Estimates of solutions and asymptotic symmetry for parabolic equations on bounded domains

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Abstract. We consider fully nonlinear parabolic equations on bounded domains under Dirichlet boundary condition. Assuming that the equation and the domain satisfy certain symmetry conditions, we prove that each bounded positive solution of the Dirichlet problem is asymptotically symmetric. Compared with previous results of this type, we do not assume certain crucial hypotheses, such as uniform (with respect to time) positivity of the solution or regularity of the nonlinearity in time. Our method is based on estimates of solutions of linear parabolic problems, in particular on a theorem on asymptotic positivity of such solutions.

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1 Introduction

We consider the Dirichlet problem for a fully nonlinear parabolic equation,

$$u_t = F(t, x, u, Du, D^2u), \quad x \in \Omega, t > 0, \tag{1.1}$$
$$u = 0, \quad x \in \partial\Omega, t > 0. \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$,

$$Du = (u_{x_i})_{i=1,\dots,N}, \quad D^2u = (u_{x_ix_j})_{i,j=1,\dots,N},$$

and F is an elliptic nonlinearity satisfying suitable regularity assumptions. We assume that Ω is convex in x_1 and symmetric in the hyperplane $\{x = (x_1, \dots, x_N) : x_1 = 0\}$. We also assume certain symmetry and monotonicity of the function F. A specific example to which our results apply is the equation

$$u_t = f(t, u, |\nabla u|, \Delta u), \quad x \in \Omega, t > 0,$$

where $f(t, u, \eta, \xi)$ is continuous in $(t, u, \eta, \xi) \in \mathbb{R}^4$ and Lipschitz in (u, η, ξ), f_ξ exists everywhere and $f_\xi \geq \alpha_0$ for some positive constant α_0. Our goal is to prove that each global bounded solution $u(x, t)$ is asymptotically symmetric (even) in x_1 and nonincreasing in $x_1 > 0$. In other words, we want to prove that all limit profiles of $u(\cdot, t)$ as $t \to \infty$ are symmetric in x_1 and nonincreasing in $x_1 > 0$. Results of this type have been available for parabolic equations for about a decade (for elliptic equations for much longer). However, using new ideas, inspired by our recent considerations of similar problems on $\mathbb{R}^N$, we are now able to remove some essential restrictions imposed in the existing theorems. We shall be more specific about these theorems, but first let us briefly discuss similar elliptic results.

There is vast literature on symmetry and monotonicity of positive solutions of elliptic equations. In the classical paper [19], Gidas, Ni and Nirenberg proved the following result. If Ω is as above (convex and symmetric in x_1), then each positive solution u of the Dirichlet problem

$$\begin{aligned} \Delta u + f(u) = 0, &\qquad x \in \Omega, \\ u = 0, &\qquad x \in \partial\Omega, \end{aligned} \tag{1.3}$$

has the following symmetry and monotonicity properties:

$$\begin{aligned} &u(-x_1, x_2, \ldots, x_N) = u(x_1, x_2, \ldots, x_N) \quad (x \in \Omega), \\ &u_{x_1}(x_1, x_2, \ldots, x_N) < 0 \quad (x \in \Omega, x_1 > 0). \end{aligned} \tag{1.4}$$

The method of moving hyperplanes, which is the basic geometric technique in this and most of the related papers, was introduced by Alexandrov [1] and further developed by Serrin [36] ([36] also contains a related result on radial symmetry). Generalizations and extensions of the original theorems have been made by many authors, see the surveys [5, 26, 32] for an account of results, methods and references. In particular, Li [30] considered fully nonlinear equations on smooth domains and Berestycki and Nirenberg [6] found a way to deal with general equations on nonsmooth domains (see also [15]). The following theorem is proved in [6]. Assume u is a positive solution of the problem

$$0 = F(x, u, Du, D^2u), \quad x \in \Omega, \tag{1.5}$$

$$u = 0, \qquad x \in \partial\Omega, \tag{1.6}$$

where F satisfies assumptions (F1)-(F3) formulated in Section 2 (of course, here F is independent of t). Then u has the properties (1.4).

In a very fruitful direction of research, initiated in [20], many results of a similar flavor were proved for elliptic equations on various unbounded domains; we again refer the reader to the surveys [5, 32]. Other directions include extensions to more general classes of equations, including degenerate ones [14, 37], to elliptic systems [9, 38], and to parabolic equations on bounded and unbounded domains [2, 3, 4, 8, 13, 16, 18, 23, 33, 34, 35].

Let us discuss in more detail the existing results on asymptotic symmetry of solutions of nonautonomous parabolic equations. As above, we assume that Ω is convex in x_1 and symmetric in the hyperplane $\{x \in \mathbb{R}^N : x_1 = 0\}$. Assume $u(x,t)$ is a solution of a parabolic problem (1.1), (1.2) such that its orbit $\{u(\cdot,t) : t > 0\}$ is relatively compact in $C(\bar{\Omega})$ and let us introduce the ω-limit set of u in this space:

$$\omega(u) := \{\phi : \phi = \lim u(\cdot, t_n) \text{ for some } t_n \to \infty\}.$$

By asymptotic monotonicity and symmetry of u we mean the property that each $\phi \in \omega(u)$ is monotone nonincreasing in $x_1 > 0$ and

$$\phi(-x_1, x_2, \ldots, x_N) = \phi(x_1, x_2, \ldots, x_N) \quad (x \in \Omega).$$

In [23], the asymptotic symmetry was proved for positive bounded solutions of the following semilinear problem

$$\begin{aligned} u_t &= \Delta u + f(t, u) \quad && x \in \Omega, t > 0, \\ u &= 0, && x \in \partial\Omega, t > 0. \end{aligned} \tag{1.7}$$

It was assumed in [23] that f is locally Lipschitz in u (uniformly with respect to $t \geq 0$) and that for some $\vartheta > 0$ and each M, the $(\vartheta/2, \vartheta)$-Hölder norm of f on $[T, T+1] \times [0, M]$ is bounded above by a constant independent of $T > 0$. In addition to the symmetry of Ω, it is assumed there that $\dim \Omega \geq 2$, that Ω is of class $C^{2+\alpha}$, for some $\alpha > 0$, and that it is strictly convex in x_1 (for each $x \in \partial\Omega$ with $x_1 > 0$, the exterior normal vector $\nu(x)$ has a positive first component $\nu_1(x)$).

In an independent work, Babin [2, 3] proved the asymptotic symmetry for positive solutions of (1.1), (1.2), assuming that F is independent of t, it satisfies assumptions (F1)-(F3) of Section 2, and $F(x, 0, 0, 0) \geq 0$ $(x \in \Omega)$. It is also assumed in [2, 3] that the positive solution u has its "orbit" $\{u(\cdot, \cdot + \tau) : \tau \geq 1\}$ relatively compact in $C(\bar{\Omega} \times [0, 1])$ and in $C^{2,1}(\bar{D} \times [0, 1])$ for any subdomain $D \subset \bar{D} \subset \Omega$. Moreover, a uniform positivity assumption is made which guarantees that all $\phi \in \omega(u)$ are strictly positive in Ω. In a more recent paper, Babin and Sell [4] extended these results to nonautonomous equations (1.1), retaining the condition $F(t, x, 0, 0, 0) \geq 0$ $(x \in \Omega, t > 0)$, the compactness condition on u and its derivatives, and the uniform positivity condition. Also, the t-dependence is assumed to be regular: $F(t, x, u, p, q)$ is uniformly continuous on any set where the (u, p, q)-components of (t, x, u, p, q) are bounded. The assumptions of F being Lipschitz continuous and elliptic is replaced by a weaker assumptions, which still guarantees that the difference of two solutions of (1.1) satisfies a linear uniformly parabolic equation (or inequality).

The above discussion indicates the following limitations of the existing asymptotic symmetry results.

(I) Regularity requirements: in [23] the domain is smooth, and in both [4] and [23] some regularity of the time-dependence of the nonlinearities is assumed.

(II) Compactness requirements: in the fully nonlinear case, compactness assumptions are made on $\{D_x^j u(\cdot,t) : t \geq 1\}$, involving spatial derivatives $D_x^j u(\cdot,t)$ up to order $j = 2$.

(III) Strong positivity requirements: in the case of nonsmooth domains, a positivity assumption on the nonlinearity is made and only the solutions staying away from zero are considered.

Thus, because of (I) and (III), no asymptotic symmetry result was available so far even for semilinear equations (1.7) on a nonsmooth domain if $f(t,0)$ takes negative values for arbitrarily large values of t. Also, the time regularity assumptions in the available results rule out nonlinearities like $f(t,u) = \sin(e^t)g(u)$. Now, even letting compactness aside, the boundedness requirements in (II) are too restrictive as well. While it is often relatively easy to find a priori bounds on a solution u, using the maximum principle for example, finding bounds on its derivatives is typically much harder. Although without such bounds it might not be possible to guarantee global existence, still it is of interest to understand the behavior of a bounded solution which is *assumed* global. Because of (II), the symmetry results for such solutions were so far available for semilinear equations only. Finally, when one wants to use asymptotic symmetry results in a further investigation of the asymptotic behavior (for example, when attempting to prove a convergence property of positive solutions, as in [22, 10]) it is important to have the symmetry without a priori restricting the asymptotic behavior of the solutions as in (III).

These drawbacks are in the nature of the methods used in [23, 4]. In particular, they rely on the fact that limits of time translations of a solution, $u(\cdot, \cdot + \tau_k)$, solve a suitable limit parabolic equation. Mainly for this reason the regularity requirements (I), (II) are made. The strong positivity assumptions (III) are used for relating the asymptotic symmetry of a solution to the symmetry of positive solutions (of a limit equation) defined for all $t \in \mathbb{R}$.

Although our present method shares the basic technical tools, such as moving hyperplanes, maximum principles and Harnack inequalities, with the methods of [23, 4], it is substantially different in several aspects. We make no use of limit equations; the method is based on direct estimates of supersolutions of linear parabolic problems. Some of the results we obtain for linear equations might be of independent interest. Such is, we believe, an asymptotic positivity result which states that supersolutions that are initially positive except for a set of small measure (and they are not large in that set) are asymptotically nonnegative.

Building on estimates for linear equations, we prove general asymptotic symmetry results where the restrictions discussed above are removed. In the most general case, we do need the assumption that at least one function in $\omega(u)$ be strictly positive to get the asymptotic monotonicity and symmetry about $\{x : x_1 = 0\}$. This assumption cannot be removed, as we illustrate by an example, but without it we can still establish some asymptotic symmetry and monotonicity, although the symmetry hyperplane may not be the canonical one. Some of our theorems provide sufficient conditions for the existence of a positive element of $\omega(u)$. For example, the asymptotic nonnegativity of $F(x, t, 0, 0, 0)$, as $t \to \infty$, or Ω being a ball are such sufficient conditions. In case Ω is a ball, we can in addition establish, in a usual way, the asymptotic radial symmetry of positive solutions. In our general results we assume more than just boundedness of the solution (we need a certain equicontinuity property). We show, however, that this assumption can be removed under minor regularity assumptions on $\partial\Omega$.

In this paper, we consider bounded domains only. For symmetry results for nonautonomous equations on $\mathbb{R}^N$, the reader can consult [33, 34]. Also for now we leave aside several interesting results that can be proved under additional conditions involving in particular smoothness of the domain. Examples of such results are the exponential convergence of positive solutions to the space of symmetric functions, as proved in [4], or eventual (not just asymptotic) monotonicity properties of positive solutions as obtained in [23].

The remainder of the paper is organized as follows. In Section 2 we state our main symmetry results. The estimates of solutions of underlying linear problems, including an asymptotic positivity result, are stated and proved in Subsection 3.2. In Subsection 3.1, which bridges the linear estimates and the nonlinear problems (1.1), (1.2), we introduce reflections of solutions using moving hyperplanes. The proofs of the symmetry results are given in Section 4. Examples showing importance of some hypotheses are given in Section 5.

We included two appendices. In the first one we prove the asymptotic symmetry of positive solutions u such that $u(x, t)$ stays away from 0 for each $x \in \Omega$. This is essentially a result of [4], however, we prove it without the regularity assumption on the t-dependence of F and without compactness requirements on the derivatives of u. The reason for separating this result from the rest of our main theorems is that its proof is much simpler; in fact most of the estimates from Section 3.2 are not needed in it. The second appendix contains details regarding the proof of a Harnack inequality from Section 3.2.

2 Main results

We assume the following hypotheses.

(D1) $\Omega \subset \mathbb{R}^N$ is a bounded domain which is convex in x_1 and symmetric about the hyperplane $H_0 = \{x = (x_1, x') \in \mathbb{R}^N : x_1 = 0\}$:

$$\{(-x_1, x') : (x_1, x') \in \Omega\} = \Omega.$$

(D2) For each $\lambda > 0$, the set

$$\Omega_\lambda := \{x \in \Omega : x_1 > \lambda\}$$

has only finitely many connected components.

After the formulation of our first theorem below, we include a few comments on hypothesis (D2).

Via a canonical isomorphism, we identify the space of $N \times N$-matrices with $\mathbb{R}^{N^2}$. The nonlinearity F is defined on $[0, \infty) \times \Omega \times \mathcal{B}$, where $\mathcal{B}$ is an open convex set in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2}$ which is invariant under the transformation Q defined by

$$Q(u, p, q) = (u, -p_1, p_2, \dots, p_N, \bar{q}), \tag{2.1}$$
$$\bar{q}_{ij} = \begin{cases} -q_{ij} & \text{if exactly one of } i,\ j \text{ equals } 1, \\ q_{ij} & \text{otherwise.} \end{cases}$$

We assume that F satisfies the following conditions:

(F1) (Regularity) F is continuous on $[0, \infty) \times \bar{\Omega} \times \mathcal{B}$ and Lipschitz in (u, p, q), uniformly with respect to (x, t): there is $L > 0$

$$\sup_{x \in \Omega, t \geq 0} |F(t, x, u, p, q) - F(t, x, \tilde{u}, \tilde{p}, \tilde{q})| \leq L|(u, p, q) - (\tilde{u}, \tilde{p}, \tilde{q})|$$
$$((u, p, q), (\tilde{u}, \tilde{p}, \tilde{q}) \in \mathcal{B}). \tag{2.2}$$

Moreover, F is differentiable with respect to q on $[0, \infty) \times \Omega \times \mathcal{B}$.

(F2) (Ellipticity) There is a constant $\alpha_0 > 0$ such that

$$F_{q_{ij}}(t,x,u,p,q)\xi_i\xi_j \geq \alpha_0|\xi|^2 \quad ((t,x,u,p,q) \in [0,\infty)\times\Omega\times\mathcal{B},\ \xi \in \mathbb{R}^N). \tag{2.3}$$

Here and below we use the summation convention (summation over repeated indices).

(F3) (Symmetry and Monotonicity) For any $(t,u,p,q) \in [0,\infty)\times\mathcal{B}$ and any $(x_1,x'),(\tilde{x}_1,x') \in \Omega$ with $\tilde{x}_1 > x_1 \geq 0$ one has

$$F(t,\pm x_1,x',Q(u,p,q)) = F(t,x_1,x',u,p,q) \geq F(t,\tilde{x}_1,x',u,p,q).$$

Note that the previous relations imply

$$F(t,2\lambda - x_1,x',Q(u,p,q)) \geq F(t,x_1,x',u,p,q) \qquad ((x_1,x') \in \Omega_\lambda, (t,u,p,q) \in [0,\infty)\times\mathcal{B}) \tag{2.4}$$

for all $\lambda \geq 0$.

Remark 2.1. (i) We assume the global Lipschitz continuity of F (and similarly the ellipticity) on the set $\mathcal{B}$ only. This is no more than local Lipschitz continuity, uniform in x and t, in case $\mathcal{B}$ is bounded. However, we shall make no assumption on the boundedness of the derivatives of the solutions in question. Thus the range of (u, Du, D^2u) may be unbounded and then we need global Lipschitz continuity of F on this range.

(ii) The differentiability assumption on F with respect to q can be relaxed. For example, it is sufficient to assume that each derivative $F_{q_{ij}}$ (which is defined almost everywhere by (H1)) extends to a bounded function $\tilde{F}_{ij}$ defined on $[0,\infty)\times\bar{\Omega}\times\mathcal{B}$ which is the pointwise limit of a sequence of continuous functions. In the ellipticity condition (F2) and in the Hadamard formulas in Section 3.1, the functions $F_{q_{ij}}$ are then replaced by $\tilde{F}_{ij}$.

We consider global classical solutions u of (1.1), (1.2). By this we mean functions $u \in C^{2,1}(\Omega\times(0,\infty)) \cap C(\bar{\Omega}\times[0,\infty))$ such that

$$(u(x,t), Du(x,t), D^2u(x,t)) \in \mathcal{B} \quad (x \in \Omega, t > 0)$$

and (1.1), (1.2) are satisfied everywhere. We assume the following conditions on u.

(U1) $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded uniformly in t.

(U2) The functions $u(\cdot,\cdot+s)$, $s \geq 1$, are equicontinuous on $\Omega \times [0,1]$:

$$\lim_{h\to 0} \sup_{\substack{x,\bar{x}\in\bar{\Omega},\, t,\bar{t}\in[s,s+1],\\ |x-\bar{x}|,|t-\bar{t}|<h,\\ s\geq 1}} |u(x,t) - u(\bar{x},\bar{t})| = 0.$$

Under a minor boundedness assumption on F, it can be proved, see Proposition 2.7 below, that if (U1) holds, then (U2) follows from, and hence is equivalent to, the following stronger form of the boundary condition

(U2)' $u(x,t) \to 0$ as $\operatorname{dist}(x,\partial\Omega) \to 0$, uniformly with respect to $t \geq 1$.

If one is willing to make regularity assumptions on Ω, then condition (U2)' is a consequence of (U1) and Hölder boundary estimates for solutions of (1.1), (1.2), see Proposition 2.7. In that situation, assumption (U2) can be omitted in our theorems. Note that no assumption on compactness or even boundedness of derivatives of u is made or can be derived from boundedness without extra assumptions on F.

Hypotheses (U1), (U2) imply in particular that $\{u(\cdot,t) : t > 0\}$ is relatively compact in $C(\bar{\Omega})$. We introduce the ω-limit set of u in this space:

$$\omega(u) = \{z \in C(\bar{\Omega}) : \|u(\cdot,t_k) - z\|_{L^\infty(\Omega)} \to 0 \text{ for some } t_k \to \infty\}.$$

Observe that $\{u(\cdot,t) : t > 0\}$ is also compact in $C_0(\bar{\Omega})$, the closed subspace of $C(\bar{\Omega})$ consisting of all continuous functions on $\bar{\Omega}$ vanishing on $\partial\Omega$. Hence $\omega(u) \subset C_0(\bar{\Omega})$.

Our first result is as follows.

Theorem 2.2. *Let* (D1), (D2), (F1)–(F3) *hold and let u be a nonnegative global solution of* (1.1), (1.2) *satisfying* (U1), (U2)*. Assume that there is $\phi \in \omega(u)$ such that $\phi > 0$ on Ω. Then u is asymptotically symmetric and monotone in x_1. More specifically, for each $z \in \omega(u)$ one has*

$$z(-x_1, x') = z(x_1, x') \quad ((x_1, x') \in \Omega), \tag{2.5}$$

and either $z \equiv 0$ or else z is strictly decreasing in x_1 on $\Omega_0 := \{x \in \Omega : x_1 > 0\}$. The latter holds in the form $z_{x_1} < 0$ on Ω_0, provided $z_{x_1} \in C(\Omega_0)$.

We remark that $z_{x_1} \in C(\Omega_0)$ for each $z \in \omega(u)$, provided $\{u_{x_1}(\cdot, t) : t > 1\}$ is relatively compact in $C(\bar{B})$ for each closed ball $\bar{B} \subset \Omega_0$.

The assumption that ϕ be positive everywhere can be relaxed somewhat, but it cannot be completely removed. This is illustrated in Example 2.3 below. One can give several sufficient conditions for the existence of a positive limit profile of u, see for example Theorem 2.5 and Corollary 2.6 below. Observe that Theorem 2.2 in particular implies that if some $\phi \in \omega(u)$ is positive in Ω then each $\phi \in \omega(u) \setminus \{0\}$ is positive in Ω.

Hypotheses (D2) can be relaxed so as to require that the number of connected components of Ω_λ is infinite for all $\lambda > 0$ if it is infinite for some $\lambda > 0$. If all elements of $\omega(u)$ are positive (that is, $u(x, t)$ stays away from 0 for each x), then the hypothesis can be dropped (see Appendix I). We do not know whether in the general fully nonlinear setting (D2) is merely a technical condition.

Example 2.3. *Let $\Omega = [-1, 1] \times [-1, 1]$. There is a Lipschitz function f on $[-1, 1] \times \mathbb{R}$ such that the problem*

$$u_t = \Delta u + f(y, u) \quad (x, y) \in \Omega, t > 0, \tag{2.6}$$

$$u = 0, \qquad (x, y) \in \partial\Omega, t > 0, \tag{2.7}$$

has a global bounded positive solution u with $\omega(u) = \{z\}$, where $z \in C_0(\bar{\Omega})$ is a nonnegative function satisfying $z(x, y) > 0$ if $(x, y) \in \Omega$, $x > 0$, and $z(0, y) = 0$ for each $y \in [-1, 1]$. In particular, z is not monotone in $x > 0$.

See Section 5 for details regarding this example. As the next theorem shows, without the assumption of the existence of a positive $\phi \in \omega(u)$, the elements in $\omega(u)$ still have some monotonicity and partial symmetry properties, but not necessarily with respect to the hyperplane $\{x_1 = 0\}$ (in the above example the hyperplane is $\{x_1 = 1/2\}$).

Theorem 2.4. *Let (D1), (D2), (F1)–(F3) hold and let u be a nonnegative global solution of (1.1), (1.2) satisfying (U1), (U2). Then there exists $\lambda \geq 0$ such that for each $z \in \omega(u)$ the following is true: z is monotone nonincreasing in x_1 on Ω_λ and there is a connected component U of Ω_λ such that*

$$z(2\lambda - x_1, x') = z(x_1, x') \quad ((x_1, x') \in U). \tag{2.8}$$

If Ω_λ is connected, then for each $z \in \omega(u)$ ((2.8) holds with $U = \Omega_\lambda$ and) either $z \equiv 0$ in Ω_λ or else z is strictly decreasing in x_1 on Ω_λ. The latter holds in the form $z_{x_1} < 0$ if $z_{x_1} \in C(\Omega_\lambda)$.

In the next theorem we show, that under an asymptotic nonnegativity condition on F, the existence of a positive ϕ in $\omega(u)$ can be omitted in hypotheses of Theorem 2.2. Note that under that condition it is still possible for $\omega(u)$ to contain 0, as no uniform positivity on u or F is assumed.

Theorem 2.5. *Assume that* (D1), (D2), (F1)–(F3) *hold. Further assume that* $(0,0,0) \in \mathcal{B}$ *and*

$$\liminf_{x\in\Omega,\, t\to\infty} F(t,x,0,0,0) \geq 0. \tag{2.9}$$

Let u be a global nonnegative solution of (1.1), (1.2) *satisfying* (U1), (U2). *Then either $\omega(u) = \{0\}$ or else there exists $\phi \in \omega(u)$ with $\phi > 0$ in Ω. Consequently, the conclusion of Theorem 2.2 holds.*

It is a standard consequence of the above symmetry results that if the domain and the equation are invariant under all rotations of $\mathbb{R}^N$ (or a subspace of $\mathbb{R}^N$), then one gets the corresponding asymptotic rotational symmetry of positive solutions. Such consequences are proved by applying the reflectional symmetry in any admissible direction. We formulate just one result of this sort, leaving formulations of other radial symmetry results to the reader.

Assume Ω is the unit ball centered at the origin and consider the problem

$$u_t = f(t,|x|,u,|\nabla u|,\Delta u) \quad x \in \Omega, t > 0, \tag{2.10}$$

$$u = 0, \qquad x \in \partial\Omega, t > 0, \tag{2.11}$$

where $f(t,r,u,\eta,\xi)$ is defined on $[0,\infty) \times [0,1] \times \mathcal{B}$, where $\mathcal{B}$ is an open ball in $\mathbb{R}^3$ centered at the origin. We assume that f satisfies the following conditions:

$(\mathrm{F1})_{rad}$ $f : [0,\infty) \times [0,1] \times \mathcal{B} \to \mathbb{R}$ is continuous in all variables, Lipschitz in (u,η,ξ), uniformly with respect to (x,t) and differentiable with respect to ξ.

$(\mathrm{F2})_{rad}$ $\tilde{f}_\xi \geq \alpha_0$ on $[0,\infty) \times [0,1] \times \mathcal{B}$ for some positive constant α_0.

$(\mathrm{F3})_{rad}$ f is monotone nonincreasing in r.

Corollary 2.6. *Let Ω be the unit ball,* $(\mathrm{F1})_{rad}$–$(\mathrm{F3})_{rad}$ *hold, and let u be a nonnegative global solution of* (2.10), (2.11) *satisfying* (U1), (U2). *Then u is asymptotically radially symmetric and radially nonincreasing. More specifically, each $z \in \omega(u)$ is a radial function and either $z \equiv 0$ or else z is strictly decreasing in $r = |x|$. If $z \in C^1(\Omega)$ and $z \not\equiv 0$ then $z_r < 0$ for $r \in (0,1)$.*

Proof. The statement is trivial if $\omega(u) = \{0\}$, so we assume that $\phi \not\equiv 0$ for some $\phi \in \omega(u)$. It is sufficient to prove that $\phi > 0$ in Ω; an application of Theorem 2.2 after an arbitrary rotation of the coordinate system then gives the radial symmetry.

By Theorem 2.4, there is $\lambda \geq 0$ such that $V_\lambda \phi \equiv 0$ in Ω_λ and ϕ is decreasing in $x_1 > 0$. In particular, $\phi > 0$ in $\bar{\Omega}_\lambda \setminus \partial\Omega$, as $\phi = 0$ on $\partial\Omega$. If $\lambda = 0$ we are done. We show that $\lambda > 0$ is impossible. First we apply Theorem 2.4, after any rotation, to infer that $\varphi > 0$ in Ω near $\partial\Omega$. Now if $\lambda > 0$ then $P_\lambda(\partial\Omega)$ contains points in Ω arbitrarily close to $\partial\Omega$. Since $V_\lambda \phi \equiv 0$ implies that $\phi = 0$ on $P_\lambda(\partial\Omega)$ we have a contradiction.

□

We end this section with a regularity result on bounded solutions of (1.1), (1.2), which is a consequence of well-known Hölder estimates. It implies that if $F(\cdot,\cdot,0,0,0)$ is bounded and u satisfies (U1), (U2)', then it has the equicontinuity property (U2). If, in addition, Ω satisfies the exterior cone condition or a more general condition (A) stated below, mere boundedness of u implies (U2). Thus under these conditions on Ω and F, the assumption (U2) can be omitted in all the above theorems (or it can be replaced by (U2)' without the additional regularity requirement on Ω).

We say that Ω satisfies condition (A) if there exist numbers $\varsigma \in (0,1)$ and $R > 0$ such that for each $x \in \partial\Omega$, $\rho \in (0,R))$ one has

$$|\Omega \cap B(x,\rho)| \leq \varsigma |B(x,\rho)|,$$

where $B(x,\rho)$ is the ball of radius ρ centered at x and $|A|$ stands for the (Lebesgue) measure of a set A.

Proposition 2.7. *Let Ω be a bounded domain in $\mathbb{R}^N$ and let* (F1), (F2) *hold. Further assume that $(0,0,0) \in \mathcal{B}$ and the function $F(\cdot,\cdot,0,0,0)$ is bounded on $[0,\infty) \times \Omega$. Then for any global solution u of* (1.1) *satisfying* (U1) *the following holds. For any domain $G \subset \Omega$ with $\bar{G} \subset \Omega$, there are constants $\alpha > 0$ and C such that*

$$\sup_{\substack{x,\bar{x} \in \bar{G}, x \neq \bar{x} \\ t,\bar{t} \in [s,s+1], t \neq \bar{t} \\ s > 0}} \frac{|u(x,t) - u(\bar{x},\bar{t})|}{|x-\bar{x}|^\alpha + |t-\bar{t}|^{\frac{\alpha}{2}}} \leq C. \tag{2.12}$$

If, in addition, Ω satisfies condition (A) then the above conclusion holds with $G = \Omega$.

Proof. We have

$$u_t - F(t,x,u,Du,D^2u) + F(t,x,0,0,0) = F(t,x,0,0,0), \quad x \in \Omega, t > 0.$$

It follows, by (F1), (F2) and the Hadamard integral formulas (cp. Subsection 3.1), that u is a solution of a linear uniformly parabolic equation with bounded coefficients and bounded right-hand side. Interior Hölder estimates for such equations (see [27, Sect. IV.2] or [31, Sect. VII.8]) imply that (2.12) holds for some $C > 0$ and $\alpha > 0$. If condition (A) is satisfied, the interior Hölder estimates combined with boundary Hölder estimates found in [31, Sect. VII.9] (also in [27], under more restrictive conditions; a recent work of Cho and Safonov, see [11, 12], contains more general results) give (2.12) for $G = \Omega$. □

3 Reflection in hyperplanes and linear equations

In this section we introduce linear parabolic inequalities associated with the method of moving hyperplanes. A major part of the section is devoted to estimates of solutions and supersolutions of such inequalities.

The following general notation is used throughout the paper. For $x_0 \in \mathbb{R}^N$ and $r > 0$, $B(x_0, r)$ stands for the ball centered at x_0 with radius r. For a set $\Omega \subset \mathbb{R}^N$ and functions v and w on Ω, the inequalities $v \geq 0$ and $w > 0$ are always understood in the pointwise sense:

$$v(x) \geq 0, \; w(x) > 0 \quad (x \in \Omega).$$

For a function z, $\sup_Q z$ denotes the supremum of z over a set Q, similarly for the infimum; z^+, z^- stand for the positive and negative parts of z, respectively:

$$z^+(x) = (|z(x)| + z(x))/2 \geq 0,$$
$$z^-(x) = (|z(x)| - z(x))/2 \geq 0.$$

If D_0 and D are bounded sets in $\mathbb{R}^N$ or $\mathbb{R}^{N+1}$, the notation $D_0 \subset\subset D$ means $\bar{D}_0 \subset D$, $\mathrm{diam}(D)$ stands for the diameter of D, and $|D|$ for the (Lebesgue) measure of D (if D is measurable). For $D_0 \subset\subset D$ we denote

$$\mathrm{dist}(\bar{D}_0, \partial D) = \inf\{|x-y| : x \in \overline{D}_0, y \in \partial D\}.$$

The parabolic boundary of a cylindrical domain $U \times (\tau, T)$ is the set

$$\partial_P(U \times (\tau, T)) = (\partial U \times (\tau, T]) \cup (\bar{U} \times \{\tau\}).$$

3.1 Reflections in hyperplanes

Assume that the hypotheses (D1), (F1)–(F3) are satisfied and u is a positive global solution of (1.1), (1.2) satisfying (U1), (U2).

Let

$$\ell := \max\{x_1 : (x_1, x') \in \bar{\Omega} \text{ for some } x' \in \mathbb{R}^{N-1}\},$$

and, for $\lambda \in [0, \ell)$,

$$\begin{aligned} \Omega_\lambda &:= \{x \in \Omega : x_1 > \lambda\}, \\ H_\lambda &:= \{x \in \mathbb{R}^N : x_1 = \lambda\}, \\ \Gamma_\lambda &:= H_\lambda \cap \bar{\Omega}. \end{aligned} \tag{3.1}$$

Let P_λ denote the reflection in the hyperplane H_λ. For a function $z(x) = z(x_1, x')$, let z^λ and $V_\lambda z$ be defined by

$$\begin{aligned} z^\lambda(x) &= z(P_\lambda x) = z(2\lambda - x_1, x'), \\ V_\lambda z(x) &= z^\lambda(x) - z(x) \quad (x \in \Omega_\lambda). \end{aligned} \tag{3.2}$$

For the positive solution u, let us now consider the function $u^\lambda(x, t) = u(P_\lambda x, t)$ for any $\lambda \in [0, \ell)$. Hypothesis (F3) implies

$$u_t^\lambda \geq F(t, x, u^\lambda, Du^\lambda, D^2 u^\lambda) \quad x \in \Omega_\lambda, t > 0. \tag{3.3}$$

Moreover, we obviously have

$$\begin{aligned} u(x, t) &= 0 \quad (x \in \Gamma_\lambda, t > 0), \\ u(x, t) &\geq 0 \quad (x \in \partial\Omega_\lambda \setminus \Gamma_\lambda, t > 0). \end{aligned} \tag{3.4}$$

It follows that the function $v = V_\lambda u = u^\lambda - u$ satisfies

$$v_t \geq a_{ij}(x, t) v_{x_i x_j} + b_i(x, t) v_{x_i} + c(x, t) v, \quad x \in \Omega_\lambda,\ t > 0, \tag{3.5}$$

$$v \geq 0, \qquad x \in \partial\Omega_\lambda,\ t > 0, \tag{3.6}$$

where the coefficients are obtained from the Hadamard formula. Specifically, we define (omitting the argument (x,t) of of u and u^λ)

$$c(x,t) = \begin{cases} \displaystyle\int_0^1 F_u(t,x,u+s(u^\lambda-u),Du,D^2u)\,ds & \text{if } u^\lambda(x,t)\neq u(x,t), \\ 0 \quad \text{if } u^\lambda(x,t)=u(x,t), \end{cases}$$

$$b_i(x,t) = \begin{cases} \displaystyle\int_0^1 F_{p_i}(t,x,u^\lambda,\dots,u^\lambda_{x_{i-1}},u_{x_i}+s(u^\lambda_{x_i}-u_{x_i}),u_{x_{i+1}},\dots,D^2u)\,ds \\ \hfill \text{if } u^\lambda_{x_i}(x,t)\neq u_{x_i}(x,t), \\ 0 \quad \text{if } u^\lambda_{x_i}(x,t)=u_{x_i}(x,t), \end{cases}$$

$$\begin{aligned} a_{ij}(x,t) = \int_0^1 & F_{q_{ij}}(t,x,u^\lambda,Du^\lambda,\dots, \\ & u^\lambda_{x_{i^-}x_{j^-}},u_{x_ix_j}+s(u^\lambda_{x_ix_j}-u_{x_ix_j}),u_{x_{i^+}x_{j^+}},\dots,u_{x_Nx_N})\,ds \end{aligned}$$

where (i^-,j^-), (i^+,j^+) stand for the pairs of indices preceding, respectively following, (i,j) in the identification of $N\times N$ matrices with $\mathbb{R}^{N^2}$. By (F1), the integrals make sense and give the right quotients for the right hand side of (3.5) to be equal to the difference of $F(t,x,u^\lambda,Du^\lambda,D^2u^\lambda)$ and $F(t,x,u,Du,D^2u)$. For example:

$$c(x,t) = \frac{F(t,x,u^\lambda,Du,D^2u)-F(t,x,u,Du,D^2u)}{u^\lambda-u}$$

if $u^\lambda(x,t)\neq u(x,t)$. This also implies that the coefficients are (everywhere-defined) measurable functions. Note that under the relaxed assumption mentioned in Remark 2.1, one replaces $F_{q_{ij}}$ with $\tilde F_{ij}$ in the definition of a_{ij}. The measurability of a_{ij} then follows from the assumption on $\tilde F_{ij}$ (which is clearly satisfied by $\tilde F_{ij} = F_{q_{ij}}$ if $F_{q_{ij}}$ are defined everywhere). By (F1), (F2), there is $\beta_0>0$ independent of λ such that (with α_0 as in (F2))

$$|a_{ij}(x,t)|,|b_i(x,t)|,|c(x,t)|<\beta_0 \quad (x\in\Omega_\lambda,\ t>0) \tag{3.7}$$

$$a_{ij}(x,t)\xi_i\xi_j \geq \alpha_0|\xi|^2 \quad (\xi\in\mathbb{R}^N, x\in\Omega_\lambda, t>0). \tag{3.8}$$

The proofs of our symmetry results depend on the fact that $V_\lambda u$ satisfies (3.5), (3.6) and on the estimates of solutions of such linear problems that we now derive.

3.2 Estimates of solutions of linear equations

In this subsection we derive several estimates of solutions of the linear problem (3.5). The relation of the coefficients a_{ij}, b_i, c to the nonlinearity F is irrelevant here. We consider a general linear problem

$$
\begin{aligned}
&v_t = a_{ij}(x,t)v_{x_i x_j} + b_i(x,t)v_{x_i} + c(x,t)v, && (x,t) \in U \times (\tau, T), && (3.9)\\
&v = 0 && (x,t) \in \partial U \times (\tau, T), && (3.10)
\end{aligned}
$$

where U is an open subset of some fixed bounded domain $\Omega \subset \mathbb{R}^N$, $0 \le \tau < T \le \infty$, the coefficients are defined on $U \times (\tau, T)$ and for some positive constants α_0, β_0 they satisfy the following conditions

(L1) a_{ij}, b_i, c are measurable and

$$
\begin{aligned}
|a_{ij}(x,t)|,\ |b_i(x,t)|,\ |c(x,t)| < \beta_0 \quad & (i,j = 1,\dots,N, x \in U, t \in (\tau,T)),\\
a_{ij}(x,t)\xi_i\xi_j \ge \alpha_0|\xi|^2 \quad & (\xi \in \mathbb{R}^N, x \in U, t \in (\tau,T)).
\end{aligned}
$$

When referring to a *solution*, respectively *supersolution*, of (3.9), we mean a function v in the Sobolev space $W^{2,1}_{N+1,loc}(U \times (\tau,T))$ such that (3.9), respectively (3.9) with "=" replaced by "$\ge$", is satisfied almost everywhere. A *solution*, respectively *supersolution*, of (3.9), (3.10), is in addition continuous on $\bar U \times [\tau, T)$ and satisfies (3.10), respectively (3.10) with "=" replaced by "$\ge$". A *subsolution* is defined as supersolution with both inequality signs reversed. Below we often use standard maximum and comparison principles for super and subsolutions of (3.9) as found in [31], for example.

The main technical results of this section are Lemmas 3.1, 3.3, and 3.4. At the end of this subsection, building on these lemmas, we prove a theorem on asymptotic positivity of supersolutions that plays a prominent role in our proof of the symmetry results.

Lemma 3.1. *Given any $k > 0$ there is a constant δ determined only by α_0, β_0, N,* diam(Ω), *and k such that for any open set $U \subset \Omega$ with $|U| < \delta$ the following holds. If $v \in C(\bar U \times [\tau,T))$ is a supersolution of a problem* (3.9), (3.10) *whose coefficients satisfy* (L1), *then*

$$
\|v^-(\cdot,t)\|_{L^\infty(U)} \le 2e^{-k(t-\tau)}\|v^-(\cdot,\tau)\|_{L^\infty(U)} \quad (t \in (\tau,T)). \tag{3.11}
$$

In the proof we employ, similarly as in [2, 4], the following result of Varadhan (as attributed to in [6]).

Lemma 3.2. *Given any $a_0 > 0$, $b_0 \geq 1$, there exists $\delta > 0$ determined only by a_0, b_0, N, and* $\mathrm{diam}(\Omega)$ *such that for any closed set $K \subset \Omega$ with $|K| \leq \delta$ there exists a smooth function g on Ω such that $1 \leq g \leq 2$ and for any symmetric positive definite matrix (a_{ij}) with*

$$\det(a_{ij}) \geq a_0^N \tag{3.12}$$

one has

$$a_{ij} g_{x_i x_j} + b_0(|\nabla g| + g) < 0 \quad (x \in K). \tag{3.13}$$

The result follows from [6, Theorem 3.1] which differs from our formulation in that $K \subset B(0,1)$ (rather than $K \subset \Omega$) and $\det(a_{ij}) \geq 1$ instead of (3.12) is assumed. Our statement is easily derived from this result by scaling the variable x and then multiplying the matrix (a_{ij}) by a suitable constant.

Proof of Lemma 3.1. Fix any $k \geq 1$ and let δ be as in Lemma 3.2 with $a_0 = \alpha_0$ and $b_0 = \sqrt{N}\beta_0 + k$. We claim that the conclusion holds for this δ. Indeed, let $U \subset \Omega$ and v satisfy the assumptions in Lemma 3.1. Suppose that (3.11) fails. Then there are $T_1 \in (\tau, T)$ and $x_0 \in U$ such that

$$r_0 := e^{kT_1} v^-(x_0, T_1) > 2e^{k\tau} \|v^-(\cdot, \tau)\|_{L^\infty(U)}. \tag{3.14}$$

Since $v \in C(\bar{U} \times [\tau, T))$ and $v \geq 0$ on $\partial U \times [\tau, T)$, we can find an open set $D \subset\subset U$ such that

$$e^{kt} v^-(x,t) < \frac{1}{2} r_0 \quad (x \in U \setminus D, t \in [\tau, T_1]). \tag{3.15}$$

Now let g be as in Lemma 3.2 with $K = \bar{D}$ ($|K| < \delta$ as $D \subset\subset U$). A simple calculation shows that $w := v/g$ is a supersolution of

$$w_t = a_{ij}(x,t) v_{x_i x_j} + \hat{b}_i(x,t) v_{x_i} + \hat{c}(x,t) v, \quad (x,t) \in D \times (\tau, T_1], \tag{3.16}$$

where

$$\hat{b}_i = 2a_{ij} \frac{g_{x_i}}{g} + b_i,$$
$$\hat{c} = a_{ij} \frac{g_{x_i x_j}}{g} + b_i \frac{g_{x_i}}{g} + c.$$

We have

$$\hat{c} \leq \frac{a_{ij}g_{x_ix_j} + \sqrt{N}\beta_0(g + |\nabla g|)}{g}$$
$$= \frac{a_{ij}g_{x_ix_j} + (b_0 - k)(g + |\nabla g|)}{g} \leq -\frac{k(g + |\nabla g|)}{g} \leq -k.$$

The maximum principle implies that the function $e^{kt}w$ assumes its nonpositive minimum on $\partial_P(D \times (\tau, T_1))$. However, by (3.14) we have

$$e^{kT_1}w^-(x_0, T_1) \geq \frac{1}{2}r_0 > e^{k\tau}\|v^-(\cdot, \tau)\|_{L^\infty(D)} \geq e^{k\tau}\|w^-(\cdot, \tau)\|_{L^\infty(D)}.$$

Similarly, by (3.15),

$$e^{kT_1}w^-(x_0, T_1) \geq \frac{1}{2}r_0 > v^-(x, t) \geq w^-(x, t) \quad (x \in U \setminus D, t \in [\tau, T_1]).$$

Since $x_0 \in D_1$, we have $(x_0, T_1) \notin \partial_P(D \times (\tau, T_1))$, a contradiction proving (3.11). □

The above proof can be simplified a little if $|\bar{U}| < \delta$. However, with no assumption on the boundary, $|U| < \delta$ does not imply $|\bar{U}| < \delta$.

Lemma 3.3. *For any $\rho > 0$ there exist a constant $\gamma > 0$ determined only by N, ρ, α_0, and β_0, and a smooth function h_ρ on $B(0, \rho)$ with*

$$h_\rho(x) > 0 \quad (x \in B(0, \rho)), \qquad h_\rho(x) = 0 \quad (x \in \partial B(0, \rho)) \tag{3.17}$$

such that the following holds. For any $x_0 \in \Omega$ with $U := B(x_0, \rho) \subset \Omega$, and any coefficients satisfying (L1), *the function $\varphi(x, t) = e^{-\gamma t}h_\rho(x - x_0)$ is a (strict) subsolution of* (3.9)*:*

$$\varphi_t - (a_{ij}\varphi_{x_ix_j} + b_i\varphi_{x_i} + c\varphi) < 0 \quad \textit{in } B(x_0, \rho) \times [\tau, T]. \tag{3.18}$$

Proof. Let η be a smooth function on $[0, 1]$ such that

$$\eta(1) = \eta'(1) = 0 < \eta''(1), \tag{3.19}$$
$$\eta \equiv 1 \text{ near } 0, \text{ and } \eta > 0 \text{ in } [0, 1). \tag{3.20}$$

Set $h_\rho(x) = \eta(|x|/\rho)$. We verify that h_ρ and a suitable $\gamma = \gamma(N, \rho, \alpha_0, \beta_0) > 0$ have the stated properties. Let $x_0 \in \Omega$ be such that $B(x_0, \rho) \subset \Omega$. Using a translation, we may assume without loss of generality that $x_0 = 0$. Taking $\varphi(x,t) = e^{-\gamma t} h_\rho(r)$, $r = |x|$, a simple computation shows that the left-hand side of (3.18) is equal to

$$\begin{aligned} &- e^{-\gamma t} \left((\gamma + c)\eta + a_{ij}x_i x_j \left(\frac{\eta''}{\rho^2 r^2} - \frac{\eta'}{\rho r^3} \right) + a_{ii}\frac{\eta'}{\rho r} + b_i x_i \frac{\eta'}{\rho r} \right) \\ &\leq e^{-\gamma t} \left((-\gamma + \beta_0)\eta - \alpha_0 \frac{\eta''}{\rho^2} + \beta_0 (N^2 + N)\frac{|\eta'|}{\rho r} + \frac{\beta_0 N |\eta'|}{\rho} \right), \end{aligned}$$

where we have omitted the argument r/ρ of η. Assuming ρ is fixed, the expression is negative (for any $\gamma > 0$) if $r/\rho > 1 - \epsilon$ and $\epsilon > 0$ is sufficiently small. In the remaining region, $\eta \geq \delta_1$, for some $\delta_1 > 0$, and η'', η'/r are bounded (by (3.20)). Choosing $\gamma = \gamma(\rho, \alpha_0, \beta_0, N)$ large enough, we make the expression negative in that region, as well. □

If Q is an open bounded set in $\mathbb{R}^{N+1}$, u is a bounded continuous function on Q and $p > 0$, we denote

$$[u]_{p,Q} = \left(\frac{1}{|Q|} \int_Q |u(x,t)|^p \, dxdt \right)^{\frac{1}{p}}.$$

Also we set

$$[u]_{\infty,Q} = \sup_Q |u|.$$

Lemma 3.4. *Given $d > 0$, $\varepsilon > 0$, $\theta > 0$, there are positive constants κ, p determined only by N, $\operatorname{diam}(\Omega)$, α_0, β_0, d, ε, and θ with the following property. If D, U are domains in Ω with $D \subset\subset U$, $\operatorname{dist}(\bar{D}, \partial U) \geq d$, $|D| > \varepsilon$, and $v \in C(\bar{U} \times [\tau, \tau + 4\theta])$ is a supersolution of an equation* (3.9) *(with $T = \tau + 4\theta$) whose coefficients satisfy* (L1)*, then*

$$\inf_{D\times(\tau+3\theta,\tau+4\theta)} v(x,t) \;\geq\; \kappa [v^+]_{p,D\times(\tau+\theta,\tau+2\theta)} - \sup_{\partial_P(U\times(\tau,\tau+4\theta))} e^{4m\theta} v^-, \quad (3.21)$$

where $m = \sup_{U\times(\tau,\tau+4\theta)} c$. If v is a solution of (3.9)*, then one can take $p = \infty$ and κ is independent of ε.*

For solutions the lemma is proved, in a slightly weaker form, in [33]. The above formulation is the one that we use below. For the proof, however, it is more convenient to have a more general formulation dealing with nonhomogeneous equation

$$v_t = a_{ij}(x,t)v_{x_ix_j} + b_i(x,t)v_{x_i} + c(x,t)v + g(x,t), \quad (x,t) \in U \times (\tau, T), \tag{3.22}$$

where $g \in L^{N+1}(U \times (\tau, T))$. Lemma 3.4 readily follows from the following result.

Lemma 3.5. *Given numbers $d > 0$, $\varepsilon > 0$, $\theta > 0$, $0 < \tau_1 < \tau_2 < \tau_3 < \tau_4$, there are positive constants κ, κ_1, p determined only by N, $\operatorname{diam}(\Omega)$, α_0, β_0, d, ε, θ, $\tau_2 - \tau_1$, $\tau_3 - \tau_2$, and $\tau_4 - \tau_3$, with the following property. If D, U are domains in Ω with $D \subset\subset U$, $\operatorname{dist}(\bar{D}, \partial U) \geq d$, $|D| > \varepsilon$, and $v \in C(\bar{U} \times [\tau, T])$ is a supersolution of an equation (3.22), where $\tau_1 - 2\theta \leq \tau \leq \tau_1 - \theta$, $T \geq \tau_4$, the coefficients satisfy (L1) and $g \in L^{N+1}(U \times (\tau, \tau_4))$, then*

$$v(x,t) \geq \kappa [v^+]_{p, D \times (\tau_1, \tau_2)} - \kappa_1 \|g^-\|_{L^{N+1}(U \times (\tau, \tau_4))} - \sup_{\partial_P(U \times (\tau, \tau_4))} e^{m(\tau_4 - \tau)} v^-,$$
$$((x,t) \in D \times (\tau_3, \tau_4)), \tag{3.23}$$

with $m = \sup_{U \times (\tau, \tau_4)} c$. If v is a solution of (3.22), then one can take $p = \infty$ and κ, κ_1 are independent of ε.

In the proof we shall use the following weak Harnack inequality for nonnegative supersolutions.

Lemma 3.6. *Lemma 3.5 holds under the additional assumptions that $g \equiv 0$ and $v \geq 0$.*

The statement regarding solutions is a form of the Krylov-Safonov Harnack inequality [27, 28]. For supersolutions the statement is proved (without the restriction $g \equiv 0$) in [21], see also [31, Section VII.8]. Although only standard parabolic cylinders are considered there, the extension to general cylindrical domains follows by a construction of a chain of parabolic cylinders. The arguments are rather standard, however, we were unable to locate them in literature. For the reader's convenience, we give the details in Appendix II.

Proof of Lemma 3.5. First we prove the result under the extra assumptions that the coefficients a_{ij} are continuous on $\bar{U} \times [\tau, T]$ and U has smooth boundary.

Set $\sigma := \sup_{\partial_P(U\times(\tau,\tau_4))} v^- \geq 0$. We write v as $v = v_1 + v_2 + v_3$, where $v_1 := v - v_2 - v_3$, v_2 is the solution of (3.9) on $U \times (\tau, \tau_4)$ satisfying $v_2 \equiv -\sigma$ on $\partial_P(U \times (\tau, \tau_4))$, and v_3 is the solution of the problem

$$\begin{aligned} &v_t = a_{ij}(x,t)v_{x_ix_j} + b_i(x,t)v_{x_i} + c(x,t)v - g^-(x,t), && (x,t) \in U \times (\tau, \tau_4), \\ &v = 0, && (x,t) \in \partial_P(U \times (\tau, \tau_4)). \end{aligned} \tag{3.24}$$

The (unique) solvability of these boundary value problem follows from standard theorems (see [29, 31]) thanks to our extra regularity assumptions. A simple computation shows that v_1 is a supersolution of (3.9) satisfying

$$v_1(x,t) = v(x,t) + \sigma \geq 0 \quad ((x,t) \in \partial_P(U \times (\tau, \tau_4)).$$

By the maximum principle,

$$v_1(x,t) \geq 0,\ 0 \geq e^{-mt}v_2(x,t) \geq -e^{-m\tau}\sigma,\ 0 \geq v_3(x,t) \quad (x \in U, t \in (\tau, \tau_4)).$$

This implies in particular that $v_1 \geq v^+$. Applying Lemma 3.6 to v_1, a nonnegative supersolution of the homogeneous equation, we obtain

$$v_1(x,t) \ \geq\ \kappa[v_1]_{p,D\times(\tau_1,\tau_2)} \ \geq\ \kappa[v^+]_{p,D\times(\tau_1,\tau_2)} \quad ((x,t) \ \in\ D \times (\tau_3, \tau_4)),$$

for some constants κ and p determined by N, $\operatorname{diam}(\Omega)$, α_0, β_0, d, ε, and θ, $\tau_2 - \tau_1$, $\tau_3 - \tau_2$, and $\tau_4 - \tau_3$.

Next, the Alexandrov-Krylov estimate for solutions of (3.24) (see [27, Theorem III.3.9] or [31, Theorem VII.7.1]) states that there is a constant κ_1 determined by N, $\operatorname{diam}(\Omega)$, α_0, β_0, and $\tau_4 - \tau_1 + 2\theta \geq \tau_4 - \tau$ such that

$$\sup_{U\times(\tau,\tau_4)} |v_3| \leq \kappa_1 \|g^-\|_{L^{N+1}(U\times(\tau,\tau_4))}.$$

Therefore, for any $(x,t) \in D \times (\tau_3, \tau_4)$,

$$\begin{aligned} v(x,t) &= v_1(x,t) + v_2(x,t) + v_3(x,t) \\ &\geq \kappa[v^+]_{p,D\times(\tau_1,\tau_2)} - e^{m(\tau_4-\tau)}\sigma - \kappa_1\|g^-\|_{L^{N+1}(U\times(\tau,\tau_4))} \end{aligned}$$

as stated in (3.23).

We now remove the restrictions imposed on U and a_{ij}. For that we use an approximation argument. Choose (employing a mollification, for example) sequences $(a^n_{ij})_n$, $i,j = 1,\dots,N$, of continuous functions such that (L1) is satisfied with a_{ij} replaced by a_{ij}^n and β_0 replaced by $2\beta_0$ (we leave b_i and c unchanged) and such that $a_{ij}^n \to a_{ij}$ almost everywhere as $n \to \infty$. Further choose a real sequence $\tau^k \searrow \tau$ and a sequence of domains U^k with smooth boundary such that $D \subset\subset U^k \subset\subset U$ and ∂U^k approaches ∂U in the Hausdorff metric as $k \to \infty$. A supersolutions v of (3.22) also satisfies

$$v_t \geq a_{ij}^n(x,t)v_{x_ix_j}+b_i(x,t)v_{x_i}+c(x,t)v+g^n(x,t)+g(x,t), \ \ (x,t) \in U^k\times(\tau^k,T),$$

where $g_n(x,t) = (a_{ij}(x,t) - a_{ij}^n(x,t))v_{x_ix_j}$. Note that, since $v \in W^{2,1}_{N+1}(U^k \times (\tau^k, \tau_4)$, the dominated convergence theorem implies that for each k

$$g_n \to 0 \text{ in } L^{N+1}(U^k \times (\tau^k, \tau_4)) \text{ as } n \to \infty. \tag{3.25}$$

Now, if k is sufficiently large, replacing U, τ, a_{ij} by their approximations, and further replacing d by $d/2$, θ by $\theta/2$, and β_0 by $2\beta_0$, we may apply the result proved above to obtain the following. For some positive constant κ, κ_1 and p determined as specified in the lemma, we have

$$\begin{aligned}\sup_{D\times(\tau_3,\tau_4)} v(x,t) \geq &\kappa[v^+]_{p,D\times(\tau_1,\tau_2)} \\ &- \kappa_1(\|g^-\|_{L^{N+1}(U\times(\tau^k,\tau_4))} + \|g_n^-\|_{L^{N+1}(U\times(\tau^k,\tau_4))}) \\ &- \sup_{\partial_P(U^k\times(\tau^k,\tau_4))} e^{m(\tau_4-\tau^k)}v^-.\end{aligned}$$

Taking first the limit as $n \to \infty$, making use of (3.25), and then the limit as $k \to \infty$, using continuity of v, we obtain (3.23). $\square$

The above lemmas are used at several places in the proofs of the symmetry results in the next section. Attempting to make the proofs more transparent, we decided to single out one of our key arguments and formulate it still in the context of general linear equations (3.9), (3.10) without burdening it with additional notation and relations of Section 4. Although the formulation is somewhat lengthy, see Theorem 3.7 below, having it prepared will make the proofs of the symmetry results considerably simpler. Also the result might be of independent interest.

Intuitively, the statement can be described as follows. If v is a supersolution of (3.9), (3.10) with $T = \infty$, v is positive on a *bounded* cylinder $D\times[\tau,\tau+8\theta]$, where $D \subset\subset U$ is sufficiently "large in U", and $\sup_{U\setminus D} v^-(\cdot,\tau)$ is not too large (compared with an integral of v in $D \times [t + \theta, \tau + 2\theta]$), then

v is positive in the *unbounded* cylinder $D \times [\tau, \infty)$ and the positivity domain of $v(\cdot, t)$ spreads all over U as $t \to \infty$. We remark that using recent results of [25], a stronger result can be proved if ∂U is Lipschitz. One then has $v(\cdot, t) > 0$ in U for all sufficiently large t, that is, $v(\cdot, t)$ is eventually positive. On nonlipschitz domains the eventual positivitity cannot be expected in general (see [24] for a relevant example in this regard).

For a domain $D \subset \Omega$, we define the inner radius of D to be

$$\mathrm{inrad}(D) := \sup\{\rho > 0 : B(x_0, \rho) \subset D \text{ for some } x_0 \in D\}.$$

If $D \subset \Omega$ is an open set, we let $\mathrm{inrad}(D)$ stand for the infimum of the inner radii of all connected components of D. Note that $\mathrm{inrad}(D) \geq \rho > 0$ for $D \subset \Omega$ necessarily means that D has only finitely many connected components, each of them having measure at least $|B(0, \rho)|$.

Theorem 3.7. *Fix $\rho \in (0, \mathrm{diam}(\Omega)/2)$, let $\gamma = \gamma(N, \alpha_0, \beta_0, \rho)$ be as in Lemma 3.3, and let $\delta > 0$ be such that the conclusion of Lemma 3.1 holds with $k = \gamma + 1$ (thus $\delta = \delta(N, \mathrm{diam}(\Omega), \alpha_0, \beta_0, \rho)$). Given any $d > 0$, $\theta > 0$, there exist positive constants $p = p(N, \mathrm{diam}(\Omega), \alpha_0, \beta_0, d, \theta, \rho)$ and $\mu = \mu(N, \mathrm{diam}(\Omega), \alpha_0, \beta_0, d, \theta, \rho)$ with the following properties. If $D \subset U$ are open sets in Ω satisfying*

$$\mathrm{inrad}(D) > \rho, \quad |U \setminus \bar{D}| < \delta, \tag{3.26a}$$

$$\mathrm{dist}(\bar{D}, \partial U) > d, \tag{3.26b}$$

if $v \in C(\bar{U} \times [\tau, \infty))$ is a supersolution of a problem (3.9), (3.10) *whose coefficients satisfy* (L1) *(with $T = \infty$), and if*

$$v(x, t) > 0 \quad ((x, t) \in \bar{D} \times [\tau, \tau + 8\theta)), \tag{3.27a}$$

$$\|v^-(\cdot, \tau)\|_{L^\infty(U \setminus \bar{D})} \leq \mu [v]_{p, D_0 \times (\tau + \theta, \tau + 2\theta)} \tag{3.27b}$$

for each connected component D_0 of D, then the following statements hold true:

(s1) $v(x, t) > 0 \quad ((x, t) \in \bar{D} \times [\tau, \infty))$*;*

(s2) $\|v^-(\cdot, t)\|_{L^\infty(U)} \to 0$ *as $t \to \infty$, in fact,*

$$\|v^-(\cdot, t)\|_{L^\infty(U)} \leq 2 e^{-(\gamma + 1)(t - \tau)} \|v^-(\cdot, \tau)\|_{L^\infty(U)} \quad (t > \tau); \tag{3.28}$$

(s3) *for any domain* $G \subset\subset U$ *with* $G \cap D \neq \emptyset$ *there is* T_G *such that* $v(\cdot, t) > 0$ *on* G *for all* $t \geq T_G$.

We remark that once δ is fixed, for the conditions in (3.26) to be compatible (i.e. to apply to some $D \subset\subset U$), d must be sufficiently small. If v is a solution, then one can take $p = \infty$ in (3.27b).

It will become clear from the proof that (3.26b) can be replaced by the following set of conditions

$$\begin{aligned} \partial D = \Gamma_1 \cup \Gamma_2, \quad \operatorname{dist}(\bar{\Gamma}_1, \partial U) > d, \\ v(x,t) > 0 \quad (x \in \Gamma_2 \times (\tau, \infty)). \end{aligned}$$

Proof of Theorem 3.7. Fix $\rho \in (0, \operatorname{diam}(\Omega)/2)$ and let γ and δ be as stated. Also let h_ρ be as in Lemma 3.3.

Next take $\varepsilon = |B(0, \rho)|$ and fix arbitrary $\theta > 0$, $d > 0$. Corresponding to these numbers, let κ, p be as in Lemma 3.4; they are determined by N, $\operatorname{diam}(\Omega)$, α_0, β_0, d, θ, ρ. Finally, we define $\mu = C_2/(2C_3)$, where C_2 and C_3 are certain constants specified below, see (3.33), which are determined by the same quantities as κ and p. We prove that the conclusion of the theorem holds with this choice of p and μ.

To this end, let D, U and v satisfy all the stated conditions. The proof of (s1) is by contradiction. Assume it does not hold. Then there exists $T \geq \tau + 8\theta$ such that

$$v(x,t) > 0 \quad ((x \in \bar{D},\, t \in [\tau, T)), \tag{3.29a}$$

$$v(x^*, T) = 0 \text{ for some } x^* \in \partial D. \tag{3.29b}$$

Let D_0 be a connected component of D such that $x^* \in \partial D_0$.

Since $v \geq 0$ on $\partial(U \setminus D) \times [\tau, T]$, and $|U \setminus \bar{D}| < \delta$, the conclusion of Lemma 3.1 (with U replaced by $U \setminus \bar{D}$ and with our choice $k = \gamma + 1$) gives

$$\|v^-(\cdot,t)\|_{L^\infty(U)} = \|v^-(\cdot,t)\|_{L^\infty(U\setminus\bar{D})} \leq 2e^{-(\gamma+1)(t-\tau)} \|v^-(\cdot,\tau)\|_{L^\infty(U\setminus\bar{D})} \qquad (t \in [\tau, T]). \tag{3.30}$$

Denote

$$r_0 := \|v^-(\cdot,\tau)\|_{L^\infty(U\setminus\bar{D})}.$$

From Lemma 3.4 (replacing D by D_0 and noting that $c \le \beta_0$ and $|D_0| \ge \varepsilon = |B(0,\rho)|$) we have

$$\begin{aligned}
v(x,t) &\ge \kappa[v]_{p,D_0\times(\tau+\theta,\tau+2\theta)} - \sup_{\partial_P(U\times(\tau,\tau+4\theta))} e^{4\beta_0\theta}v^- \\
&\ge \kappa[v]_{p,D_0\times(\tau+\theta,\tau+2\theta)} - 2e^{4\beta_0\theta}r_0 \\
&\ge r_0\left(\frac{\kappa}{\mu} - 2e^{4\beta_0\theta}\right) =: r_1 \quad ((x,t)\in \bar{D}_0\times(\tau+3\theta,\tau+4\theta)),
\end{aligned} \tag{3.31}$$

where we have used (3.30) and (3.27b).

Since $\mathrm{inrad}(D) > \rho$, there is $x_0 \in D_0$ such that $B_0 := B(x_0,\rho) \subset D_0$. We next use a comparison between the supersolution v and the subsolution $\varphi(x,t) = e^{-\gamma t}h_\rho(x-x_0)$ (cf. Lemma 3.3). Since,

$$\begin{aligned}
&v > 0 \text{ in } D_0\times[\tau+4\theta,T) \\
&\varphi = 0 \text{ on } \partial B_0\times[\tau+4\theta,T),
\end{aligned}$$

and, by (3.31),

$$v(x,\tau+4\theta) \ge r_1\frac{\varphi(x,\tau+4\theta)}{\|\varphi(\cdot,\tau+4\theta)\|_{L^\infty(B_0)}} \quad (x\in B_0),$$

we have

$$\begin{aligned}
v(x,t) &\ge r_1\frac{\varphi(x,t)}{\|\varphi(\cdot,\tau+4\theta)\|_{L^\infty(B_0)}} \\
&= r_1 e^{-\gamma(t-\tau-4\theta)}\frac{h_\rho(x-x_0)}{\|h_\rho\|_{L^\infty(B(0,\rho))}} \quad (x\in B_0, t\in[\tau+4\theta,T]).
\end{aligned} \tag{3.32}$$

Next we use the estimate from Lemma 3.4 on the time interval $[T-4\theta,T]$ and subsequently we use (3.32), (3.30):

$$\begin{aligned}
v(x,T) &\ge \kappa[v]_{p,D_0\times(T-3\theta,T-2\theta)} - \sup_{\partial_P(U\times(T-4\theta,T))} e^{4\beta_0\theta}v^- \\
&\ge \kappa e^{-\gamma(T-\tau-4\theta)}r_1C_1(p,\rho) - 2e^{4\beta_0\theta}e^{-\gamma(T-\tau-4\theta)}r_0 \quad (x\in\bar{D}_0),
\end{aligned}$$

where

$$C_1(p,\rho) = \left(\frac{1}{|D_0|}\int_{B(0,\rho)} h_\rho^p\right)^{\frac{1}{p}}\frac{1}{\|h_\rho\|_{L^\infty(B(0,\rho))}} > 0.$$

Substituting for r_1 from (3.31), we obtain

$$\begin{aligned} v(x,T) &\geq r_0 e^{-\gamma(T-\tau-4\theta)} \left(\frac{C_1(p,\rho)\kappa^2}{\mu} - 2C_1(p,\rho)\kappa e^{4\beta_0\theta} - 2e^{4\beta_0\theta} \right) \\ &= r_0 e^{-\gamma(T-\tau-4\theta)} \left(\frac{C_2}{\mu} - C_3 \right) \qquad (x \in \bar{D}_0), \end{aligned}$$

where

$$C_2 = C_1(p,\rho)\kappa^2, \quad C_3 = 2C_1(p,\rho)\kappa e^{4\beta_0\theta} + 2e^{4\beta_0\theta} \tag{3.33}$$

(thus C_2, C_3 depend only on N, α_0, β_0, d, θ and ρ). It follows that with μ defined by $\mu = C_2/(2C_3)$, the above estimates give

$$v(x,T) > 0 \quad (x \in \bar{D}_0)$$

contradicting (3.29b). This contradiction proves (s1).

Once (s1) is proved, we also know that (3.30) holds for all $T > \tau + 4\theta$, which proves (3.28).

To prove the last statement, let $G \subset\subset U$ be a domain intersecting a component D_0 of D. Replacing G by $G \cup D_0$, we may assume $D_0 \subset G$. By (s1), estimate (3.32) is valid for any $T > \tau + 8\theta$ and any ball $B_0 = B(x_0, \rho)$ contained in D_0. This gives a lower estimate on $[v^+]_{p,G\times(T-3\theta,T-2\theta)}$ for any $p, T > 0$. On the other hand, we still have the upper estimate (3.30) on v^- valid for all t. Since the exponent $\gamma + 1$ in (3.30) is larger than the the exponent in (3.32), an application of Lemma 3.4 to $G \subset\subset U$ (with $d := \operatorname{dist}(\bar{G}, \partial U)$) shows that $v(\cdot, t) > 0$ on G if t is sufficiently large. This proves (s3) and completes the proof. □

4 Proofs of the symmetry results

We use the notation introduced in Subsection 3.1. Without further notice, we shall also use the fact if u is a solution of (1.1), (1.2), then $v = V_\lambda u$ satisfies (3.5), (3.6), and the coefficients in (3.5) satisfy (3.7), (3.8).

In the whole section we assume that hypotheses (D1), (D2), (F1)–(F3) are satisfied and u is a positive global solution of (1.1), (1.2) satisfying (U1), (U2). We want to prove that for some $\lambda \geq 0$, we have $\|V_\lambda u(\cdot, t)\|_{L^\infty(\Omega_\lambda)} \to 0$. By relative compactness of $\{u(\cdot, t) : t > 0\}$ in $C(\bar{\Omega})$, this is equivalent to

$$V_\lambda z \equiv 0 \text{ on } \Omega_\lambda \qquad (z \in \omega(u)). \tag{4.1}$$

Moreover, under the extra assumption of Theorem 2.2, we prove that this is true for $\lambda = 0$. This gives the symmetry part of the statements of Theorems 2.2, 2.4. Then we prove the monotonicity part.

For $\lambda \in [0, \ell)$, consider the following statement.

$$(\mathrm{S})_\lambda \qquad \|(V_\lambda u)^-(\cdot, t)\|_{L^\infty(\Omega_\lambda)} \to 0 \text{ as } t \to \infty.$$

We use moving hyperplanes to carry out the following steps of the proofs.

Lemma 4.1. *There is $\delta_1 > 0$ such that for each $\lambda \in [0, \ell)$ the following statement holds. If K is a closed subset of Ω_λ with $|\Omega_\lambda \setminus K| < \delta_1$ and there is $t_0 \geq 0$ such that $V_\lambda u(\cdot, t) \geq 0$ on K for $t \geq t_0$, then*

$$\|(V_\lambda u)^-(\cdot, t)\|_{L^\infty(\Omega_\lambda)} \leq 2e^{-(t-t_0)}\|(V_\lambda u)^-(\cdot, t_0)\|_{L^\infty(\Omega_\lambda)} \quad (t \geq t_0). \tag{4.2}$$

In particular, $(\mathrm{S})_\lambda$ holds if $\lambda < \ell$ is sufficiently close to ℓ.

Lemma 4.2. *Suppose $\lambda_0 \in [0, \ell)$ is such that $(\mathrm{S})_\lambda$ holds for all $\lambda \in (\lambda_0, \ell)$. Then $(\mathrm{S})_{\lambda_0}$ holds and for each $z \in \omega(u)$ and each connected component U of Ω_{λ_0} either $V_{\lambda_0} z > 0$ on U or else $V_{\lambda_0} z \equiv 0$ on U.*

In the previous two lemmas, hypothesis (D2) is not needed. It is mainly because of the next lemma that it is assumed.

Lemma 4.3. *If $\lambda_0 > 0$ is as in Lemma 4.2 and $V_{\lambda_0} z > 0$ on Ω_{λ_0} for some $z \in \omega(u)$, then there exists $\epsilon > 0$ such that $(\mathrm{S})_\lambda$ holds for each $\lambda \in (\lambda_0 - \epsilon, \lambda_0]$.*

In the following lemmas we set

$$\lambda_0 = \inf\{\mu > 0 : \ (\mathrm{S})_\lambda \text{ holds for all } \lambda > \mu\}. \tag{4.3}$$

Lemma 4.4. *If $\phi > 0$ in Ω for some $\phi \in \omega(u)$, then $\lambda_0 = 0$.*

Lemma 4.5. *If $\lambda_0 > 0$ or if $\phi > 0$ in Ω for some $\phi \in \omega(u)$, then for each $z \in \omega(u)$, $V_{\lambda_0} z \equiv 0$ on some connected component of Ω_{λ_0}.*

Lemma 4.6. *Each $z \in \omega(u)$ is monotone nonincreasing in x_1 on Ω_{λ_0}. Moreover, if Ω_{λ_0} is connected (in particular if $\lambda_0 = 0$) then either $z \equiv 0$ on Ω_{λ_0} or else z is strictly decreasing in x_1 on Ω_{λ_0}. If $z \in C^1(\Omega_{\lambda_0})$ and $z \not\equiv 0$ on Ω_{λ_0}, then $z_{x_1} < 0$ on Ω_{λ_0}.*

Proof of Lemma 4.1. Let $\delta_1 = \delta$ be as in Lemma 3.1 with $k = 1$. Taking $U = \Omega_\lambda \setminus \bar{K}$ and using the assumption on K, the conclusion follows upon application of Lemma 3.1. The last statement follows since $|\Omega_\lambda| < \delta_1$ for $\lambda \approx \ell$. $\square$

Proof of Lemma 4.2. First observe that $(S)_{\lambda_0}$ holds. Indeed, by compactness of $\{u(\cdot,t) : t > 0\}$ in $C(\bar{\Omega})$, $(S)_\lambda$ is equivalent to

$$V_\lambda z \geq 0 \text{ on } \Omega_\lambda \quad (z \in \omega(u)).$$

Taking the limit $\lambda \searrow \lambda_0$ we obtain $(S)_{\lambda_0}$.

Fix any $z \in \omega(u)$ and any connected component U of Ω_{λ_0}. Assume $V_{\lambda_0} z \not\equiv 0$ on U. Since $V_{\lambda_0} z \geq 0$, we have

$$V_{\lambda_0} z > 0 \text{ on } \bar{B}_0$$

for some ball $B_0 \subset\subset U$. We prove that $V_{\lambda_0} z > 0$ on U. Let $t_n \to \infty$ be such that $u(\cdot,t_n) \to z$ in $C(\bar{\Omega})$. Then also $V_{\lambda_0} u(\cdot,t_n) \to V_{\lambda_0} z$, hence there are $r_0 > 0$, n_0 such that

$$V_{\lambda_0} u(\cdot,t_n) > 2r_0 \quad (x \in \bar{B}_0, n > n_0).$$

By the equicontinuity property (U2), there is $\vartheta > 0$ such that

$$V_{\lambda_0} u(\cdot,t) > r_0 \quad (x \in \bar{B}_0, t \in [t_n - 4\vartheta, t_n], n > n_0).$$

Since $(S)_{\lambda_0}$ holds, applying Lemma 3.4 (with $v = V_{\lambda_0} u$, $\tau = t_n - 4\vartheta$, $\theta = \vartheta/4$), one shows that for any open set $D \subset\subset \Omega_{\lambda_0}$ there exists $r_1 > 0$ such that

$$V_{\lambda_0} u(\cdot,t) > r_1 \quad (x \in \bar{D}, t \in [t_n - \vartheta, t_n], n > n_0). \tag{4.4}$$

Taking $t = t_n$ and sending n to ∞, we in particular obtain

$$V_{\lambda_0} z \geq r_1 \quad (x \in \bar{D}).$$

This shows that $V_\lambda z > 0$ in U, as claimed. $\square$

Proof of Lemma 4.3. By (D2), Ω_{λ_0} has only finitely many connected components, hence $\rho := \mathrm{inrad}(\Omega_{\lambda_0})/2 > 0$. With this choice of ρ, let δ, γ be as in Theorem 3.7 (so they are some constants determined by ρ, N, α_0, β_0

and diam(Ω)). Choose an open set $D \subset\subset \Omega_{\lambda_0}$ such that D intersects each connected component of Ω_{λ_0} and

$$|\Omega_{\lambda_0} \setminus \bar{D}| < \delta/2 \tag{4.5}$$

$$\operatorname{inrad}(D) > \rho. \tag{4.6}$$

Let $z \in \omega(u)$ be such that $V_{\lambda_0} z > 0$ on $\Omega_{\lambda_0} \supset \bar{D}$. Taking a sequence $t_n \to \infty$ such that $u(\cdot, t_n) \to z$ and using the equicontinuity as in the proof of Lemma 4.2, we find $r_1 > 0$, $\vartheta \in (0,1)$, and n_0 such that

$$V_{\lambda_0} u(\cdot, t) > 2r_1 \quad (x \in \bar{D}, t \in [t_n - \vartheta, t_n], n \geq n_0). \tag{4.7}$$

Set $d := \operatorname{dist}(\bar{D}, \partial\Omega_{\lambda_0})$, $\theta := \vartheta/8$, and take the corresponding μ and p as in Theorem 3.7. Using Theorem 3.7 we want to show that (S)$_\lambda$ holds for all $\lambda \in [\lambda_0 - \epsilon, \lambda_0]$, provided $\epsilon > 0$ is sufficiently small. As the first requirement on ϵ, we postulate that

$$|\Omega_\lambda \setminus \Omega_{\lambda_0}| < \delta/2 \quad (\lambda \in [\lambda_0 - \epsilon, \lambda_0)). \tag{4.8}$$

Assuming $\lambda \in [\lambda_0 - \epsilon, \lambda_0]$ (making ϵ smaller, as necessary), we now verify that Theorem 3.7 applies with $U = \Omega_\lambda$, D as chosen above, $v = V_\lambda u$, and $\tau = t_n - \vartheta$ with n is sufficiently large. We have $\operatorname{dist}(\bar{D}, \partial U) \geq \operatorname{dist}(\bar{D}, \partial\Omega_{\lambda_0}) = d$. By (4.5) and (4.8), $|U \setminus D| < \delta$. We have thus verified conditions (3.26a), (3.26b). Clearly, $v \in C(\bar{U} \times [\tau, \infty))$ is a supersolution of a problem (3.9), (3.10), as required in Theorem 3.7. It remains to verify conditions (3.27).

By the equicontinuity,

$$\sup_{D \times [t_n - \vartheta, t_n]} |V_\lambda u - V_{\lambda_0} u| \to 0 \text{ as } \lambda \to \lambda_0.$$

This and (4.7) imply that, making ϵ smaller if necessary,

$$V_\lambda u(x,t) > r_1 \quad (x \in \bar{D}, t \in [t_n - \vartheta, t_n], n \geq n_0, \lambda \in [\lambda_0 - \epsilon, \lambda_0]). \tag{4.9}$$

Since $[\tau, \tau + 8\theta] = [t_n - \vartheta, t_n]$, (3.27a) is satisfied. From (4.9) we also see that the right hand side of (3.27b) is bounded below by μr_1. We prove that for all large n and small ϵ

$$\|(V_\lambda u)^-(\cdot, s_n)\|_{L^\infty(U \setminus \bar{D})} = \|(V_\lambda u)^-(\cdot, s_n)\|_{L^\infty(\Omega_\lambda \setminus \bar{D})} \leq \mu r_1, \tag{4.10}$$

where $s_n = t_n - \vartheta$ $(= \tau)$.

By $(\mathrm{S})_{\lambda_0}$, there is $n_1 > n_0$ such that for each $n > n_1$ we have

$$\|(V_{\lambda_0}u)^-(\cdot, s_n)\|_{L^\infty(\Omega_{\lambda_0}\setminus\bar{D})} \le \frac{\mu r_1}{2}.$$

Consequently, by the equicontinuity, if $\epsilon > 0$ is sufficiently small,

$$\|(V_\lambda u)^-(\cdot, s_n)\|_{L^\infty(\Omega_{\lambda_0}\setminus\bar{D})} \le \mu r_1 \quad (n > n_1, \lambda \in [\lambda_0 - \epsilon, \lambda_0]). \tag{4.11}$$

Next, by (U2)' (which clearly follows from (U2) and (1.2)), there exists a neighborhood $\mathcal{E}$ of $\partial\Omega$ independent of λ such that

$$(V_\lambda u)^-(\cdot, t) = (u(2\lambda - x_1, x', t) - u(x_1, x', t))^- \le u(x_1, x', t) < \mu r_1$$
$$(x = (x_1, x') \in \mathcal{E} \cap \Omega_\lambda, t > 0). \tag{4.12}$$

Finally, the remaining set $\bar{\Omega}_\lambda \setminus (\mathcal{E} \cup \Omega_{\lambda_0})$ is contained in an arbitrarily small neighborhood G_0 of $\Gamma_{\lambda_0} \setminus \mathcal{E}$ if $\lambda \approx \lambda_0$. Since $V_{\lambda_0}u(\cdot, t)$ vanishes on $\Gamma_{\lambda_0} \setminus \mathcal{E} \subset\subset \Omega$, using the equicontinuity we can choose G_0 so that $G_0 \subset\subset \Omega$ and

$$\|(V_{\lambda_0}u)^-(\cdot, s_n)\|_{L^\infty(G_0)} \le \|V_{\lambda_0}u(\cdot, s_n)\|_{L^\infty(G_0)} \le \frac{\mu r_1}{2}.$$

For $\lambda \approx \lambda_0$ we have $\bar{\Omega}_\lambda \setminus (\mathcal{E} \cup \Omega_{\lambda_0}) \subset G_0$ and, again by equicontinuity,

$$\|(V_\lambda u)^-(\cdot, s_n)\|_{L^\infty(G_0)} \le \mu r_1. \tag{4.13}$$

With (4.11), (4.12), (4.13), we have shown (4.10) and thus all conditions (3.27) are satisfied. We have verified all the hypotheses of Theorem 3.7. Statement (s2) of that theorem gives $(\mathrm{S})_\lambda$. The lemma is proved. $\square$

Remark 4.7. Theorem 3.7, as applied in the above proof, also gives the following property (see statement (s3)). For any open set $G \subset\subset \Omega_\lambda$ such that $G \cap D \neq \emptyset$ there exists $T = T(\lambda, G)$ such that $V_\lambda u(\cdot, t) > 0$ on G for all $t \ge T$.

Proof of Lemma 4.4. We show that if $\lambda > 0$ then $V_\lambda\phi \not\equiv 0$ on any connected component of Ω_λ. Lemmas 4.2 and 4.3 then readily imply that $\lambda_0 = 0$. If $V_\lambda\phi \equiv 0$ on some component U of Ω_λ, then, since $\phi = 0$ on $\partial\Omega$, ϕ vanishes on $P_\lambda(\partial\Omega_\lambda) \cap \bar{U}$. But for $\lambda > 0$ this intersection contains points of Ω and we have a contradiction with $\phi > 0$. $\square$

Proof of Lemma 4.5. By Lemma 4.2, it is sufficient to show that $V_{\lambda_0} z > 0$ on Ω_{λ_0} is impossible. If $\lambda_0 > 0$, this follows directly from Lemma 4.3 and the definition of λ_0.

Assume that $\phi > 0$ for some $\phi \in \omega(u)$ so that by Lemma 4.4 we have $\lambda_0 = 0$. Using analogous arguments as above, moving the hyperplanes from the left starting with $\lambda \approx -\ell$, one shows that

$$V_\lambda z \geq 0 \text{ on } \Omega_\lambda^- := \{x \in \Omega : x_1 < \lambda\} \quad (z \in \omega(u)), \tag{4.14}$$

for all $\lambda \leq 0$. (The fact that the process can be continued up to $\lambda = 0$ follows from the positivity of ϕ.) Thus for $\lambda = 0$ and each $z \in \omega(u)$ we have $V_0 z \leq 0$ in Ω_0 and also in Ω_0^- which gives $V_0 z \equiv 0$. $\square$

Proof of Lemma 4.6. Let $z \in \omega(u)$ be arbitrary. We have $V_\lambda z \geq 0$ for each $\lambda \in [\lambda_0, \ell)$. This readily implies that z is monotone nonincreasing in x_1 on Ω_{λ_0}. Next assume that Ω_{λ_0} is connected (this is clearly the case, by the symmetry assumption, if $\lambda_0 = 0$). If z is constant in x_1 in Ω_{λ_0} then $z \equiv 0$ on Ω_{λ_0} by the boundary condition. Assume z is not constant and let $d_h z$ denote its difference quotient:

$$d_h z(x) := \frac{z(x_1 + h, x') - z(x_1, x')}{h}.$$

By the monotonicity and continuity of z, for each sufficiently small $h > 0$, we can find a ball $\bar{B}_0 \subset\subset \Omega_{\lambda_0}$ such that

$$d_h z(x) < -2r_0 \quad (x = (x_1, x') \in \bar{B}_0). \tag{4.15}$$

Here B_0 and r_0 depend on h in general, but if $z_{x_1} \in C(\Omega_{\lambda_0})$, we can choose them independent of h (for h sufficiently small).

Let $t_n \to \infty$ be such that $u(\cdot, t_n) \to z$ in $C(\bar{\Omega})$. Given any sufficiently small $h > 0$ (and the corresponding B_0, r_0), there is n_0 such that for $n > n_0$ we have

$$d_h u(x, t_n) < -2r_0 \quad (x \in \bar{B}_0).$$

By the equicontinuity, there is $\theta > 0$ such that

$$d_h u(x, t) < -r_0 \quad (x \in \bar{B}_0, t \in [t_n - 4\theta, t_n], n > n_0). \tag{4.16}$$

Let $U \subset\subset \Omega_{\lambda_0}$ be any domain such that $U + he_1 := \{x + he_1 : x \in U\} \subset\subset \Omega_{\lambda_0}$ ($e_1 = (1, 0, \ldots, 0)$) and $B_0 \subset\subset U$. Then, by the monotonicity of F in x_1,

$v := -d_h u$ is a supersolution of a linear equation (3.9) with coefficients satisfying condition (L1) of Subsection 3.2, with α_0 and β_0 independent of h. By the monotonicity of all $\tilde{z} \in \omega(u)$ we have

$$\|v^-(\cdot,t)\|_{L^\infty(U)} \to 0 \text{ as } t \to \infty. \tag{4.17}$$

For an arbitrary domain $D \subset\subset U$ containing B_0 set $d := \operatorname{dist}(\bar{D}, U)$ and let κ, p be as in Lemma 3.4 (they depend only on α_0, β_0 d, N, θ and d). Then, using (3.21), (4.16) and (4.17), we obtain

$$v(x,t_n) \geq \frac{\kappa r_0}{2}\left(\frac{|B_0|}{|D|}\right)^{\frac{1}{p}} > 0 \quad (x \in \bar{D})$$

for all sufficiently large n. Taking $n \to \infty$ gives

$$-d_h z(x) \geq \frac{\kappa r_0}{2}\left(\frac{|B_0|}{|D|}\right)^{\frac{1}{p}} \quad (x \in \bar{D}).$$

We have thus shown that $d_h z$ is negative in $\bar{D}$ and, since D was arbitrary, also in U. This holds for all sufficiently small h showing that z is strictly decreasing in Ω_{λ_0}. Moreover, if $z_{x_1} \in C(\Omega_{\lambda_0})$, then the above arguments apply with B_0 and r_0 independent of $h > 0$. This implies $z_{x_1} < 0$ in Ω_{λ_0}. The lemma is proved. □

Proof of Theorems 2.2 and 2.4. Let λ_0 be as in (4.3). By Lemma 4.1, $\lambda_0 < \ell$ and, by Lemma 4.4, $\lambda_0 = 0$ under the assumptions of Theorem 2.2. The symmetry and monotonicity properties stated in the theorems follow from Lemmas 4.5, 4.6 (using again that Ω_0 is connected). □

Proof of Theorem 2.5. We have

$$u_t \geq F(t,x,u,Du,D^2u) - F(t,x,0,0,0) - (F(t,x,0,0,0))^- \quad (x \in \Omega, t > 0).$$

Hence, denoting $g(x,t) := -(F(t,x,0,0,0))^-$, u is a supersolution of (3.22) with coefficients satisfying (L1). If $\omega(u) = \{0\}$, there is nothing to prove. Assume that for some $t_n \to \infty$ we have

$$u(\cdot,t_n) \to z$$

and $z > 0$ in some ball $\bar{B}_0 \subset \Omega$. Then there is a constant r_0 such that for all large n

$$u(x,t_n) \geq 2r_0 \quad (x \in \bar{B}_0).$$

Consequently, by the equicontinuity,

$$u(x,t) \geq r_0 \quad (x \in \bar{B}_0, t \in [t_n, t_n + 4\theta]), \tag{4.18}$$

where $\theta > 0$ is independent of n. Given any domain $D \subset\subset \Omega$, we now obtain, using (4.18) and Lemma 3.5 (remembering that $u > 0$), that

$$u(\cdot, t_n + 3\theta) \geq r_1 - \kappa_1 \|g\|_{L^{N+1}(\Omega \times (t_n, t_n+4\theta)} \quad (x \in \bar{D}), \tag{4.19}$$

where r_1 and κ_1 are independent of n (they depend on $\mathrm{dist}(\bar{D}, \partial\Omega)$). Passing to a subsequence we may assume that $u(\cdot, t_n + 3\theta) \to \phi \in \omega(u)$. Then (4.19) and (2.9) give $\phi \geq r_1 = r_1(D) > 0$ in D. Since D is arbitrary, we have $\phi > 0$ is Ω. Thus Theorem 2.2 applies and the conclusion follows. $\square$

5 Examples

First consider the one dimensional problem

$$u_t = u_{xx} + g(u), \quad x \in (-1, 1) \tag{5.1}$$

$$u(-1,t) = u(1,t) = 0, \quad t > 0. \tag{5.2}$$

We show that for a suitable nonlinearity, there is a positive solution which approaches an equilibrium which is not monotone in (0,1).

Choose a smooth bounded Lipschitz function g such that $g(0) < 0$ and for some $a > 0$

$$G(u) := \int_0^u g(\xi)\, d\xi < 0 \quad (u \in (0,a))$$
$$G(a) = 0 \text{ and } g(a) > 0.$$

By elementary phase plane analysis, the equation

$$\phi_{xx} + g(\phi) = 0$$

has a solution ϕ such that, for some $L > 0$

$$\phi(0) = \phi(\pm L) = \phi'(0) = \phi'(\pm L) = 0, \tag{5.3}$$

$$\phi > 0 \text{ in } (-L, 0) \cup (0, L). \tag{5.4}$$

Also ϕ is an even function. Rescaling x and replacing g by $L^2 g$, we may assume $L = 1$.

Consider the linear eigenvalue problem

$$
\begin{aligned}
&v_{xx} + g'(\phi(x))v + \mu v = 0, \quad x \in (-1,1),\\
&v(-1) = v(1) = 0,\\
&v \text{ is even.}
\end{aligned}
\tag{5.5}
$$

This is equivalent to the eigenvalue problem with the same equation and boundary conditions

$$v_x(0) = v(1) = 0. \tag{5.6}$$

Let $\mu_k \to \infty$, denote the eigenvalues of the problem and v_k, $k = 1, 2, \ldots$ the corresponding eigenfunctions normalized such that $v_k(0) = 1$ (the normalization is always possible by (5.6)). Let k be odd and so large that $\mu_k > 0$. Since v_k has k zeros in $(0, 1]$, we have

$$v_k(0) = 1, \quad v_k'(1) < 0, \quad v_k'(-1) > 0. \tag{5.7}$$

There exist a solution of (5.1), (5.2) which is even in x (equivalently, this is a solution of the equation under the boundary conditions $u_x(0,t) = 0 = u(1,t)$) such that

$$u(x,t) = \phi(x) + q(t)(v_k(x) + R(x,t)), \tag{5.8}$$

where $q(t) = \|u(\cdot,t) - \phi\|_{C^1[-1,1]} \to 0$, as $t \to \infty$, and R is a function satisfying

$$R(-1,t) = R(1,t) = 0 \quad (t \geq 0), \tag{5.9}$$

$$R(t,\cdot) \to 0 \text{ as } t \to \infty \text{ in } C^1[-1,1]. \tag{5.10}$$

The existence can be established using the simplicity of the eigenvalue μ_k and employing suitable invariant manifolds, see [7].

By (5.10) and (5.7) there is $\epsilon > 0$ such that for all large t we have

$$
\begin{aligned}
v_k(x) + R(x,t) &> 0 \quad (x \in (-\epsilon, \epsilon)),\\
v_k'(x) + R_x(x,t) &< 0 \quad (x \in (1-\epsilon, 1)),\\
v_k'(x) + R_x(x,t) &> 0 \quad (x \in (-1, -1+\epsilon)).
\end{aligned}
$$

Consequently, by the boundary conditions,

$$v_k(x) + R(x,t) > 0 \quad (x \in [-1, -1+\epsilon) \cup (-\epsilon, \epsilon) \cup (1-\epsilon, 1])$$

for all sufficiently large t. Since $\phi \geq 0$ everywhere and $\phi > 0$ in $[-1+\epsilon, -\epsilon] \cup [\epsilon, 1-\epsilon]$ we conclude that $u(x,t) > 0$ in $(-1,1)$ for all large t. Thus we have a positive solution $u(\cdot,t) \to \phi$ as $t \to \infty$ and ϕ is not monotone in $(0,1)$.

We now give the details for Example 2.3. Let $\Omega = [-1,1] \times [-1,1]$. Let g, u and ϕ be as above. For a large T set

$$U(x,y,t) = u(x,t+T)\psi(y) \quad ((x,y) \in \Omega, t > 0)),$$

where $\psi = \cos(\pi y/2)$, that is, ψ solves

$$\begin{aligned}
&\psi'' + \mu\psi = 0, \quad (y \in (-1,1)),\\
&\psi(-1) = \psi(1) = 0,\\
&\psi > 0 \text{ in } (-1,1),
\end{aligned}$$

with $\mu = (\pi/2)^2$. Then $U > 0$ in $\Omega \times (0,\infty)$ and it satisfies

$$U_t = \Delta U + f(y,U) \quad (x,y) \in \Omega, t > 0, \tag{5.11}$$

$$U = 0, \qquad (x,y) \in \partial\Omega, t > 0, \tag{5.12}$$

where

$$f(y,U) = \begin{cases} g\left(\dfrac{U}{\psi(y)}\right)\psi(y) + \mu U & \text{if } |y| \neq 1,\\ \mu U \quad \text{if } |y| = 1. \end{cases}$$

Clearly f is continuous and its derivative

$$f_U(y,U) = \begin{cases} g'\left(\dfrac{U}{\psi(y)}\right) + \mu & \text{if } |y| \neq 1,\\ \mu \quad \text{if } |y| = 1, \end{cases}$$

is bounded. We have $U(\cdot,t) \to \phi\psi$ in $C^1(\bar{\Omega})$ and $\phi\psi$ has the nodal set as stated in Example 2.3. In particular, it is not monotone in $x > 0$ (it is monotone in $x > 1/2$ and symmetric about the hyperplane $\{x = 1/2\}$).

6 Appendix I: Positive limit profiles

Here we prove the conclusion of Theorem 2.2 assuming that $z > 0$ in Ω for all $z \in \omega(u)$. The proof is much simpler in this case, as Theorem 3.7 is not needed. Moreover, hypothesis (D2) is not needed and hypotheses (U1), (U2) can be relaxed somewhat. We shall assume the following instead.

(U3) The set $\{u(\cdot,t) : t \geq 1\}$ is relatively compact in $C(\bar{\Omega})$ and hypothesis (U2) holds with Ω replaced by any domain $G \subset\subset \Omega$.

Theorem 6.1. *Let* (D1), (F1)–(F3) *hold and let u be a nonnegative global solution of* (1.1), (1.2) *satisfying* (U3). *Assume that $z > 0$ on Ω for each $z \in \omega(u)$. Then for each $z \in \omega(u)$*

$$z(-x_1, x') = z(x_1, x') \quad ((x_1, x') \in \Omega), \tag{6.1}$$

and z is strictly decreasing in x_1 on $\Omega_0 = \{x \in \Omega : x_1 > 0\}$. The latter holds in the form $z_{x_1} < 0$, provided $z_{x_1} \in C(\Omega_0)$.

Proof. Consider the statement $(\mathrm{S})_\lambda$ as in Section 4. Lemma 4.1 and 4.2 remain valid under the present hypotheses. Define λ_0 as in (4.3). We show that $\lambda_0 = 0$. Assume $\lambda_0 > 0$. Then for each $z \in \omega(u)$, the assumption $z > 0$ in Ω implies $V_{\lambda_0} z \not\equiv 0$ on each connected component of Ω_{λ_0} (see the proof of Lemma 4.4), hence $V_{\lambda_0} z > 0$ in Ω_{λ_0} by Lemma 4.2. With δ_1 as in Lemma 4.1, choose any closed set $K \subset \Omega_{\lambda_0}$ with $|\Omega_{\lambda_0} \setminus K| < \delta_1/2$. By compactness of $\omega(u)$ in $C(\bar{\Omega})$, there is $\epsilon_0 > 0$ such that

$$V_{\lambda_0} z \geq 3\epsilon_0 \text{ on } K \quad (z \in \omega(u)).$$

Consequently, there is t_0 such that

$$V_{\lambda_0} u(\cdot, t) \geq 2\epsilon_0 \text{ on } K \quad (t \geq t_0),$$

and, by equicontinuity,

$$V_\lambda u(\cdot, t) \geq \epsilon_0 \text{ on } K \quad (t \geq t_0), \tag{6.2}$$

for all $\lambda \approx \lambda_0$. Now, let $\lambda < \lambda_0$ be so close to λ_0 that (6.2) holds and $|\Omega_\lambda \setminus \Omega_{\lambda_0}| < \delta_1/2$. Then $|\Omega_\lambda \setminus K| < \delta_1$ and applying Lemma 4.1 we conclude that $(\mathrm{S})_\lambda$ holds for all such λ, which is a contradiction to the definition of λ_0. We have thus proved that $\lambda_0 = 0$.

As in Lemma 4.6 one shows that each $z \in \omega(u)$ is decreasing in $x_1 > 0$ and if $z_{x_1} \in C(\Omega_0)$ then $z_{x_1} < 0$ in Ω_0. The symmetry statement is obtained by an analogous procedure, moving the hyperplanes from the left starting near $-\ell$. The proof is complete. $\square$

7 Appendix II: Proof of Lemma 3.6

Assuming the hypotheses of the lemma, we say that a constant is determined by the given quantities if it is determined by d, ε, θ, N, $\mathrm{diam}(\Omega)$, α_0, β_0, and $\tau_2-\tau_1$, $\tau_3-\tau_2$, $\tau_4-\tau_3$.

For $R, \vartheta \in \mathbb{R}$ let

$$Q(R,\vartheta) = \{(x,t) \in \mathbb{R}^{N+1} : |x| < R,\ t \in (-\vartheta R^2, 0)\}.$$

If $Q = (x_0,t_0) + Q(R,\vartheta)$, we denote

$$Q^b := (x_0,t_0) + \{(x,t) \in \mathbb{R}^{N+1} : |x| < R,\ t \in (-\vartheta R^2, -\frac{3}{5}\vartheta R^2)\},$$
$$Q^t := (x_0,t_0) + \{(x,t) \in \mathbb{R}^{N+1} : |x| < R,\ t \in (-\frac{2}{5}\vartheta R^2, 0)\}.$$

We use the fact, proved in [21] (and in [31, Section VII.8]) that the lemma is valid for standard parabolic cylinders. Specifically, if $R \in (0, 1/2)$ and $(x_0,t_0) + Q(2R,\vartheta)$ is contained in $U \times (\tau,\tau_4)$ (U, τ_4, $v \geq 0$ etc. are as in the lemma), then for some constants C_1 and p determined by N, α_0, β_0, R, and ϑ we have

$$\inf_{Q^t} v(x,t) \geq C_1 [v]_{p,Q^b} \tag{7.1}$$

If v is a solution, one can take $p = \infty$.

We now choose suitable R, ϑ determined by the given quantities. Assuming d and θ are given, first we take $R := \min\{1/2, 4d\}$. There is an integer σ depending on N and $\mathrm{diam}(\Omega)$, such that any subdomain D of Ω is covered by σ balls having centers in D and radius $R/2$. Define ϑ by

$$4\vartheta R^2 = \min\{\theta, \frac{\tau_3-\tau_2}{\sigma+1}\}. \tag{7.2}$$

With this choice of R and ϑ, let C_1 and p be as above (cf. (7.1)). Further, let s be the minimal positive integer with $(\tau_2-\tau_1)/s < \vartheta R^2/5$, and let

$$\tau_1^i = \tau_1 + i\frac{(\tau_2-\tau_1)}{s}, \quad i = 0,\dots,s.$$

For D given as in Lemma 3.6, let $B(y,R/2)$, $y \in \mathcal{S}$, be a system of σ balls covering D; here $\mathcal{S} \subset D$. The cylinders

$$B(y,R) \times [\tau_1^i, \tau_1^{i+1}], \quad y \in \mathcal{S}, i = 0,\dots,s,$$

cover $D \times (\tau_1, \tau_2)$, therefore

$$[v]_{p,D\times(\tau_1,\tau_2)} \leq C_2[v]_{p,B(y_0,R)\times(\tau_1^{i_0},\tau_1^{i_0+1})},$$

where (y_0, i_0) maximizes the integral

$$\int_{B(y,R)\times(\tau_1^i,\tau_1^{i+1})} v^p(x,t)\,dxdt$$

over $(y,i) \in \mathcal{S} \times \{0,\dots,s\}$ and

$$C_2 := \left(\frac{\sigma|B(y_0,R)|}{\varepsilon}\right)^{\frac{1}{p}} \geq \left(\frac{\sigma|B(y_0,R)|}{|D|}\right)^{\frac{1}{p}}$$

(thus C_2 is determined by the given quantities). It follows that the lemma will be proved if we show that for any $(x_0,t_0) \in D \times (\tau_3,\tau_4)$ we have

$$v(x_0,t_0) \geq C_0[v]_{p,B(y_0,R)\times(\tau_1^{i_0},\tau_1^{i_0+1})}, \tag{7.3}$$

where C_0 is determined by the given quantities.

To show this, we construct a chain of cylinders Q_j, $j = 1,\dots,m$, with the following properties: each Q_j is of the form $B(y,R)\times(t,t+\vartheta R^2)$ for some $t \leq \tau_4 - \vartheta R^2$ and $y \in \mathcal{S}$, and one has

$$\begin{aligned}
&(x_0,t_0) \in Q_1^t,\\
&B(y_0,R)\times(\tau_1^{i_0},\tau_1^{i_0+1}) \subset Q_m^b,\\
&|Q_j^b \cap Q_{j+1}^t| \geq \nu,
\end{aligned}$$

where $m \leq m_0$ and $m_0, \nu > 0$ are some constants determined by the given quantities. The chain can be found as follows. Take a chain of balls $B_i = B(y_i,R)$, $i = 1,\dots,\sigma$, with $y_i \in \mathcal{S}$, such that

$$x_0 \in B_1, \quad B_\sigma = B(y_0,R), \quad |B_i \cap B_{i+1}| > \mu,$$

where $\mu = \mu(R,N) > 0$. Such a chain exists as $B(y,R/2)$, $y \in \mathcal{S}$, cover D (if necessary, we repeat some balls in the sequence to have their number equal to σ). Next set $Q_1 = B_1 \times (t_1, t_1 + \vartheta R^2)$ for some $t_1 \leq \tau_4 - \vartheta R^2$ with $t_0 \in (t_1 + 3\vartheta R^2/5, t_1 + \vartheta R^2)$. If $Q_i = B_i \times (t_i, t_i + \vartheta R^2)$ is defined, we set $Q_{i+1} = B_{i+1} \times (t_i - 2\vartheta R^2/5, t_i + 3\vartheta R^2/5)$, where $B_i \equiv B_\sigma$ for $i > \sigma$. We continue the recursion until Q_i^b contains $B(y_0, R) \times (\tau_1^{i_0}, \tau_1^{i_0+1})$, which does occur since $\tau_1^{i_0+1} - \tau_1^{i_0} < \vartheta R^2/5$ and it does not occur for $i < \sigma$ due to the choice of ϑ (see (7.2)). The chain constructed this way has the desired properties with $\nu := \mu\vartheta R^2/5$ and with $m \geq \sigma$ estimated above by a constant m_0 which depends on $\tau_4 - \tau_1$.

Having constructed the chain, we can now estimate, using (7.1):

$$\begin{aligned}\inf_{Q_i^t} v(x,t) &\geq C_1 [v]_{p,Q_i^b} \geq C_1 \left(\frac{|Q_i^b \cap Q_{i+1}^t|}{|Q_i^b|} \right)^{\frac{1}{p}} [v]_{p,Q_i^b \cap Q_{i+1}^t} \\ &\geq C_1 C_3 [v]_{p,Q_i^b \cap Q_{i+1}^t} \geq C_1 C_3 \inf_{Q_{i+1}^t} v,\end{aligned}$$

where

$$C_3 = \left(\frac{5\nu}{2|B(y_i, R)|\vartheta R^2} \right)^{1/p}$$

is determined by the given quantities. Repeating these estimates m times, we obtain (7.3). This completes the proof.

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