15}$$

which is known to also possess infinitely many conservations laws. In fact, the original proof of the existence of infinitely many conservation laws of the KdV equation $u_t = K(u)$ stemmed from the remarkable transformation of Miura[8] relating the two equations. Explicitly, if

$$v = u^2 + \mu u_x, \quad \text{where } \mu = \sqrt{-6},$$

then

$$(\mu D + 2u)[u_t - \tilde{K}(u)] = v_t - K(v).$$

Let us assume for the moment that $\tilde{K}$ possesses a recursion operator $\tilde{\mathcal{D}}$ and that furthermore the higher-order analogs $u_t = \tilde{K}^{(j)} = \tilde{\mathcal{D}}_j\tilde{K}$ are related to the higher-order analogs of the KdV equation by the same formula,

$$(\mu D + 2u)[u_t - \tilde{K}^{(j)}(u)] = v_t - K^{(j)}(v).$$

We conclude that the recursion operator $\tilde{\mathcal{D}}$ must be related to the recursion operator $\mathcal{D}$ of K by the formal operator equation

$$(\mu D + 2u)\cdot\tilde{\mathcal{D}} = \mathcal{D}\cdot(\mu D + 2u).$$

A straightforward calculation shows that for this to hold,

$$\tilde{\mathcal{D}} = D^2 + \tfrac{2}{3}u^2 + \tfrac{2}{3}u_xD^{-1}\cdot u. \tag{16}$$

The last term in (16) is the operator which takes a polynomial $P \in R\{u\}$, multiplies it by u, then applies D^{-1}, and finally multiplies the result by $\frac{2}{3}u_x$. We shall check that $\tilde{\mathcal{D}}$ is indeed a recursion operator for $\tilde{K}$. We have

$$\mathcal{A}(\tilde{K}) = \tilde{A} = D^3 + u^2D + 2uu_x.$$

Note that

$$D^{-1}\cdot u = uD^{-1} - D^{-1}\cdot u_xD^{-1}.$$

Hence

$$
\begin{aligned}
u_xD^{-1}\cdot u\tilde{A} &= uu_xD^2 + u^3u_x - u_xD^{-1}\cdot(u_xD^2 + u^2u_x) \\
&= uu_xD^2 - u_x^2D + (u^3u_x + u_xu_{xx}) \\
&\quad - u_xD^{-1}\cdot(u_{xxx} + u^2u_x)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{A}\cdot u_xD^{-1}\cdot u &= [u_xD^3 + 3u_{xx}D^2 + (3u_{xxx} + u^2u_x)D \\
&\quad + (u_{xxxx} + u^2u_{xx} + 2uu_x^2)]\cdot D^{-1}\cdot u \\
&= uu_xD^2 + (2u_x^2 + 3uu_{xx})D + (3uu_{xxx} + 4u_xu_{xx} \\
&\quad + u^3u_x) + (u_{xxxx} + u^2u_{xx} + 2uu_x^2)D^{-1}\cdot u.
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\tilde{A}, \tfrac{2}{3}u_xD^{-1}\cdot u] &= 2(uu_{xx} + u_x^2)D + 2(uu_{xxx} + u_xu_{xx}) \\
&\quad + \tfrac{2}{3}(u_{xxxx} + u^2u_{xx} + 2uu_x^2)D^{-1}\cdot u \\
&\quad + \tfrac{2}{3}u_xD^{-1}\cdot(u_{xxx} + u^2u_x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
[\tilde{A}, \tilde{\mathcal{D}}] &= \tfrac{4}{3}u(u_{xxx} + u^2u_x) + \tfrac{2}{3}(u_{xxxx} + u^2u_{xx} \\
&\quad + 2uu_x^2)D^{-1}\cdot u + \tfrac{2}{3}u_xD^{-1}\cdot(u_{xxx} + u^2u_x).
\end{aligned}
$$

On the other hand,

$$D_t\cdot u_xD^{-1}\cdot u = u_{xt}D^{-1}\cdot u + u_xD^{-1}\cdot u_t + u_xD^{-1}\cdot uD_t.$$

Hence

$$[D_t, \tilde{\mathcal{D}}] = \tfrac{4}{3}uu_t + \tfrac{2}{3}u_{xt}D^{-1}\cdot u + \tfrac{2}{3}u_xD^{-1}\cdot u_t.$$

Comparing the expressions for $[\tilde{A}, \tilde{\mathcal{D}}]$ and $[D_t, \tilde{\mathcal{D}}]$ shows that condition (8) holds formally under the plausible assumption that $u_xD^{-1}\cdot u_t$ and $u_xD^{-1}\cdot(u_{xxx} + u^2u_x)$ define the same operator modulo the ideal $\{u_t - \tilde{K}(u)\}$. We conclude that the flows

$$u_t = \tilde{K}^{(j)}(u) = \tilde{\mathcal{D}}^j\tilde{K}(u)$$

all preserve the modified KdV equation. The first few of these flows are

$$
\begin{aligned}
u_t &= K^{(0)}(u) = u_{xxx} + u^2u_x, \\
u_t &= K^{(1)}(u) = u_{xxxxx} + \tfrac{5}{3}u^2u_{xxx} + \tfrac{20}{3}uu_xu_{xx} + \tfrac{5}{3}u_x^3 + \tfrac{5}{6}u^4u_x, \\
u_t &= K^{(2)}(u) = u_{xxxxxxx} + \tfrac{7}{3}u^2u_{xxxxx} + 14uu_xu_{xxxx} \\
&\quad + 21u_x^2u_{xxx} + \tfrac{70}{3}uu_{xx}u_{xxx} + \tfrac{91}{3}u_xu_{xx}^2 \\
&\quad + \tfrac{35}{18}u^4u_{xxx} + \tfrac{140}{9}u^3u_xu_{xx} + \tfrac{105}{9}u^3u_x^3 + \tfrac{35}{54}u^6u_x.
\end{aligned}
$$

Example 6. In this example we consider the sine-Gordon equation

$$u_{xt} = \sin u. \tag{17}$$

This equation is already known to possess infinitely many conservation laws and symmetry groups.[9] Although this equation does not belong to the class of evolution equations, we shall indicate how the methods used previously can be modified so as to rederive the symmetry groups of (17). The analog of Theorem 1 in this case is that the flow generated by a polynomial $P \in R\{u\}$ preserves the set of solutions of the sine-Gordon equation if and only if

$$A(P) \in \{u_{xt} - \sin u\}, \tag{18}$$

where A is the operator

$$A = DD_t - \cos u.$$

In this case we call P an (infinitesimal) symmetry of the sine-Gordon equation.

Suppose we can find operators $\mathcal{D}$ and $\mathcal{D}'$ satisfying

$$[A\mathcal{D} - \mathcal{D}'A]Q \in \{u_{xt} - \sin u\} \tag{19}$$

for any $Q \in R\{u\}$. Then condition (18) implies that $\mathcal{D}P$ is a symmetry of (17) whenever P is. In other words, $\mathcal{D}$ would be a recursion operator for the sine-Gordon equation. However, since (17) is not an evolution equation, we are left with the task of finding one symmetry of (17) in order to prove that there are infintely many symmetries. This is simplified by the observation that any partial differenital equation not explicitly involving the independent variable x or t is invariant under the flow

$$u_t = u_x.$$

This amounts to the statement that $f(x+\lambda, t)$ is a solution whenever $f(x,t)$ is. [In our previous examples, if we apply the recursion operators to the polynomial u_x, the flow we derive is just that of the original equation. For instance,

$$(D^2 + \tfrac{2}{3}u + \tfrac{1}{3}u_x D^{-1})u_x = u_{xxx} + uu_x. \,] \qquad (20)$$

Now consider the operators

$$\mathcal{D} = D^2 + u_x^2 - u_x D^{-1}\cdot u_{xx}, \quad \mathcal{D}' = D^2 + u_x^2 + u_{xx}D^{-1}\cdot u_x. \qquad (21)$$

We have

$$A\mathcal{D} = D^3 D_t - \cos u D^2 + u_x^2 D D_t + u_x u_{xx} D_t + 2u_x u_{xt} D + (u_{xx}u_{xt} + u_x u_{xxt} - u_x^2 \cos u) + (u_x \cos u - u_{xxt})D^{-1}\cdot u_{xx} - u_{xx}D^{-1}\cdot u_{xxt} - u_{xx}D^{-1}\cdot u_{xx}D_t,$$

and

$$\mathcal{D}'A = D^3 D_t - \cos u D^2 + u_x^2 D D_t + u_x u_{xx} D_t + 2u_x \sin u D + u_{xx}\sin u - u_{xx}D^{-1}\cdot u_x \cos u - u_{xx}D^{-1}\cdot u_{xx}D_t,$$

where we have used the identity

$$D^{-1}\cdot u_x D = u_x - D^{-1}\cdot u_{xx}.$$

Comparing these expressions verifies condition (18) formally. We conclude that the flows

$$u_t = \mathcal{D}^k(u_x), \quad k = 0, 1, 2, \cdots$$

are all symmetries of the sine-Gordon equation. The first few of these flows are

$$u_t = u_x,$$

$$u_t = u_{xxx} + \tfrac{1}{2}u_x^3,$$

$$u_t = u_{xxxxx} + \tfrac{5}{2}u_x^2 u_{xxx} + \tfrac{5}{2}u_x u_{xx}^2 + \tfrac{3}{8}u_x^5,$$

$$u_t = u_{xxxxxxx} + \tfrac{7}{2}u_x^2 u_{xxxxx} + 14u_x u_{xx} u_{xxxx} + \tfrac{21}{2}u_x u_{xxx}^2 + \tfrac{35}{2}u_{xx}^2 u_{xxx} + \tfrac{35}{8}u_x^4 u_{xxx} + \tfrac{35}{4}u_x^3 u_{xx}^2 + \tfrac{5}{16}u_x^7.$$

[1] R. M. Miura *et al.*, J. Math. Phys. **9**, 1204 (1968).
[2] L. V. Ovsjannikov, *Group Properties of Differential Equations*, translated by G. W. Bluman (unpublished). See also P. J. Olver, thesis, Harvard University, 1976, for a rigorous exposition.
[3] N. H. Ibragimov, "Invariance and Conservation Laws of Continuum Mechanics," in *Symmetry, Similarity and Group Theoretic Methods in Mechanics*, edited by P. G. Glockner and M. C. Singh (University of Calgary Press, Calgary, 1974), pp. 63—82.
[4] Discussed in P. D. Lax, Commun. Pure Appl. Math. **28**, 141 (1975).
[5] R. L. Anderson, S. Kumei, and C. E. Wulfman, J. Math. Phys. **14**, 1527 (1973) and S. Kumei, J. Math. Phys. **16**, 2461 (1975) also considered this concept.
[6] I. M. Gelfand and L. A. Dikii, Russ. Math. Survey, **30**, 63 (1975); Func. Anal. **10**, 18 (1975).
[7] See M. D. Kruskal *et al.*, J. Math. Phys. **11**, 952 (1970), for a precise discussion of this operator.
[8] R. M. Miura, J. Math. Phys. **9**, 1202 (1968).
[9] S. Kumei, J. Math. Phys. **16**, 2461 (1975).