A Plethora of Integrable Bi-Hamiltonian Equations

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November 17, 1995

1 Introduction

This paper discusses several algorithmic ways of constructing integrable evolution equations based on the use of multi-Hamiltonian structures. The recognition that integrable soliton equations, such as the Korteweg-deVries (KdV) and nonlinear Schrödinger (NLS) equations, can be constructed using a biHamiltonian method dates back to the late 1970's. An extension of the method was proposed by the first author and Fuchssteiner in the early 1980's and was used to derive integrable generalizations of the KdV and of the modified KdV. However it was not until these models reappeared in physical problems, and their novel solutions such as compactons and peakons were discovered, that the method achieved recognition. In this paper, we describe the basic approach to constructing a wide variety of integrable bi-Hamiltonian equations. In addition to usual soliton equations, these new hierarchies include equations with nonlinear dispersion which support novel types of solitonic solutions.

Let us start with the simple case of a scalar evolution equation

$$u_t = K[u], \tag{1.1}$$

where $K[u]$ is a smooth function depending on the scalar spatial variable x, the dependent variable u, and its x derivatives u_x, u_{xx}, etc. Later, we shall generalize K by allowing it to be either non-local, or depend on additional spatial variables. A second evolution equation

$$u_s = Q[u]$$

is called a **symmetry** of equation (1.1) iff the two flows commute. Cross-differentiation produces the basic symmetry condition

$$K'Q - Q'K = 0, \tag{1.2}$$

where the prime denotes Fréchet differentiation, i.e.

$$K'[u]Q = \frac{\partial}{\partial \varepsilon} K[u + \varepsilon Q]\bigg|_{\varepsilon=0}.$$

More explicitly,

$$K' = \frac{\partial K}{\partial u} + \frac{\partial K}{\partial u_x}\partial + \frac{\partial K}{\partial u_{xx}}\partial^2 + \cdots, \qquad \text{where} \qquad \partial = \frac{\partial}{\partial x}. \tag{1.3}$$

For example, if equation (1.1) does not depend explicitly on x then it is invariant under the group of x-translations, which gives rise to the symmetry $Q = u_x$. See [1] for details.

According to the general symmetry approach to integrability, [1], [2], [3], the existence of an infinite hierarchy of higher order symmetries is a manifestation of the integrability of the evolution equation (1.1). The basic method for constructing such hierarchies is through the use of a **recursion operator** which is defined as an operator $\Phi[u]$ which maps symmetries to symmetries. It was shown in [4] that if the operator $\Phi[u]$ satisfies the operator equation

$$\Phi'[K] = [K', \Phi], \tag{1.4}$$

where $[\cdot, \cdot]$ denotes the usual commutator, then Φ is a recursion operator for (1.1). If Φ is a recursion operator for (1.1), then there is an associated hierarchy of commuting flows (symmetries) taking the form

$$u_t = \Phi^n K. \tag{1.5}$$

A large class of recursion operators satisfy a remarkable property introduced by Fuchssteiner [5], namely they are **hereditary** (or Nijenhuis) operators. The operator $\Phi[u]$ is called hereditary iff $\Phi'[\Phi v]w - \Phi\Phi'[v]w$ defines a symmetric bilinear form of the functions v and w. It can be shown that if Φ is hereditary and if Φ is a recursion operator for the seed equation $u_t = K_0$, then Φ is also a recursion operator for any constant coefficient linear combination

$$u_t = c_0 K_0 + c_1 \Phi K_0 + \ldots + c_n \Phi^n K_0, \tag{1.6}$$

of the associated hierarchy of symmetries, or, even more generally,

$$u_t = B(\Phi) K_0, \tag{1.7}$$

where $B(z)$ is any rational, or even analytic function of z; see [1] for details. (Indeed, one can even regard the Bäcklund transformation as an exponential series in the higher order symmetries [6] and hence in the recursion operator Φ.) Thus the question of constructing integrable equations reduces to the question of constructing hereditary operators and "starting symmetries" K_0, i.e. functions K_0 which satisfy equation (1.4). Usually, the hereditary operator is independent of the spatial variable x, in which case the starting symmetry is $K_0 = u_x$, and the seed equation is the linear wave equation $u_t = u_x$. Indeed, $\partial(\Phi v) = \Phi'[u_x]v + \Phi\partial v$, or $\Phi'[u_x] = [\partial, \Phi]$, and equation (1.4) is satisfied with $K_0 = u_x$.

An operator $\theta[u]$ is called **Hamiltonian** if it is skew symmetric (with respect to a suitable inner product — usually a variant of the L^2 inner product), and such that the associated Poisson bracket $\{F, H\} = \int \delta F \theta \delta H \, dx$ on the space of functionals satisfies the Jacobi identity. Here δH denotes the variational derivative of the Hamiltonian functional H. An evolution equation is called **Hamiltonian** if it can be written in the form

$$u_t = \theta \delta H, \tag{1.8}$$

where θ is a Hamiltonian operator.

Following the fundamental discovery of Magri [7] that Hamiltonian integrable equations are actually bi-Hamiltonian systems, an algorithmic way of constructing hereditary operators was proposed independently in [8] and [9]. Two operators θ_1 and θ_2 are said to form a **Hamiltonian pair** if every linear combination $\alpha\theta_1 + \beta\theta_2$ for α, β constant, is a Hamiltonian operator. This requires that θ_1 and θ_2 are Hamiltonian, and, moreover, that they satisfy a certain bilinear compatibility condition [7]. Given a Hamiltonian pair θ_1, θ_2, it can be shown [8], [9] that the operator $\Phi = \theta_2\theta_1^{-1}$ is a hereditary operator. Furthermore, if θ_1 and θ_2 do not depend explicitly on x, then $K_0 = u_x$ is a seed symmetry for Φ.

If a hereditary operator Φ is derived from a pair of compatible Hamiltonian operators θ_1 and θ_2, then there exists an algorithmic way of constructing additional starting symmetries. Let $C[u]$ be a function which is annihilated by the Hamiltonian operator θ_1, i.e. $\theta_1 C = 0$. In most cases, C is the variational derivative of a a **Casimir** (or distinguished) functional B for the Hamiltonian operator θ_1, i.e. $C = \delta B$ [1], [10]. By abuse of terminology, we shall call all such functions $C[u]$ Casimirs. Then

$$K_0 = \theta_2 C; \qquad \text{where} \qquad \theta_1 C = 0, \tag{1.9}$$

is a starting symmetry of the hereditary operator $\Phi = \theta_2\theta_1^{-1}$. Indeed, we can write $K_0 = \Phi \cdot 0$ as the image of the trivial symmetry $u_t = 0$ under the recursion operator Φ since (formally) $C = \theta_1^{-1}0$. Clearly $K = 0$ satisfies the recursion operator criterion (1.4), which implies that K_0 does also.

Combining the above discussion with the general form (1.7) of the hierarchy generated by a recursion operator, suggests the following algorithmic construction of integrable evolution equations:

Let θ_1, θ_2 be a Hamiltonian pair which do not depend on x. Let $\Phi = \theta_2\theta_1^{-1}$. Then the following equations are integrable,

$$u_t = \Phi u_x, \tag{1.10}$$

$$\Phi u_t = u_x, \tag{1.11}$$

$$u_t = \Phi u_y, \tag{1.12}$$

$$u_t = \theta_2 C, \qquad \theta_1 C = 0, \tag{1.13}$$

$$\sum_{\kappa=1}^{i} a_\kappa \Phi^\kappa u_t = \sum_{\kappa=1}^{j} b_\kappa \Phi^\kappa u_x + \sum_{\kappa=1}^{k} d_\kappa \Phi^\kappa u_y + \sum_{\kappa=1}^{l} e_\kappa \Phi^\kappa \theta_2 C, \tag{1.14}$$

where $a_\kappa, b_\kappa, d_\kappa, e_\kappa$ are constants.

The integrability of equation (1.11) is a consequence of the fact that if Φ is hereditary then Φ^{-1} is also hereditary. In (1.12), y is an additional arbitrary independent variable that does not occur in the operator Φ, and thus u_y is a starting symmetry for Φ. The integrability of equations (1.13) has been commented upon in [11]. The integrability of equations (1.14) follows from the fact that on can take arbitrary constant coefficient linear combinations of all the equations in the hierarchies associated with (1.10–13).

In many cases, one of the operators in the Hamiltonian pair is itself a linear combination of two Hamiltonian operators, so that the linear combinations $\alpha\theta_1 + \beta\theta_2$ are actually members of a three parameter family $\tilde{\alpha}\tilde{\theta}_1 + \tilde{\beta}\tilde{\theta}_2 + \tilde{\gamma}\tilde{\theta}_3$, where $\tilde{\theta}_1$, $\tilde{\theta}_2$, $\tilde{\theta}_3$ form a Hamiltonian triple. In such cases, there are several distinct choices of the Hamiltonian pair θ_1, θ_2, that can be used to generate a hereditary operator $\Phi = \theta_2\theta_1^{-1}$. Usually this happens because there is a scaling transformation which decouples one of the operators into two components having different scaling properties, and hence decomposing into a sum of two Hamiltonian operators. In such cases, a single Hamiltonian triple can lead to a plethora of associated integrable equations. This fact was first exploited in [12], where certain generalizations of both KdV and modified KdV were presented. Analogous generalizations of the nonlinear Schrödinger equation and of the sine-Gordon equations are given in [13].

It is interesting that although equations (1.10–14) are derived from the same basic mathematical structure, namely a Hamiltonian pair or triple, they admit very different types of solutions. These include, solitons, peakons (peaked solitons) [14], compactons (solitons with compact support) [15], 2-hump solitons [13], infinitely many solitons [16], and twisted solitons [17]. It is also remarkable that many of these equations appear in physical applications. For example the generalized KdV and the generalized modified KdV appear in the modeling of unidirectional idealized water waves [13], [14], [18].

2 Integrable Generalizations of the KdV Equation

The prototypical example, leading to the Korteweg-deVries equation and its generalizations, starts with the operator

$$\theta = \alpha\partial + \beta\partial^3 + \gamma(u\partial + \partial u). \tag{2.1}$$

It can be shown that θ is a Hamiltonian operator for all values of the constant parameters α, β, γ, and hence its component parts ∂, ∂^3 and $u\partial + \partial u = 2u\partial + u_x$ form a compatible hamiltonian triple. In what follows we use this operator to illustrate the constructions (1.10–13).

(i) Let $\theta_2 = \theta$, $\theta_1 = \partial + \nu\partial^3$. Then equation (1.10) becomes

$$u_t = (\alpha\partial + \beta\partial^3 + \gamma(u\partial + \partial u))(\partial + \nu\partial^3)^{-1}u_x. \tag{2.2}$$

If $\nu = 0$, this equation becomes the celebrated KdV equation

$$u_t = \alpha u_x + \beta u_{xxx} + 3\gamma u u_x. \tag{2.3}$$

If $\nu \neq 0$, equation (2.2) can be written in a local form by letting $(\partial + \nu\partial^3)^{-1}u_x = q$, or $u = q + \nu q_{xx}$. Then equation (2.2) becomes

$$(q + \nu q_{xx})_t = \Big(\alpha\partial + \beta\partial^3 + 2\gamma(q + \nu q_{xx})\partial + \gamma(q_x + \nu q_{xxx})\Big)\, q,$$

or

$$q_t + \nu q_{xxt} = \alpha q_x + \beta q_{xxx} + 3\gamma q q_x + \gamma\nu(q q_{xxx} + 2 q_x q_{xx}). \tag{2.4}$$

Equation (2.2) is equation (26.e) of [12], while equation (2.4) is equation (5.3) of [20]. Equation (2.4) with $\nu > 0$ was derived from physical considerations in [14] where also its Lax pair as well as its peakon solutions were given. For $\nu < 0$ (2.4) admits compacton solutions [15]. The inverse spectral method for equation (2.4) is discussed in [16]; other interesting aspects of this equation are discussed in [13], [20], [21].

Letting $\alpha = \beta = 0$, $\nu = \varepsilon^{-1}$, $\varepsilon = 0$, in equation (2.4), and integrating once, one finds

$$q_{xt} = \gamma(q q_{xx} + \frac{1}{2} q_x^2). \tag{2.5}$$

Equation (2.5) was first shown to be integrable by Calogero [22]; its relation to the generalized KdV was pointed out in [11].

(ii) Let $\theta_2 = \theta$, $\theta_1 = \partial + \nu\partial^3$, $u = q_x + \nu q_{xxx}$, then equation (1.11) becomes

$$\alpha q_{xt} + \beta q_{xxxt} + \gamma(2 q_x q_{xt} + q_{xx} q_t) + \gamma\nu(q_{xxxx} q_t + 2 q_{xxx} q_{xt}) = q_{xx} + \nu q_{xxxx}. \tag{2.6}$$

The particular case when $\nu = 0$ is given in [12].

(iii) Let $\theta_2 = \theta$, $\theta_1 = \partial + \nu\partial^3$, $u = q_x + \nu q_{xxx}$, then equation (1.12) becomes

$$q_{xt} + \nu q_{xxxt} = \alpha q_{xy} + \beta q_{xxxy} + \gamma(q_{xx} q_y + 2 q_x q_{xy}) + \gamma\nu(q_{xxxx} q_y + 2 q_{xxx} q_{xy}). \tag{2.7}$$

The particular case of $\nu = 0$ has been discussed earlier, and it has been shown that in this case equation (2.7) supports breaking solitons.

(iv) Let $\theta_2 = \theta$, $\theta_1 = u\partial + \partial u$. Then, since equation $\theta_1 C = 0$ implies $C = u^{-1/2}$, equation (1.13) becomes

$$u_t = (\alpha\partial + \beta\partial^3) u^{-1/2}. \tag{2.8}$$

This equation, which admits the hereditary operator $\Phi = \theta(u\partial + \partial u)^{-1}$, where θ is defined by equation (2.1), was derived in [11]. If $\alpha = 0$, equation (2.8) reduces to the Harry–Dym equation.

3 Integrable Generalizations of the mKdV Equation

A second collection of integrable Hamiltonian systems starts with the non-local Hamiltonian operator

$$\theta = \alpha\partial + \beta\partial^3 + \gamma\partial u \partial^{-1} u \partial, \tag{3.1}$$

which is a Hamiltonian operator for all values of the constant parameters α, β, γ. This operator leads to a similar set of integrable equations associated with the modified Korteweg-deVries equation.

(i) Let $\theta_2 = \theta$, $\theta_1 = \partial + \nu\partial^3$. Then equation (1.10) becomes

$$u_t = (\alpha\partial + \beta\partial^3 + \gamma\partial u\partial^{-1}u\partial)(\partial + \nu\partial^3)^{-1}u_x. \tag{3.2}$$

If $\nu = 0$, this equation reduces to the mKdV equation

$$u_t = \alpha u_x + \beta u_{xxx} + \frac{3}{2}\gamma u^2 u_x. \tag{3.3}$$

If $\nu \neq 0$, equation (3.2) can be written in a local form by letting $u = q + \nu q_{xx}$. Then equation (3.2) becomes

$$(q + \nu q_{xx})_t = \alpha q_x + \beta q_{xxx} + \frac{1}{2}\gamma\partial[(q + \nu q_{xx})(q^2 + \nu q_x^2)]. \tag{3.4}$$

(ii) Let $\theta_2 = \theta$, $\theta_1 = \partial + \nu\partial^3$, then equation (1.11) becomes the nonlocal equation

$$(\alpha + \beta\partial^2 + \gamma\partial u\partial^{-1}u)(1 + \nu\partial^2)^{-1}u_t = u_x. \tag{3.5}$$

Setting $u = q + \nu q_{xx}$ removes the second non-locality, but there appear to be no direct way to remove the first ∂^{-1}.

(iii) Similarly, setting $\theta_2 = \theta$, $\theta_1 = \partial + \nu\partial^3$, then equation (1.12) leads to a non-local $2+1$ dimensional equation

$$u_t = (\alpha + \beta\partial^2 + \gamma\partial u\partial^{-1}u)(1 + \nu\partial^2)^{-1}u_y. \tag{3.6}$$

Again, as in (3.5) the non-locality appears essential.

(iv) Let $\theta_2 = \partial + \lambda\partial^3$, $\theta_1 = \partial^3 + \sigma^2\partial u\partial^{-1}u\partial$. If $\sigma \neq 0$, then the Casimir functional for θ_1 is found to be $\int \sigma^{-1}\cos(\sigma\partial^{-1}u)\,dx$ with variational derivative $C = \partial^{-1}\sin(\sigma\partial^{-1}u)$. Then equation (1.13) becomes

$$u_t = (\partial + \lambda\partial^3)\sin(\sigma\partial^{-1}u). \tag{3.7}$$

Setting $u = q_x + \lambda q_{xxx}$, we see that (3.7) can be rewritten in the local form

$$q_{xt} = \sin\sigma(q + \lambda q_{xx}) \tag{3.8}$$

an equation whose integrability was first noted in [13]. In particular, for $\lambda = 0$, (3.8) reduces to the well-known sine–Gordon equation.

On the other hand, if we consider the "singular limit" $\sigma \to \infty$, then the Casimir for $\theta_1 = \partial u\partial^{-1}u\partial$ has variational derivative $C = u^{-2}$, leading to the equation

$$u_t = (\partial + \lambda\partial^3)\,u^{-2}. \tag{3.9}$$

According to the formal symmetry approach of Shabat the two Casimir equations (2.8), (3.9), are the only two integrable cases of the general class of equations $u_t = \partial(1 + \lambda\partial^2)u^k$. As with (2.8), equation (3.9) admits solitons, whereas replacing u by $r = 1/u$ in (3.9), we obtain

$$r_t = r^2(\partial + \partial^3)r^2, \tag{3.10}$$

which admits both traveling and stationary compactons [23].

4 The Nonlinear Schrödinger Equation

As our final example, we consider the integrable Hamiltonian systems that are associated with the nonlinear Schrödinger equation. In this case, the function u is complex-valued, as are the associated evolution equations. We use bars to denote complex conjugates, and $i = \sqrt{-1}$. Consider the non-local Hamiltonian operator

$$\theta F = \alpha i F + \beta \partial F + \gamma u \partial^{-1}(\overline{u} F - u \overline{F}), \tag{4.1}$$

which is Hamiltonian for all values of the constant parameters α, β, γ.

Let $\theta_2 = \theta$, $\theta_1 = i + \nu\partial$. Then equation (1.7) becomes

$$u_t = (\alpha i + \beta\partial)(i + \nu\partial)^{-1} u_x + \gamma u \partial^{-1}[\overline{u}(i + \nu\partial)^{-1} u_x - u(-i + \nu\partial)^{-1}\overline{u}_x]. \tag{4.2}$$

Setting $u = (i + \nu\partial)q$, equation (4.2) becomes the local equation

$$i q_t + \nu q_{xt} = \alpha i q_x + \beta q_{xx} - i\gamma(iq + \nu q_x)|q|^2, \tag{4.3}$$

first noted in [13] (see also [11]). If $\alpha = \nu = 0$, $\beta = \gamma = 1$, this equation reduces to the nonlinear Schrödinger equation

$$u_t = i(u_{xx} + |u|^2 u). \tag{4.3}$$

On the other hand, if $\alpha = \beta = 0$, and $v = q e^{-ix}$, it follows that

$$-i v_{xt} = |v|^2 v_x. \tag{4.4}$$

This equation has a first integral $|v_x|^2$. Here, in contrast to the KdV and mKdV cases, the dispersion remains linear due to the fact that the Hamiltonian operator θ_2 is a pure integral operator. The construction of an associated hierarchy is more problematic in this case due to nonlocalities.

Acknowledgement

A.S. Fokas was partially supported by an NSF grant DMS 9111611 and by an AFOSR grant F49620-93. P.J. Olver was partially supported by an NSF grant DMS 92-04192. P. Rosenau was partially supported by an AFOSR grant F49620-95.

References

[1] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Second Ed., Vol. 107, Springer-Verlag, NY, 1993.

[2] A.S. Fokas, J. Math. Phys. **2** 1318 (1980); Stud. Appl. Math. **77** 253 (1987).

[3] A.V. Mikhailov, A.B. Shabat and V.V. Sokolov, in *What is Integrability?* V.E. Zakharov, ed., Springer Verlag, New York, 1990, pp. 115–184.

[4] P.J. Olver, J. Math. Phys., **18**, 1212 (1977).

[5] B. Fuchssteiner, Non. Anal. Theory Meth. Appl., **3**, 849 (1979).

[6] S. Kumei, J. Math. Phys., **16**, 2461 (1975).

[7] F. Magri, J. Math. Phys., **19**, 1156 (1978).

[8] A.S. Fokas and B. Fuchssteiner, Lett. Nuovo Cimento, **28**, 299 (1980).

[9] I.M. Gel'fand, and I.Y. Dorfman, Funct. Anal. Appl., **3**, 14 (1980).

[10] I.Y. Dorfman, *Dirac Structures and Integrabilities of Nonlinear Evolution Equations*, J. Wiley, NY 1993.

[11] P.J. Olver and P. Rosenau, Tri-Hamiltonian Duality Between Solitons and Compactons, preprint (1995).

[12] B. Fuchssteiner and A.S. Fokas, Physica **4D**, 47 (1981).

[13] A.S. Fokas, Acta Appl. Math., **34**, 295 (1995); Physica **87D**, 145 (1995).

[14] D. Holm and R. Camassa, Phys. Rev. Lett. **71**, 1671 (1993).

[15] P. Rosenau, Phys. Rev. Lett. **73**, 1737 (1994).

[16] A.S. Fokas and P.M. Santini, preprint (1995).

[17] M. Wadati, K. Konno, and Y.H. Ichikawa, J. Phys. Soc. Japan, **46**, 1965 (1979).

[18] A.S. Fokas and Q. Liu, Asymptotic Integrability of Water Waves, 1995.

[19] B. Fuchssteiner, Prog. Theor. Phys., **65**, 3 (1981).

[20] M.S. Alber, R. Camassa, D.D. Holm, J.E. Marsden, Proc. Roy. Soc., (1995).

[21] M.S. Alber, R. Camassa, D.D. Holm, J.E. Marsden, Lett. Math. Phys. **32**, 137 (1994).

[22] F. Calogero, Stud. Appl. Math. **70**, 189 (1984).

[23] P. Rosenau, Phys. Rev. Lett., to appear.