Higher Order Symmetries of Underdetermined Systems of Partial Differential Equations and Noether's Second Theorem

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Abstract. Every underdetermined system of partial differential equations arising from a variational principle admits an infinite hierarchy of higher order generalized symmetries. These symmetries are a consequence of the Noether dependencies among the Euler–Lagrange equations that follow from Noether's Second Theorem. This result is a consequence of a more general theorem on the existence of higher order generalized symmetries for any system of differential equations that admits an infinitesimal symmetry generator depending on an arbitrary function of the independent variables.

There are two well known classes of partial differential equations that admit infinite hierarchies of higher order generalized symmetries, [**11**, **13**]. The first consists of linear systems of partial differential equations that admit a nontrivial point symmetry group, as well as systems that can be linearized into one of these by a change of variables (a point transformation). A second consists of integrable nonlinear partial differential equations such as the Korteweg–de Vries equation, the nonlinear Schrödinger equation, Burgers' equation, etc. Indeed, an interesting question, [**4**, **5**, **7**, **11**], is whether, under certain conditions, the existence of higher order symmetries or, more generally, an infinite hierarchy of higher order symmetries, implies integrability. The purpose of this note is to introduce

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a third general class: underdetermined systems of partial differential equations that admit an infinite-dimensional symmetry algebra[†] depending on one or more arbitrary functions of the independent variables — although, as shown in the final example, not every underdetermined system admits such a symmetry algebra. A cautionary consequence of the latter fact is that any integrability or linearizability criterion that is based on higher order symmetries will not be applicable to such systems.

An important subclass of the third category are the underdetermined systems arising from a variational principle that admits an infinite-dimensional variational symmetry algebra depending on one or more arbitrary functions of the independent variables. Noether's Second Theorem, [**12**, **13**], tells us that the conservation laws associated with these infinitesimal symmetries, as provided by the Noether integration by parts identity, are all trivial. On the other hand, the associated Euler–Lagrange equations are underdetermined, meaning that they admit a *Noether dependency*: a nontrivial linear combination of them and their derivatives that vanishes identically. This result, when specialized to Einstein's equations of general relativity based on Hilbert's variational principle, allowed Noether to explain why the energy conservation law in relativistic theories is trivial: it is a consequence of the fact that the corresponding time translational symmetry generator is contained in such an infinite-dimensional variational symmetry algebra, whose associated Noether dependencies are the relativistic Bianchi identities. Noether's remarkable result resolved an issue that perplexed both Einstein and Hilbert in the early days of general relativity, and we refer the reader to Kosmann-Schwarzbach's book, [**10**], for the historical details.

The main theorems in this paper are relatively easy consequences of the basic calculus of symmetries and conservation laws of differential equations, as developed, for example, in [**13**], whose results and notation we will use throughout. In particular, all functions are assumed to be smooth, i.e., C^∞. The first main result appears not to have been noticed previously, whereas a result similar to our second theorem for variational problems can be found in a paper by Fulp, Lada, and Stasheff, [**8**], in which they allow arbitrary differential functions in their definition of gauge symmetries, and then apply Noether's identity to derive the Noether dependencies; see also A. Kiselev's lecture notes, [**9**]. Later, in a recent survey paper, [**1**], Anco proved that an underdetermined system of differential equations possesses adjoint symmetries that depend on an arbitrary differential function; as with Noether's result, the associated conservation laws are trivial. In the case of a system of Euler–Lagrange equations, adjoint symmetries coincide with ordinary variational symmetries, and hence Anco's result includes that in [**8**]. Apparently, the first place in the literature where generalized symmetries depending upon an arbitrary differential function appear is in a paper of I. Anderson and C. Torre, [**2**, **3**], in which they prove that the only generalized symmetries of Einstein's vacuum field equations are scalings of the metric tensor and what they call "infinitesimal generalized diffeomorphisms". The latter arise from the invariance of relativity under space-time diffeomorphisms, and so are considered "physically trivial" and not indicative of any underlying integrability property of the Einstein equations.

[†] Recall, [**13**], that the infinitesimal symmetries of a system of differential equations form a Lie algebra under the commutator bracket.

Given independent variables $x = (x^1, \ldots, x^p)$ and dependent variables $(u^1, \ldots, u^q)$ forming local coordinates on an open subset[†] $M = X \times U \subset \mathbb{R}^{p+q}$, by a *differential function* we mean a scalar-valued function, denoted

$$F[u] = F(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ ),$$

depending smoothly on the independent variables, the dependent variables, and finitely many of their derivatives, denoted by u^α_J, i.e., the induced coordinate of the jet spaces over M. Here $1 \leq \alpha \leq q$, while $J = (j_1, \ldots, j_n)$ with $1 \leq j_\nu \leq p$ is a symmetric multi-index of order[‡] $0 \leq n = \#J$.

To start with, suppose that we have a system of differential equations

$$\Delta_\kappa[u] = 0, \qquad \kappa = 1, \ldots, q, \tag{1}$$

determined by the differential functions $\Delta_1, \ldots, \Delta_q$. As is usual in symmetry analysis — see [**13**; Definitions 2.30, 2.70] — we assume that the system (1) is of maximal rank, meaning that its Jacobian matrix with respect to all variables has rank q, and locally solvable, meaning that every point in jet space that satisfies the algebraic constraints imposed by system (1) belongs to the jet of at least one solution $u = f(x)$.

Definition 1. A system of differential equations (1) is called *underdetermined* if there exist differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_q$ that do not simultaneously vanish on solutions, such that the differentiated linear combination

$$\mathcal{D}_1 \Delta_1 + \ \cdots\ + \mathcal{D}_q \Delta_q \equiv 0 \tag{2}$$

vanishes identically.

Examples of underdetermined systems arising in basic physics include Maxwell's equations for electromagnetism and Einstein's equations for general relativity.

Remark: Note that we are assuming that the system (1) contains the same number of equations as unknowns $u^1, \ldots, u^q$, and our definition of underdetermined is a slight reformulation of that in [**13**; p. 171]. While the analysis in this paper can be applied to systems involving any number of equations, the rigorous general definition of an underdetermined system of differential equations is considerably more technical, relying on the system being in involution. See Seiler, [**15**], for full details on what this entails and how it is related to Definition 1.

We will work exclusively with infinitesimal symmetries in characteristic form, [**13**]. Thus, a (generalized) evolutionary vector field

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha[u] \frac{\partial}{\partial u^\alpha}, \tag{3}$$

† More generally, M could be a fiber bundle over a base manifold X, or even a general smooth manifold, [**13**]. Since we work in local coordinates throughout, we are not losing generality by assuming $X \subset \mathbb{R}^p$ and $U \subset \mathbb{R}^q$ are open subsets of Euclidean space.

‡ When the order $\#J = 0$, by convention we set $u^\alpha_J = u^\alpha$.

where $Q = (Q_1, \ldots, Q_q)$ is a q-tuple of differential functions known as the *characteristic* of (3), forms an infinitesimal symmetry of (1) if and only if Q satisfies the infinitesimal *determining equations*

$$\operatorname{pr} \mathbf{v}_Q[\Delta] = 0 \qquad \text{whenever} \qquad \Delta = 0. \tag{4}$$

In other words, the infinitesimal invariance criterion in (4) holds on all solutions to the system (1), taking into account the equations and all their derivatives. Further,

$$\operatorname{pr} \mathbf{v}_Q = \sum_{\alpha=1}^{q} \sum_{\#J \geq 0} D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha} \tag{5}$$

denotes the standard jet space prolongation of the evolutionary vector field (3) whose coefficients are obtained by applying the iterated total derivative operators $D_J = D_{j_1} \cdots D_{j_n}$ to the individual components of the characteristic. Similarly, if $h(x)$ is a smooth function of the independent variables, we denote its partial derivatives (which coincide with its total derivatives) by $h_K(x) = \partial_K h(x) = D_K h$, where again K is a symmetric multi-index.

Let us now state the first main result.

Theorem 2. *Suppose that a system of differential equations* (1) *admits an infinitesimal symmetry* $\mathbf{v}_Q$ *whose characteristic*

$$Q[u,h] = Q(\ \ldots\ x^i\ \ldots\ u_J^\alpha\ \ldots\ h_K(x)\ \ldots\ )$$

depends on finitely many derivatives of an arbitrary function $h(x)$ *of the independent variables. Let* $F[u]$ *be an arbitrary differential function. Then the characteristic*

$$\widehat{Q}[u] = Q(\ \ldots\ x^i\ \ldots\ u_J^\alpha\ \ldots\ D_K F\ \ldots\ ) \tag{6}$$

obtained by replacing the derivatives of h *by the corresponding total derivatives of* F *is also the characteristic of an infinitesimal symmetry* $\mathbf{v}_{\widehat{Q}}$ *of the system. Thus, any such system of differential equations automatically admits an infinite family of higher order symmetries depending upon an arbitrary function* F *of the independent variables, the dependent variables, and their derivatives of arbitrarily high order.*

Proof: This is a consequence of Kiselev's general Substitution Principle, [**9**; Exercise 9.2]. In detail, since $h(x)$ is an arbitrary function of all the independent variables, its partial derivatives $h_K(x)$ can assume any values[†]. The h_K can therefore be treated as algebraically independent quantities appearing in the algebraic relations prescribed by the symmetry determining equations (4). Since they are independent, they can be replaced by any other quantities, $h_K \mapsto c_K$, independent or not, without affecting these algebraic relations.

[†] All expressions only involve finitely many of the h_K, and so no convergence issues arise.

Furthermore, according to the prolongation formula (5), the coefficients of $\operatorname{pr}\mathbf{v}_Q$ are obtained by total differentiation and, as noted above, the partial derivatives of h coincide with its total derivatives. If we write out their explicit formulas

$$D_I Q_\alpha = R_{\alpha,I}(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ \partial_K h(x)\ \ldots\ ),$$

where $R_{\alpha,I}$ are certain functions of the jet coordinates and the partial (total) derivatives of h, then, replacing h by F in Q as in (6) leads to the same algebraic expressions for its total derivatives,

$$D_I \widehat{Q}_\alpha = R_{\alpha,I}(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ D_K F\ \ldots\ ),$$

in terms of the jet coordinates and the total derivatives of F. By the preceding remarks, we can thus replace each partial derivative $h_K(x)$ appearing in the determining equations (4) by the corresponding total derivative $D_K F$ without affecting their validity. We conclude that the evolutionary vector field $\mathbf{v}_{\widehat{Q}}$ with characteristic (6) also satisfies the symmetry determining equations for the system of differential equations. *Q.E.D.*

Remark: Systems that satisfy the hypothesis of Theorem 2 are necessarily underdetermined, although, as we will see, not every underdetermined system will admit such a symmetry generator. The preceding statement is easily established when the system arises from a variational principle — see below — but to rigorously prove it for more general systems appears to be nontrivial, requiring the technical machinery of involutivity, [**15**]. In outline, one first shows that the admission of a symmetry generator involving an arbitrary function of the independent variables implies that the last Cartan character of the system of differential equations is nonzero — on an infinitesimal level this will be equivalent to the statement that general solution to the system involves an arbitrary function of the independent variables — and hence, according to [**15**; Proposition 7.5.7 and Lemma 8.2.1], the system is necessarily underdetermined. Filling in the details would take us too far afield, and is thus left as a challenge for the motivated reader.

Next, suppose we have a variational problem

$$I[u] = \int L[u]\,dx \tag{7}$$

whose *Lagrangian* $L[u]$ is a differential function, with corresponding Euler–Lagrange equations

$$\Delta_\alpha = E_\alpha(L) = 0, \qquad \alpha = 1, \ldots, q. \tag{8}$$

Here E_α denotes the variational derivative or *Euler operator* corresponding to the dependent variable u^α, [**13**]. According to Noether's Second Theorem, $I[u]$ admits an infinitesimal symmetry $\mathbf{v}_Q$ whose characteristic depends linearly[†] on an arbitrary function $h(x)$ — meaning that

$$Q_\alpha = \mathcal{D}_\alpha h, \qquad \alpha = 1, \ldots, q, \tag{9}$$

[†] If Q depends nonlinearly on h, then, using the argument in the proof of Noether's Second Theorem given in [**12, 13**], its linearization with respect to h also forms a symmetry, and so there is no loss in generality assuming linearity of Q in h.

where $\mathcal{D}_1, \ldots, \mathcal{D}_q$ are linear differential operators, which may depend on the jet coordinates x^i, u^α_J — if and only if its Euler–Lagrange equations satisfy the corresponding *Noether dependency*

$$\sum_{\alpha=1}^{q} \mathcal{D}^*_\alpha E_\alpha(L) = 0, \tag{10}$$

where $\mathcal{D}^*_\alpha$ denotes the formal adjoint of the differential operator $\mathcal{D}_\alpha$; details can be found in [**13**; pp. 329, 343]. The existence of a nontrivial Noether dependency implies that the Euler–Lagrange equations of such a variational problem form an underdetermined system of differential equations.

Remark: Except in completely trivial cases, the existence of a Noether dependency and thus such a symmetry generator requires that the number of dependent variables $q \geq 2$.

According to [**13**; Theorem 4.14], any variational symmetry is also a symmetry of the Euler–Lagrange equations. Thus, one can adapt the argument used to justify Theorem 2 to prove that, when we replace h by an arbitrary differential function $F[u]$ as in (6), the resulting vector field $\mathbf{v}_{\widehat{Q}}$ remains a variational symmetry and a symmetry of the Euler–Lagrange equations. Since the argument leading to the *Noether dependency* (10) can be reversed, every linear dependency of the Euler–Lagrange equations produces a corresponding infinite-dimensional variational symmetry algebra whose characteristics are of the form (9). Thus, we deduce the second main theorem, which can be viewed as a counterpart of the aforementioned results in [**1**, **8**].

Theorem 3. *If $E(L) = 0$ is any underdetermined system of Euler–Lagrange equations, then it admits generalized symmetries of arbitrarily high order depending upon one or more arbitrary differential functions.*

This result resolves a mystery concerning Noether's Second Theorem, which relies on infinitesimal symmetries that involve one or more arbitrary functions of the p independent variables. But one can always perform a "hodograph-type" change of variables in which the roles of independent and dependent variables are interchanged; see [**13**; Theorem 4.8] for how the Euler operators behave. Consequently, such a change of variables does not affect the existence of Noether dependencies for the transformed variational problem. On the other hand, the transformed symmetries no longer involve functions solely of the new independent variables, and so Noether's Second Theorem implies the existence of an ostensibly different symmetry generator depending on arbitrary functions of the new independent variables. What Theorems 2 and 3 imply is that, if the variational problem or system of differential equations admits an infinite-dimensional symmetry algebra depending on arbitrary functions of any p of the independent and dependent variables $(x, u) = (x^1, \ldots, x^p, u^1, \ldots, u^q)$, then it automatically admits the enlarged symmetry algebra depending, in the same manner, on *all* of the independent and dependent variables and, in fact, on any finite collection of jet variables also. In other words, Noether's Second Theorem does not, in fact, rely on any artificial distinction between independent and dependent variables!

Let us finish by illustrating the preceding results with two examples.

Systems of differential equations or variational problems for curves, surfaces, etc., that do not depend on any underlying parametrization thereof are called *parameter-independent* or *parametric*, cf. [**13**]. In other words, a system of differential equations involving the independent variables $x = (x^1, \ldots, x^p) \in X$ is parameter-independent if it admits the symmetry pseudo-group consisting of all local diffeomorphisms of the base space X. Its infinitesimal generators are all the locally defined vector fields on X:

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x) \frac{\partial}{\partial x^i}\,, \tag{11}$$

where the $\xi^i(x)$ are arbitrary functions of the independent variables. Their evolutionary representative takes the form

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} \left(\sum_{i=1}^{p} \xi^i(x)\, u_i^\alpha \right) \frac{\partial}{\partial u^\alpha}\,, \qquad \text{where} \qquad u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}\,, \tag{12}$$

and, by a general principle, is also an infinitesimal symmetry, [**13**]. Theorem 2 immediately implies that any parameter-independent system of differential equations admits hierarchies of generalized symmetries depending on $p = \dim X$ arbitrary differential functions.

Theorem 4. *A system of differential equations* $\Delta[u] = 0$ *is parameter-independent if and only if it admits all generalized infinitesimal symmetry generators of the form*

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} \left(\sum_{i=1}^{p} u_i^\alpha\, F_i[u] \frac{\partial}{\partial u^\alpha} \right), \tag{13}$$

where $F_1[u], \ldots, F_p[u]$ *are arbitrary differential functions.*

In particular, any system of Euler–Lagrange equations (8) arising from a parameter-independent variational problem admits the generalized symmetries (13) along with the p consequential Noether dependencies

$$\sum_{\alpha=1}^{q} u_i^\alpha\, E_\alpha(L) = 0, \qquad i = 1, \ldots, p. \tag{14}$$

We remark that one can explicitly characterize all nondegenerate parameter-independent systems of differential equations and variational problems in terms of the differential invariants and invariant volume form of the diffeomorphism pseudo-group, cf. [**6, 13, 14**].

Example 5. As in [**13**; Example 5.70], let $p = 1$ and $q = 2$, so there is a single independent variable x and two dependent variables u, v. The solution to a parameter-independent differential equation or variational problem can be regarded as a plane curve $C \subset \mathbb{R}^2$ which is independent of any specific parametrization $x \mapsto (u(x), v(x))$ thereof. In particular, a second order system of differential equations is parameter-independent if and only if it is equivalent to one in the form

$$H_1\left(u, v, \frac{v_x}{u_x}, \frac{u_x v_{xx} - u_{xx} v_x}{u_x^3}\right) = H_2\left(u, v, \frac{v_x}{u_x}, \frac{u_x v_{xx} - u_{xx} v_x}{u_x^3}\right) = 0. \tag{15}$$

If we use u to parametrize the curve, so $v = v(u)$, then the system (15) should reduce to a single second order ordinary differential equation:

$$H\left(u, v, \frac{dv}{du}, \frac{d^2v}{du^2}\right) = 0. \tag{16}$$

In other words, unless the system (15) is overdetermined, one of its parameter-independent equations is redundant.

According to Theorem 4, any such system (15) admits the higher order generalized symmetries

$$\mathbf{v}_Q = u_x F[u,v] \frac{\partial}{\partial u} + v_x F[u,v] \frac{\partial}{\partial v},$$

where $F[u,v]$ is an arbitrary differential function. On the other hand, these symmetries do not carry over to the reduced ordinary differential equation (16), which only admits ordinary Lie symmetries. If the system (15) is the Euler–Lagrange equations for a parametric (parameter-independent) variational problem, which must take the form

$$I[u,v] = \int L(x,u,v,u_x,v_x)\,dx = \int G\left(u, v, \frac{v_x}{u_x}\right) u_x\,dx = \int G\left(u, v, \frac{dv}{du}\right) du,$$

for some function G, then it admits the usual Noether dependency

$$u_x E_u(L) + v_x E_v(L) = 0.$$

Example 6. Following [**13**; Example 5.71], consider the variational problem

$$I[u,v] = \iint \tfrac{1}{2}(u_x + v_y)^2\,dx\,dy, \qquad \text{with Lagrangian} \qquad L[u,v] = \tfrac{1}{2}(u_x+v_y)^2, \tag{17}$$

involving two independent variables x, y, and two dependent variables u, v. Its Euler–Lagrange equations are

$$E_u(L) = -u_{xx} - v_{xy} = 0, \qquad E_v(L) = -u_{xy} - v_{yy} = 0. \tag{18}$$

The variational problem (17) admits the infinite-dimensional abelian symmetry group with generator

$$\mathbf{v} = -\frac{\partial h}{\partial y}\frac{\partial}{\partial u} + \frac{\partial h}{\partial x}\frac{\partial}{\partial v}, \tag{19}$$

where $h(x,y)$ is an arbitrary function of the independent variables. Noether's Second Theorem produces the evident linear dependency among the Euler–Lagrange equations:

$$D_y E_u(L) - D_x E_v(L) = 0. \tag{20}$$

Theorem 2 implies that, for any differential function $F[u,v]$ depending on x, y, u, v and their derivatives, the evolutionary vector field

$$\widehat{\mathbf{v}} = -D_y F \frac{\partial}{\partial u} + D_x F \frac{\partial}{\partial v}$$

also forms a variational symmetry, and thus a symmetry of the Euler–Lagrange equations — which is easy to check by direct computation. Thus, the underdetermined system (18) admits an infinite hierarchy of generalized symmetries of arbitrarily high order. On the one hand, as explained in [**13**; Proposition 5.22], since the system is linear, this fact is not so surprising. On the other hand, the same result holds for more complicated variational problems admitting the same variational symmetry (19). For example, the second order variational problem

$$\widetilde{I}[u,v] = \iint \left[\tfrac{1}{2}\,(u_{xx}+v_{xy})\,(u_{xy}+v_{yy}) + \tfrac{1}{6}\,(u_x+v_y)^3 \right] dx\,dy,$$

with underdetermined nonlinear fourth order Euler–Lagrange equations

$$u_{xxxy}+v_{xxyy} = (u_x+v_y)\,(u_{xx}+v_{xy}), \qquad u_{xxyy}+v_{xyyy} = (u_x+v_y)\,(u_{xy}+v_{yy}),$$

possesses the aforementioned properties.

While Theorem 3 implies the existence of higher order symmetries of any underdetermined system of Euler–Lagrange equations, this result does not extend to general underdetermined systems of nonlinear partial differential equations. Indeed, in the present context, if $H[u,v]$ is any differential function, then the underdetermined system

$$\Delta_1 = D_x H = 0, \qquad \Delta_2 = D_y H = 0, \tag{21}$$

satisfies the same linear dependency:

$$D_y \Delta_1 - D_x \Delta_2 = 0.$$

An evolutionary infinitesimal generator $\mathbf{v} = Q[u,v]\,\partial_u + R[u,v]\,\partial_v$ will be an infinitesimal symmetry of (21) provided

$$D_x[\operatorname{pr}\mathbf{v}(H)] = D_y[\operatorname{pr}\mathbf{v}(H)] = 0$$

whenever (21) holds. It is clear that, by making $H[u,v]$ sufficiently complicated, one can ensure that there are no symmetries. Thus, such an underdetermined system does not admit an infinite-dimensional symmetry algebra of the required form, and hence Theorem 2 does not apply.

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