# On the Symmetry Group of a Linear Partial Differential Equation

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## Abstract

We investigate the symmetry group of a single linear partial differential equation in p independent variables $(x^1, \ldots, x^p) = x$ and one dependent variable u, extending the results of Ovsjannikov for a second order linear equation. It is first shown that all the symmetries of an equation of order $\geq 3$ must be projectable, i.e., of the form $(x, u) \mapsto (f(x), g(x, u))$. Using the fact that the symmetries form a subgroup of the conformal group of the top-order symbol of the equation, a bound for the number of symmetries is obtained. Precisely, if G is the full symmetry group and T the trivial normal subgroup that results from the linearity of the equation, then $\dim G/T \leq p+1$ for a "nondegenerate" linear partial differential equation.

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## 1. Introduction

In this paper some results on the symmetry group of a linear partial differential equation are derived. The Lie-Ovsjannikov theory of symmetry groups of partial differential equations is discussed in great detail in a number of references, for instance [1], [6], and [8]. Roughly speaking, the symmetry group of a partial differential equation is the largest local Lie group of transformations acting on both the independent and dependent variables, which transforms solutions of the equation to other solutions of the equation. In section 2 a brief introduction to the Lie-Ovsjannikov theory is sketched. Another version of the symmetry group concept, to which this paper also applies, may be found in the papers of Miller, Kalnins et. al., [5].

Suppose that

$$\Delta[u] = \sum_I \sigma^I(x)\partial_I u = 0 \tag{1.1}$$

is an n-th order linear partial differential equation in p independent variables $x = (x^1, \ldots, x^p)$ and one dependent variable $u$. The summation in (1.1) is taken over multi-indices $I = (i_1, \ldots, i_p)$ and $\partial_I$ denotes the corresponding partial derivative. Note that (1.1) is always invariant under the trivial symmetry group $T$ generated by all transformations of the form

$$(x, u) \mapsto (x, \lambda u + \varphi(x))$$

for $0 \neq \lambda \in \mathbb{R}$ and $\varphi$ an arbitrary solution of (1.1).

A transformation will be called projectable if it is of the form

$$(x, u) \mapsto (f(x), g(x, u)).$$

In [6], Ovsjannikov showed that a second order linear equation that was not "strongly degenerate" (see section 3) possessed only projectable symmetries. In section 3 this result is extended to show that any linear equation of order $\geq 3$ possesses only projectable symmetries.

Let G denote the full symmetry group of (1.1). Ovsjannikov has shown that the trivial subgroup T is a normal subgroup of G. The factor group G/T forms the main object of interest. In [6] Osvjannikov demonstrated that for a not strongly degenerate second order equation with $p > 2$,

$$\dim G/T \leq \frac{(p+1)(p+2)}{2}.$$

In Theorem 6.8 it is shown that for a linear equation with "nondegenerate" symbol,

$$\dim G/T \leq p+1.$$

The adjective <u>nondegenerate</u> roughly means that for every system of coordinates $(x^1, \ldots, x^p)$, the set

$$\{I : |I| = n, \ \sigma^I(x) \neq 0\}$$

spans a (p-1)-dimensional affine subspace of $\mathbb{R}^p$. For $n \geq 3$, the non-degenerate symbols form a dense open subset of the set of all symbols. (For $n = 2$, all symbols are degenerate in our terminology.) The method of proof of this theorem parallels Ovsjannikov's method for second order equations. Once the possibility of nonprojectable symmetries is disposed of, section 4 shows that the symmetry group may be viewed as a subgroup of the conformal group of the symbol of the equation. Section 5 discusses the conformal groups of symmetric homogeneous polynomials and elucidates the concept of nondegeneracy. Section 6 ties together these results to get the bound on the dimension of $G/T$.

Some of the results of this paper appeared in my Ph.D. Thesis, [7]. I would like to express my profound gratitude to my advisor, Professor G. Birkhoff, for his invaluable help and encouragement.

## 2. Symmetry Groups of Partial Differential Equations

This section will provide a brief review of the Lie-Ovsjannikov theory of symmetry groups of partial differential equations. For simplicity we shall restrict our attention to equations in one dependent variable defined over an open subset of Euclidean space. Completely expositions of the general theory in Euclidean space may be found in references [1] and [6]. The generalization of the theory to arbitrary smooth manifolds is done in [8].

Consider the Euclidean space $Z = \mathbb{R}^p \times \mathbb{R}$ with coordinates $(x, u)$. Here $x = (x^1, \ldots, x^p)$ are the independent variables and $u$ the dependent variable in the partial differential equation under consideration. We construct the Euclidean space $J_n Z$, called the <u>n-jet bundle</u> of $Z$, which is of dimension $p + N_{n,p}$, where

$$N_{n,p} = 1 + p + \binom{p+1}{2} + \ldots + \binom{p+n-1}{n} .$$

Before proceeding further, some standard multi-index notation is presented.

<u>Definition. 2.1.</u> Define the set of p-multi-indices of rank n to be

$$\mathcal{S}^p_n = \{I = (i_1, \ldots, i_p) \colon 0 \leq i_\nu \in Z,\ |I| = i_1 + \ldots + i_p = n\} .$$

Let $\mathcal{S}^p = \bigcup_{n=0}^{\infty} \mathcal{S}^p_n$ be the set of all p-multi-indices. Given $I \in \mathcal{S}^p_n$, $J \in \mathcal{S}^p_n$, let $I + J$ be the multi-index with components $i_\nu + j_\nu$.

Introduce a partial ordering on $\mathcal{J}^p$ by defining $I \leq J$ whenever $i_\nu \leq j_\nu$ for all $\nu = 1, \ldots, p$. For $I \leq J$, let $J - I$ be the multi-index with components $j_\nu - i_\nu$. Define

$$I! = i_1! i_2! \ldots i_p! , \quad I \in \mathcal{J}^p ,$$

$$\binom{J}{I} = \frac{J!}{I!(J-I)!} , \quad I \leq J \in \mathcal{J}^p .$$

Let $\delta^j \in \mathcal{J}^p_1$ be the Kronecker multi-index, with components $\delta^j_\nu$.

For convenience the following notations for partial derivatives will be used:

$$\partial_i = \frac{\partial}{\partial x^i} , \quad \partial_u = \frac{\partial}{\partial u} ,$$

$$\partial_I = \partial_1^{i_1} \partial_2^{i_2} \ldots \partial_p^{i_p} , \quad I \in \mathcal{J}^p .$$

The coordinates on $J_n Z$ shall be written $(x, u, u^{(n)})$, where $u^{(n)}$ has components $u_I$ for $I \in \mathcal{J}^p$, $|I| \leq n$. If $f: \mathbb{R}^p \to \mathbb{R}$ is a smooth ($C^\infty$) function $j_n f: \mathbb{R}^p \to \mathbb{R}^{N_{n,p}}$, called the n-jet of $f$, given by $u_I = \partial_I f(x)$. (These concepts may of course be made coordinate free, cf. [4] or [8].) In this context, an n-th order partial differential equation is regarded as a closed subvariety $\Delta_0 \subset J_n Z$ given by the vanishing of a smooth function $\Delta: J_n Z \to \mathbb{R}$, i.e.,

$$\Delta_0 = \{(x, u, u^{(n)}) : \Delta(x, u, u^{(n)}) = 0\} .$$

A (smooth) solution of $\Delta_0$ is therefore a function $f: \mathbb{R}^p \to \mathbb{R}$ such that the graph of its n-jet, $j_n f$, is entirely contained in the subvariety $\Delta_0$.

Now suppose $G$ is a local Lie group of transformations acting on $Z$, cf. [2] or [9]. An element $g \in G$ is a local diffeomorphism $g: U_g \to Z$, where $U_g \subset Z$ is open. If $f: \mathbb{R}^p \to \mathbb{R}$ is a smooth function its graph is a smooth p-dimensional submanifold $M_f = \{(x, f(x)) : x \in \mathbb{R}^p\} \subset Z$, which is transversal to the fibers $\{x = \text{constant}\}$. Now $g \in G$ transforms $M_f \cap U_g$ into another p-dimensional submanifold of $Z$, but this transformed submanifold is not necessarily the graph of a function since the condition of transversality is not maintained. (These difficulties are dealt with in detail in [8].) However, transformations $g \in G$ sufficiently close to the identity locally preserve the transversality condition when transforming $M_f$. In other words for such $g$, there is an open subset $V \subset Z$ such that $g[M_f \cap V]$ is part of the graph of a function $g \bullet f: \mathbb{R}^p \to \mathbb{R}$. This local action on functions induces a local Lie group action of $G$ on the jet space $J_n Z$, called the n-th <u>prolongation</u> of $G$, and denoted by $pr^{(n)}G$. Thus

$$pr^{(n)}g(j_n f) = j_n(g \bullet f) .$$

The actual formulas for the prolonged group action are very complicated. However, there is a fairly simple formula relating the infinitesimal generators of $G$ to the infinitesimal generators of $pr^{(n)}G$. If $v$ is a vector field on $Z$, with local one-parameter group $\exp(tv)$, then its n-th prolongation is the vector field

$$\mathrm{pr}^{(n)}v = \frac{d}{dt}\Big|_{t=0} \mathrm{pr}^{(n)} \exp(tv)$$

on $J_n Z$.

Recall that the total derivative in the $x^i$ direction of a function $\varphi: J_k Z \to \mathbb{R}$ is the function $D_i\varphi: J_{k+1} Z \to \mathbb{R}$ given by

$$D_i\varphi(x, u, u^{(k+1)}) = \partial_i\varphi(x, u, u^{(k)}) + \sum_J u_{J+\delta^i} \frac{\partial\varphi}{\partial u_J}(x, u, u^{(k)}). \qquad (2.1)$$

Given $I \in \mathcal{S}^p$, let

$$D_I = D_1^{i_1} D_2^{i_2} \cdots D_p^{i_p} .$$

<u>Theorem 2.2,</u> [8]. Suppose

$$v = \sum_{i=1}^{p} \xi^i(x, u)\partial_i + \varphi(x, u)\partial_u \qquad (2.2)$$

is a smooth vector field on Z. Then the n-th prolongation of v is the vector field

$$\mathrm{pr}^{(n)}v = v + \sum_{|I| \le n} \varphi^I(x, u, u^{(n)}) \frac{\partial}{\partial u_I} ,$$

whose coefficient functions are

$$\varphi^I(x, u, u^{(n)}) = D_I\varphi - \sum_{0 < K \le I} \sum_{i=1}^{p} \binom{I}{K} u_{I-K+\delta^i} D_K \xi^i . \qquad (2.3)$$

An n-th order partial differential equation $\Delta_0 \subset J_n Z$ is said to be invariant under G (or G is a <u>symmetry group</u> of $\Delta_0$) if all the prolonged transformations in $\mathrm{pr}^{(n)}G$ leave the subvariety $\Delta_0$ invariant. Note that this implies that G transforms solutions of $\Delta_0$ to solutions. The converse of this statement is true providing that for any point in $\Delta_0$ there is a solution whose n-jet passes through that point. The next proposition follows from the standard infinitesimal criterion of invariance of a subvariety.

Proposition 2.3. Suppose $\Delta_0$ is a partial differential equation given by the vanishing of a smooth function $\Delta: J_n Z \to \mathbb{R}$. A connected local group of transformations $G$ acting on $Z$ is a symmetry group of $\Delta$ if and only if

$$\mathrm{pr}^{(n)} v[\Delta(x, u, u^{(n)})] = 0 \tag{2.4}$$

for all $(x, u, u^{(n)}) \in \Delta_0$ and all infinitesimal generators $v$ of $G$.

The symmetry group of a given partial differential equation will mean the largest connected local group of transformations whose prolongation leaves the equation invariant. Note that we are excluding discrete symmetries of the equation by requiring the group to be connected. Special care must be exercised when this "group" becomes infinite dimensional.

Now we consider a linear partial differential equation. Suppose

$$\Delta = \sum_{|I| \leq n} \sigma^I(x) \partial_I$$

is a n-th order linear differential operator, with corresponding equation

$$\Delta_0 = \{(x, u, u^{(n)}) : \Delta[u] = \sum \sigma^I(x) u_I = 0\} .$$

Suppose $v$ is a vector field on $Z$, with k-th order prolongation given by Theorem 2.2. The symmetry conditions of proposition 2.3 for $\Delta_0$ are

$$\sum_I (\sigma^I \varphi^I + \sum_{\nu=1}^{p} \xi^\nu \partial_\nu \sigma^P u_I) = \mu \sum_I \sigma^I u_I . \tag{2.5}$$

Here $\mu: J_k Z \to \mathbb{R}$ is a multiplier, which may a priori depend on $(x, u, u^{(n)})$, and reflects the fact that (2.4) only needs to hold on $\Delta_0$. Formula (2.3) shows that all the coordinates $u_I$ for $I > 0$ occur polynomially in (2.4), hence $\mu$ must be a polynomial in these variables. The results of this paper will come from a detailed analysis of the symmetry equations (2.5). The first symmetries we consider are the so-called nonprojectable symmetries. Projectable transformations are those of the form

$$g(x, u) = (g_1(x), g_2(x, u)).$$

A vector field v given by (2.2) generates a projectable one-parameter group if and only if

$$\partial_u \xi^i(x, u) = 0 \qquad i = 1, \ldots, p.$$

## 3. Nonprojectable Symmetries.

In this section we dispose of the nonprojectable symmetries of linear equations. The main result is that a linear equation of order $n \geq 3$ possesses only projectable symmetries. In the proof of this result, we rederive the result of Ovsjannikov, [6], that a second order linear equation possesses a nonprojectable symmetry if and only if it is strongly degenerate, i.e., is, under a change of coordinates, of the form

$$\sigma_2(x)\partial_1^2 u + \sigma_1(x)\partial_1 u + \sigma_0(x)u = 0.$$

More generally, let us make the following definition.

Definition 3.1. A linear partial differential operator $\Delta$ of order $n$ is partially degenerate if there is a local coordinate system $(x^1, \ldots, x^p)$ on $\mathbb{R}^p$ such that

$$\Delta = \sigma_n(x)\partial_1^n + \Delta',$$

where $\Delta'$ is a linear partial differential operator of order $n' < n$. An n-th order linear partial differential operator is strongly degenerate if there is a coordinate system $(x^1, \ldots, x^p)$ on $\mathbb{R}^p$ such that

$$\Delta = \sigma_n(x)\partial_1^n + \sigma_{n-1}(x)\partial_1^{n-1} + \ldots + \sigma_1(x)\partial_1 + \sigma_0(x).$$

In other words, $\Delta$ is equivalent under a change of independent variables to a parametrized ordinary differential operator.

The plan of proof is to first show that if $\Delta$ is of order $n \geq 2$, then $\Delta_0$ being invariant under a nonprojectable symmetry implies that $\Delta'$ is strongly degenerate. Then if $n \geq 3$, the strongly degenerate equations are also shown to have only projectable symmetries. The proof involves a detailed consideration of the symmetry equations (2.5) based on the formulas for the prolongations of vector fields given in Theorem 2.2. Throughout this section, the symbol $\mathcal{L}(k)$ shall be used to denote a polynomial expression in the partial derivatives $u_J$ for $|J| < k$.

Lemma 3.2. Let $\varphi: \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$ be smooth. Let $I \in \mathcal{S}^p_m$ be a multi-index. Then

$$D_I\varphi = u_I \frac{\partial\varphi}{\partial u} + \sum_{\nu=1}^{p} i_\nu u_{I-\delta^\nu} D_\nu \frac{\partial\varphi}{\partial u} + \mathcal{L}(m-1), \tag{3.1}$$

for $m > 2$. For $m = 2$, we have

$$D_jD_k\varphi = u_{jk}\frac{\partial\varphi}{\partial u} + u_k u_j \frac{\partial^2\varphi}{\partial u^2} + u_k \frac{\partial^2\varphi}{\partial x^j \partial u} + u_j \frac{\partial^2\varphi}{\partial x^k \partial u} + \mathcal{L}(1). \tag{3.1'}$$

The proof is an easy induction using the formula for total derivative (2.1). As a corollary of this, we see that the coordinates $\varphi^I$ of the prolongation of a vector field $v$ given by (2.3) are of the form

$$\varphi^I = \frac{\partial\varphi}{\partial u} u_I - \sum_{k=1}^{p}\sum_{\nu=1}^{p}\left[(i_k + \delta^k_\nu)\frac{\partial\xi^\nu}{\partial u} u_k + i_k \frac{\partial\xi^\nu}{\partial x^k}\right] u_{I-\delta^k+\delta^\nu} + \mathcal{L}(m), \tag{3.2}$$

for $I \in \mathcal{S}^p_m$. To see this, it suffices to note that the only terms in the sum in (2.3) that contribute m-th order derivatives of u are when $K = I$ or $K = \delta^k$ for some k.

Lemma 3.3. If $\Delta_0$ is invariant under the one parameter group generated by the vector field v with multiplier $\mu$, then

$$\mu = \mu_0(x,u) + \sum_{k=1}^{p} \mu_k(x,u)u_k . \tag{3.3}$$

Moreover,

$$\mu_k(x,u) = \alpha_k \frac{\partial \xi^k}{\partial u} , \quad k = 1, \ldots, p , \tag{3.4}$$

where $\alpha_k$ is the integer defined by

$$\alpha_k = -1 - \sup\{j_k : J \in \mathcal{S}_n^p , \sigma^J(x) \neq 0\} . \tag{3.5}$$

Proof. Using (3.2), the left-hand side of the symmetry equation (2.5) is

$$\sum_{I \in \mathcal{S}_n^p} [(\sigma^I \partial_u \varphi + \sum_\nu \xi^\nu \partial_\nu \sigma^I) u_I - \sum_{k,\nu} ((i_k + \delta_\nu^k)\partial_u \xi^\nu u_k + i_k \partial_k \xi^\nu) \cdot u_{I-\delta^k+\delta^\nu}] + \mathcal{L}(n). \tag{3.6}$$

This proves $\mu$ must be of the form (3.3). Equating the coefficients of $u_k u_J$ yields the following important equations:

$$- \sum_{\nu=1}^{p} (j_k + 1)\sigma^{J-\delta^\nu+\delta^k} \partial_u \xi^\nu = \mu_k \sigma^J , \tag{3.7}$$

for all $1 \leq k \leq p$, $J \in \mathcal{S}_n^p$. Given k, let $\ell = \sup\{j_k : J \in \mathcal{S}_n^p , \sigma^J(x) \neq 0\}$. For this value of k and some J with $j_k = \ell$, $\sigma^J(x) \neq 0$, equation (3.7) reduces to

$$-(\ell+1)\sigma^J \partial_u \xi^k = \mu_k \sigma^J ,$$

proving the lemma.

Combining equations (3. 4) and (3. 7), we have

$$\sum_{\nu=1}^{p} (j_k + 1 + \alpha_k \delta^k_\nu) \sigma^{J-\delta^\nu+\delta^k} \partial_u \xi^\nu = 0, \qquad (3.8)$$

for all $1 \leq k \leq p$, $J \in \mathcal{S}^p_n$.

Lemma 3.4. For each $k = 1, \ldots, p$ and each $(x, u)$, either $\partial_u \xi^k(x,u) = 0$ or $\sigma^{n\delta^k}(x) \neq 0$.

Proof. Assume that $\sigma^{n\delta^k}(x) = 0$. Let $\ell = -(\alpha_k + 1) < n$, so that $\sigma^{\ell\delta^k + I}(x) \neq 0$ for some $I \in \mathcal{S}^p_{n-\ell}$ with $i_k = 0$. Choose $j$ so that $i_j \neq 0$. Then (3.8) in the case $J = (\ell + 1)\delta^k + I - \delta^j$ and $j$ replacing $k$ is

$$i_j \sigma^{\ell\delta^k + I}(x) \partial_u \xi^k(x,u) = 0,$$

hence $\partial_u \xi^k = 0$.

Lemma 3. 5. An n-th order linear partial differential operator is partially degenerate if and only if its n-th order coefficient functions are of the form

$$\sigma^J(x) = \frac{n!}{J!} \rho^J(x) \hat{\sigma}(x), \qquad J \in \mathcal{S}^p_n, \qquad (3.9)$$

for some real-valued functions $\rho^1, \ldots, \rho^p, \hat{\sigma}$. (Here $\rho^J$ denotes $(\rho^1)^{j_1} (\rho^2)^{j_2} \ldots (\rho^p)^{j_p}$.)

The proof follows directly from the way the operator $\partial/\partial x^1$ behaves under a change of coordinates, cf. [7]. We are now in a position to prove our first main result:

Theorem 3.6. If $\Delta_0$ is a linear partial differential equation of order $n \geq 2$ which is invariant under a nonprojectable group of transformations, then $\Delta$ is strongly degenerate.

Proof. We first show that $\Delta$ must be partially degenerate via the criterion of Lemma 3.5. Let

$$\rho^i(x,u) = \partial_u \xi^i(x,u).$$

By the assumption of nonprojectability, $\rho^k(x,u) \neq 0$ for at least one $k$, hence by Lemma 3.4, $\sigma^{n\delta^k}(x) \neq 0$ for this value of $k$. Define

$$\hat{\sigma}(x,u) = \sigma^{n\delta^k}(x)[\rho^k(x,u)]^{-n} . \tag{3.10}$$

Suppose $J = (n-\ell)\delta^k + I$, where $I \in \mathcal{S}^p_{n-\ell}$ with $i_k = 0$. Criterion (3.9) shall be proven for any fixed value of $u$ by induction on $\ell$. Equation (3.10) constitutes the case $\ell = 0$. Now (3.8) implies

$$-\ell\sigma^J \partial_u \xi^k + (n-\ell+1) \sum_{\nu \neq k} \sigma^{(n-\ell+1)\delta^k + I - \delta^\nu} \partial_u \xi^\nu = 0 .$$

Therefore, by induction,

$$\ell\sigma^J \rho^k = (n-\ell+1) \sum_{\substack{\nu \neq k \\ i_\nu \neq 0}} \frac{n!}{(I-\delta^\nu)!\,(n-\ell+1)!} \rho^J \rho^k \hat{\sigma}$$

$$= \frac{n!}{J!} \Big[ \sum_{\nu \neq k} i_\nu \Big] \rho^J \rho^k \hat{\sigma} = \frac{n!}{J!} \ell \rho^J \hat{\sigma} \rho^k .$$

This proves the partial degeneracy of $\Delta$.

To prove the strong degeneracy, we work in the coordinate system $(x^1, \ldots, x^p)$ such that

$$\Delta = \sigma_n(x)\,\partial_1^{\,n} + \sigma_{n-1}(x)\partial_1^{\,n-1} + \ldots + \sigma_0(x) + \Delta' ,$$

where $\Delta'$ is a linear partial differential operator of order $n' < n$. By Lemma 3.2, $\partial_u \xi^k = 0$ for $k = 2, \ldots, n$, hence

$$\mu = \mu_0(x,u) - (n+1)\partial_u \xi^1(x,u)\,u_1 .$$

Using Lemma 3.2 on the symmetry equations (2.5), the only terms containing $u_k u_J$ for $k \neq 1$, $|J| = n'$, are

$$\sum \sigma^J \sum_{k=2}^{p} \partial_u \xi^1 u_k u_{J-\delta^k+\delta^1} = 0 ,$$

where the sum is over all $J \in \mathcal{S}^p_{n'}$ except $n'\delta^1$. This immediately implies $\sigma^J(x) = 0$ for all such $J$, hence $\Delta'$ is in reality of order $n'-1$. This completes our proof.

<u>Theorem 3.7.</u> The symmetry group of a linear partial differential equation of order $n \geq 3$ contains no nonprojectable symmetries.

<u>Proof.</u> By Theorem 3.6, it only remains to consider the strongly degenerate equations. Suppose $v$ is an infinitesimal symmetry of

$$\Delta[u] = \sum_{i=0}^{n} \sigma_i(x)\partial_1^{\,i} u .$$

Using the prolongation formula (2.3) and Lemma 3.2, it can be seen that the only terms in the symmetry equation (2.5) involving $u_{2\delta^1} u_{(n-1)\delta^1}$ are

$$-\sigma^n(x)\left[\binom{n}{2} + \binom{n}{n-1}\right]\partial_u \xi^1 u_{2\delta^1} u_{(n-1)\delta^1} = 0 .$$

(This relies heavily on the fact that $n > 2$.) We conclude that $\partial_u \xi^1 = 0$, and hence v is projectable.

To show that this theorem is not true when $n = 2$, we compute the symmetry group of the equation

$$u_{xx} = 0$$

on $\mathbb{R}^2 \times \mathbb{R}$. If $v = \xi\, \partial/\partial x + \eta\, \partial/\partial y + \varphi \partial/\partial u$ is an arbitrary vector field, then the symmetry equations (2.5) become

$$0 = \varphi_{xx} + u_x(2\varphi_{xu} - \xi_{xx}) + u_x^2(\varphi_{uu} - 2\xi_{xu})$$
$$- u_x^3 \xi_{uu} - u_y(\eta_{xx} + 2u_x \eta_{xu} + u_x^2 \eta_{uu}) - 2u_{xy}(\eta_x + u_x \eta_u).$$

Therefore

$$\varphi_{xx} = 0$$
$$2\varphi_{xu} = \xi_{xx}$$
$$\varphi_{uu} = 2\xi_{xu}$$
$$\xi_{uu} = 0$$
$$\eta_y = 0 .$$

We conclude that the infinitesimal symmetry algebra is generated by all vector fields of the form

$$[c_1 + c_4 x + c_5 u]\frac{\partial}{\partial x} + c_2 \frac{\partial}{\partial y} + [c_3 + c_6 x + (c_7 + 2c_4 x)u]\frac{\partial}{\partial u} ,$$

where $c_1, \ldots, c_7$ are arbitrary functions of y.

## 4. Conformal Transformations

We now make the connection between the projectable symmetries of a linear partial differential equation and the group of conformal transformations of the top order symbol of the equation. We first recall some standard notations from differential geometry. If $X$ is a smooth manifold, we let $TX$ (resp. $T^*X$) denote the tangent (resp. cotangent) bundle of $X$. If $f: X \to X$ is any smooth map, then there are induced maps $df: TX|_x \to TX|_{f(x)}$ and $\delta f: T^*X|_{f(x)} \to T^*X|_x$, cf. [10]. Given a vector space $V$, we let $\odot_n V$ denote the n-th symmetric power of $V$, cf. [3]. The elements of $\odot_n V$ are all of the form $v_1 \odot v_2 \odot \dots \odot v_n$ for $v_i \in V$, where $\odot$ is a symmetric product, i.e., $v \odot v' = v' \odot v$. If $\{e_1, \dots, e_p\}$ forms a basis of $V$, then $\{e_I = (e_1)^{i_1} \odot \dots \odot (e_p)^{i_p} : I \in \mathcal{S}_n^p\}$ forms a basis of $\odot_n V$. We may identify $\odot_n V$ with the space of all n-th order homogeneous polynomials on the dual space $V^*$ by evaluating $\Phi \in \odot_n V$ at the element $\omega \odot \omega \odot \dots \odot \omega \in \odot_n V^*$. The inverse map is given by the standard polarization process. In a similar fashion, we may define the symmetric powers $\odot_n TX$ and $\odot_n T^*X$ of the tangent and cotangent bundles of $X$. If $\{\partial_i = \partial/\partial x^i : i = 1, \dots, p\}$ is a basis of $TX$ corresponding to a local coordinate system on $X$, then $\{\partial_I : I \in \mathcal{S}_n^p\}$ is a basis of $\odot_n TX$. Similarly, a basis of $\odot_n T^*X$ is given by $\{dx^I : I \in \mathcal{S}_n^p\}$. These bases are dual to within a factor:

$$dx^I(\partial_I) = I! \quad , \quad dx^I(\partial_J) = 0 \quad , \quad I \neq J.$$

Suppose

$$\Delta = \sum \sigma^I(x)\partial_I$$

is a linear partial differential operator of order $n$. The (top order) symbol of $\Delta$ is the map

$$\sigma : \mathbb{R}^p \to \odot_n T\mathbb{R}^p$$

given by

$$\sigma(x) = \sum_{|I|=n} \sigma^I(x)\,\partial_I\,|_x \;.$$

(This definition can of course be made in a coordinate free fashion.)

Definition 4.1 Let $X$ be a smooth manifold and $\sigma: X \to \odot_n TX$ a smooth section. A map $f: X \to X$ is called a conformal transformation of $\sigma$ if for each $x \in X$ and any covectors $\omega_1, \ldots, \omega_k \in T^*X|_{f(x)}$ we have

$$\sigma[\delta f(\omega_1)\odot \ldots \odot \delta f(\omega_k)\,;x] = \mu(x)\sigma[\omega_1\odot \ldots \odot \omega_k; f(x)]$$

for some real-valued function $\mu: X \to \mathbb{R}$.

A vector field $v$ will be called an infinitesimal conformal transformation if the local transformations $\exp(tv)$ are conformal for $t$ sufficiently small.

Lemma 4.2. Let $x = (x^1, \ldots, x^p)$ be local coordinates on $X$. The vector field $v = \sum \xi^i(x)\partial_i$ is an infinitesimal conformal transformation of $\sigma = \sum \sigma^I(x)\partial_I$ if and only if there is a function $\mu: X \to \mathbb{R}$ such that

$$\sum_{k=1}^{p}\sum_{j=1}^{p}(i_j + 1 - \delta_j^k)\sigma^{I-\delta^j+\delta^k}(x)\partial_j\xi^k(x) = \mu(x)\sigma^I(x) + \sum_{k=1}^{p} \partial_k\sigma^I(x)\xi^k(x)$$

for every multi-index $I \in \mathcal{S}_n^p$.

Proof. Let $f_t = \exp(tv)$. We have

$$\delta f_t(dx^i) = \sum_j \partial_j f_t^i(x) dx^j .$$

Therefore by the duality of the bases of the tangent and cotangent spaces,

$$\frac{d}{dt}\Big|_{t=0} \sigma[\delta f_t(dx^I); x] = \sum_{j,k} (I-\delta^j+\delta^k)!\, i_j \partial_j \xi^k(x) \sigma^{I-\delta^j+\delta^k}(x) .$$

On the other hand, this must equal

$$\frac{d}{dt}\Big|_{t=0} \mu_t(x)\sigma[dx^I; f_t(x)] = I!\, [\mu(x)\sigma^I(x) + \sum_{k=1}^{p} \xi^k(x)\partial_k \sigma^I(x)] ,$$

where $\mu(x) = \frac{d}{dt}\Big|_{t=0} \mu_t(x)$. The proof of the converse follows via the usual properties of the exponential map.

Now suppose $\Delta_0$ is an n- th order linear partial differential equation and $v = \sum_i \xi^i(x)\partial_i + \varphi(x,u)\partial_u$ is a projectable symmetry of $\Delta_0$. We shall call the vector field $v' = \sum_i \xi^i(x)\partial_i$ on $\mathbb{R}^p$ the projection of $v$. (In general, if $f(x,u) = (f_1(x), f_2(x,u))$ is any projectable transformation of $\mathbb{R}^p \times \mathbb{R}$, the transformation $f_1(x)$ on $\mathbb{R}^p$ shall be called the projection of f.)

Proposition 4.3. If a linear partial differential equation $\Delta_0$ is invariant under a projectable one parameter group with generator $v = \sum \xi^i\partial_i + \varphi\partial_u$, then $\varphi(\cdot, 0)$ is a solution of $\Delta_0$. If $\Delta_0$ contains no term of order zero, then $\varphi(\cdot, u)$ forms a one parameter family of solutions of $\Delta_0$.

Proof. Note first that if $(x, u, u^{(n)})$ is any solution of $\Delta_0$, so is $(x, \lambda u, \lambda u^{(n)})$ for any $\lambda \in \mathbb{R}$. Applying $pr^{(n)}v$ to the equation and evaluating at $(x, \lambda u, \lambda u^{(n)})$ we get

$$\sum \sigma^I(x)\varphi^I(x, \lambda u, \lambda u^{(n)}) + \lambda \sum v(\sigma^I(x))u_I = 0.$$

Letting $\lambda \to 0$ and noting that $\varphi^I(x, u, 0) = \partial_I\varphi(x, u)$ gives the first result. The second follows from considering $\Delta_0$ at the point $(x, u, \lambda u^{(n)})$.

Corollary 4.4. If $\sum \xi^i\partial_i + \varphi(x, u)\partial_u$ is a projectable symmetry of $\Delta_0$, then

$$\varphi(x, u) = \alpha(x) + \beta(x)u$$

where $\alpha$ (and, if $\Delta_0$ has no zeroth order term, $\beta$ ) is a solution of $\Delta_0$.

The proof is a direct consequence of Lemmas 3.2, 3.3 and the symmetry equations (2.5). The details are omitted. Now consider the top order terms in the symmetry equations for a projectable symmetry of $\Delta_0$. By Lemma 3.2 and Corollary 4.4,

$$\varphi^I = \beta u_I - \sum_{j=1}^{p} \sum_{k=1}^{p} i_j \partial_j \xi^k u_{J-\delta^k+\delta^k} + \mathcal{L}(n)$$

for each $I \in \mathcal{S}_n^p$. Comparing the coefficients of $u_J$ for each $J \in \mathcal{S}_n^p$ in (2.5) shows first that $\mu = \mu(x)$ and second that the conformal equations of Lemma 4.2 are satisfied for the projection of $v$ with $\lambda = \beta - \mu$.

Summaring we have

Theorem 4.5. The group of projectable symmetries of a linear partial differential equation, when projected, is a subgroup of the conformal group of the symbol of the equation.

## 5. Symmetries of Homogeneous Polynomials

Consider the space of all n-th order homogeneous polynomials defined on the vector space $(\mathbb{C}^p)^*$. According to section 4, we may identify this space with the n-th order symmetric power $\odot_n \mathbb{C}^p$. Let $e_1, \ldots, e_p$ be a basis of $\mathbb{C}^p$, so that $\{e_I : I \in \mathcal{S}_n^p\}$ forms a basis of $\odot_n \mathbb{C}^p$. A polynomial $\sigma \in \odot_n \mathbb{C}^p$ is given by

$$\sigma = \sum_I \sigma^I e_I ,$$

and the complex numbers $\sigma^I$ shall be thought of as the "local coordinates" of $\sigma$. If $\zeta \in (\mathbb{C}^p)^*$, with coordinates $\zeta^i = \zeta(e_i)$, then

$$\sigma(\zeta) = \sigma(\zeta^p) = \sum \sigma^I \zeta^I ,$$

explicitly showing the polynomial nature of $\sigma$.

There is a natural action of $GL(p)$, the general linear group of $\mathbb{C}^p$, on $\odot_n C^p$:

$$A(v_1 \odot \ldots \odot v_p) = A(v_1) \odot \ldots \odot A(v_p) , \quad A \in GL(p), \quad v_1, \ldots, v_p \in V.$$

Let $\gamma : \odot_n \mathbb{C}^p \times gl(p) \rightarrow T\odot_n \mathbb{C}^p$ denote the infinitesimal version of this action:

$$\gamma(\alpha)\big|_\sigma = \frac{d}{dt}\Big|_{t=0} \exp(t\alpha)(\sigma) \quad , \alpha \in gl(n), \quad t \in \mathbb{R}, \ \sigma \in \odot_n \mathbb{C}^p .$$

With respect to the basis $\{e_1, \ldots, e_p\}$, $\alpha$ has the matrix form $(\alpha_j^i)$, and in terms of the induced coordinates $\sigma^I$ on $\odot_n \mathbb{C}^p$,

$$\gamma(\alpha) = \sum_I \left[ \sum_{k=1}^{p} \sum_{j=1}^{p} (i_k + 1 - \delta_k^j) \alpha_j^k \sigma^{I - \delta^j + \delta^k} \right] \frac{\partial}{\partial \sigma^I} \quad . \tag{5.1}$$

Definition 5.1. Let

$$n_\sigma = \dim \gamma[gl(n)]|_\sigma \quad , \sigma \in \odot_n \mathbb{C}^p .$$

A polynomial $\sigma$ is called nondegenerate if $n_\sigma = n^2$. Otherwise $\sigma$ is called degenerate.

A (conformal) symmetry of a polynomial $\sigma \in \odot_n \mathbb{C}^p$ is a linear transformation $A \in GL(n)$ satisfying

$$A(\sigma) = \lambda\sigma$$

for some $\lambda \in \mathbb{C}$. A matrix $\alpha \in gl(n)$ is called an infinitesimal symmetry of $\sigma$ if the matrices $\exp(t\alpha)$ are symmetries for $t \in \mathbb{R}$ sufficiently small. Note that the infinitesimal symmetries of a given $\sigma$ form a Lie subalgebra of $gl(n)$, called the symmetry algebra of $\sigma$.

Theorem 5.2. A matrix $\alpha$ is an infinitesimal symmetry of a polynomial $\sigma \in \odot_n \mathbb{C}^p$ if and only if $\gamma(\alpha)|_\sigma = \gamma(\lambda \underset{\sim}{1})|_\sigma$ for some $\lambda \in \mathbb{C}$. (Here $\underset{\sim}{1}$ is the identity matrix.)

Corollary 5.3. The dimension of the symmetry algebra of a polynomial $\sigma$ is one if and only if the polynomial is nondegenerate. In fact, the only symmetries of a nondegenerate $\sigma$ are the multiples of the identity matrix.

Note that Corollary 5.3 is true in $\mathbb{R}^p$ as well as $\mathbb{C}^p$. The reason for using complex space is in the following useful characterization of the nondegenerate polynomials. We regard the multi-indices $I = (i_1, \ldots, i_p)$ as points in $\mathbb{R}^p$. Note that all the multi-indices in $\mathcal{S}_n^p$ lie in the (n-1)-plane $\{x : x_1 + \ldots + x_p = n\}$. Let $\sigma \in \odot_n \mathbb{C}^p$. For each basis $E = \{e_1, \ldots, e_p\}$ of $\mathbb{C}^p$, let $N_\sigma(E)$ denote the dimension of the smallest affine subspace of $\mathbb{R}^p$ containing the set $\{I : \sigma^I \neq 0\}$. Let $N_\sigma$ denote the minimum of all $N_\sigma(E)$ for all possible choices of bases of $\mathbb{C}^p$. Note that $0 \leq N_\sigma \leq n-1$.

<u>Theorem 5.4.</u> A homogeneous polynomial $\sigma \in \odot_n \mathbb{C}^p$ is nondegenerate if and only if $N_\sigma < n-1$.

<u>Proof.</u> First suppose that $N_\sigma < n-1$. This means that for some basis of $\mathbb{C}^p$ there exist real constants $\alpha_1, \ldots, \alpha_p$, not all equal, and a real constant c such that $\sigma^I = 0$ unless $\sum \alpha_k i_k = c$. But this implies, for $\alpha = \text{diag}(\alpha_1, \ldots, \alpha_p)$,

$$\gamma(\alpha)\big|_\sigma = \gamma(c\underset{\sim}{1})\big|_\sigma ,$$

with $\alpha$ not being a multiple of $\underset{\sim}{1}$.

Conversely, suppose $\sigma$ is degenerate. This implies there exists $0 \neq \alpha \in gl(n)$ with $\gamma(\alpha)\big|_\sigma = 0$. By a suitable choice of basis in $\mathbb{C}^p$, we can assume that $\alpha$ is in Jordan canonical form, i.e.,

$$\begin{pmatrix} \alpha_1 & \varepsilon_1 & & & \bigcirc \\ & \alpha_2 & \varepsilon_2 & & \\ & & \ddots & \ddots & \\ & & & \alpha_{p-1} & \varepsilon_{p-1} \\ \bigcirc & & & & \alpha_p \end{pmatrix}$$

where $\varepsilon_i = 0$ or $1$ and if $\varepsilon_i = 1$, then $\alpha_i = \alpha_{i+1}$. Substituting into (5.1) we see that

$$\sum_{k=1}^{p} [\alpha_k i_k \sigma^I + \varepsilon_k (i_k + 1)\sigma^{I-\delta^{k+1}+\delta^k}] = 0 \tag{5.2}$$

for each $I \in \mathcal{S}_n^p$. Now define

$$f(I) = i_2 + 2i_3 + 3i_4 + \ldots + (p-1)i_p .$$

Note that

$$f(I - \delta^{k+1} + \delta^k) = f(I) - 1$$

for any k. Let

$$\ell_0 = \min \{f(I) : \sigma^I \neq 0\} .$$

For any I with $f(I) = \ell_0$, equation (5.2) implies $(\sum \alpha_k i_k)\sigma^I = 0$, hence $\sigma^I \neq 0$ implies $h(I) = \sum \alpha_k i_k = 0$. In particular, this implies all the $\alpha_k$'s cannot be equal unless they are all 0. However, this special case can be dealt with by noting that at least one $\varepsilon_k$ must be nonzero, and then (5.2) implies $\sigma^I = 0$ unless $i_k = 0$, which proves the theorem.

We therefore may assume all the $\alpha_k$'s are not equal. Let us assume by induction that for all I with $f(I) < \ell$, $\sigma^I = 0$ unless $h(I) = 0$. Suppose I is given with $f(I) = \ell$, and $h(I) \neq 0$. Since

$$h(I - \delta^{k+1} + \delta^k) = h(I) + (\alpha_k - \alpha_{k+1}) ,$$

equation (5.2) only contains a coefficient $\sigma^J$ with $J \neq I$ when $f(J) < \ell$ and $h(J) = h(I)$. By induction, these $\sigma^J = 0$, hence (5.2) reduces to

$$h(I)\sigma^I = 0.$$

This completes the induction on $\ell$ and thereby proves the result.

Suppose $p > 1$. It is not hard to see that if $n = 1$ or $2$, then every homogeneous polynomial is, in our terminology, degenerate. However, for $n > 2$, an open dense subset of homogeneous polynomials in $\odot_n \mathbb{C}^p$ are nondegenerate. In particular,

$$\sigma(\zeta) = \zeta_1^p + \zeta_2^p + \ldots + \zeta_n^p$$

is nondegenerate for $p > 2$.

## 6. The Triviality of the Symmetry Group

We now apply the main result of the preceding section to prove the bound on dim $G/T$ for a linear partial differential equation with non-degenerate symbol. This shall be accomplished in two stages. The case when the symbol is constant shall be proven directly from section 5. The general case will then be reduced to this special case. In this section we shall return to working in $\mathbb{R}^p$, although the results hold equally well in $\mathbb{C}^p$.

To keep the notation straight, we define

$$\mathcal{V} = \{v : \mathbb{R}^p \to T\mathbb{R}^p\}$$

to be the Lie algebra of all smooth vector fields on $\mathbb{R}^p$. Let

$$\mathcal{M} = \{\alpha : \mathbb{R}^p \to gl(p)\}$$

be the space of all smooth matrix valued maps on $\mathbb{R}^p$. We define a map

$$\rho : \mathcal{V} \to \mathcal{M}$$

as follows: if $v = \sum \xi^i(x)\, \partial_i \in \mathcal{V}$, then $\rho(v)(x)$ is the matrix with components $\alpha^i_j = \partial_j \xi^i(x)$. In other words $\rho(v)$ is just the Jacobian matrix of $\xi = (\xi^1, \ldots, \xi^p)$. Let $\mathcal{V}_0 \subset \mathcal{V}$ denote the subalgebra of all constant vector fields, and let $\mathcal{V}_1 \subset \mathcal{V}$ denote the subalgebra of all vector fields whose coefficient functions are linear homogeneous polynomials in $(x^1, \ldots, x^p)$. Let $\mathcal{M}_0 \subset \mathcal{M}$ denote the subspace of all constant maps. We then have $\ker \rho = \mathcal{V}_0$, and

$$\rho : \mathcal{V}_1 \xrightarrow{\sim} \mathcal{M}_0$$

is a Lie algebra isomorphism if we use the reversed Lie bracket on $\mathfrak{M}_0$:

$$[\alpha, \beta] = \beta\alpha - \alpha\beta \quad , \quad \alpha, \beta \in \mathfrak{M}_0 .$$

More generally, if $\alpha \in \mathfrak{M}$, then $\alpha \in \operatorname{im} \rho$ if and only if $\alpha$ satisfies the integrability conditions

$$\partial_k \alpha_j^i(x) = \partial_j \alpha_k^i(x) \quad , \qquad i, j, k = 1, \ldots, p. \tag{6.1}$$

Now suppose that $\Delta_0$ is an n-th order linear partial differential equation on $\mathbb{R}^p \times \mathbb{R}$, whose top order symbol is a constant map into $\odot_n \mathbb{R}^p$. We shall identity $\sigma$ with its image. From Lemma 4.2 and formula (5.1) we have

<u>Lemma 6.1.</u> A vector field $v \in \mathcal{V}$ is an infinitesimal conformal transformation of $\sigma$ (in the sense of section 4) if and only if $\gamma(\rho(v))$ is an infinitesimal symmetry of $\sigma$ (in the sense of section 5).

<u>Lemma 6.2.</u> Let $0 \neq \alpha_0 \in gl(n)$ and let $\alpha(x) = f(x)\alpha_0$ for some smooth function $f: \mathbb{R}^n \to \mathbb{R}$. Then $\alpha \in \operatorname{im} \rho$ if and only if $f$ is constant or $\alpha_0$ is a rank one matrix and, under a linear change of coordinates, $f(x) = f(x^1)$.

<u>Proof.</u> Let $\alpha_0 = (\alpha_j^i)$. The integrability conditions (6.1) for $\alpha$ are

$$\partial_k f(x)\alpha_j^i = \partial_j f(x)\alpha_k^i .$$

Therefore, if $f$ is not constant, there exist vectors $a = (a^1, \ldots, a^p)$ and $b = (b_1, \ldots, b_p)$ with $\alpha_j^i = a^i b_j$ and

$$b_j \partial_k f(x) = b_k \partial_j f(x) \quad , \quad j, k = 1, \ldots, p .$$

These equations imply

$$f(x) = f\Big(\sum_{i=1}^{p} \big(\prod_{j\neq i} b_j\big)x^i\Big),$$

proving the lemma.

Definition 6.3. We define the Lie group $G_0$ acting on $\mathbb{R}^p$ to be generated by the translations

$$T_t^i : (x^1, \ldots, x^p) \mapsto (x^1, \ldots, x^i + t, \ldots, x^p), \quad t \in \mathbb{R},$$

for $i = 1, \ldots, p$, and the single scale transformation

$$S_t : (x^1, \ldots, x^p) \mapsto (e^t x^1, \ldots, e^t x^p), \quad t \in \mathbb{R}.$$

$G_0$ may be thought of as the semidirect product of $\mathbb{R}$ and $\mathbb{R}^p$, its elements being all matrices of the form

$$\begin{pmatrix} \lambda \underset{\sim}{1} & v \\ 0 & 1 \end{pmatrix} \qquad \lambda \in \mathbb{R}, \quad v \in \mathbb{R}^p,$$

where $\underset{\sim}{1}$ is the $p \times p$ identity matrix.

Theorem 6.4. If $\Delta_0$ is a linear partial differential equation with constant, nondegenerate symbol, then the symmetry group of $\Delta_0$, when projected onto $\mathbb{R}^p$, is contained in $G_0$.

Proof. Since the symmetries of $\Delta_0$, when projected, are contained in the conformal symmetries of $\sigma$, it suffices to prove that $G_0$ is the conformal group of $\sigma$. It is obvious that all the transformations in $G_0$ are conformal transformations of $\sigma$. Since $\sigma$ is nondegenerate all its symmetries in $\mathcal{M}$ must be of the form $f(x)\underset{\sim}{1}$. However, according to Lemma 3, the only ones of these contained in im $\rho$ are those with $f$ constant. This proves the theorem.

We now turn to the more general case of partial differential equations with nonconstant symbols. Let $(x^1, \ldots, x^p)$ be a local coordinate system on $\mathbb{R}^p$. Let $\sigma: \mathbb{R}^p \to \odot_n T\mathbb{R}^p$ be the symbol of the equation $\Delta_0$.

Lemma 6.5. Suppose $\partial_1, \ldots, \partial_\ell$ are infinitesimal conformal symmetries of $\sigma$. Then there exists a smooth function $h: \mathbb{R}^p \to \mathbb{R} - \{0\}$ such that

$$\sigma(x) = h(x)\hat{\sigma}(x^{\ell+1}, \ldots, x^p),$$

where $\sigma: \mathbb{R}^p \to \odot_n T\mathbb{R}^p$ depends only on the last $p-\ell$ coordinates, i.e., $\hat{\sigma}$ is constant along the leaves of the involutive differential system spanned by $\partial_1, \ldots, \partial_\ell$.

Proof. If $\partial_1$ is a conformal symmetry of $\sigma$, then $\partial_1 \sigma^I(x) = \lambda(x)\sigma^I(x)$ for each multi-index $I \in \mathcal{S}_n^p$. Therefore

$$\sigma^I(x) = h(x)\hat{\sigma}^I(x^2, \ldots, x^p),$$

where $h$ is nonvanishing and $\partial_1 \log h = \lambda$. This proves the lemma for $\ell = 1$. Next suppose $\partial_1$ and $\partial_2$ are conformal symmetries of $\sigma$. Then

$$\sigma^I(x) = h_1(x)\hat{\sigma}_1^I(x^2, x^3, \ldots, x^p) = h_2(x)\hat{\sigma}_2^I(x^1, x^3, \ldots, x^p).$$

Therefore

$$\sigma^I(x) = h(x)\hat{\sigma}^I(x^3, \ldots, x^p)$$

where

$$h(x) = h_1(x)\hat{\sigma}_1^J(x) = h_2(x)\hat{\sigma}_2^J(x),$$

$$\hat{\sigma}^I(x) = \frac{\hat{\sigma}_1^I(x)}{\hat{\sigma}_1^J(x)} = \frac{\hat{\sigma}_2^I(x)}{\hat{\sigma}_2^J(x)},$$

where $J$ is chosen so that $\sigma^J(x)$ does not vanish locally. This proves the lemma for $\ell = 2$. The proof for more general $\ell$ is clear. (The passage from a local to a global result is made via a partition of unity.)

Lemma 6.6. Suppose $\sigma$ satisfies the hypotheses of Lemma 6.5. Then a vector field of the form

$$v = \xi^1(x)\partial_1 + \dots + \xi^\ell(x)\partial_\ell$$

is an infinitesimal conformal transformation of $\sigma$ if and only if it is an infinitesimal conformal transformation of $\hat{\sigma}$.

Proof. Using the formula of Lemma 3.2,

$$h(x)\sum (i_k + 1 - \delta_j^k)\hat{\sigma}^{I-\delta^j+\delta^k}\partial_j\xi^k = [h(x)\mu(x) + v(h(x))]\hat{\sigma}^I .$$

Thus $\sigma$ satisfies the same formula with $\hat{\mu} = \mu + v(\log h)$ replacing $\mu$.

Theorem 6.7. Suppose $\Delta_0$ is a linear partial differential equation whose symbol is nondegenerate on a dense open subset of $\mathbb{R}^p$. Then the symmetry group of $\Delta_0$ is projectable and its projection onto $\mathbb{R}^p$ has dimension at most $p+1$.

Proof. By the remarks at the end of section 5, we know that $\Delta$ must be at least a third order equation on a dense open subset of $\mathbb{R}^p$, hence its symmetry group, when projected, is contained in the conformal group of its symbol, $\sigma$. For each $x \in \mathbb{R}^p$, let $\mathcal{D}|_x$ denote the subspace of $T\mathbb{R}^p|_x$ spanned by all infinitesimal conformal symmetries of $\sigma$ at x. The differential system $\mathcal{D}$ is involutive since the Lie bracket of two symmetries is again a symmetry. Let $U_\ell \subset \mathbb{R}^n$ be a maximal open subset where $\mathcal{D}$ has constant dimension $\ell$ and $\sigma$ is nondegenerate. We know that $\bigcup_{\ell=0}^{p} U_\ell$ is dense in $\mathbb{R}^p$. Using Frobenius' theorem, cf. [10], we have local coordinates on $U_\ell$ so that $\mathcal{D}$ is spanned by the conformal symmetries $\partial_1, \ldots, \partial_\ell$. Using Lemma 6.5, in these coordinates $\sigma(x) = h(x)\hat{\sigma}(x^{\ell+1}, \ldots, x^p)$. Note that since $\sigma$ is nondegenerate, so is $\hat{\sigma}$, since h is nonvanishing. Since we are in $U_\ell$, the only symmetries of $\hat{\sigma}$ are of the form $v = \xi^1(x)\partial_1 + \ldots + \xi^\ell(x)\partial_\ell$. Moreover, since $\mathcal{D}$ is involutive, each $\xi^i$ can only depend on $(x^1, \ldots, x^\ell)$. But since $\hat{\sigma}$ depends only on $(x^{\ell+1}, \ldots, x^p)$, we are reduced to the constant coefficient case, to which Theorem 6.4 is applicable. This proves the theorem.

Now we return to consider the symmetry group G of $\Delta_0$ before projection onto $\mathbb{R}^p$. As noted in section 2, G always contains the subgroup T generated by the infinitesimal symmetries $[cu + \varphi(x)]\partial_u$ for $c \in \mathbb{R}$ and $\varphi(x)$ a solution of $\Delta_0$. T is a normal subgroup and we let $G' = G/T$ be the factor group. By Corollary 4.4, its infinitesimal generators are of the form $v = \sum \xi^i(x)\partial_i + \beta(x)u\partial_u$, and two generators are identified if the $\xi^i$'s are all the same and the $\beta$'s differ by a constant.

<u>Theorem 6.8.</u> Suppose $\Delta_0$ satisfies the hypothesis of Theorem 6.7. Then

$$\dim G' \leq p+1.$$

<u>Proof.</u> From Theorem 6.7 it suffices to prove that if $v$ is an infinitesimal symmetry in G', then $\beta$ is determined up to a constant. By choosing local coordinates on $\mathbb{R}^p$, we may assume that $v = \partial_1 + \beta(x)u\partial_u$. The coefficients of the prolonged vector field are

$$\varphi^I = D_I[\beta(x)u] = \beta(x)u_I + \sum_{k=1}^{p} i_k \partial_k \beta(x) u_{I-\delta^k} + \ldots$$

Substituting this into the symmetry equations (2.5), the coefficients of $u_{I-\delta^k}$ for $I \in \mathcal{J}_n^p$ are

$$\sigma^I i_k \partial_k \beta + \partial_1 \sigma^{I-\delta^k} = (\mu - \beta)\sigma^{I-\delta^k} .$$

Since $\sigma$ is nondegenerate and since the multiplier $\mu-\beta$ is prescribed from the conformal equation, all the partial derivatives of $\beta$ are determined from these equations, proving the theorem. (Note that the coefficients of derivatives of $u$ of lower order prescribe more conditions on $\beta$ which must be met.)

Definition 6.9. Suppose $\Delta_0$ and $\Delta_0'$ are two linear partial differential equations on $\mathbb{R}^p \times \mathbb{R}$. We say $\Delta_0$ and $\Delta_0'$ are equivalent if one can be obtained from the other through a combination of the following operations:

i) Change of coordinates in the independent variables: $x' = \varphi(x)$, $\varphi: \mathbb{R}^p \to \mathbb{R}^p$.

ii) Multiplication of the dependent variable by a nonvanishing function:

$$u' = \psi(x)u \, , \quad \psi: \mathbb{R}^p \to \mathbb{R} - \{0\} .$$

iii) Multiplication of the entire equation by a nonvanishing function:

$$\Delta_0' = \chi(x)\Delta_0 \quad , \quad \chi: \mathbb{R}^p \to \mathbb{R} - \{0\} .$$

Theorem 6.10. A linear equation $\Delta_0$ is invariant under a one parameter group of transformations if and only if $\Delta_0$ is equivalent to a partial differential equation all of whose coefficients do not depend on on one of the coordinates (i.e., $\sigma^I(x) = \sigma^I(x^2, \ldots, x^p)$).

Proof. By a change of coordinate in the independent variable, we may assume that the infinitesimal generator of the one-parameter group is of the form

$$v = \partial_1 + [\beta(x)u + \alpha(x)]\partial_u .$$

As in the proof of Theorem 6.8, the symmetry equations (2.5) reduce to

$$\sum (\sigma^I D_I(\beta u) + \partial_1 \sigma^I u_I) = \mu \sum \sigma^I u_I ,$$

$$\sum \sigma^I D_I \alpha = 0 .$$

Therefore

$$\partial_1 \sigma^I = \mu\sigma^I - \sum_{K\geq 0} \binom{I+K}{K} \sigma^{I+K} \partial_K \beta \; .$$

Choose functions $\gamma$ and $\nu$ with $\partial_1 \gamma = \beta$ and $\partial_1 \nu = \lambda$. Let $\Delta' = \sum \sigma^I \partial_I$ be the linear operator whose coefficients are recursively defined by

$$\Delta(u) = e^{\nu} \Delta'(e^{-\gamma} u)$$

for all u. Thus $\Delta'_0$ is an equivalent equation to $\Delta_0$. Explicitly we have

$$\sigma^I = e^{\nu} \sum_{L\geq 0} \binom{I+L}{L} \sigma'^{I+L} \partial_L (e^{-\gamma}) \; .$$

Assume by induction that $\partial_1 \sigma'^I = 0$ for $|I| > m$. Then for $|I| = m$,

$$\begin{aligned}
\partial_1 \sigma^I &= \mu\sigma^I + e^{\nu} \sum \binom{I+L}{L} [\sigma'^{I+L} \partial_L(-\beta e^{-\gamma}) + \partial_1 \sigma'^{I+L} \partial_L e^{-\gamma}] \\
&= \mu\sigma^I - e^{\nu} \sum_{L\geq 0} \binom{I+L}{L} \sigma'^{I+L} \sum_{0\leq K\leq L} \partial_K \beta \partial_{L-K} e^{-\gamma} + e^{\nu-\gamma} \partial_1 \sigma'^I \\
&= \mu\sigma^I - \sum_{K\geq 0} \binom{I+K}{I} e^{\nu} \sum_{M\geq 0} \binom{I+K+M}{M} \partial_M e^{-\gamma} \partial_K \beta + e^{\nu-\gamma} \partial_1 \sigma'^I \\
&= \mu\sigma^I - \sum_{K\geq 0} \binom{I+K}{I} \sigma^{I+K} \partial_K \beta + e^{\nu-\gamma} \partial_1 \sigma'^I \; .
\end{aligned}$$

Comparing this with the previous expression for $\partial_1 \sigma^I$ shows that $\partial_1 \sigma'^I = 0$, proving the theorem.

This theorem shows that invariance under a one-parameter group is in a limited sense equivalent to the question of separation of variables. An analogue of this theorem for more than one-parameter groups is much more complicated and cannot be answered here. The work of Miller, Kalnins, et. al., [5], on separation of variables should provide some indication of the complexity of this question.

## References


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