

CHEN-KHOVANOV SPECTRA FOR TANGLES

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ABSTRACT. We note that our stable homotopy refinements of Khovanov’s arc algebras and tangle invariants induce refinements of Chen-Khovanov and Stroppel’s platform algebras and tangle invariants, and discuss the topological Hochschild homology of these refinements.

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1. INTRODUCTION

In this paper, we continue our homotopical journey through Khovanov homology by giving spectral refinements of Chen-Khovanov and Stroppel’s platform algebras.

Khovanov homology [Kho00] associates a bigraded abelian group $Kh^{i,j}(L)$ to an oriented link $L \subset \mathbb{R}^3$, so that the graded Euler characteristic of $Kh^{i,j}(L)$ is the Jones polynomial. Khovanov [Kho02] extended his invariant to $(2m, 2n)$ -tangles by associating a graded algebra \mathcal{H}^n to each non-negative integer n and a chain complex of graded $(\mathcal{H}^m, \mathcal{H}^n)$ -bimodules \mathcal{C}_T to an oriented $(2m, 2n)$ -tangle diagram T , such that:

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- (K-1) the chain homotopy type of \mathcal{C}_T is an isotopy invariant of the tangle represented by T ,
- (K-2) composition of tangles corresponds to the tensor product of bimodules, i.e., $\mathcal{C}_{T_1 T_2} \simeq \mathcal{C}_{T_1} \otimes_{\mathcal{H}^n} \mathcal{C}_{T_2}$, and
- (K-3) $\mathcal{H}^0 \cong \mathbb{Z}$ and for a $(0, 0)$ -tangle L , $H_*(\mathcal{C}_L) = Kh(L)$.

(See Table 2.1 for a dictionary of notation between this paper and the references. Note that our tangles are drawn left-to-right, not top-to-bottom as is more usual.)

Let V be the fundamental (2-dimensional) representation of $U_q(\mathfrak{sl}(2))$. Khovanov further showed that the Grothendieck group of the category of finitely-generated, graded \mathcal{H}^n -modules (and of the category of finitely-generated complexes of \mathcal{H}^n -modules) is canonically isomorphic to $\text{Inv}(V^{\otimes 2n})$, the subspace of $U_q(\mathfrak{sl}(2))$ -invariants in $V^{\otimes 2n}$.

From a representation-theoretic standpoint, it is more interesting to categorify $V^{\otimes n}$ itself. This was accomplished by Chen-Khovanov and Stroppel [CK14, Str09], who defined an algebra $\mathcal{A}^n = \bigoplus_{k=0}^n \mathcal{A}^{n-k, k}$ for each $n \in \mathbb{N}$ and associated to each (m, n) -tangle an $(\mathcal{A}^m, \mathcal{A}^n)$ -bimodule satisfying Properties (K-1)–(K-3) and so that the Grothendieck group of graded, projective \mathcal{A}^n -modules is canonically isomorphic to $V^{\otimes n}$. These *platform algebras* were further studied and generalized by Brundan-Stroppel [BS11]. A related, though more geometric, tangle invariant was also introduced by Bar-Natan [Bar05] and, recently, another extension of Khovanov homology to tangles has been given by Roberts [Rob16b, Rob16a].

In a previous paper [LLSb], we gave stable homotopy refinements of Khovanov’s algebras and modules. That is, for each integer n we constructed a ring spectrum $\mathcal{A}_{2n}^{\text{ring}}$ and for each $(2m, 2n)$ -tangle diagram T an $(\mathcal{A}_{2m}^{\text{ring}}, \mathcal{A}_{2n}^{\text{ring}})$ -bimodule spectrum $\mathcal{X}_T^{\text{mod}}$ such that:

- (K-1) the weak equivalence class of $\mathcal{X}_T^{\text{mod}}$ is an isotopy invariant of the tangle represented by T [LLSb, Theorem 4],
- (K-2) composition of tangles corresponds to the tensor product of bimodule spectra, i.e., $\mathcal{X}_{T_1 T_2}^{\text{mod}} \simeq \mathcal{X}_{T_1}^{\text{mod}} \otimes_{\mathcal{A}_{2n}^{\text{ring}}}^L \mathcal{X}_{T_2}^{\text{mod}}$ [LLSb, Theorem 5],
- (K-3) $\mathcal{A}_0^{\text{ring}} \simeq \mathbb{S}$, the sphere spectrum, and for a $(0, 0)$ -tangle L , $\mathcal{X}_T^{\text{mod}}$ is weakly equivalent to the previously-constructed [LS14, HKK16, LLSa] Khovanov spectrum, and
- (K-4) the singular chains on $\mathcal{A}_{2n}^{\text{ring}}$ (respectively $\mathcal{X}_T^{\text{mod}}$) is quasi-isomorphic to Khovanov’s algebra \mathcal{H}^n (respectively complex of bimodules \mathcal{C}_T) [LLSb, Proposition 4.2].

In this paper, we modify $\mathcal{A}_{2n}^{\text{ring}}$ and $\mathcal{X}_T^{\text{mod}}$ to give stable homotopy refinements of the platform algebras and modules.

This paper is organized as follows. Section 2 reviews the platform algebras and bimodules and the spectral refinements of the arc algebras and bimodules. Section 3 constructs spectral refinements of the platform algebras and bimodules and proves their basic properties. In Section 4 we show that the topological Hochschild homology of the spectral platform bimodules is homotopy equivalent to the naive spectral refinement of annular Khovanov homology.

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2. THE PLATFORM ALGEBRAS AND MODULES

The platform algebras are subquotients of Khovanov’s arc algebras. In this section, we review both collections of algebras. For the platform algebras, we will expand on some details in Chen-Khovanov’s proofs, so that it is easier to see how they adapt to the spectral case.

2.1. Some notation. In order to keep track of the quantum gradings throughout, it is convenient to work in “graded” versions of various well-known categories. We list them below.

- Let \mathbb{Z}^{Sets} denote the category of finite, graded sets, whose objects are finite sets X together with set maps $\text{gr}: X \rightarrow \mathbb{Z}$, and morphisms $f: (X, \text{gr}_X) \rightarrow (Y, \text{gr}_Y)$ are set maps $f: X \rightarrow Y$ so that $\text{gr}_X = \text{gr}_Y \circ f$.
- Let Kom be the category of freely generated chain complexes over \mathbb{Z} . (The Khovanov complex is usually presented as a cochain complex; we will view it as a chain complex by negating the homological grading, see [LLSb, §2.1].) Let \mathbb{Z}^{Kom} denote the full subcategory of $\prod_{\mathbb{Z}} \text{Kom}$ where all but finitely many of the chain complexes are zero. So, \mathbb{Z}^{Kom} is isomorphic to the category of bigraded chain complexes—the first grading being the homological grading and the second grading being an additional grading—that are bounded in the second grading.
- Let \mathbb{Z}^{Ab} denote the category of freely and finitely generated graded abelian groups. We can (and will) identify \mathbb{Z}^{Ab} with the full subcategory of \mathbb{Z}^{Kom} with objects the finitely generated chain complexes supported in homological grading 0.
- Let \mathcal{S} be the category of symmetric spectra [HSS00]. Let $\mathbb{Z}^{\mathcal{S}}$ denote the full subcategory of $\prod_{\mathbb{Z}} \mathcal{S}$ where all but finitely many of the spectra are trivial (that is, just the basepoint). Taking reduced singular chain complexes gives a functor $C_*: \mathbb{Z}^{\mathcal{S}} \rightarrow \mathbb{Z}^{\text{Kom}}$, cf. [LLSb, §2.7].

Our notation for the arc and platform algebras and modules differs slightly from Khovanov’s and Chen-Khovanov’s; see Table 2.1 for a summary.

2.2. Arc algebras and modules. Let $V = H^*(S^2) = \mathbb{Z}[X]/(X^2)$ denote Khovanov’s Frobenius algebra. Explicitly, the comultiplication is given by $1 \mapsto 1 \otimes X + X \otimes 1$ and $X \mapsto X \otimes X$, and the counit map is given by $1 \mapsto 0$ and $X \mapsto 1$. We view V as a graded abelian group with 1 in grading (-1) and X in grading 1; this grading is called the quantum grading. (See [LLSb, Remark 2.55] for a brief discussion of gradings.) Given a compact 1-manifold Z , let $V(Z) = \bigotimes_{\pi_0(Z)} V$, the tensor product over the connected components of Z . Via the equivalence between Frobenius algebras and 2-dimensional topological field theories, we can view V as a topological field theory which assigns V to the circle.

We prefer to view the arc and platform algebras as linear categories, rather than algebras. Given a category \mathcal{C} , we will write $\mathcal{C}(a, b)$ for $\text{Hom}_{\mathcal{C}}(a, b)$. Then Khovanov’s *arc algebra* \mathcal{H}^n is the category with objects the crossingless matchings of $2n$ points, \mathbb{B}_{2n} , and

$$\mathcal{H}^n(a, b) := V(a\widehat{b})\{n\},$$

where \widehat{b} denotes the horizontal reflection of b and $\{n\}$ denotes an upward quantum grading shift by n . Composition is induced by the *canonical cobordisms* $\widehat{b} \amalg b \rightarrow \text{Id}$, which gives cobordisms $a\widehat{b} \amalg \widehat{bc} \rightarrow a\widehat{c}$, and the topological field theory associated to V .

Our notation	[Kho02, CK14] notation	Meaning
V	\mathcal{A}	Frobenius algebra $H^*(S^2)$.
\mathbf{B}_{2n}	B^n	Isotopy classes of crossingless matchings of $2n$ points.
\widehat{b}	$W(b)$	Reflection of b .
$\mathcal{A}_{2n} = \mathcal{H}^n$	H^n	Algebra associated to n points.
\mathcal{C}_T	$V_D = \bigoplus_{v \in 2^N} \mathcal{F}(D(v))\{-v\}$	Khovanov tangle invariant, a complex of $(\mathcal{A}_{2m}, \mathcal{A}_{2n})$ -bimodules.
$\mathbf{B}_n^{k_1, k_2} = \mathbf{B}_n^{\mathbf{k}}$	$B^{n-k, k}$	Crossingless matchings of $n + k_1 + k_2$ points with no matching among first k_1 or last k_2 points. Chen-Khovanov require $k_1 + k_2 = n$.
$\overline{\mathcal{A}}_n^{k_1, k_2} = \overline{\mathcal{A}}_n^{\mathbf{k}}$	$\widetilde{A}^{n-k, k}$	Subring of \mathcal{A}_{2n} induced by $\mathbf{B}_n^{k_1, k_2}$.
$\mathcal{I}_n^{k_1, k_2} = \mathcal{I}_n^{\mathbf{k}}$	$I^{n-k, k}$	A particular ideal in $\overline{\mathcal{A}}_n^{k_1, k_2}$.
$\mathcal{A}_n^{k_1, k_2} = \mathcal{A}_n^{\mathbf{k}}$	$A^{n-k, k}$	Platform algebra, $\overline{\mathcal{A}}_n^{k_1, k_2} / \mathcal{I}_n^{k_1, k_2}$.
$\mathcal{A}_n^{\mathcal{P}}$	A^n	Total platform algebra, $\bigoplus_{k=0}^n \mathcal{A}_n^{n-k, k}$.
$\mathcal{J}_T^{h_1, h_2; k_1, k_2} = \mathcal{J}_T^{\mathbf{h}; \mathbf{k}}$	$I(T)$	A particular submodule of $\mathcal{C}_{\overline{T}}$.
$\mathcal{C}_T^{h_1, h_2; k_1, k_2} = \mathcal{C}_T^{\mathbf{h}; \mathbf{k}}$	$\mathcal{F}(T)$	Platform tangle invariant, $\mathcal{C}_{\overline{T}} / \mathcal{J}_T^{h_1, h_2; k_1, k_2}$.
$\mathcal{C}_T^{\mathcal{P}}$	$\mathcal{F}(T)$	Total platform bimodule, $\bigoplus_{\substack{h, k \\ m-n=2(h-k)}} \mathcal{C}_T^{m-h, h; n-k, k}$.

TABLE 2.1. Comparison of notation with Khovanov and Chen-Khovanov.

To keep notation consistent later in this paper, let

$$\mathcal{A}_{2n} = \mathcal{H}^n.$$

Given a flat $(2m, 2n)$ -tangle T there is an induced $(\mathcal{A}_{2m}, \mathcal{A}_{2n})$ -bimodule, i.e., functor of linear categories

$$\mathcal{C}_T: (\mathcal{A}_{2m})^{\text{op}} \times \mathcal{A}_{2n} \rightarrow \mathbb{Z}^{\text{Ab}},$$

which on objects is given by $\mathcal{C}_T(a, b) = V(aT\widehat{b})\{n\}$ and on morphisms is induced by the canonical cobordisms. More generally, given a non-flat oriented tangle diagram T , there is a functor

$$\underline{\mathbb{Z}}^N \times (\mathcal{A}_{2m})^{\text{op}} \times \mathcal{A}_{2n} \rightarrow \mathbb{Z}^{\text{Ab}}$$

given on objects by

$$(v, a, b) \mapsto \mathcal{C}_{T_v}(a, b)\{-|v| - N_+ + 2N_-\} = V(aT_v\widehat{b})\{n - |v| - N_+ + 2N_-\}.$$

Here, $\underline{\mathbb{Z}}^N = \{0 \rightarrow 1\}^N$, and N_+, N_-, N are the number of positive, negative, and total crossings in T , respectively. This induces a functor

$$\mathcal{C}_T: (\mathcal{A}_{2m})^{\text{op}} \times \mathcal{A}_{2n} \rightarrow \mathbb{Z}^{\text{Kom}},$$

by taking iterated mapping cones along $\underline{2}^N$, and then shifting the homological grading down by N_+ .

2.3. Platform algebras. Let $\mathbf{B}_n^{k_1, k_2} \subset \mathbf{B}_{n+k_1+k_2}$ be the set of crossingless matchings with no matchings among the first k_1 points and no matchings among the last k_2 points. So, $\mathbf{B}_{2n} = \mathbf{B}_{2n}^{0,0}$. The cases used for the platform algebras and modules are $\mathbf{B}_n^{n-k, k}$ (for $0 \leq k \leq n$), but the definitions generalize to $\mathbf{B}_n^{k_1, k_2}$ for any $k_1 + k_2 \equiv n \pmod{2}$, and it is convenient to work in the more general setting.

Given $a, b \in \mathbf{B}_{n+k_1+k_2}$, for each $1 \leq i \leq n + k_1 + k_2$ there is a corresponding circle Z_i of \widehat{ab} containing the point i . This induces an equivalence relation $\sim_{a,b}$ on $1, \dots, n + k_1 + k_2$ by $i \sim_{a,b} j$ if and only if $Z_i = Z_j$. (Equivalently, this equivalence relation is generated by $i \sim j$ if i is matched to j in either a or b .) So, $a \in \mathbf{B}_n^{k_1, k_2}$ if and only if $i \not\sim_{a,a} j$ whenever $i < j \leq k_1$ or $i > j > n + k_1$.

For $a, b \in \mathbf{B}_n^{k_1, k_2}$ define $\mathcal{I}_n^{k_1, k_2}(a, b) \subset \mathcal{A}_{n+k_1+k_2}(a, b)$ by

- (I-1) $\mathcal{I}_n^{k_1, k_2}(a, b) = \mathcal{A}_{n+k_1+k_2}(a, b)$ if there is some pair $i < j \leq k_1$ or $i > j > n + k_1$ so that $i \sim_{a,b} j$.
- (I-2) the span of the set of generators in $\mathcal{A}_{n+k_1+k_2}(a, b)$ which label some circle Z_i with $i \leq k_1$ or $i > n + k_1$ by X .

Then $\mathcal{A}_n^{k_1, k_2}$ is the category with objects $\mathbf{B}_n^{k_1, k_2}$ and

$$\mathcal{A}_n^{k_1, k_2}(a, b) = \mathcal{A}_{n+k_1+k_2}(a, b) / \mathcal{I}_n^{k_1, k_2}(a, b).$$

In other words, if we let $\overline{\mathcal{A}}_n^{k_1, k_2}$ be the full subcategory of $\mathcal{A}_{n+k_1+k_2}$ spanned by objects in $\mathbf{B}_n^{k_1, k_2}$ then

$$\mathcal{A}_n^{k_1, k_2} = \overline{\mathcal{A}}_n^{k_1, k_2} / \mathcal{I}_n^{k_1, k_2}.$$

Chen-Khovanov encode the points $1, \dots, k_1$ and $n + k_1 + 1, \dots, n + k_1 + k_2$ by drawing two vertical line segments, *platforms*, in \widehat{ab} where a and \widehat{b} meet, one containing the first k_1 points and the other containing the last k_2 points. Then, the points in Case (I-1) lie on a circle that meets one of the platforms more than once; following Chen-Khovanov, we will sometimes call such a circle a *type III circle*. The points in Case (I-2) lie on a circle that meets at least one platform, which we will sometimes follow Chen-Khovanov in calling a *type II circle* (if it is not already a type III circle).

There is an inclusion

$$\iota: \mathbf{B}_{n+k_1+k_2} \hookrightarrow \mathbf{B}_{n+k_1+k_2+2}$$

where $\iota(a)$ is obtained by matching the first and last points, and then matching the remaining points by a (shifted up by 1). (That is, if we think of a as an involution of $\{1, \dots, n + k_1 + k_2\}$ then $\iota(a)(1) = n + k_1 + k_2 + 2$ and $\iota(a)(i) = a(i - 1) + 1$ if $2 \leq i \leq n + k_1 + k_2 + 1$.) The map ι sends $\mathbf{B}_n^{k_1, k_2}$ to $\mathbf{B}_n^{k_1+1, k_2+1}$. If $k_1 + k_2 \geq n$ then $\iota: \mathbf{B}_n^{k_1, k_2} \rightarrow \mathbf{B}_n^{k_1+1, k_2+1}$ is a bijection.

There is an induced map

$$\iota: \mathcal{A}_{n+k_1+k_2} \rightarrow \mathcal{A}_{n+k_1+k_2+2}$$

which labels the new circle in $\iota(a)\widehat{\iota(b)}$ (containing the points 1 and $n + k_1 + k_2 + 2$) by 1.

Lemma 2.1. *Given n, k_1, k_2 with $k_1 + k_2 \equiv n \pmod{2}$, $\mathcal{I}_n^{k_1, k_2} \subset \overline{\mathcal{A}}_n^{k_1, k_2}$ is a 2-sided ideal, so $\mathcal{A}_n^{k_1, k_2}$ is a linear category.*

Proof. The proof is spelled out by Chen-Khovanov [CK14, Proof of Lemma 1], but we repeat it here for completeness. Suppose x is a labeling of \widehat{ab} so that some circle passing through a platform, at a point P , is labeled X . Let y be any labeling of \widehat{bc} . In the sequence of merges and splits relating $\widehat{ab} \amalg \widehat{bc}$ to \widehat{ac} , the circle containing P is always labeled X , so, in particular, in the product xy a circle passing through a platform is labeled X . If there is a circle in \widehat{ab} which passes through two points, P and Q , on the same platform then in the cobordism from $\widehat{ab} \amalg \widehat{bc}$ to \widehat{ac} , either P and Q stay on the same circle throughout, in which case \widehat{ac} has a type III circle, or some split occurs to the circle containing P and Q , in which case either P or Q is labeled X after the split. In the latter case, the circle containing that point continues to be labeled X throughout the cobordism, giving a type II or III circle labeled X . \square

Lemma 2.2. *Given n, k_1, k_2 with $k_1 + k_2 \equiv n \pmod{2}$ and $a, b \in \mathcal{B}_n^{k_1, k_2} = \text{Ob}(\mathcal{A}_n^{k_1, k_2})$,*

$$\iota^{-1}(\mathcal{I}_n^{k_1+1, k_2+1}(\iota(a), \iota(b))) = \mathcal{I}_n^{k_1, k_2}(a, b).$$

So, ι induces a homomorphism (functor of linear categories)

$$\iota: \mathcal{A}_n^{k_1, k_2} \rightarrow \mathcal{A}_n^{k_1+1, k_2+1}$$

which is always full and faithful and injective on objects, and is an isomorphism if $k_1 + k_2 \geq n$.

Proof. This is immediate from the definitions. \square

2.4. Platform modules. Given an (m, n) -flat tangle T , Chen-Khovanov define an $(\mathcal{A}_m^{m-h, h}, \mathcal{A}_n^{n-k, k})$ -bimodule for all (h, k) with $0 \leq h \leq m$, $0 \leq k \leq n$, and $m - n = 2(h - k)$. Their construction extends immediately to give $(\mathcal{A}_m^{h_1, h_2}, \mathcal{A}_n^{k_1, k_2})$ -bimodules for any h_1, h_2, k_1, k_2 , with $h_1 - h_2 = k_1 - k_2$, and some details are easier in the more general setting, so we sketch their construction there.

Fix $a \in \mathcal{B}_m^{h_1, h_2}$ and $b \in \mathcal{B}_n^{k_1, k_2}$ with $h_1 - h_2 = k_1 - k_2$, and an (m, n) -flat tangle T . Assume that $k_1 - h_1 \geq 0$; the other case is symmetric. Form a closed 1-manifold $\langle aT\widehat{b} \rangle$ by:

- (1) Adding k_1 horizontal strands below T and k_2 horizontal strands above T , to obtain \overline{T} .
- (2) Gluing $\iota^{k_1-h_1}(a)$ to \overline{T} to \widehat{b} , to obtain

$$\langle aT\widehat{b} \rangle = \iota^{k_1-h_1}(a) \overline{T} \widehat{b}.$$

See Figure 2.1. There is a subset $aT\widehat{b} \subset \langle aT\widehat{b} \rangle$ which we call the *partial closure*. The endpoints of the arcs in $aT\widehat{b}$ are on four *platforms*.

Define a submodule $\mathcal{J}_T^{h_1, h_2; k_1, k_2}(a, b) \subset \mathcal{C}_{\overline{T}}(\iota^{k_1-h_1}(a), b)$ by declaring that

- (J-1) If $aT\widehat{b}$ has an arc with both ends on the same platform then $\mathcal{J}_T^{h_1, h_2; k_1, k_2}(a, b) = \mathcal{C}_{\overline{T}}(\iota^{k_1-h_1}(a), b)$.
- (J-2) Otherwise, $\mathcal{J}_T^{h_1, h_2; k_1, k_2}(a, b)$ is spanned by the generators of $\mathcal{C}_{\overline{T}}(\iota^{k_1-h_1}(a), b)$ which label at least one arc component of the partial closure $aT\widehat{b}$ by X .

Define

$$\mathcal{C}_T^{h_1, h_2; k_1, k_2}(a, b) = \mathcal{C}_{\overline{T}}(\iota^{k_1-h_1}(a), b) / \mathcal{J}_T^{h_1, h_2; k_1, k_2}(a, b).$$

Following Chen-Khovanov, we will call the arcs in Case (J-1) *type III arcs* and the arcs in Case (J-2) *type II arcs* (if they are not already type III arcs). We call a circle in $\langle aT\widehat{b} \rangle$ containing

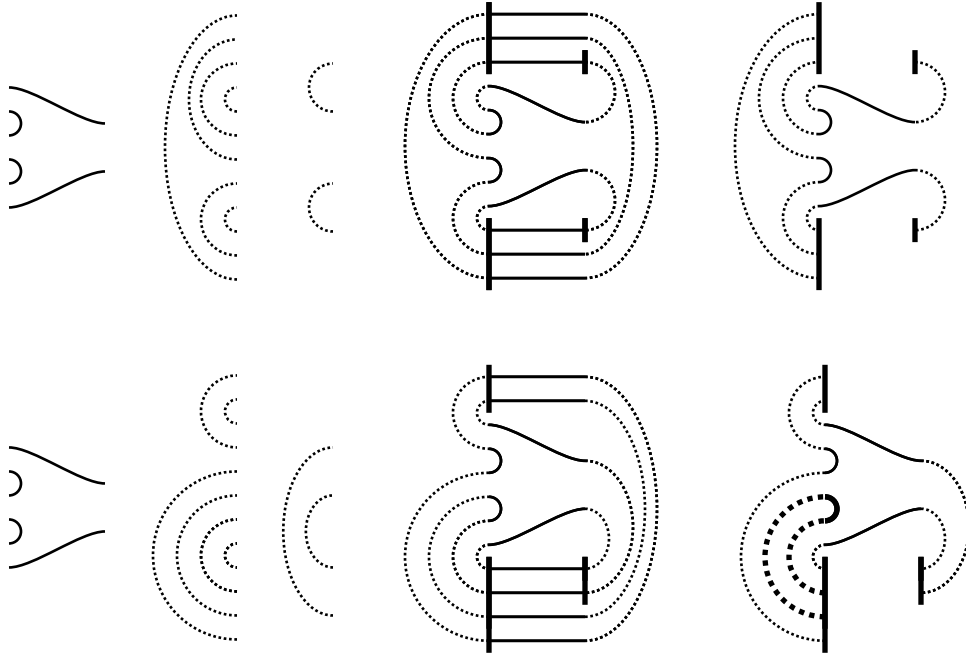


FIGURE 2.1. **Capping off a flat tangle.** Each row, left to right: a tangle T , crossingless matchings a, b , the closure $\langle aT\hat{b} \rangle$, and the partial closure $aT\hat{b}$. On the first row, $m = 6$, $n = 2$, $h_1 = 3$, $h_2 = 3$, $k_1 = 1$, and $k_2 = 1$. On the second row, $m = 6$, $n = 2$, $h_1 = 4$, $h_2 = 2$, $k_1 = 2$, and $k_2 = 0$. On the first row, there are no arcs with both endpoints on the same platform (type III arcs); on the second row, there is one, drawn in bold.

a type III arc a *type III circle* and a circle in $\langle aT\hat{b} \rangle$ containing a type II arc a *type II circle* (if it is not already a type III circle). The following reformulation of these conditions will be useful:

Lemma 2.3. *A circle Z in $\langle aT\hat{b} \rangle$ is a type III circle if and only if Z intersects some platform more than once. A circle Z in $\langle aT\hat{b} \rangle$ is a type II circle if and only if Z intersects each platform at most once and some platform at least once.*

Proof. We start with the statement about type III circles. That a type III circle intersects some platform more than once is immediate from the definition. For the other direction, observe first that Z contains a type III arc if and only if there is an embedded bigon in \mathbb{R}^2 with one edge on Z , one edge on a platform, and interior disjoint from Z and the platforms. Let P be the union of the top-left and bottom-left platforms. Since the two top platforms are connected by horizontal lines in \overline{T} , as are the two bottom platforms, existence of a bigon as above is equivalent to existence of a bigon with one edge on Z , one edge on P , and interior disjoint from $Z \cup P$.

Let D be the disk inside Z , i.e., the bounded region of $\mathbb{R}^2 \setminus Z$. The platforms P cut D into a collection of polygons, two of which might contain the *tips* of P —the segments of $P \setminus Z$ adjacent to

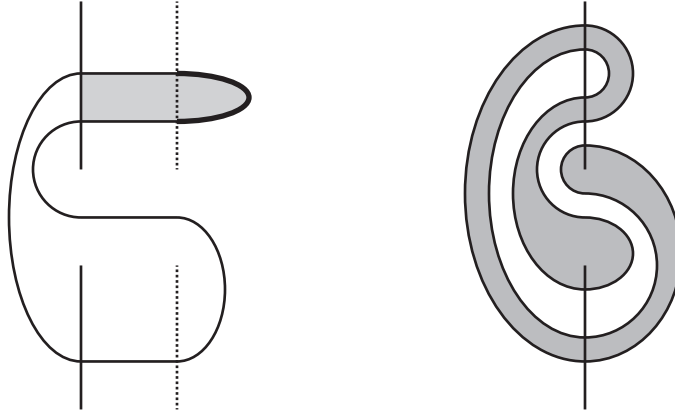


FIGURE 2.2. **Type III circles and arcs.** Left: a bigon (shaded) and the corresponding type III arc (bold). The platforms not in P are dotted. Right: a case where both bigons in D (shaded) contain tips.

the endpoints of P . (See Figure 2.2.) Let P' be the complement of the tips in P . Since D intersects some platform more than once, $D \setminus P'$ has at least two connected components. Every polygon in $D \setminus P'$ has an even number of edges. Consequently, the only polygons in $D \setminus P'$ with positive Euler measure are bigons. Since the Euler measure of P' is 1 and Euler measure is additive, at least two of the components of $D \setminus P'$ must be bigons, and if $D \setminus P'$ has exactly two bigons then all other components of $D \setminus P'$ are rectangles.

If some component of $D \setminus P'$ is a bigon not containing a tip of P then we are done. In the remaining case, $D \setminus P'$ consists of two bigons, both of which contain tips of P , and some number of rectangles.

In this last case, consider the complement $D' = \mathbb{R}^2 \setminus D$. View the platforms as extending to infinity in \mathbb{R}^2 . Since D contains both tips, the platforms P cut the punctured disk D' into two non-compact regions and some polygons. At least one of the non-compact regions has more than 2 corners, for otherwise, Z will be a circle passing through each platform in P once, contradicting the hypothesis. It follows by considering the Euler measure that at least one of the regions in D' is a bigon, which implies that there is a type III arc.

The statement about type II circles is immediate from the statement about type III circles. \square

Remark 2.4. In the special case that $h_1 = h_2 = k_1 = k_2 = 0$, $\mathcal{J}_T^{0,0;0,0}(a, b) = \{0\}$, so $\mathcal{C}_T^{0,0;0,0} = \mathcal{C}_T$, the arc algebra bimodule. The case considered by Chen-Khovanov is $h_2 = m - h_1$ and $k_2 = n - k_1$.

Convention 2.5. To shorten notation, we will often write $\mathbf{h} = (h_1, h_2)$ and $\mathbf{k} = (k_1, k_2)$, so

$$\mathcal{A}_m^{\mathbf{h}} = \mathcal{A}_m^{h_1, h_2} \quad \mathcal{J}_T^{\mathbf{h}; \mathbf{k}} = \mathcal{J}_T^{h_1, h_2; k_1, k_2} \quad \mathcal{C}_T^{\mathbf{h}; \mathbf{k}} = \mathcal{C}_T^{h_1, h_2; k_1, k_2}.$$

Composing the ring map $\iota^{k_1-h_1} : \overline{\mathcal{A}}_m^{\mathbf{h}} \hookrightarrow \overline{\mathcal{A}}_m^{\mathbf{k}}$ with the inclusion $j : \overline{\mathcal{A}}_m^{\mathbf{k}} \hookrightarrow \mathcal{A}_{m+k_1+k_2}$ gives a map $j \circ \iota^{k_1-h_1}$. Given an $(\mathcal{A}_{m+k_1+k_2}, \mathcal{A}_{n+k_1+k_2})$ -bimodule M , we can restrict scalars by $j \circ \iota^{k_1-h_1} \otimes j$ to obtain an $(\overline{\mathcal{A}}_m^{\mathbf{h}}, \overline{\mathcal{A}}_n^{\mathbf{k}})$ -bimodule $(j \circ \iota^{k_1-h_1} \otimes j)^* M$.

Proposition 2.6. [CK14] *The subsets*

$$\mathcal{J}_T^{\mathbf{h};\mathbf{k}}(a, b) \subset \mathcal{C}_{\overline{T}}(\iota^{k_1-h_1}(a), b)$$

form a submodule of the $(\overline{\mathcal{A}}_m^{\mathbf{h}}, \overline{\mathcal{A}}_n^{\mathbf{k}})$ -bimodule $(j \circ \iota^{k_1-h_1} \otimes j)^* \mathcal{C}_{\overline{T}}$, and

$$\begin{aligned} \mathcal{I}_m^{\mathbf{h}} \cdot ((j \circ \iota^{k_1-h_1} \otimes j)^* \mathcal{C}_{\overline{T}}) &\subset \mathcal{J}_T^{\mathbf{h};\mathbf{k}} \\ ((j \circ \iota^{k_1-h_1} \otimes j)^* \mathcal{C}_{\overline{T}}) \cdot \mathcal{I}_n^{\mathbf{k}} &\subset \mathcal{J}_T^{\mathbf{h};\mathbf{k}}, \end{aligned}$$

so $\mathcal{C}_T^{\mathbf{h};\mathbf{k}}$ inherits the structure of an $(\mathcal{A}_m^{\mathbf{h}}, \mathcal{A}_n^{\mathbf{k}})$ -bimodule.

Proof. The same argument as in the proof of Lemma 2.1 shows that the subsets

$$\mathcal{J}_T^{\mathbf{k};\mathbf{k}}(a, b) \subset \mathcal{C}_{\overline{T}}(a, b),$$

for $a \in \mathbf{B}_m^{\mathbf{k}}$ and $b \in \mathbf{B}_n^{\mathbf{k}}$, form a submodule of $\mathcal{C}_{\overline{T}}$ and that

$$\begin{aligned} \mathcal{I}_m^{\mathbf{k}} \cdot ((j \otimes j)^* \mathcal{C}_{\overline{T}}) &\subset \mathcal{J}_T^{\mathbf{k};\mathbf{k}} \\ ((j \otimes j)^* \mathcal{C}_{\overline{T}}) \cdot \mathcal{I}_n^{\mathbf{k}} &\subset \mathcal{J}_T^{\mathbf{k};\mathbf{k}}. \end{aligned}$$

By Lemma 2.2, $(\iota^{k_1-h_1})^{-1}(\mathcal{I}_m^{\mathbf{k}}) = \mathcal{I}_m^{\mathbf{h}}$. The result follows. \square

Proposition 2.7. [CK14] *Given a non-flat tangle T , the maps in the cube of resolutions*

$$\mathcal{C}_{\overline{T}_v}(\iota^{k_1-h_1}(a), b) \rightarrow \mathcal{C}_{\overline{T}_w}(\iota^{k_1-h_1}(a), b)$$

send $\mathcal{J}_{T_v}^{\mathbf{h};\mathbf{k}}(a, b)$ to $\mathcal{J}_{T_w}^{\mathbf{h};\mathbf{k}}(a, b)$ and hence descend to homomorphisms (natural transformations)

$$\mathcal{C}_{T_v}^{\mathbf{h};\mathbf{k}}\{-|v|\} \rightarrow \mathcal{C}_{T_w}^{\mathbf{h};\mathbf{k}}\{-|w|\}.$$

Proof. Suppose y is a labeling of $\iota^{k_1-h_1}(a)\overline{T}_w\widehat{b}$ and x is a labeling of $\iota^{k_1-h_1}(a)\overline{T}_v\widehat{b}$ so that (w, y) occurs as a term in $\partial(v, x)$. If there is a circle in $\iota^{k_1-h_1}(a)\overline{T}_v\widehat{b}$ passing through a point P on a platform which is labeled X by x (a type II circle labeled X) then the circle in $\iota^{k_1-h_1}(a)\overline{T}_w\widehat{b}$ containing P is also labeled X .

The more interesting case is that there is an arc in the partial closure $aT_v\widehat{b}$ with both endpoints on the same platform (a type III arc). Let $Z \subset \langle aT_v\widehat{b} \rangle$ be the corresponding circle. By Lemma 2.3, if $\langle aT_w\widehat{b} \rangle$ does not have a type III circle then the circle Z must split into two circles, each of which intersects a platform. Since the split map sends X to $X \otimes X$ and 1 to $1 \otimes X + X \otimes 1$, one of these circles must be labeled X by y , as desired. \square

Definition 2.8. By Propositions 2.6 and 2.7, associated to a non-flat tangle T is a cube of $(\mathcal{A}_m^{\mathbf{h}}, \mathcal{A}_n^{\mathbf{k}})$ -bimodules: $v \mapsto \mathcal{C}_{T_v}^{\mathbf{h};\mathbf{k}}\{-|v|-N_++2N_-\}$. Let $\mathcal{C}_T^{\mathbf{h};\mathbf{k}}$ be the total complex of this cube, with homological grading shifted down by N_+ .

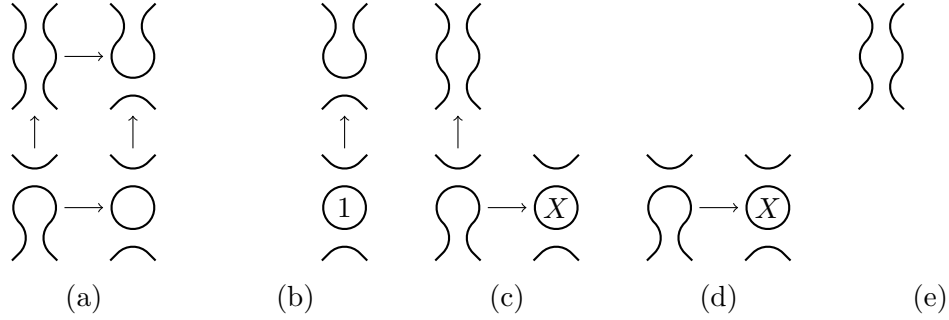


FIGURE 2.3. **Invariance under Reidemeister II moves.** (a) $\mathcal{C}_{\overline{T}}(a, b)$, the square of resolutions at a Reidemeister II move. (b) The acyclic subcomplex $C_1(a, b)$. (c) The quotient complex $C_2(a, b)$. (d) The acyclic quotient complex $C_3(a, b)$ of $C_2(a, b)$. (e) The corresponding subcomplex $C_4(a, b) \cong \mathcal{C}_{\overline{T'}}(a, b)$.

Lemma 2.9. *There are isomorphisms*

$$\mathcal{C}_T^{h_1, h_2; k_1, k_2} \cong (\iota \otimes \text{Id})^* \mathcal{C}_T^{h_1+1, h_2+1; k_1, k_2} \cong (\text{Id} \otimes \iota)^* \mathcal{C}_T^{h_1, h_2; k_1+1, k_2+1}.$$

Proof. This is immediate from the definitions. \square

Theorem 1. [CK14] *Up to homotopy equivalence of chain complexes of bimodules, $\mathcal{C}_T^{\mathbf{h}; \mathbf{k}}$ is invariant under Reidemeister moves.*

Proof. By Lemma 2.9, it suffices to prove the result when $\mathbf{h} = \mathbf{k}$. We will spell out the proof for a Reidemeister II move; the cases of Reidemeister I moves and braid-like Reidemeister III moves (see, e.g., [Ball11, §7.3] or [LS14, §6]) are similar, and we comment on them briefly at the end of the proof.

Suppose T and T' are related by a Reidemeister II move, and T has two more crossings than T' . Given $a \in \mathbf{B}_m^{\mathbf{k}}$, $b \in \mathbf{B}_n^{\mathbf{k}}$, Figure 2.3 (a) shows the complex $\mathcal{C}_{\overline{T}}(a, b)$, where the two crossings involved in the Reidemeister II move are resolved in the four possible ways. Figure 2.3 (b) shows an acyclic subcomplex $C_1(a, b)$ consisting of two of the resolutions, and where one closed circle is labeled 1. The quotient complex $C_2(a, b) = \mathcal{C}_{\overline{T}}(a, b)/C_1(a, b)$ (shown in Figure 2.3 (c)) has a subcomplex $C_4(a, b)$ (shown in Figure 2.3 (e)), so that $C_3(a, b) = C_2(a, b)/C_4(a, b)$ (shown in Figure 2.3 (d)) is acyclic, and $C_4(a, b) \cong \mathcal{C}_{\overline{T'}}(a, b)$.

Since each $C_i(a, b)$ is defined by restricting to certain vertices of the cube and certain labels of the closed circles in the resolutions of T (not $a\overline{T}b$), each C_i restricts to an $(\overline{\mathcal{A}}_m^{\mathbf{k}}, \overline{\mathcal{A}}_n^{\mathbf{k}})$ -bimodule. The inclusion maps $C_1(a, b) \hookrightarrow \mathcal{C}_{\overline{T}}(a, b)$ and $C_4(a, b) \hookrightarrow C_2(a, b)$, and the identification $C_4(a, b) \cong \mathcal{C}_{\overline{T'}}(a, b)$ respect the labels of type II and III circles. Hence, if we let $C'_i(a, b) = C_i(a, b)/\mathcal{J}_T^{\mathbf{k}; \mathbf{k}}$ then there are induced maps $C'_1 \hookrightarrow \mathcal{C}_T^{\mathbf{k}; \mathbf{k}}$, $C'_4 \hookrightarrow C'_2$, and $C'_4 \cong \mathcal{C}_{T'}^{\mathbf{k}; \mathbf{k}}$.

It remains to verify that C'_1 and C'_3 are acyclic. Let T_v and T_w be resolutions of T which agree at all of the crossings not involved in the Reidemeister II move and have the two forms allowed in C_1 near the Reidemeister II move. Notice that $a\overline{T}_v b$ has a type III arc if and only if $a\overline{T}_w b$ does, and

the type II arcs of $a\overline{T_v}\widehat{b}$ and $a\overline{T_w}\widehat{b}$ correspond. It follows that C'_1 is acyclic. Similarly, if T_v and T_w are resolutions as in C_3 which agree away from the Reidemeister II move, the type II (respectively III) arcs of T_v correspond to the type II (respectively III) arcs of T_w , so it follows that C'_3 is acyclic. This completes the proof for Reidemeister II moves.

As Chen-Khovanov note, there are two key points to this argument. First, the sub-complexes involved are defined locally near the Reidemeister move by placing restrictions on the labels of closed circles only, hence descend to the Chen-Khovanov quotient. Second, the acyclic subcomplexes stay acyclic after one quotient by the Chen-Khovanov submodule, essentially because type II and III arcs at the different relevant vertices correspond. Inspecting the proofs of Reidemeister I and III invariance (see, e.g., [LS14, §6]), both properties hold for them as well. Indeed, in all cases there are two kinds of cancellations that occur: merging on a circle labeled 1 or splitting off a circle labeled X . Neither operation changes the labels of the other components nor which points in the boundary of the tangle lie on the same strand, so each of the sub- and quotient complexes remains acyclic after quotienting by the Chen-Khovanov submodule. This completes the proof. \square

The gluing theorem for the tangle invariants only holds in Chen-Khovanov's generality:

Theorem 2. [CK14] *Let T_1 be an (m, n) -tangle and T_2 an (n, p) -tangle. Fix integers $h_1, h_2, k_1, k_2, \ell_1, \ell_2$ satisfying:*

$$h_1 - h_2 = k_1 - k_2 = \ell_1 - \ell_2$$

and

$$h_1 + h_2 \geq m, \quad k_1 + k_2 \geq n, \quad \ell_1 + \ell_2 \geq p.$$

Then,

$$\mathcal{C}_{T_1 T_2}^{\mathbf{h}; \mathbf{l}} \simeq \mathcal{C}_{T_1}^{\mathbf{h}; \mathbf{k}} \otimes_{A_n^{\mathbf{k}}}^{\mathbb{L}} \mathcal{C}_{T_2}^{\mathbf{k}; \mathbf{l}}.$$

As Chen-Khovanov note, the proof of Theorem 2 is essentially the same as the arc algebra case [Kho02, Proposition 13]. For the spectral refinements, we need an explicit description of the homotopy equivalence in Theorem 2, so as usual we give a few more details.

Let T_1 be an (m, n) -tangle and T_2 an (n, p) -tangle. By Lemma 2.2, Theorem 2 is equivalent to the same statement with (h_1, h_2) replaced by $(h_1 + 1, h_2 + 1)$ (or \mathbf{k}, \mathbf{l} increased similarly). Thus, for the rest of the section, we will assume that

$$h_1 = k_1 = \ell_1$$

so also

$$h_2 = k_2 = \ell_2.$$

Given $a \in \mathbf{B}_m^{\mathbf{k}}$, $b \in \mathbf{B}_n^{\mathbf{k}}$, and $c \in \mathbf{B}_p^{\mathbf{k}}$, the canonical cobordism from $\widehat{b} \amalg b$ to the identity braid induces a map

$$(2.1) \quad \widetilde{\Psi}(a, b, c): \mathcal{C}_{\underline{T}_1}(a, b) \otimes_{\mathbb{Z}} \mathcal{C}_{\underline{T}_2}(b, c) \rightarrow \mathcal{C}_{\underline{T}_1 T_2}(a, c)$$

Lemma 2.10. *The gluing map $\widetilde{\Psi}$ is a chain map of bimodules. Further, for $a \in \mathbf{B}_m^{\mathbf{k}}$, $b \in \mathbf{B}_n^{\mathbf{k}}$, and $c \in \mathbf{B}_p^{\mathbf{k}}$, Φ takes the submodule*

$$\mathcal{J}_{T_1}^{\mathbf{k}; \mathbf{k}}(a, b) \otimes \mathcal{C}_{\underline{T}_2}(b, c) + \mathcal{C}_{\underline{T}_1}(a, b) \otimes \mathcal{J}_{T_2}^{\mathbf{k}; \mathbf{k}}(b, c) \subset \mathcal{C}_{\underline{T}_1}(a, b) \otimes \mathcal{C}_{\underline{T}_2}(b, c)$$

to the submodule

$$\mathcal{J}_{T_1 T_2}^{\mathbf{k}; \mathbf{k}}(a, c) \subset \mathcal{C}_{\overline{T_1 T_2}}(a, c)$$

and hence induces a homomorphism

$$(2.2) \quad \overline{\Psi}: \mathcal{C}_{T_1}^{\mathbf{k}; \mathbf{k}} \otimes_{\mathbb{Z}} \mathcal{C}_{T_2}^{\mathbf{k}; \mathbf{k}} \rightarrow \mathcal{C}_{T_1 T_2}^{\mathbf{k}; \mathbf{k}}.$$

Proof. It is immediate from far-commutativity that the gluing map is a chain map of bimodules (see also [Kho02]). It remains to check that its restriction preserves the submodules. This is clear for elements coming from type III arcs on the left of T_1 or the right of T_2 . Next, suppose we are considering a generator of $\mathcal{C}_{\overline{T_1}}(a, b)$ in which some type II circle is labeled X . Let P be any intersection of that circle with a platform for a . Then the cobordism map takes this generator to one where the circle containing P is labeled X , so there is a type II circle labeled X as desired.

Next, suppose there is a type III arc in $aT_1\hat{b}$ to the right of T_1 , and let P, Q be its endpoints. If P and Q are on the same circle of $a\overline{T_1 T_2}\hat{c}$ then by Lemma 2.3, $aT_1 T_2\hat{c}$ has a type III arc and we are done. Otherwise, at some point in the canonical cobordism, the circle containing P and Q splits. After the split, one of the components is a circle labeled X passing through a platform, implying that the corresponding generator for $a\overline{T_1 T_2}\hat{c}$ has a type II (or perhaps III) circle labeled X . \square

Lemma 2.11. *The map $\tilde{\Psi}$ descends to the tensor product*

$$\mathcal{C}_{\overline{T_1}} \otimes_{\mathcal{A}_n^{\mathbf{k}}} \mathcal{C}_{\overline{T_2}} \rightarrow \mathcal{C}_{\overline{T_1 T_2}}$$

and hence $\overline{\Psi}$ descends to a map

$$\Psi: \mathcal{C}_{T_1}^{\mathbf{k}; \mathbf{k}} \otimes_{\mathcal{A}_n^{\mathbf{k}}} \mathcal{C}_{T_2}^{\mathbf{k}; \mathbf{k}} \rightarrow \mathcal{C}_{T_1 T_2}^{\mathbf{k}; \mathbf{k}}.$$

Proof. This follows from far-commutativity of the cobordism maps and the fact that the action of $\overline{\mathcal{A}_n^{\mathbf{k}}}$ is itself induced by the canonical cobordism $\hat{b} \amalg b \rightarrow \text{Id}$ [Kho02, Proof of Theorem 1]. \square

Lemma 2.12. *Suppose T_1 is a flat tangle and $a \in \mathbf{B}_m^{\mathbf{k}}$ is a crossingless matching so that aT_1 has no type III arcs. Then $a\overline{T_1}$ is (isotopic rel endpoints to) the union of an element $a' \in \mathbf{B}_n^{\mathbf{k}}$ and a collection of unknots.*

Proof. Since any flat $(0, n + k_1 + k_2)$ -tangle is the disjoint union of a crossingless matching and some unknots, all that remains is to verify that none of the points on the same platform are connected by the matching; but if two points on the same platform were connected then the corresponding arc of aT_1 would be a type III arc. \square

Lemma 2.13. *For T_1 an (m, n) -tangle, $\mathcal{C}_{T_1}^{\mathbf{k}; \mathbf{k}}$ is a projective left module over $\mathcal{A}_m^{\mathbf{k}}$ and is a projective right module over $\mathcal{A}_n^{\mathbf{k}}$.*

That is, in Khovanov's language [Kho02], $\mathcal{C}_{T_1}^{\mathbf{k}; \mathbf{k}}$ is a sweet bimodule. This is weaker than $\mathcal{C}_{T_1}^{\mathbf{k}; \mathbf{k}}$ being bi-projective (projective as a bimodule).

Proof of Lemma 2.13. It suffices to prove the result when T_1 is a flat tangle. We prove $\mathcal{C}_{T_1}^{\mathbf{k}; \mathbf{k}}$ is right projective. As a right module,

$$\mathcal{C}_{T_1}^{\mathbf{k}; \mathbf{k}} = \bigoplus_{a \in \mathbf{B}_m^{\mathbf{k}}} \mathcal{C}_{T_1}^{\mathbf{k}; \mathbf{k}}(a, \cdot),$$

so it suffices to prove each summand is right projective. Let a' be the crossingless matching from Lemma 2.12, and suppose that $\overline{aT_1}$ has q circle components. Then, it is immediate from the definitions that

$$\mathcal{C}_{T_1}^{\mathbf{k};\mathbf{k}}(a, \cdot) = V^{\otimes q} \otimes (a' \cdot \mathcal{A}_n^{\mathbf{k}}).$$

This implies the result. \square

Proof of Theorem 2. As noted above, by Lemma 2.2, we may assume $\mathbf{h} = \mathbf{k} = \mathbf{1}$. So, by Lemma 2.13, we can replace the derived tensor product with the ordinary tensor product. So, it suffices to prove that the gluing map Ψ is a chain isomorphism. By Lemma 2.10, we know Ψ is a well-defined chain map, so it suffices to prove that for single vertices v, w for the cube of resolutions of T_1, T_2 , and crossingless matchings a, c , the gluing map

$$(2.3) \quad \Psi: \mathcal{C}_{T_{1,v}}^{\mathbf{k};\mathbf{k}}(a, \cdot) \otimes_{\mathcal{A}_n^{\mathbf{k}}} \mathcal{C}_{T_{2,w}}^{\mathbf{k};\mathbf{k}}(\cdot, c) \rightarrow \mathcal{C}_{T_{1,v}T_{2,w}}^{\mathbf{k};\mathbf{k}}(a, c)$$

is an isomorphism of free abelian groups.

If $\overline{aT_{1,v}}$ or $\overline{T_{2,w}\hat{c}}$ has a type III arc then both sides vanish, so there is nothing to prove.

In the remaining case, by Lemma 2.13 and its proof, we have

$$\mathcal{C}_{T_{1,v}}^{\mathbf{k};\mathbf{k}}(a, \cdot) \otimes_{\mathcal{A}_n^{\mathbf{k}}} \mathcal{C}_{T_{2,w}}^{\mathbf{k};\mathbf{k}}(\cdot, c) \cong V^{\otimes q+q'} \otimes (a' \cdot \mathcal{A}_n^{\mathbf{k}}) \otimes_{\mathcal{A}_n^{\mathbf{k}}} (\mathcal{A}_n^{\mathbf{k}} \cdot c'),$$

where a' is as in Lemma 2.12 and c' is defined similarly. This is isomorphic to

$$V^{\otimes q+q'} \otimes (a' \cdot \mathcal{A}_n^{\mathbf{k}} \cdot c') = V^{\otimes q+q'} \otimes \mathcal{A}_n^{\mathbf{k}}(a', c').$$

This already proves that the two sides of Equation (2.3) are isomorphic.

So, to see that Ψ is an isomorphism, it suffices to verify that the map $\overline{\Psi}$ from Equation (2.2) is surjective. Fix a generator $y \in \mathcal{C}_{T_{1,v}T_{2,w}}^{\mathbf{k};\mathbf{k}}(a, c)$. Consider generators $y_1 \in \mathcal{C}_{T_{1,v}}^{\mathbf{k};\mathbf{k}}(a, c')$ and $y_2 \in \mathcal{C}_{T_{2,w}}^{\mathbf{k};\mathbf{k}}(c', c)$, so that:

- y_1 and y_2 label the closed components of $\overline{aT_{1,v}}$ and $\overline{T_{2,w}\hat{c}}$ the same way as y .
- y_1 labels each arc of \hat{c}' the same way as y labels the corresponding arc of \hat{c} .
- y_2 labels each arc of c' by 1.

Since the merge map from $\hat{c}' \amalg c'$ to Id induces multiplication in the algebra, and 1 is a unit, it follows that $\overline{\Psi}(y_1 \otimes y_2) = y$. \square

In light of Theorem 2, Lemma 2.2, and Lemma 2.9, it is natural to restrict to the case that $h_2 = m - h_1$ and $k_2 = n - k_1$, exactly as Chen-Khovanov do. So, let

$$\mathcal{A}_n^{\mathcal{P}} = \bigoplus_{k=0}^n \mathcal{A}_n^{n-k,k} \quad \mathcal{C}_T^{\mathcal{P}} = \bigoplus_{\substack{h,k \\ m-n=2(h-k)}} \mathcal{C}_T^{m-h,h;n-k,k}.$$

3. REVIEW OF THE SPECTRAL ARC ALGEBRAS AND MODULES

Hu-Kriz-Kriz used functors to the Burnside category and Elmendorf-Mandell's K -theory multifunctor to give a new construction of a Khovanov homotopy type [HKK16]. One can show [LLSa] that their construction is equivalent to our original Khovanov stable homotopy type (from [LS14]). The canonical nature of the Elmendorf-Mandell construction makes Hu-Kriz-Kriz's approach well-suited to functorial aspects of Khovanov homotopy, like tangle invariants. Therefore, our spectral refinement of the tangle invariants [LLSb] was carried out using the Hu-Kriz-Kriz viewpoint, and we continue to use that approach here.

This section provides concise background on some of the relevant topics. (For more detailed background, see [LLSb, §2].) Section 3.1 recalls the notion of a multicategory, the basic framework for the construction. Section 3.2 introduces a particular target multicategory, the graded Burnside category (of the trivial group). Section 3.3 discusses how one can reinterpret the tangle invariants (both combinatorial and spectral) in terms of multifunctors. Section 3.4 reviews key properties of the spectral refinements of the arc algebras and modules (from [LLSb]). Section 3.5 introduces two analogues of ideals, submodules, and quotient rings and modules in the setting of multifunctors, one of which is used to define the spectral platform algebras and bimodules and the other of which is used to prove the spectral bimodules are invariants. Section 3.6 recalls a notion of equivalence for multifunctors which guarantees that the corresponding spectral bimodules are weakly equivalent.

To avoid excessive repetition, the material presented in this section omits many details from our previous paper [LLSb], with which the reader is assumed to be familiar or be willing to accept as a black box.

3.1. Multicategories. A *multicategory* (or *colored operad*) is an operad with many objects or, equivalently, a category with n -input morphism sets and the obvious kinds of compositions; so

$$\begin{array}{ccc} \text{monoid} & \longrightarrow & \text{category} \\ \downarrow & \lrcorner & \downarrow \\ \text{operad} & \longrightarrow & \text{multicategory.} \end{array}$$

(See [EM06, Definition 2.1].) Any monoidal category can be viewed as a multicategory (cf. \star -*categories* from [HKK16]). For instance, the category of abelian groups forms a multicategory, with $\text{Hom}(G_1, \dots, G_n; H)$ the group homomorphisms $G_1 \otimes \dots \otimes G_n \rightarrow H$. Similarly, topological spaces (or simplicial sets) forms a multicategory, with $\text{Hom}(X_1, \dots, X_n; Y)$ the maps $X_1 \times \dots \times X_n \rightarrow Y$. Similarly, symmetric spectra \mathcal{S} [HSS00] forms a multicategory and is naturally a simplicial multicategory (multicategory enriched in simplicial sets). As in categories, given a multicategory \mathcal{C} we will typically write $\mathcal{C}(x_1, \dots, x_n; y)$ for $\text{Hom}_{\mathcal{C}}(x_1, \dots, x_n; y)$.

A *multifunctor* from \mathcal{C} to \mathcal{D} consists of a map $F: \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$ and, for each $x_1, \dots, x_n, y \in \text{Ob}(\mathcal{C})$, a map $\mathcal{C}(x_1, \dots, x_n; y) \rightarrow \mathcal{D}(F(x_1), \dots, F(x_n), F(y))$, respecting the identity maps and composition. Note that while we sometimes consider multicategories enriched in groupoids or simplicial sets, composition is always strictly associative and has strict units for multicategories and multifunctors always strictly respect the identity maps and compositions; we never consider a multicategorical analogue of lax 2-categories.

3.2. The Burnside multicategory. The *graded Burnside multicategory* \mathcal{B} is the multicategory enriched in categories with:

- Objects finite graded sets X .
- $\mathcal{B}(X; Y) = \text{Hom}_{\mathcal{B}}(X; Y)$ the category with objects finite correspondences $(A, s: A \rightarrow X, t: A \rightarrow Y)$ of graded sets and morphisms diagrams of graded set maps

$$\begin{array}{ccc} & A & \xrightarrow{\cong} B \\ & \searrow s & \nearrow f \\ X & & Y \\ & \nearrow s & \searrow t \end{array}$$

with f being a bijection.

- More generally, $\mathcal{B}(X_1, \dots, X_n; Y) = \mathcal{B}(X_1 \times \dots \times X_n; Y)$, where $X_1 \times \dots \times X_n$ is a graded set with

$$\text{gr}_{X_1 \times \dots \times X_n}(x_1, \dots, x_n) = \text{gr}_{X_1}(x_1) + \dots + \text{gr}_{X_n}(x_n)$$

(like the grading on a tensor product).

- Composition is induced by the fiber product.

(We elide some technicalities about associativity of the Cartesian product of sets; see, e.g., [LLSb, §3.2.1] for one way of handling these.) Taking the nerve of each multimorphism category turns \mathcal{B} into a simplicial multicategory, which we also denote \mathcal{B} and call the Burnside multicategory. We will typically drop the word “graded” and simply call \mathcal{B} the Burnside category.

There is a multifunctor $\mathcal{F}_{\text{orget}}: \mathcal{B} \rightarrow \mathbb{Z}^{\text{Ab}}$ which sends a set X to $\bigoplus_{x \in X} \mathbb{Z}$ and a correspondence A to the $Y \times X$ -matrix $[\#s^{-1}(x) \cap t^{-1}(y)]_{x \in X, y \in Y}$.

The Elmendorf-Mandell machine can be used to lift this functor to spectra. In more detail, Elmendorf-Mandell’s K -theory is a multifunctor from the multicategory of all permutative categories to spectra, and there is a multifunctor from the Burnside category to the multicategory of permutative categories which sends $X \in \mathcal{B}$ to Sets/X . Composing these two functors, we get a (simplicially enriched) functor $\mathcal{B} \rightarrow \mathbb{Z}^{\mathcal{S}}$, which we will still denote K . Moreover, the following diagram commutes up to quasi-isomorphism:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{K} & \mathbb{Z}^{\mathcal{S}} \\ \mathcal{F}_{\text{orget}} \downarrow & & \downarrow C_* \\ \mathbb{Z}^{\text{Ab}} & \hookrightarrow & \mathbb{Z}^{\text{Kom}} \end{array}$$

In particular, for any $X \in \mathcal{B}$, $K(X)$ is weakly equivalent to $\bigvee_{x \in X} \mathbb{S}$.

3.3. Tangle invariants as multifunctors. As a final step before using the Elmendorf-Mandell machine, we reformulate the tangle invariants in terms of multifunctors. Consider the *tangle shape multicategory* $\mathcal{T}_{2m; 2n}$ with:

- Three kinds of objects: pairs $(a_1, a_2) \in \mathbf{B}_{2m} \times \mathbf{B}_{2m}$, pairs $(b_1, b_2) \in \mathbf{B}_{2n} \times \mathbf{B}_{2n}$, and pairs $(a, b) \in \mathbf{B}_{2m} \times \mathbf{B}_{2n}$. For reasons that will become clear, we will denote the third kind of object as (a, T, b) , where T is a placeholder.

- A unique multimorphism in each of the following cases, and no other multimorphisms:

$$(3.1) \quad (a_1, a_2), (a_2, a_3), \dots, (a_{\alpha-1}, a_\alpha) \rightarrow (a_1, a_\alpha)$$

$$(3.2) \quad (b_1, b_2), (b_2, b_3), \dots, (b_{\beta-1}, b_\beta) \rightarrow (b_1, b_\beta)$$

$$(3.3) \quad (a_1, a_2), \dots, (a_{\alpha-1}, a_\alpha), (a_\alpha, T, b_1), (b_1, b_2), \dots, (b_{\beta-1}, b_\beta) \rightarrow (a_1, T, b_\beta)$$

The special cases $\alpha = 1$ and $\beta = 1$ are allowed, and they correspond to 0-input morphisms:

$$(3.4) \quad \emptyset \rightarrow (a, a)$$

$$(3.5) \quad \emptyset \rightarrow (b, b)$$

This category has several full subcategories:

- The category $\mathcal{S}_{2m} \subset \mathcal{T}_{2m;2n}$ spanned by the objects in $\mathbf{B}_{2m} \times \mathbf{B}_{2m}$. This is isomorphic to the subcategory of $\mathcal{T}_{2n;2m}$ spanned by the objects in $\mathbf{B}_{2m} \times \mathbf{B}_{2m}$.
- The category $\mathcal{S}_m^{h_1, h_2} = \mathcal{S}_m^{\mathbf{h}} \subset \mathcal{S}_{m+h_1+h_2}$ spanned by the objects in $\mathbf{B}_m^{\mathbf{h}} \times \mathbf{B}_m^{\mathbf{h}} \subset \mathbf{B}_{m+h_1+h_2} \times \mathbf{B}_{m+h_1+h_2}$.
- The category $\mathcal{T}_{m;n}^{h_1, h_2; k_1, k_2} = \mathcal{T}_{m;n}^{\mathbf{h}; \mathbf{k}} \subset \mathcal{T}_{m+h_1+h_2; n+k_1+k_2}$ spanned by objects in

$$(\mathbf{B}_m^{\mathbf{h}} \times \mathbf{B}_m^{\mathbf{h}}) \amalg (\mathbf{B}_n^{\mathbf{k}} \times \mathbf{B}_n^{\mathbf{k}}) \amalg (\mathbf{B}_m^{\mathbf{h}} \times \mathbf{B}_n^{\mathbf{k}}).$$

(with $h_1 - h_2 = k_1 - k_2$).

The arc algebras determine a multifunctor

$$F_{\mathcal{A}_{2m}} : \mathcal{S}_{2m} \rightarrow \mathbb{Z}^{\text{Ab}}$$

by defining $F_{\mathcal{A}_{2m}}(a_1, a_2) = \mathcal{A}_{2m}(a_1, a_2)$ and

$$F_{\mathcal{A}_{2m}}((a_1, a_2), (a_2, a_3), \dots, (a_{\alpha-1}, a_\alpha) \rightarrow (a_1, a_\alpha))$$

to be the iterated composition (multiplication) map

$$f_{12}, f_{23}, \dots, f_{\alpha-1, \alpha} \mapsto f_{\alpha-1, \alpha} \circ \dots \circ f_{23} \circ f_{12}.$$

Conversely, the functor $F_{\mathcal{A}_{2m}}$ determines the arc algebra in an obvious way. Extending further, Khovanov's invariant of a tangle T (with ordered crossings) is equivalent to a functor

$$\mathcal{T}_{2m;2n} \rightarrow \mathbb{Z}^{\text{Kom}}.$$

(See also [LLSb, §2.3].)

Similarly, the platform algebra and invariant of a tangle T are equivalent to functors

$$\mathcal{S}_m^{\mathbf{h}} \rightarrow \mathbb{Z}^{\text{Ab}}, \quad \mathcal{T}_{m;n}^{\mathbf{h}; \mathbf{k}} \rightarrow \mathbb{Z}^{\text{Kom}},$$

respectively.

For the homotopical refinement we need two variants on these constructions. First, we need a *canonical thickening* $\tilde{\mathcal{T}}_{2m;2n}$ of $\mathcal{T}_{2m;2n}$, which is a simplicial multicategory with the same objects as $\mathcal{T}_{2m;2n}$ but in which each Hom-set is replaced by a (particular) simplicial set [LLSb, §2.4], defined in terms of labeled trees of *basic multimorphisms*.

Second, we need to be able to multiply by a cube. Define $\underline{2}^N \tilde{\times} \mathcal{T}_{2m;2n}$ to have objects

$$(\mathbf{B}_{2m} \times \mathbf{B}_{2m}) \amalg (\mathbf{B}_{2n} \times \mathbf{B}_{2n}) \amalg (\text{Ob}(\underline{2}^N) \times \mathbf{B}_{2m} \times \mathbf{B}_{2n}).$$

where, again, $\underline{2}^N = \{0 \rightarrow 1\}^N$.

The multimorphisms in $\underline{2}^N \tilde{\times} \mathcal{T}_{2m;2n}$ are the same as in $\mathcal{T}_{2m;2n}$ except that Formula (3.3) is replaced with

$$(3.6) \quad (a_1, a_2), \dots, (a_{\alpha-1}, a_\alpha), (v, a_\alpha, T, b_1), (b_1, b_2), \dots, (b_{\beta-1}, b_\beta) \rightarrow (w, a_1, T, b_\beta)$$

whenever $v \leq w \in \underline{2}^N$ (with the partial order induced by $0 < 1$). This category still has \mathcal{S}_{2m} and \mathcal{S}_{2n} as subcategories. The Khovanov invariant of an N -crossing tangle T is induced by a functor

$$F_C: \underline{2}^N \tilde{\times} \mathcal{T}_{2m;2n} \rightarrow \mathbb{Z}^{\text{Ab}}.$$

There is also a canonical thickening $\underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{2m;2n}$ [LLSb, §3.2.4].

The category $\underline{2}^N \tilde{\times} \mathcal{T}_{m+h_1+h_2;n+k_1+k_2}$ (respectively $\underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m+h_1+h_2;n+k_1+k_2}$) has a full subcategory $\underline{2}^N \tilde{\times} \mathcal{T}_{m;n}^{\mathbf{h};\mathbf{k}}$ (respectively $\underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}}$).

3.4. The homotopical refinement of Khovanov's arc algebras and bimodules. By using a particular multicategory of divided cobordisms, in our previous paper [LLSb] we defined multifunctors

$$\begin{aligned} \Phi_{2n}: \tilde{\mathcal{S}}_{2n} &\rightarrow \mathcal{B} \\ \Psi_T: \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{2m;2n} &\rightarrow \mathcal{B}. \end{aligned}$$

(These multifunctors were denoted $\underline{\mathbf{MB}}_n$ and $\underline{\mathbf{MB}}_T$.) We will not need the details of these constructions, just the following properties:

(Ψ-1) The compositions $\mathcal{F}_{\text{forget}} \circ \Phi_{2n}$ and $\mathcal{F}_{\text{forget}} \circ \Psi_T$ agree with the Khovanov arc algebra and module $F_{\mathcal{A}_{2n}}$ and F_C , respectively. Further, Φ_{2m} and Φ_{2n} are the restrictions of Ψ_T . More precisely, the following diagrams commute:

$$\begin{array}{ccccc} \tilde{\mathcal{S}}_{2n} & \xrightarrow{\Phi_{2n}} & \mathcal{B} & & \tilde{\mathcal{S}}_{2m} & \hookrightarrow & \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{2m;2n} & \hookleftarrow & \tilde{\mathcal{S}}_{2n} \\ \downarrow & & \downarrow \mathcal{F}_{\text{forget}} & & \downarrow & & \downarrow \Psi_T & & \downarrow \\ \mathcal{S}_{2n} & \xrightarrow{F_{\mathcal{A}_{2n}}} & \mathbb{Z}^{\text{Ab}} & & \tilde{\mathcal{S}}_{2m} & \xrightarrow{\Phi_{2m}} & \mathcal{B} & \xleftarrow{\Phi_{2n}} & \tilde{\mathcal{S}}_{2n} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{2m;2n} & \xrightarrow{\Psi_T} & \mathcal{B} & & \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{2m;2n} & \xrightarrow{F_{C_T}} & \mathbb{Z}^{\text{Ab}} & & \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{2m;2n} \end{array}$$

(Ψ-2) On objects, $\Phi_{2n}(a, b)$ is the set of labelings of the components of $\widehat{a\hat{b}}$ by elements of $\{1, X\}$, i.e., the set of Khovanov generators for $\widehat{a\hat{b}}$. Similarly, $\Psi_T(v, a, T, b)$ (respectively $\Psi_T(a_1, a_2)$, $\Psi_T(b_1, b_2)$) is the set of Khovanov generators for $aT_v\widehat{b}$ (respectively $a_1\widehat{a_2}$, $b_1\widehat{b_2}$), where T_v is the v -resolution of T .

(Ψ-3) Given a multi-morphism $f \in \tilde{\mathcal{T}}_{2m;2n}(x_1, \dots, x_\ell; y)$ with associated canonical cobordism Σ , as well as Khovanov generators $c_i \in \Psi_T(x_i)$ and $d \in \Psi_T(y)$, the correspondence $\Psi_T(f)$ satisfies

$$s^{-1}(c_1, \dots, c_\ell) \cap t^{-1}(d) = \emptyset \subset \Psi_T(f)$$

unless all of the following are satisfied:

- (a) Every component of Σ has genus 0 or 1.
- (b) For each genus 0 component of Σ , either:

- (i) all circles in the incoming boundary are labeled 1 by (c_1, \dots, c_ℓ) and exactly one circle in the outgoing boundary is labeled 1 by d or
 - (ii) exactly one circle in the incoming boundary is labeled X by (c_1, \dots, c_ℓ) and all circles in the outgoing boundary are labeled X by d .
 - (c) For each genus 1 component of Σ , all incoming circles are labeled 1 by (c_1, \dots, c_ℓ) and all outgoing circles are labeled X by d .
- (Since Φ_{2n} is a restriction of Ψ_T , this also applies to Φ_{2n} .)

Now, the spectral arc algebra is defined as follows. Composing the functor Φ_{2n} with the map $K: \mathcal{B} \rightarrow \mathbb{Z}^{\mathcal{S}}$ coming from Elmendorf-Mandell's K -theory gives a functor $\tilde{\mathcal{S}}_{2n} \rightarrow \mathbb{Z}^{\mathcal{S}}$. Their rectification results [EM06, Theorems 1.3 and 1.4] then give a multifunctor $\mathcal{S}_{2n} \rightarrow \mathbb{Z}^{\mathcal{S}}$. Finally, reinterpreting this as in Section 3.3 gives a spectral category \mathcal{A}_{2n} with one object per crossingless matching; if one prefers, one can take the wedge sum of all morphisms spaces in this category to obtain a ring spectrum $\mathcal{A}_{2n}^{\text{ring}}$.

Similarly, given an oriented tangle diagram T with N crossings, of which N_+ are positive, we can consider the composition

$$\underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{2m;2n} \xrightarrow{\Psi_T} \mathcal{B} \xrightarrow{K} \mathbb{Z}^{\mathcal{S}}.$$

Rectifying this composition gives a multifunctor $G: \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{2m;2n} \rightarrow \mathbb{Z}^{\mathcal{S}}$. We turn this into a functor $H: \mathcal{T}_{2m;2n} \rightarrow \mathbb{Z}^{\mathcal{S}}$ by letting $H(a_1, a_2) = G(a_1, a_2)$, $H(b_1, b_2) = G(b_1, b_2)$, and $H(a, T, b)$ be the iterated mapping cone, over the cube $\underline{2}^N$, of $G(v, a, T, b)$, formally desuspended N_+ times. We can then reinterpret H as a spectral bimodule \mathcal{X}_T over \mathcal{A}_{2m} and \mathcal{A}_{2n} , as in Section 3.3. This can also be viewed as a spectral bimodule $\mathcal{X}_T^{\text{mod}}$ over $\mathcal{A}_{2m}^{\text{ring}}$ and $\mathcal{A}_{2n}^{\text{ring}}$.

Abstracting somewhat, let \mathcal{C} be either $\mathcal{T}_{2m;2n}$ or $\mathcal{T}_{m;n}^{\mathbf{h};\mathbf{k}}$, and let $\mathcal{D} \subset \mathcal{C}$ be one of \mathcal{S}_{2m} or \mathcal{S}_{2n} (if \mathcal{C} is $\mathcal{T}_{2m;2n}$), or $\mathcal{S}_m^{\mathbf{h}}$ or $\mathcal{S}_n^{\mathbf{k}}$ (if \mathcal{C} is $\mathcal{T}_{m;n}^{\mathbf{h};\mathbf{k}}$). A *stable functor* is a pair (F, S) where $F: \underline{2}^N \tilde{\times} \mathcal{C} \rightarrow \mathcal{B}$ and $S \in \mathbb{Z}$. For instance, given a tangle T , (Ψ_T, N_+) is a stable functor. The procedure in the previous paragraph (rectifying, taking mapping cone along $\underline{2}^N$, then desuspending S times) produces a functor $|F|: \mathcal{C} \rightarrow \mathbb{Z}^{\mathcal{S}}$. If $A = F|_{\mathcal{D}}$ then let $\|A\|$ be the spectral category obtained by restricting $|F|$ to \mathcal{D} and reinterpreting as in Section 3.3. Also as in Section 3.3, the functor $|F|$ may be viewed as a spectral bimodule $\|F\|$ over the two spectral categories $\|A\|$. For $F = \Psi_T$, the spectral category $\|A\|$ is the category \mathcal{A}_{2m} or \mathcal{A}_{2n} and the spectral bimodule $\|F\|$ is \mathcal{X}_T , as defined earlier. So, to spectrify the Chen-Khovanov tangle invariants, all that remains is to choose the right stable functor $(F: \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}} \rightarrow \mathcal{B}, S)$.

3.5. Subfunctors and quotient functors. The following will serve as an analogue of an ideal I in a ring R :

Definition 3.1. Given a multicategory \mathcal{C} and a multifunctor $F: \mathcal{C} \rightarrow \mathcal{B}$, an *absorbing subfunctor* of \mathcal{C} is a collection of subsets $G(a) \subset F(a)$, $a \in \mathcal{C}$, so that for any $p_1, \dots, p_\ell, q \in \mathcal{C}$, $f \in \mathcal{C}(p_1, \dots, p_\ell; q)$, $x_i \in F(p_i)$, and $y \in F(q)$, if some $x_i \in G(p_i)$ and $y \notin G(q)$ then

$$(3.7) \quad s^{-1}(x_1, \dots, x_\ell) \cap t^{-1}(y) = \emptyset \subset F(f).$$

Extend G to a multifunctor $G: \mathcal{C} \rightarrow \mathcal{B}$ by defining, for $f \in \mathcal{C}(p_1, \dots, p_\ell; q)$,

$$G(f) = s^{-1}(G(p_1) \times \dots \times G(p_\ell)) \cap t^{-1}(G(q)) \subset F(f)$$

with the obvious source and target maps, and 2-morphisms induced by F in the obvious way. The fact that G respects multi-composition, i.e., that for an ℓ -input morphism f ,

$$G(f \circ (g_1, \dots, g_\ell)) = G(f) \circ (G(g_1), \dots, G(g_\ell)),$$

follows from Equation (3.7).

Given an absorbing subfunctor G of F , there is a corresponding *quotient functor* F/G defined by

- On objects $a \in \mathcal{C}$, $(F/G)(a) = F(a) \setminus G(a)$.
- On morphisms $f \in \mathcal{C}(p_1, \dots, p_\ell; q)$, $(F/G)(f) = s^{-1}((F/G)(p_1) \times \dots \times (F/G)(p_\ell)) \cap t^{-1}((F/G)(q))$.
- On 2-morphisms, F/G is induced from F .

Lemma 3.2. *If G is an absorbing subfunctor then F/G is a multifunctor.*

Proof. This is straightforward from the definition and Equation (3.7). \square

For us, the quotient by an absorbing subfunctor will correspond to quotienting both a ring and a module by an ideal. There is another kind of quotient that corresponds to leaving the ring unchanged but taking the quotient of a module, which is useful in proving invariance under Reidemeister moves:

Definition 3.3. [LLSb, Definition 3.25] Let \mathcal{C} be a multicategory and let $X \subset \text{Ob}(\mathcal{C})$ be a subset of the objects so that there are no multi-morphisms out of X , in the sense that if $p_1, \dots, p_\ell, q \in \mathcal{C}$ and some $p_i \in X$ and $q \notin X$ then $\mathcal{C}(p_1, \dots, p_\ell; q) = \emptyset$. (In the application, X will be the collection of objects of the form (v, a, T, b) in $\underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{2m;2n}$ or $\underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}}$.)

Given a functor $F: \mathcal{C} \rightarrow \mathcal{B}$, an *insular subfunctor* of F (relative to X) is a collection of subsets $G(a) \subset F(a)$ for $a \in X$, such that for any objects $p_1, \dots, p_\ell, q \in \mathcal{C}$ with some $p_i \in X$, morphism $f \in \mathcal{C}(p_1, \dots, p_\ell; q)$, and elements $x_j \in F(p_j)$ and $y \in F(q)$, if $x_i \in G(p_i)$ and $y \notin G(q)$ then

$$(3.8) \quad s^{-1}(x_1, \dots, x_\ell) \cap t^{-1}(y) = \emptyset \subset F(f).$$

Extend G to a functor $G: \mathcal{C} \rightarrow \mathcal{B}$ by defining $G(p) = F(p)$ for $p \in \text{Ob}(\mathcal{C}) \setminus X$ and, for $f \in \mathcal{C}(p_1, \dots, p_\ell; q)$,

$$G(f) = s^{-1}(G(p_1) \times \dots \times G(p_\ell)) \cap t^{-1}(G(q)) \subset F(f)$$

with the obvious source and target maps, and 2-morphisms induced by F in the obvious way. The fact that G respects multi-composition follows from Equation (3.8).

Given an insular subfunctor G of F there is a *quotient functor* $F/G: \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n} \rightarrow \mathcal{B}$ defined by:

- $(F/G)(p) = F(p)$ if $p \notin X$,
- $(F/G)(p) = F(p) \setminus G(p)$ if $p \in X$,
- $(F/G)(f) = s^{-1}((F/G)(p_1) \times \dots \times (F/G)(p_\ell)) \cap t^{-1}((F/G)(q)) \subset F(f)$ for $f \in \mathcal{C}(p_1, \dots, p_\ell; q)$, and
- the value of F/G on 2-morphisms is induced by F .

Lemma 3.4. *If G is an insular subfunctor then F/G is a multifunctor.*

Proof. Again, this is straightforward from the definitions and Equation (3.8). \square

If \mathcal{C} is either $\underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{2m;2n}$ or $\underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}}$, X is the collection of objects of the form $\{(v, a, T, b)\}$, $F: \mathcal{C} \rightarrow \mathcal{B}$, and G is an insular subfunctor of F , then there is a cofibration sequence of

$$\|G\| \rightarrow \|F\| \rightarrow \|F/G\|$$

of bimodules (over $\|F\|_{\tilde{\mathcal{S}}_{2m}}$ and $\|F\|_{\tilde{\mathcal{S}}_{2n}}$ if $\mathcal{C} = \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{2m;2n}$ or over $\|F\|_{\tilde{\mathcal{S}}_m^{\mathbf{h}}}$ and $\|F\|_{\tilde{\mathcal{S}}_n^{\mathbf{k}}}$ if $\mathcal{C} = \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}}$).

These two notions of subfunctor are compatible, in the following sense:

Lemma 3.5. *Fix a multicategory \mathcal{C} and a subset $X \subset \text{Ob}(\mathcal{C})$ as in Definition 3.3. Suppose $F: \mathcal{C} \rightarrow \mathcal{B}$ is a multifunctor, G is an insular subfunctor of F , and H is an absorbing subfunctor of F . Then G induces an insular subfunctor \overline{G} of F/H via the formula*

$$\overline{G}(a) = G(a) \setminus H(a).$$

for $a \in X$.

Proof. Fix objects $p_1, \dots, p_\ell, q \in \text{Ob}(\mathcal{C})$ with $p_i \in X$, a morphism $f \in \text{Hom}(p_1, \dots, p_\ell; q)$, and elements $x_j \in (F/H)(p_j) = F(p_j) \setminus H(p_j)$ and $y \in (F/H)(q) = F(q) \setminus H(q)$. Suppose $x_i \in \overline{G}(p_i) = G(p_i) \setminus H(p_i)$ and $y \in (F/H)(q) \setminus \overline{G}(q) = F(q) \setminus (G(q) \cup H(q))$. Then, in particular, $x_i \in G(p_i)$ and $y \in F(q) \setminus G(q)$, so since G is insular,

$$s^{-1}(x_1, \dots, x_\ell) \cap t^{-1}(y) = \emptyset,$$

as desired. \square

3.6. Equivalent functors. To prove the spectral refinements are invariant under Reidemeister moves, we use a notion of equivalence of multifunctors to the Burnside category, which we spell out here.

Let \mathcal{C} be either $\tilde{\mathcal{T}}_{2m;2n}$ or $\tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}}$.

Definition 3.6. A *face inclusion* is a functor $i: \underline{2}^M \rightarrow \underline{2}^N$ which is injective on objects and preserves the relative grading. Given a face inclusion i , let $|i|$ be the amount by which i shifts the absolute grading.

The restriction of a functor $\underline{2}^N \tilde{\times} \mathcal{C} \rightarrow \mathcal{B}$ under a face inclusion is a functor $\underline{2}^M \tilde{\times} \mathcal{C} \rightarrow \mathcal{B}$. We can also extend functors under face inclusions:

Definition 3.7. [LLSb, Definition 3.24] Let $i: \underline{2}^M \rightarrow \underline{2}^N$ be a face inclusion and $F: \underline{2}^M \tilde{\times} \mathcal{C} \rightarrow \mathcal{B}$. There is an induced functor $i_!F: \underline{2}^N \tilde{\times} \mathcal{C} \rightarrow \mathcal{B}$ defined on objects by $(i_!F)(a, b) = F(a, b)$ and

$$(i_!F)(v, a, T, b) = \begin{cases} F(u, a, T, b) & \text{if } v = i(u) \text{ is in the image of } i, \\ \emptyset & \text{otherwise.} \end{cases}$$

On multimorphisms, $i_!F$ is induced by F ; see our previous paper for an explicit description.

Given a stable functor $(F, S): \underline{2}^N \tilde{\times} \mathcal{C} \rightarrow \mathcal{B}$, for each pair of crossingless matchings (a, b) we have a cube of abelian groups by restricting $\mathcal{F}_{\text{orget}} \circ F: \underline{2}^N \tilde{\times} \mathcal{C} \rightarrow \mathbb{Z}^{\text{Ab}}$ to the full subcategory spanned by the objects of the form (v, a, T, b) . Let $\text{Tot}(\mathcal{F}_{\text{orget}} \circ F, S)$ be the direct sum over a, b of the total complex of this cube, with grading shifted down by S . So, for instance, if $(F, S) = (\Psi_T, N_+)$ then $\text{Tot}(\mathcal{F}_{\text{orget}} \circ F, S)$ is the chain complex underlying the Khovanov tangle invariant.

Definition 3.8. [LLSb, §3.5.2] We say that two stable functors $(F, S), (G, T): \underline{2}^N \tilde{\times} \mathcal{C} \rightarrow \mathcal{B}$ are *simply stably equivalent* if either:

- (1) There is a face inclusion $i: \underline{2}^M \rightarrow \underline{2}^N$ so that $G = i_!F$ and $T = S + N - M - |i|$, or
- (2) $S = T$ and there is a functor $H: \underline{2}^{N+1} \tilde{\times} \mathcal{C} \rightarrow \mathcal{B}$ so that $F = H|_{\{0\} \times \underline{2}^N}$, $G = H|_{\{1\} \times \underline{2}^N}$, and the chain complex $\text{Tot}(\mathcal{F}_{\text{orget}} \circ H, S)$ is acyclic.

Two functors are *stably equivalent* if they can be connected by a sequence of simple stable equivalences (i.e., stable equivalence is the symmetric, transitive closure of simple stable equivalence).

Lemma 3.9. *If F_1 and F_2 are stably equivalent multifunctors then $\|F_1\|$ and $\|F_2\|$ are equivalent spectral bimodules.*

Proof. In the case $\mathcal{C} = \tilde{\mathcal{T}}_{2m;2n}$ this was proved in our previous paper [LLSb, Proposition 4.7]. The proof for $\mathcal{C} = \tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}}$ is exactly the same. \square

One way to produce stable equivalences is to produce insular subfunctors. Suppose $G \subset F$ is an insular subfunctor. Fix an integer S .

- If $\|F/G\|$ is contractible then $(F, S) \simeq (G, S)$.
- If $\|G\|$ is contractible then $(F, S) \simeq (F/G, S)$.
- If $\|F\|$ is contractible then $(G, S) \simeq (F/G, S + 1)$.

4. SPECTRAL PLATFORM ALGEBRAS AND MODULES

Just like the definitions of the ordinary platform algebras and modules, we define the spectral platform algebras by first restricting and then quotienting. We start by restricting—the analogues of $\overline{\mathcal{A}}_n^{\mathbf{k}}$ and $(j \circ i^{k_1-h_1} \otimes j)^* \mathcal{C}_{\overline{T}}$.

Convention 4.1. In Section 2 we recalled how to associate an algebra $\mathcal{A}_n^{k_1, k_2}$ to a tuple of integers (n, k_1, k_2) , so $\mathcal{A}_{2n}^{0,0}$ is Khovanov’s platform algebra and $\mathcal{A}_n^{k_1, k_2}$ is isomorphic to one of the Chen-Khovanov algebras whenever $k_1 + k_2 \geq n$. In the rest of the paper, we only consider the case that $k_1 + k_2 \geq n$ (and similarly, tuples (m, h_1, h_2) with $h_1 + h_2 \geq m$), so we are only considering the Chen-Khovanov case.

Recall from Section 3.3 that we have full submulticategories

$$\tilde{\mathcal{S}}_n^{\mathbf{k}} \subset \tilde{\mathcal{S}}_{n+k_1+k_2}, \quad \tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}} \subset \tilde{\mathcal{T}}_{m+h_1+h_2;n+k_1+k_2}, \quad \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}} \subset \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m+h_1+h_2;n+k_1+k_2}.$$

The multifunctor $\Phi_{n+k_1+k_2}$ restricts to a multifunctor

$$\overline{\Phi}_n^{\mathbf{k}}: \tilde{\mathcal{S}}_n^{\mathbf{k}} \rightarrow \mathcal{B}.$$

Next, fix a flat (m, n) -tangle T and pairs $\mathbf{h} = (h_1, h_2)$ and $\mathbf{k} = (k_1, k_2)$ with $h_1 - h_2 = k_1 - k_2$. Without loss of generality, assume that $h_1 \leq k_1$. Recall that \overline{T} is the result of adding k_1 horizontal strands below T and k_2 horizontal strands above T . The multifunctor $\Psi_{\overline{T}}$ restricts to a multifunctor

$$\overline{\Psi}_T^{\mathbf{k};\mathbf{k}}: \tilde{\mathcal{T}}_{m;n}^{\mathbf{k};\mathbf{k}} \rightarrow \mathcal{B}.$$

The map $\iota: \mathbf{B}_{m+h_1+h_2} \hookrightarrow \mathbf{B}_{m+h_1+h_2+2}$ induces an embedding

$$\iota: \tilde{\mathcal{T}}_{m;n}^{h_1, h_2; k_1, k_2} \hookrightarrow \tilde{\mathcal{T}}_{m;n}^{h_1+1, h_2+1; k_1, k_2}.$$

Notation	Meaning
\mathcal{B}	Graded Burnside multicategory.
\mathcal{S}	Multicategory of symmetric spectra.
\mathcal{S}_{2n}	Strict arc algebra shape multicategory.
$\tilde{\mathcal{S}}_{2n}$	Groupoid-enriched arc algebra shape multicategory.
Φ_{2n}	Functor $\tilde{\mathcal{S}}_{2n} \rightarrow \mathcal{B}$ refining arc algebra.
\mathcal{A}_{2n}	Spectral category refining \mathcal{A}_{2n} .
$\mathcal{T}_{2m;2n}$	Strict tangle shape multicategory.
$\tilde{\mathcal{T}}_{2m;2n}$	Groupoid-enriched tangle shape multicategory.
Ψ_T	Functor $\underline{\mathbb{Z}}^N \tilde{\times} \tilde{\mathcal{T}}_{2m;2n} \rightarrow \mathcal{B}$ refining Khovanov cube of bimodules.
\mathcal{X}_T	Spectral bimodule refining \mathcal{C}_T .
$\mathcal{S}_n^{\mathbf{k}}$	Strict platform algebra shape multicategory.
$\tilde{\mathcal{S}}_n^{\mathbf{k}}$	Groupoid-enriched platform algebra shape multicategory.
$\bar{\Phi}_n^{\mathbf{k}}$	Restriction of $\Phi_{n+k_1+k_2}$ to $\tilde{\mathcal{S}}_n^{\mathbf{k}}$.
l	Absorbing subfunctor of $\bar{\Phi}_n^{\mathbf{k}}$ corresponding to $\mathcal{I}_n^{\mathbf{k}}$.
$\Phi_n^{\mathbf{k}}$	Quotient functor $\bar{\Phi}_n^{\mathbf{k}}/l$ refining platform algebra.
$\mathcal{A}_n^{\mathbf{k}}$	Spectral category refining $\mathcal{A}_n^{\mathbf{k}}$.
$\mathcal{A}_n^{\mathcal{P}}$	Spectral category refining $\mathcal{A}_n^{\mathcal{P}}$, given by $\coprod_{k=0}^n \mathcal{A}_n^{n-k, k}$.
$\mathcal{T}_{m;n}^{\mathbf{h};\mathbf{k}}$	Strict platform tangle shape multicategory.
$\tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}}$	Groupoid-enriched platform tangle shape multicategory.
$\bar{\Psi}_T^{\mathbf{h};\mathbf{k}}$	Restriction of $\Psi_{\bar{T}}$ to $\underline{\mathbb{Z}}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}}$.
J	Absorbing subfunctor of $\bar{\Psi}_T^{\mathbf{h};\mathbf{k}}$ corresponding to $\mathcal{J}_T^{\mathbf{h};\mathbf{k}}$.
$\tilde{\Psi}_T^{\mathbf{h};\mathbf{k}}$	Quotient functor $\bar{\Psi}_T^{\mathbf{h};\mathbf{k}}/J$ refining platform cube of bimodules.
$\Psi_T^{\mathbf{h};\mathbf{k}}$	Minor modification of $\tilde{\Psi}_T^{\mathbf{h};\mathbf{k}}$ taking into account certain natural isomorphisms.
$\mathcal{X}_T^{\mathbf{h};\mathbf{k}}$	Spectral bimodule refining $\mathcal{C}_T^{\mathbf{h};\mathbf{k}}$.
$\mathcal{X}_T^{\mathcal{P}}$	Spectral bimodule refining $\mathcal{C}_T^{\mathcal{P}}$, given by $\coprod_{\substack{h,k \\ m-n=2(h-k)}} \mathcal{X}_T^{m-h, h; n-k, k}$.

TABLE 4.1. **Notation used in the homotopical refinements.** The notions in the top are used for spectral refinements of arc algebras and bimodules, and the bottom ones are used for spectral refinements of platform algebras and bimodules. Compare with their homological versions in Table 2.1.

Let

$$\bar{\Psi}_T^{\mathbf{h};\mathbf{k}} = \bar{\Psi}_T^{\mathbf{k};\mathbf{k}} \circ \iota^{k_1-h_1} : \tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}} \rightarrow \mathcal{B}.$$

More generally, given an (m, n) -tangle T with N crossings, $\Psi_{\underline{T}}$ restricts to a multifunctor

$$\overline{\Psi}_T^{\mathbf{k};\mathbf{k}}: \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{\mathbf{k};\mathbf{k}} \rightarrow \mathcal{B}.$$

The map ι induces an embedding

$$\iota: \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{h_1, h_2; k_1, k_2} \hookrightarrow \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{h_1+1, h_2+1; k_1, k_2},$$

and we again let

$$\overline{\Psi}_T^{\mathbf{h};\mathbf{k}} = \overline{\Psi}_T^{\mathbf{k};\mathbf{k}} \circ \iota^{k_1-h_1}: \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}} \rightarrow \mathcal{B}.$$

Given $(a_1, a_2) \in \text{Ob}(\widehat{\mathcal{S}}_n^{\mathbf{k}})$ define $\mathsf{l}(a_1, a_2) \subset \overline{\Phi}_n^{\mathbf{k}}(a_1, a_2)$ to be the set of Khovanov generators for $a_1 \widehat{a}_2$ which are in the ideal $\mathcal{I}_n^{\mathbf{k}}(a_1, a_2)$. Similarly, given $o \in \text{Ob}(\underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}})$, if o has the form (a_1, a_2) or (b_1, b_2) define $\mathsf{J}(o) \subset \overline{\Psi}_T^{\mathbf{h};\mathbf{k}}(o)$ to be $\mathsf{l}(o)$. If o has the form (v, a, T, b) , define $\mathsf{J}(v, a, T, b)$ to be the set of Khovanov generators for $\iota^{k_1-h_1}(a) \widehat{T}_v b$ which are in the submodule $\mathcal{J}_{T_v}^{\mathbf{h};\mathbf{k}}(a, b)$.

Lemma 4.2. *The subsets l and J form absorbing subfunctors.*

Proof. The proof is the same as the proofs of Lemma 2.1 and Proposition 2.6. Alternatively, we can deduce this lemma from the combinatorial case. By Property $(\Psi-3)$, for the multi-functor Ψ , $s^{-1}(c_1, \dots, c_\ell) \cap t^{-1}(d) = \emptyset$ unless d appears with non-zero coefficient in the product $c_1 \cdots c_\ell$ in the combinatorial Khovanov bimodule. So, it follows from the fact that $\mathcal{I}_n^{\mathbf{k}}$ is an ideal and $\mathcal{J}_T^{\mathbf{h};\mathbf{k}}$ is a submodule (Lemma 2.1 and Proposition 2.6), in conjunction with the fact that the functor $\mathcal{F}_{\text{orget}}: \mathcal{B} \rightarrow \mathbb{Z}^{\text{Ab}}$ takes each morphism to a non-negative matrix, that l and J are absorbing subfunctors. \square

By Lemma 4.2, there are quotient functors

$$\begin{aligned} \Phi_n^{\mathbf{k}} &= (\overline{\Phi}_n^{\mathbf{k}}/\mathsf{l}): \widehat{\mathcal{S}}_n^{\mathbf{k}} \rightarrow \mathcal{B} \\ \tilde{\Psi}_T^{\mathbf{h};\mathbf{k}} &= (\overline{\Psi}_T^{\mathbf{h};\mathbf{k}}/\mathsf{J}): \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}} \rightarrow \mathcal{B}. \end{aligned}$$

Now, we fiddle around a little to avoid having to keep track of natural isomorphisms between functors. Given $h_1 + h_2 \geq m$ and $k_1 + k_2 \geq n$, there is a canonical isomorphism of multicategories

$$\gamma: \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{m-h, h; n-k, k} \rightarrow \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{h_1, h_2; k_1, k_2}$$

where $m - 2h = h_1 - h_2$ and $n - 2k = k_1 - k_2$. We would like to say that $\tilde{\Psi}_T^{\mathbf{h};\mathbf{k}} \circ \gamma|_{\widehat{\mathcal{S}}_m^{m-h, h}} = \Phi_m^{m-h, h}$ and $\tilde{\Psi}_T^{\mathbf{h};\mathbf{k}} \circ \gamma|_{\widehat{\mathcal{S}}_n^{n-k, k}} = \Phi_n^{n-k, k}$. This is not quite true, but is true up to the following notion of natural isomorphism:

Definition 4.3. [LLSb, Definition 3.22] Let \mathcal{C} be a multicategory enriched in groupoids and $F, G: \mathcal{C} \rightarrow \mathcal{B}$ multifunctors. A *natural isomorphism* $\eta: F \rightarrow G$ consists of:

- For each object $x \in \mathcal{C}$ a bijection of graded sets $\eta_x: F(x) \rightarrow G(x)$, and
- For each multimorphism $f \in \mathcal{C}(x_1, \dots, x_n; y)$, a bijection of graded sets $\eta_f: F(f) \rightarrow G(f)$

such that:

- (1) For any objects $x_1, \dots, x_n, y \in \mathcal{C}$ and multimorphism $f \in \mathcal{C}(x_1, \dots, x_n; y)$, the following diagram commutes:

$$\begin{array}{ccccc}
 & & F(f) & & \\
 & \swarrow & | & \searrow & \\
 F(x_1) \times \dots \times F(x_n) & & & & F(y) \\
 \eta_{x_1} \times \dots \times \eta_{x_n} \downarrow & & \eta_f & & \downarrow \eta_y \\
 G(x_1) \times \dots \times G(x_n) & & & & G(y) \\
 & \swarrow & | & \searrow & \\
 & & G(f) & &
 \end{array}$$

- (2) For every $f, g \in \mathcal{C}(x_1, \dots, x_n; y)$ and $\phi \in \text{Hom}(f, g)$,

$$\eta_g \circ F(\phi) = G(\phi) \circ \eta_f : F(f) \rightarrow G(g).$$

- (3) For every $g \in \mathcal{C}(y_1, \dots, y_n; z)$ and f_1, \dots, f_n with $f_i \in \mathcal{C}(x_{i,1}, \dots, x_{i,m_i}, y_i)$,

$$\eta_{g \circ (f_1, \dots, f_n)} = \eta_g \circ (\eta_{f_1}, \dots, \eta_{f_n}).$$

Lemma 4.4. *If ι denotes the canonical isomorphism $\tilde{\mathcal{S}}_m^{h_1, h_2} \xrightarrow{\cong} \tilde{\mathcal{S}}_m^{h_1+1, h_2+1}$ induced by $\iota : \mathbf{B}_{m+h_1+h_2} \hookrightarrow \mathbf{B}_{m+h_1+h_2+2}$ then there is a natural isomorphism between $\Phi_m^{h_1, h_2}$ and $\Phi_m^{h_1+1, h_2+1} \circ \iota$. Further, this isomorphism sends $\text{I}(o) \subset \overline{\Phi}_m^{h_1, h_2}(o)$ bijectively to $\text{I}(\iota(o)) \subset \overline{\Phi}_m^{h_1+1, h_2+1}(\iota(o))$ for each $o \in \text{Ob}(\tilde{\mathcal{S}}_m^{h_1, h_2})$. Thus, $\tilde{\Psi}_T^{\mathbf{h}; \mathbf{k}} \circ \gamma|_{\tilde{\mathcal{S}}_m^{m-h, h}}$ and $\Phi_m^{m-h, h}$ (respectively $\tilde{\Psi}_T^{\mathbf{h}; \mathbf{k}} \circ \gamma|_{\tilde{\mathcal{S}}_n^{n-k, k}}$ and $\Phi_n^{n-k, k}$) are naturally isomorphic.*

Proof. This is immediate from the definitions. \square

Definition 4.5. Define $\Psi_T^{\mathbf{h}; \mathbf{k}} : \mathbb{Z}^N \tilde{\times} \tilde{\mathcal{T}}_{m; n}^{m-h, h; n-k, k} \rightarrow \mathcal{B}$ as follows. Let η (respectively ξ) be the natural isomorphism from $\Phi_m^{m-h, h}$ to $\tilde{\Psi}_T^{\mathbf{h}; \mathbf{k}} \circ \gamma|_{\tilde{\mathcal{S}}_m^{m-h, h}}$ (respectively $\Phi_n^{n-k, k}$ to $\tilde{\Psi}_T^{\mathbf{h}; \mathbf{k}} \circ \gamma|_{\tilde{\mathcal{S}}_n^{n-k, k}}$) from Lemma 4.4.

- For objects $x \in \text{Ob}(\tilde{\mathcal{S}}_m^{m-h, h})$, define $\Psi_T^{\mathbf{h}; \mathbf{k}}(x) = \Phi_m^{m-h, h}(x)$.
- For objects $x \in \text{Ob}(\tilde{\mathcal{S}}_n^{n-k, k})$, define $\Psi_T^{\mathbf{h}; \mathbf{k}}(x) = \Phi_n^{n-k, k}(x)$.
- For all other objects x , define $\Psi_T^{\mathbf{h}; \mathbf{k}}(x) = \tilde{\Psi}_T^{\mathbf{h}; \mathbf{k}}(\gamma(x))$.
- Given objects $x_1, \dots, x_n, y \in \text{Ob}(\mathbb{Z}^N \tilde{\times} \tilde{\mathcal{T}}_{m; n}^{m-h, h; n-k, k})$ and a basic multimorphism (see [LLSb, §2.4]) $f \in \text{Hom}_{\mathbb{Z}^N \tilde{\times} \tilde{\mathcal{T}}_{m; n}^{m-h, h; n-k, k}}(x_1, \dots, x_n; y)$:

- if all the x_i (and hence y) are in $\text{Ob}(\tilde{\mathcal{S}}_m^{m-h, h})$ define $\Psi_T^{\mathbf{h}; \mathbf{k}}(f) = \Phi_m^{m-h, h}(f)$,
- if all the x_i (and hence y) are in $\text{Ob}(\tilde{\mathcal{S}}_n^{n-k, k})$ define $\Psi_T^{\mathbf{h}; \mathbf{k}}(f) = \Phi_n^{n-k, k}(f)$,
- and otherwise, if $x_1, \dots, x_i \in \text{Ob}(\tilde{\mathcal{S}}_m^{m-h, h})$ and $x_{i+2}, \dots, x_n \in \text{Ob}(\tilde{\mathcal{S}}_n^{n-k, k})$, define $\Psi_T^{\mathbf{h}; \mathbf{k}}(f) = \tilde{\Psi}_T^{\mathbf{h}; \mathbf{k}}(\gamma(f))$, viewed as a correspondence from

$$\Phi_m^{m-h, h}(x_1) \times \dots \times \Phi_m^{m-h, h}(x_i) \times \tilde{\Psi}_T^{\mathbf{h}; \mathbf{k}}(\gamma(x_{i+1})) \times \Phi_n^{n-k, k}(x_{i+2}) \times \dots \times \Phi_n^{n-k, k}(x_n)$$

to $\tilde{\Psi}_T^{\mathbf{h}; \mathbf{k}}(\gamma(y))$ by composing the source map with $\eta_{x_1}^{-1} \times \dots \times \eta_{x_i}^{-1} \times \text{Id} \times \xi_{x_{i+1}}^{-1} \times \dots \times \xi_{x_n}^{-1}$.

- On general multimorphisms, $\Psi_T^{\mathbf{h}; \mathbf{k}}$ is the composition of its values on basic multimorphisms.
- On 2-morphisms, $\Psi_T^{\mathbf{h}; \mathbf{k}}$ is induced by the values $\tilde{\Psi}_T^{\mathbf{h}; \mathbf{k}}$, $\Phi_m^{\mathbf{h}}$, $\Phi_n^{\mathbf{k}}$, and the η_f and ξ_f .

Consider the stable functor $(\Psi_T^{\mathbf{h};\mathbf{k}}, N_+)$ (where N_+ is the number of positive crossings of T). Define

$$\mathcal{A}_n^{k_1, k_2} = \|\Phi_n^{k_1, k_2}\| \qquad \mathcal{X}_T^{\mathbf{h};\mathbf{k}} = \|\Psi_T^{\mathbf{h};\mathbf{k}}\|.$$

Lemma 4.6. *The spectra $\mathcal{X}_T^{\mathbf{h};\mathbf{k}}(a, b)$ form a spectral bimodule over $\mathcal{A}_m^{\mathbf{h}}$ and $\mathcal{A}_n^{\mathbf{k}}$.*

Proof. In view of Lemma 4.4 and the discussion in Section 3.3, all that remains is to verify that rectifying $K \circ \Psi_T^{\mathbf{h};\mathbf{k}}$ and then restricting to the sub-multicategory $\mathcal{S}_n^{\mathbf{k}}$ is the same as first restricting to $\tilde{\mathcal{S}}_n^{\mathbf{k}}$ and then rectifying. This follows from the fact that $\tilde{\mathcal{S}}_n^{\mathbf{k}} \subset \tilde{\mathcal{T}}_{m;n}^{\mathbf{h};\mathbf{k}}$ has no morphisms in (i.e., is *blockaded* in the language of [LLSb, Proposition 2.39]) [LLSb, Lemma 2.44]. \square

Let

$$\mathcal{A}_n^{\mathcal{P}} = \prod_{k=0}^n \mathcal{A}_n^{n-k, k}.$$

Let $\mathcal{X}_T^{\mathcal{P}}$ denote the bimodule over $\mathcal{A}_m^{\mathcal{P}}$ and $\mathcal{A}_n^{\mathcal{P}}$ induced by the various bimodules $\mathcal{X}_T^{m-h, h; n-k, k}$.

Composing the singular chain functor C_* with $\mathcal{A}_n^{\mathbf{k}}$ gives a differential graded category (category enriched in chain complexes), and composing C_* with $\mathcal{X}_T^{\mathbf{h};\mathbf{k}}$ gives a bimodule over $C_*\mathcal{A}_m^{\mathbf{h}}$ and $C_*\mathcal{A}_n^{\mathbf{k}}$. Since $\mathcal{A}_n^{\mathbf{k}}$ has finitely many objects, by taking the direct sum of the morphism spaces we can view $C_*\mathcal{A}_n^{\mathbf{k}}$ as a differential graded algebra and $C_*\mathcal{X}_T^{\mathbf{h};\mathbf{k}}$ as a differential graded module over it.

The three main theorems are:

Theorem 3. *There is a quasi-isomorphism $C_*\mathcal{A}_n^{\mathbf{k}} \simeq \mathcal{A}_n^{\mathbf{k}}$ and, for any tangle T , a quasi-isomorphism $C_*\mathcal{X}_T^{\mathbf{h};\mathbf{k}} \simeq \mathcal{C}_T^{\mathbf{h};\mathbf{k}}$ intertwining the module structures in the obvious sense.*

Theorem 4. *If T_1 and T_2 are equivalent tangles then $\mathcal{X}_{T_1}^{\mathbf{h};\mathbf{k}}$ and $\mathcal{X}_{T_2}^{\mathbf{h};\mathbf{k}}$ are weakly equivalent spectral modules.*

Theorem 5. *If T_1 is an (m, n) -tangle and T_2 is an (n, p) -tangle then for any $h_1, h_2, k_1, k_2, \ell_1, \ell_2$ with*

$$h_1 - h_2 = k_1 - k_2 = \ell_1 - \ell_2, \quad h_1 + h_2 \geq m, \quad k_1 + k_2 \geq n, \quad \text{and} \quad \ell_1 + \ell_2 \geq p$$

there is a weak equivalence of spectral bimodules

$$\mathcal{X}_{T_1 T_2}^{\mathbf{h};\mathbf{l}} \simeq \mathcal{X}_{T_1}^{\mathbf{h};\mathbf{k}} \otimes_{\mathcal{A}_n^{n-k, k}}^{\mathbb{L}} \mathcal{X}_{T_2}^{\mathbf{k};\mathbf{l}},$$

where the right side denotes the (derived) tensor product of bimodule spectra.

Proof of Theorem 3. The proof is the same as for the arc algebras [LLSb, Proposition 4.2] and is left to the reader. \square

Lemma 4.7. *There are equivalences*

$$\mathcal{X}_T^{h_1, h_2; k_1, k_2} \simeq \mathcal{X}_T^{h_1+1, h_2+1; k_1, k_2} \simeq \mathcal{X}_T^{h_1, h_2; k_1+1, k_2+1}$$

of bimodules over $\mathcal{A}_m^{m-h, h}$ and $\mathcal{A}_n^{n-k, k}$.

Proof. It is immediate from the definitions that there is a natural isomorphism $\Psi_T^{h_1, h_2; k_1, k_2} \rightarrow \Psi_T^{h_1+1, h_2+1; k_1, k_2}$ of multifunctors from $\underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{m-h, h; n-k, k}$ to \mathcal{B} . This, in turn, implies that $\Psi_T^{h_1, h_2; k_1, k_2}$ and $\Psi_T^{h_1+1, h_2+1; k_1, k_2}$ are stably equivalent. So, the first statement follows from Lemma 3.9. The proof of the second statement is similar. \square

Proof of Theorem 4. Reordering crossings induces an automorphism of the cube and a corresponding equivalence of homotopy colimits. For invariance under Reidemeister moves, we lift the proof of Theorem 1. By Lemma 4.7, we may assume $(h_1, h_2) = (k_1, k_2)$. As in Theorem 1, we focus on a Reidemeister II move; the other cases are similar. With notation as in the proof of Theorem 1, it follows from the definitions that the subcomplex C_1 corresponds to an insular subfunctor F_1 of $\Psi_{\overline{T}}$ (Definition 3.3). The quotient functor $F_2 = \Psi_{\overline{T}}/F_1$ corresponds to the complex C_2 , and has a further insular subfunctor $F_4 \subset F_2$ naturally isomorphic to $\Psi_{\overline{T}}$ so that $F_3 = F_2/F_4$ corresponds to the acyclic complexes C_3 .

Each F_i restricts to a functor $G_i: \underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{m;n}^{\mathbf{k}; \mathbf{k}} \rightarrow \mathcal{B}$, and G_1 is an insular subfunctor of $\overline{\Psi}_T^{\mathbf{k}; \mathbf{k}}$ with quotient functor G_2 , while G_4 is an insular subfunctor of G_2 with quotient functor G_3 . By Lemma 3.5, G_1 induces an insular subfunctor \overline{G}_1 of $\overline{\Psi}_T^{\mathbf{k}; \mathbf{k}}/I = \Psi_T^{\mathbf{k}; \mathbf{k}}$, and G_4 induces an insular subfunctor \overline{G}_4 of $\overline{G}_2 = \Psi_T^{\mathbf{k}; \mathbf{k}}/\overline{G}_1$. Applying the realization procedure gives a zig-zag of spectral bimodules $\mathcal{X}_T^{\mathbf{k}; \mathbf{k}} \rightarrow \|\overline{G}_2\| \leftarrow \mathcal{X}_{T'}^{\mathbf{k}; \mathbf{k}}$. From the proof of Theorem 1, these maps induce isomorphisms on homology, and hence are stable homotopy equivalences, as desired. \square

Proof of Theorem 5. By Lemma 4.7, it suffices to prove Theorem 5 when $h_1 = k_1 = \ell_1$ (and so $h_2 = k_2 = \ell_2$).

We start by recalling the proof of the gluing theorem for the spectral Khovanov bimodules [LLSb, §5]. We introduced a *gluing shape multicategory* $\mathcal{U}_{2m; 2n; 2p}$ (denoted $\mathcal{U}_{m, n, p}^0$ in [LLSb]) with six kinds of objects: pairs $(a_1, a_2) \in \mathbf{B}_{2m} \times \mathbf{B}_{2m}$, $(b_1, b_2) \in \mathbf{B}_{2n} \times \mathbf{B}_{2n}$, $(c_1, c_2) \in \mathbf{B}_{2p} \times \mathbf{B}_{2p}$, triples (a, T_1, b) with $a \in \mathbf{B}_{2m}$ and $b \in \mathbf{B}_{2n}$, triples (b, T_2, c) with $b \in \mathbf{B}_{2n}$ and $c \in \mathbf{B}_{2p}$, and triples $(a, T_1 T_2, c)$ with $a \in \mathbf{B}_{2m}$ and $c \in \mathbf{B}_{2p}$. The categories $\mathcal{T}_{2m; 2n}$, $\mathcal{T}_{2n; 2p}$, and $\mathcal{T}_{2m; 2p}$ are full subcategories of $\mathcal{U}_{2m; 2n; 2p}$, and there is also a unique multimorphism

$$(a_1, a_2), \dots, (a_{\alpha-1}, a_{\alpha}), (a_{\alpha}, T_1, b_1), (b_1, b_2), \dots, (b_{\beta-1}, b_{\beta}), (b_{\beta}, T_2, c_1), (c_1, c_2), \dots, (c_{\gamma-1}, c_{\gamma}) \\ \rightarrow (a_1, T_1 T_2, c_{\gamma}).$$

There is also a thickened version $\tilde{\mathcal{U}}_{2m; 2n; 2p}$ and a thickened product with the cube $\underline{2}^{N_1+N_2} \tilde{\times} \tilde{\mathcal{U}}_{2m; 2n; 2p}$. We then construct a functor $F: \underline{2}^{N_1+N_2} \tilde{\times} \tilde{\mathcal{U}}_{2m; 2n; 2p} \rightarrow \mathcal{B}$ extending Ψ_{T_1} , Ψ_{T_2} , and $\Psi_{T_1 T_2}$. The functor F induces a map of spectral bimodules

$$\mathcal{X}_{T_1} \otimes_{\mathcal{A}_{2n}}^{\mathbb{L}} \mathcal{X}_{T_1} \rightarrow \mathcal{X}_{T_1 T_2}$$

and the induced map of singular chain complexes agrees with the map of Khovanov complexes of bimodules (and so is an equivalence by Whitehead's theorem).

Now, let $\mathcal{U}_{m; n; p}^{\mathbf{k}}$ be the full subcategory of $\mathcal{U}_{m+k_1+k_2; n+k_1+k_2; p+k_1+k_2}$ spanned by objects as above but with $a_i \in \mathbf{B}_m^{\mathbf{k}}$, $b_i \in \mathbf{B}_n^{\mathbf{k}}$, and $c_i \in \mathbf{B}_p^{\mathbf{k}}$; define the thickened version $\tilde{\mathcal{U}}_{m; n; p}^{\mathbf{k}}$ similarly. Let \tilde{F} be the restriction of F to $\underline{2}^{N_1+N_2} \tilde{\times} \tilde{\mathcal{U}}_{m; n; p}^{\mathbf{k}}$. For an object o of $\underline{2}^{N_1+N_2} \tilde{\times} \tilde{\mathcal{U}}_{m; n; p}^{\mathbf{k}}$ define $\mathbf{K}(o) \subset F(o)$ to be

$J_{T_1}(o)$ if $o \in \text{Ob}(\widetilde{\mathcal{T}}_{m;n}^{\mathbf{k};\mathbf{k}})$, $J_{T_2}(o)$ if $o \in \text{Ob}(\widetilde{\mathcal{T}}_{n;p}^{\mathbf{k};\mathbf{k}})$, and $J_{T_1T_2}(o)$ if $o \in \text{Ob}(\widetilde{\mathcal{T}}_{m;p}^{\mathbf{k};\mathbf{k}})$. (It is immediate from the definition that these definitions of \mathbf{K} agree on the overlaps of these subcategories.)

The proof of Lemma 2.10 shows that \mathbf{K} is an absorbing subfunctor.

As in Section 3.4 we can realize the quotient functor \widetilde{F}/\mathbf{K} to obtain a functor $|\widetilde{F}/\mathbf{K}|: \mathcal{U}_{m;n;p}^{\mathbf{k}} \rightarrow \mathbb{Z}^{\mathcal{S}}$. Similarly to Section 3.3, we can reinterpret $|\widetilde{F}/\mathbf{K}|$ as a map of spectral bimodules

$$(4.1) \quad \mathcal{X}_{T_1}^{\mathbf{k};\mathbf{k}} \otimes_{\mathcal{A}_n^{\mathbf{k}}}^{\mathbb{L}} \mathcal{X}_{T_2}^{\mathbf{k};\mathbf{k}} \rightarrow \mathcal{X}_{T_1T_2}^{\mathbf{k};\mathbf{k}}.$$

(This involves a little fiddling as in Definition 4.5.) As in the arc algebra case [LLSb, Lemma 5.6], taking singular chains this is the gluing map

$$\mathcal{C}_{T_1}^{\mathbf{k};\mathbf{k}} \otimes_{\mathcal{A}_n^{\mathbf{k}}}^{\mathbb{L}} \mathcal{C}_{T_2}^{\mathbf{k};\mathbf{k}} \rightarrow \mathcal{C}_{T_1T_2}^{\mathbf{k};\mathbf{k}}$$

from Theorem 2. By Whitehead's theorem, the map (4.1) is a weak equivalence, as desired. \square

5. TOPOLOGICAL HOCHSCHILD HOMOLOGY

Let T be an (n, n) -tangle. We can form the topological Hochschild homology of $\mathcal{A}_n^{\mathcal{P}}$ with coefficients in $\mathcal{X}_T^{\mathcal{P}}$, which we write $\text{THH}(\mathcal{A}_n^{\mathcal{P}}; \mathcal{X}_T^{\mathcal{P}})$ or $\text{THH}(\mathcal{X}_T^{\mathcal{P}})$. The spectral categories $\mathcal{A}_n^{\mathcal{P}}$ are pointwise cofibrant (see [LLSb, Lemma 4.5]), so the topological Hochschild homology can be obtained as the homotopy colimit of the diagram

$$\begin{aligned} \cdots & \rightrightarrows \coprod_{a_1, a_2, a_3 \in \text{Ob}(\mathcal{A}_n^{\mathcal{P}})} \mathcal{X}_T^{\mathcal{P}}(a_3, a_1) \wedge \mathcal{A}_n^{\mathcal{P}}(a_1, a_2) \wedge \mathcal{A}_n^{\mathcal{P}}(a_2, a_3) \\ & \rightrightarrows \coprod_{a_1, a_2 \in \text{Ob}(\mathcal{A}_n^{\mathcal{P}})} \mathcal{X}_T^{\mathcal{P}}(a_2, a_1) \wedge \mathcal{A}_n^{\mathcal{P}}(a_1, a_2) \rightrightarrows \coprod_{a_1 \in \text{Ob}(\mathcal{C})} \mathcal{X}_T^{\mathcal{P}}(a_1, a_1) \end{aligned}$$

where the horizontal maps are given by the compositions in $\mathcal{A}_n^{\mathcal{P}}$ and its actions on $\mathcal{X}_T^{\mathcal{P}}$ [BM12, Proposition 3.5].

Proposition 5.1. *There is an isomorphism*

$$H_* \text{THH}(\mathcal{X}_T^{\mathcal{P}}) \cong \text{HH}_*(\mathcal{C}_T^{\mathcal{P}})$$

Proof. The proof is the same as the analogous result for the spectral Khovanov bimodules [LLSb, Proposition 7.5]. \square

Recall that the Hochschild homology of the Chen-Khovanov bimodules has another interpretation. Given an (n, n) -tangle T , we can form the closure $[T]$ of T in the annulus $S^1 \times [0, 1]$. Asaeda-Przytycki-Sikora constructed a Khovanov homology for links in thickened surfaces [APS04] which, in particular, gives an invariant $\text{AKh}([T])$, the *annular Khovanov homology* of $[T]$. (This case was further studied by Roberts [Rob13], Grigsby-Wehrli [GW10], and others.) Specifically, there is a filtration on the Khovanov complex of the closure of T in \mathbb{R}^3 , coming from using the labels of circles by 1 or X to orient them and then considering the winding number around the axis. The invariant $\text{AKh}([T]; \ell)$ is the homology of the associated graded complex to this filtration, in winding number grading ℓ . As such, $\text{AKh}([T])$ is tri-graded, by the homological, quantum, and winding number gradings.

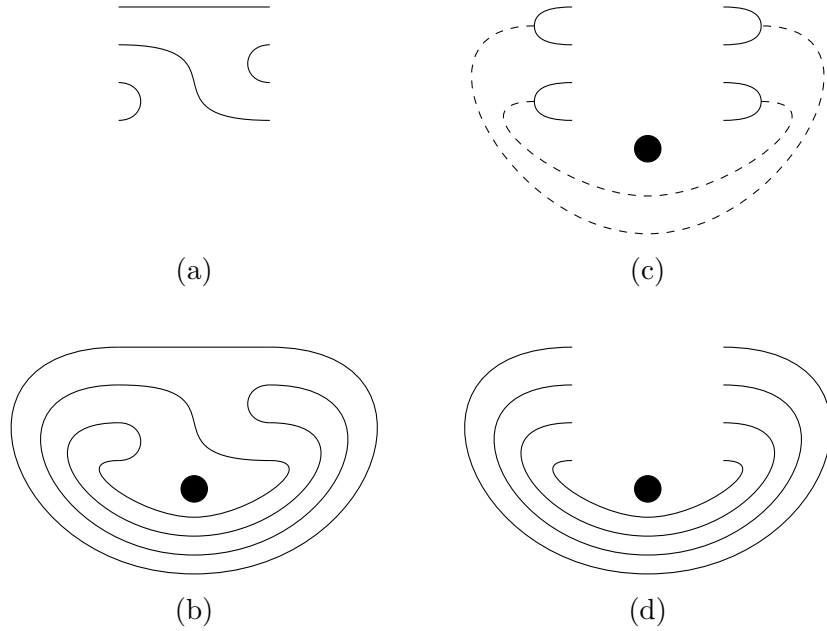


FIGURE 5.1. **The annular closure.** (a) A (flat) tangle T . (b) The annular closure $[T]$. (c) $\widehat{a} \amalg a$ for a a particular crossingless matching, together with the cores of the 1-handles (dashed) in the saddle cobordism from $\widehat{a} \amalg a$ to the identity braid. (d) The identity braid, inside the annulus.

Convention 5.2. Fix k , and let $\mathbb{k} = (n - k, k)$.

Beliakova-Putyra-Wehrli relate the annular Khovanov homology to the Chen-Khovanov invariants:

Theorem 6. [BPW, Theorem C] *There is an isomorphism*

$$(5.1) \quad \mathrm{HH}_*(\mathcal{A}_n^{\mathbb{k}}; \mathcal{C}_T^{\mathbb{k}; \mathbb{k}}) \cong \mathrm{AKh}([T]; n - 2k)\{n - 2k\}.$$

(A special case was proved earlier by Auroux-Grigsby-Wehrli in [AGW15].)

In order to prove a spectral refinement of Theorem 6, we need the explicit map

$$\mathrm{HC}_*(\mathcal{A}_n^{\mathbb{k}}; \mathcal{C}_T^{\mathbb{k}; \mathbb{k}}) \rightarrow \mathcal{C}_{\mathrm{AKh}}([T]; n - 2k)\{n - 2k\}$$

inducing the isomorphism (5.1). While Beliakova-Putyra-Wehrli's proof does not explicitly give the map, their ideas easily extend to do so. We emphasize that we do not give an independent proof of Theorem 6: for instance, the proof of Lemma 5.9 relies on Theorem 6; we merely construct an explicit isomorphism.

Let T be an (n, n) -tangle, and $[T] \subset \mathbb{R}^2 \setminus \{(0, 0)\}$ the annular closure of T . (See Figure 5.1.) We can view $[T]$ as lying in \mathbb{R}^2 and so, in particular, can consider $\mathcal{C}_{\mathrm{Kh}}([T])$. Given $a \in \mathbb{B}_n^{\mathbb{k}}$, define the map

$$A = A_{a, T}: \mathcal{C}_{\overline{T}}(a, a) \rightarrow \mathcal{C}_{\mathrm{Kh}}(\overline{[T]})$$

to be the map associated to the saddle cobordism from $\widehat{a} \amalg a$ to the identity braid around the annulus. (Again, see Figure 5.1.)

Proposition 5.3. *The maps $A_{a,T}$ satisfy the following properties:*

- (1) *Each $A_{a,T}$ is a chain map.*
- (2) *The image of $A_{a,T}$ lies in annular filtration ≤ 0 .*
- (3) *Given $a, b \in \mathbb{B}_n^k$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C}_{Kh}(a\widehat{\overline{T}}b \amalg b\widehat{a})\{2n\} & \xrightarrow{s_b} & \mathcal{C}_{Kh}(a\widehat{\overline{T}}\widehat{a})\{n\} \\ \downarrow s_a & & \downarrow A_{a,T} \\ \mathcal{C}_{Kh}(b\widehat{\overline{T}}b)\{n\} & \xrightarrow{A_{b,T}} & \mathcal{C}_{Kh}(\overline{[T]}). \end{array}$$

Here, the arrows labeled s_a and s_b are induced by the saddle cobordisms from $\widehat{a} \amalg a$ to the identity braid and $\widehat{b} \amalg b$ to the identity braid.

- (4) *Given (n, n) -tangles T_1 and T_2 and $a, b \in \mathbb{B}_n^k$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C}_{Kh}(a\widehat{\overline{T}_1}b \amalg b\widehat{\overline{T}_2}\widehat{a})\{2n\} & \xrightarrow{s_b} & \mathcal{C}_{Kh}(a\widehat{\overline{T}_1\overline{T}_2}\widehat{a})\{n\} \\ \downarrow s_a & & \downarrow A_{a,T} \\ \mathcal{C}_{Kh}(b\widehat{\overline{T}_2\overline{T}_1}b)\{n\} & \xrightarrow{A_{b,T}} & \mathcal{C}_{Kh}(\overline{[T_1T_2]} = \overline{[T_2T_1]}). \end{array}$$

Again, the arrows labeled s_a and s_b are induced by the saddle cobordisms from $\widehat{a} \amalg a$ to the identity braid and $\widehat{b} \amalg b$ to the identity braid.

Proof. Point (1) follows from far-commutativity of the cobordism maps. Point (2) follows from the facts that the cobordism maps respect the annular filtration and $a\widehat{\overline{T}}\widehat{a}$ has winding number 0. Point (3) is a special case of Point (4). Point (4) again follows from far-commutativity of the cobordism maps. \square

Let $\mathcal{F}_{\leq 0}\mathcal{C}_{Kh}(\overline{[T]})$ be the subcomplex of $\mathcal{C}_{Kh}(\overline{[T]})$ in filtration ≤ 0 . By Point (2) of Proposition 5.3, the image of $A_{a,T}$ is contained in $\mathcal{F}_{\leq 0}\mathcal{C}_{Kh}(\overline{[T]})$. Let

$$B: \mathcal{F}_{\leq 0}\mathcal{C}_{Kh}(\overline{[T]}) \rightarrow \mathcal{F}_{\leq 0}\mathcal{C}_{Kh}(\overline{[T]})/\mathcal{F}_{< 0}\mathcal{C}_{Kh}(\overline{[T]})$$

be projection to the associated graded complex.

Let P_L be the image of the lower-left platform for $\overline{[T]}$ in the annulus $\mathbb{R}^2 \setminus \{(0, 0)\}$ and let P_U be the image of the upper-left platform. In $\overline{[T]}$, there are several kinds of circles:

- (A-i) Circles which are disjoint from P_L and P_U .
- (A-ii) Circles which pass through P_L once and are disjoint from P_U . Call these *lower horizontal circles*.
- (A-iii) Circles which pass through P_U once and are disjoint from P_L . Call these *upper horizontal circles*.

Observe that the complex $\mathcal{C}_{Kh}(\overline{[T]})$ decomposes as a direct sum corresponding to the different ways of labeling the upper and lower horizontal circles. Define

$$C: \mathcal{F}_{\leq 0} \mathcal{C}_{Kh}(\overline{[T]}) / \mathcal{F}_{< 0} \mathcal{C}_{Kh}(\overline{[T]}) \rightarrow \mathcal{C}_{AKh}([T]; n - 2k) \{n - 2k\}$$

to be the result of projecting to the summand where each of the k upper horizontal circles is labeled 1 and each of the $(n - k)$ lower horizontal circles is labeled X , and then forgetting the lower and upper horizontal circles. It is clear that the image of this map lies in the summand with winding number grading $n - 2k$.

Lemma 5.4. *The composition $C \circ B \circ A$ vanishes on*

$$\mathcal{J}_T^{k;k}(a, a) \subset \mathcal{C}_{\overline{[T]}}(a, a).$$

Proof. Unsurprisingly, the proof is a case analysis. Fix a generator $y \in \mathcal{J}_T^{k;k}(a, a)$. Suppose first that $(a\overline{T}\widehat{a}, y)$ has a type II circle Z labeled X . If the circle passes through the upper platforms then $A_{a,T}(y)$ will have a upper horizontal circle labeled X , so $C(B(A_{a,T}(y))) = 0$. If the circle Z passes through the lower platforms, notice that at some point in the saddles corresponding to $A_{a,T}$, Z either splits into two essential circles labeled X or merges with an essential circle labeled 1 to form an essential circle labeled X . In either case, the annular filtration strictly decreases, so $B(A_{a,T}(y)) = 0$.

Next, suppose $(a\overline{T}\widehat{a}, y)$ has a type III circle Z passing through the upper platforms. If Z is labeled X then $A_{a,T}(y)$ will have an upper horizontal circle labeled X , so $C(B(A_{a,T}(y))) = 0$. More generally, let P and Q be two points on the intersection of Z and one of the upper platforms. During the saddles in $A_{a,T}$, if a circle containing P or Q is ever labeled X then we have $C(B(A_{a,T}(y))) = 0$. However, since P and Q end up on separate circles, at some point the saddle cobordism must be a split with P and Q ending on opposite components. Since the split map sends 1 to $1 \otimes X + X \otimes 1$, one of these components will be labeled X .

Finally, suppose $(a\overline{T}\widehat{a}, y)$ has a type III circle Z passing through the lower platforms. If Z is labeled X , the same analysis as in the type II circle case implies $C(B(A_{a,T}(y))) = 0$. So, suppose Z is labeled 1. Let P and Q be two points at the intersection of Z and a lower platform. Eventually, both P and Q must lie on circles labeled X , or else $C(B(A_{a,T}(y))) = 0$. If P (or Q) is ever on an inessential circle labeled X then P cannot later be on an essential circle labeled X without decreasing the annular filtration. Similarly, if P (or Q) is ever on an essential circle labeled 1 then P can never later be on an essential circle labeled X . If P and Q are on the same essential circle labeled X , then (using the previous two observations) there is no way for P and Q to end up on different essential circles labeled X . But now we have ruled out all possibilities: P and Q start on the same inessential circle labeled 1, and the only changes that can happen are for them to next be on the same inessential circle labeled X , the same essential circle labeled 1, the same essential circle labeled X , or different essential circles one of which is labeled 1. \square

Definition 5.5. By Lemma 5.4, $C \circ B \circ A$ descends to a map

$$(5.2) \quad \Xi_0: \bigoplus_{a \in \mathbb{B}_n^k} \mathcal{C}_T^{k;k}(a, a) \rightarrow \mathcal{C}_{AKh}([T]; n - 2k) \{n - 2k\}$$

which we call the *annular gluing map*.

If $\mathrm{HC}_*(\mathcal{A}_n^k; \mathcal{C}_T^k; \mathcal{C}_T^k)$ denotes the standard Hochschild complex, which is the total complex of the bicomplex

$$\begin{aligned} \cdots &\rightarrow \bigoplus_{a_1, a_2, a_3 \in \mathbb{B}_n^k} \mathcal{C}_T^k; \mathcal{C}_T^k(a_1, a_2) \otimes_{\mathbb{Z}} \mathcal{A}_n^k(a_2, a_3) \otimes_{\mathbb{Z}} \mathcal{A}_n^k(a_3, a_1) \\ &\rightarrow \bigoplus_{a_1, a_2 \in \mathbb{B}_n^k} \mathcal{C}_T^k; \mathcal{C}_T^k(a_1, a_2) \otimes_{\mathbb{Z}} \mathcal{A}_n^k(a_2, a_1) \rightarrow \bigoplus_{a_1 \in \mathbb{B}_n^k} \mathcal{C}_T^k; \mathcal{C}_T^k(a_1, a_1), \end{aligned}$$

then there is an induced map

$$\Xi: \mathrm{HC}_*(\mathcal{A}_n^k; \mathcal{C}_T^k; \mathcal{C}_T^k) \rightarrow \mathcal{C}_{AKh}([T]; n-2k)\{n-2k\}$$

defined by projecting to $\mathcal{C}_T^k; \mathcal{C}_T^k$ and then applying the map Ξ_0 from Equation (5.2).

Lemma 5.6. *The map Ξ is a chain map.*

Proof. This follows from the fact that Ξ_0 is a chain map (since A is a chain map by Item (1) of Proposition 5.3 and it is immediate from their definitions that B and C are chain maps), and Item (3) of Proposition 5.3, which implies that A vanishes on the image of $\bigoplus_{a_1, a_2 \in \mathbb{B}_n^k} \mathcal{C}_T^k; \mathcal{C}_T^k(a_1, a_2) \otimes_{\mathbb{Z}} \mathcal{A}_n^k(a_2, a_1)$. \square

Lemma 5.7. *Let T_1 be an (m, n) -tangle and T_2 and (n, m) -tangle. Then for any h, k with $m-n = 2(h-k)$, the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{HC}_*(\mathcal{A}_m^{m-h, h}; \mathcal{C}_{T_1 T_2}^{m-h, h; m-h, h}) & \xrightarrow{\cong} & \mathrm{HC}_*(\mathcal{A}_n^{n-k, k}; \mathcal{C}_{T_2 T_1}^{n-k, k; n-k, k}) \\ \Xi \downarrow & & \downarrow \Xi \\ \mathcal{C}_{AKh}([T_1 T_2]; m-2h)\{m-2h\} & \xrightarrow{\cong} & \mathcal{C}_{AKh}([T_2 T_1]; n-2k)\{n-2k\}. \end{array}$$

Here, the top horizontal map is induced by Theorem 2 and cyclic symmetry of Hochschild homology and the bottom by the fact that the closures of $T_1 T_2$ and $T_2 T_1$ are isotopic links (in fact, link diagrams) in the annulus. (Note that $m-2h = n-2k$.)

Proof. This follows from the definitions and Item (4) of Proposition 5.3. \square

Lemma 5.8. *Let T be an (n, n) -tangle diagram which is the union of an (n, n) -tangle T' and an unknotted circle U disjoint from T' . Then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{HC}_*(\mathcal{A}_n^k; \mathcal{C}_T^k; \mathcal{C}_T^k) & \xrightarrow{\cong} & \mathrm{HC}_*(\mathcal{A}_n^k; \mathcal{C}_{T'}^k; \mathcal{C}_{T'}^k) \otimes V \\ \Xi \downarrow & & \downarrow \Xi \otimes \mathrm{Id} \\ \mathcal{C}_{AKh}([T])\{n-2k\} & \xrightarrow{\cong} & \mathcal{C}_{AKh}([T'])\{n-2k\} \otimes V. \end{array}$$

Here, the top horizontal map is induced by the obvious isomorphism $\mathcal{C}_T^k; \mathcal{C}_T^k \cong \mathcal{C}_{T'}^k; \mathcal{C}_{T'}^k \otimes V$ and the bottom horizontal map is also the obvious isomorphism.

Proof. This is immediate from the definitions. \square

Lemma 5.9. *Suppose T is the (n, n) -tangle diagram consisting of n horizontal strands. Then the map $\Xi_*: \mathrm{HH}_*(\mathcal{A}_n^k; \mathcal{C}_T^{k;k}) \rightarrow \mathrm{AKh}([T]; n-2k)\{n-2k\}$ is an isomorphism.*

Proof. Let \mathcal{S} be the set of all subsets of $\{1, 2, \dots, n\}$ of size k . Define a partial order \preceq on \mathcal{S} by declaring $S = \{s_1 < s_2 < \dots < s_k\} \preceq T = \{t_1 < t_2 < \dots < t_k\}$ if $s_i \leq t_i$ for all $1 \leq i \leq k$.

Since T is a flat tangle, $\mathrm{AKh}([T]; n-2k) = \mathcal{C}_{\mathrm{AKh}}([T]; n-2k)$. Moreover, in order to have winding number grading $n-2k$, exactly k of the n circles in $[T]$ have to be labeled X . Therefore, after numbering the strands of T by $1, 2, \dots, n$ from bottom to top, $\mathrm{AKh}([T]; n-2k)$ can be identified with the free \mathbb{Z} -module generated by \mathcal{S} : the generator corresponding to $S \in \mathcal{S}$ labels the circles in S by X , and the remaining circles by 1 . We will view $\mathrm{AKh}([T]; n-2k)$ as a filtered group, with the filtration given by the partial order \preceq on \mathcal{S} .

Since Beliakova-Putyra-Wehrli have already established that $\mathrm{HH}_*(\mathcal{A}_n^k; \mathcal{C}_T^{k;k}) \cong \mathrm{AKh}([T]; n-2k)\{n-2k\}$, it is enough to show that the map $\Xi_*: \mathrm{HH}_*(\mathcal{A}_n^k; \mathcal{C}_T^{k;k}) \rightarrow \mathrm{AKh}([T]; n-2k)\{n-2k\}$ is surjective. Since T is a flat tangle, the chain complex $\bigoplus_{a \in \mathcal{B}_n^k} \mathcal{C}_T^{k;k}(a, a)$ has no differential, so it is enough to show that the annular gluing map Ξ_0 from Definition 5.5 is surjective. Given $a \in \mathcal{B}_n^k$, let y_a be the generator of $\mathcal{C}_T^{k;k}(a, a)$ where each circle of $a\overline{T}\widehat{a}$ is labeled 1 . Let

$$M = \langle \{y_a \mid a \in \mathcal{B}_n^k\} \rangle \subset \bigoplus_{a \in \mathcal{B}_n^k} \mathcal{C}_T^{k;k}(a, a).$$

We will show that $\Xi_0|_M$ is an isomorphism (and therefore, Ξ_0 is surjective).

Recall that \mathcal{B}_n^k is in canonical bijection with \mathcal{S} [CK14, §6], as follows: for any $a \in \mathcal{B}_n^k$, the corresponding element $S_a \in \mathcal{S}$ is the subset of the n non-platform points (numbered $1, 2, \dots, n$ from bottom to top) which are matched to a higher point by a . Therefore, \preceq induces a filtration \preceq on M by $y_a \preceq y_b$ if and only if $S_a \preceq S_b$.

Therefore, the generators of $M = \mathbb{Z}\langle \{y_a\} \rangle$ and $\mathrm{AKh}([T]; n-2k) = \mathbb{Z}\langle \{S_a\} \rangle$ are in bijection with each other, via $y_a \leftrightarrow S_a$. We will prove $\Xi_0|_M$ is a filtered map, and the associated graded piece of the map sends each generator of M to the corresponding generator of $\mathrm{AKh}([T]; n-2k)$; it follows that $\Xi_0|_M$ is an isomorphism.

Consider a generator y_a of M . Recall that Ξ_0 is a composition of three maps, $C \circ B \circ A$. The map A is a composition of n splits, each splitting a non-essential circle labeled 1 . Therefore, A preserves the winding number grading, and so we do not need the map B . The map C projects onto the summand where each circle passing through the lower (respectively upper) platform is labeled X (respectively 1).

Circles in $a\overline{T}\widehat{a}$ are of the following four types.

- Circles Z that pass through both the upper and the lower platform. Under the map Ξ_0 , Z splits into an upper horizontal circle labeled 1 and a lower horizontal circle labeled X , which are then forgotten.
- Circles Z that pass through only the upper platform. Assume Z contains the i^{th} strand of T , and consequently, $i \in S_a$. Under the map Ξ_0 , Z splits into an upper horizontal circle labeled 1 (which is then forgotten) and the i^{th} component of $[T]$ labeled X .

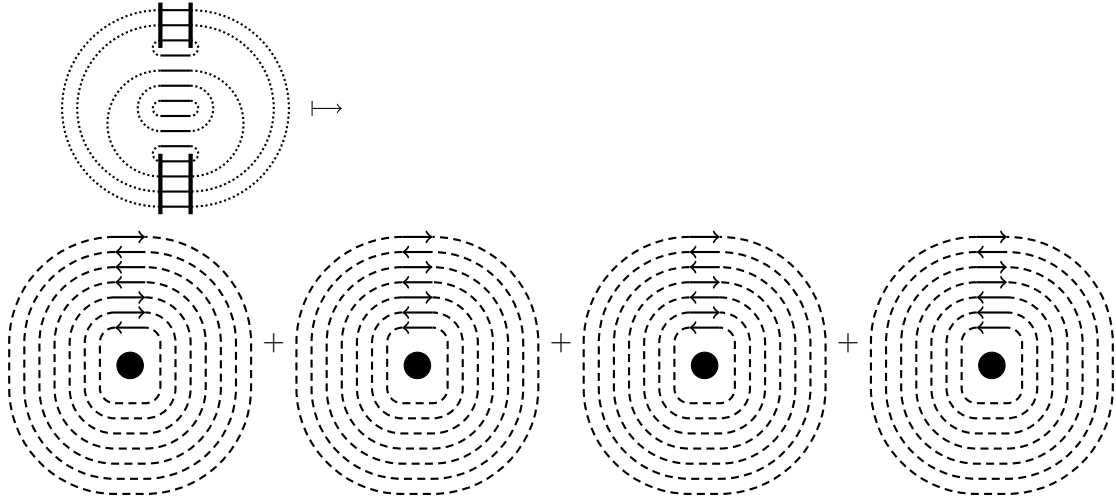


FIGURE 5.2. **An example of the map $\Xi_0|_M$.** Here, $n = 7, k = 3$ and we are starting with generator y_a corresponding to the subset $S_a = \{2, 3, 7\}$. The map sends y_a to a sum of four generators, corresponding to the subsets $\{2, 3, 7\}$, $\{3, 5, 7\}$, $\{2, 4, 7\}$, and $\{4, 5, 7\}$. (Circles labeled 1 are oriented counter-clockwise and circles labeled X are oriented clockwise. The arcs of \overline{T} are solid, the platforms are thick, the arcs in a and \hat{a} are dotted, and the new arcs in the annular closure $[T]$ are dashed.)

- Circles Z that pass through only the lower platform. Assume Z contains the i^{th} strand of T , and consequently, $i \in \{1, 2, \dots, n\} \setminus S_a$. Under the map Ξ_0 , Z splits into a lower horizontal circle labeled X (which is then forgotten) and the i^{th} component of $[T]$ labeled 1.
- Circles Z that are disjoint from the platforms. Assume Z contains the i^{th} and j^{th} strand of T , with $i < j$, and consequently, $i \in S_a$ and $j \in \{1, 2, \dots, n\} \setminus S_a$. Under the map Ξ_0 , Z splits into the i^{th} and j^{th} component of $[T]$, one labeled 1 and the other labeled X .

In the first three cases, the map Ξ_0 sends the generator y_a to the corresponding generator S_a . In the last case, Ξ_0 sends y_a to a sum of two generators—one corresponding to the same subset S_a and one corresponding to $T = S_a \cup \{j\} \setminus \{i\}$ —and we have $S_a \prec T$. Therefore, the map increases or preserves the filtration given by \preceq , and the associated graded piece of the map sends each generator to the corresponding generator. See Figure 5.2 for an example of this map. \square

Theorem 7. *The map Ξ induces the isomorphism from Theorem 6.*

Proof. First, assume that T is a flat (n, n) -tangle. We prove the result by induction on n . By Lemma 5.8, we can assume that T has no closed components. So, if $n = 0$, T is empty and the result is trivial. Next, for general n , if T is the identity braid, the result is Lemma 5.9. Otherwise, we can decompose T as $T_1 T_2$ where T_1 is a flat (n, m) -tangle, T_2 is a flat (m, n) -tangle, and $m < n$. By Lemma 5.7, Ξ is an isomorphism for $T_1 T_2$ if and only if Ξ is an isomorphism for $T_2 T_1$ which is true by induction.

Next, for a general tangle T , note that each generator of $\mathrm{HC}_*(\mathcal{A}_n^{\mathbb{k}}; \mathcal{C}_T^{\mathbb{k}; \mathbb{k}})$ lies over some vertex v of the cube. Consider the filtrations on $\mathrm{HC}_*(\mathcal{A}_n^{\mathbb{k}}; \mathcal{C}_T^{\mathbb{k}; \mathbb{k}})$ and $\mathcal{C}_{AKh}([T]; n - 2k)$ by $|v|$, the grading on the cube. The map Ξ respects this filtration and, by the previous case, induces an isomorphism at the E_1 -page of the associated spectral sequence. Thus, Ξ is a quasi-isomorphism, as desired. \square

We give a spectral refinement of this result. Before stating the main theorem, we observe:

Lemma 5.10. *The topological Hochschild homology of $\mathcal{X}_T^{\mathbb{k}; \mathbb{k}}$ is an invariant of the annular closure of T .*

Proof. It suffices to verify that $\mathrm{THH}(\mathcal{X}_T^{\mathbb{k}; \mathbb{k}})$ is invariant under Reidemeister moves and cyclic rotation of T . Invariance under Reidemeister moves is Theorem 4. The fact that

$$\mathrm{THH}(\mathcal{X}_{T_1 T_2}^{\mathbb{k}; \mathbb{k}}) \simeq \mathrm{THH}(\mathcal{X}_{T_2 T_1}^{\mathbb{k}; \mathbb{k}})$$

follows from Theorem 5 and the fact that topological Hochschild homology is a trace. \square

Given a link $L \subset S^1 \times \mathbb{D}^2$, the winding number filtration on $\mathcal{C}_{Kh}(L \subset \mathbb{R}^3)$ induces a filtration on the Khovanov spectrum \mathcal{X}_L . The associated graded spectrum $\mathcal{A}\mathcal{X}_L^\ell$ in winding number grading ℓ is a spectral refinement of annular Khovanov homology $AKh(L; \ell)$; verifying that the homotopy type of this associated graded spectrum is an invariant of the annular link is straightforward. (See also [SZ].)

Theorem 8. *There is a weak equivalence*

$$\mathrm{THH}(\mathcal{X}_T^{\mathbb{k}; \mathbb{k}}) \simeq \mathcal{A}\mathcal{X}_{[T]}^{n-2k} \{n - 2k\}$$

of bigraded spectra.

The last ingredient in the proof of Theorem 8 is a mild extension of the divided cobordism category Cob_d from our previous paper [LLSb, §3.1] to the annulus, and the Khovanov-Burnside functor to this divided cobordism category in the presence of platforms. To have strict identities and make composition strictly associative while not destroying interesting topology, we will quotient by a particular class of diffeomorphisms:

Definition 5.11. Let $\mathrm{Diff}'(S^1)$ denote the group of orientation-preserving diffeomorphisms $\phi: S^1 \rightarrow S^1$ so that there is some $\epsilon = \epsilon(\phi) > 0$ with $\phi|_{B_\epsilon(1)} = \mathrm{Id}$. (Here, $1 \in S^1 \subset \mathbb{C}$ and $B_\epsilon(1)$ is an interval around 1.)

Let $\mathrm{Diff}'([0, 1] \times S^1)$ denote the group of orientation-preserving diffeomorphisms $\phi: [0, 1] \times S^1 \rightarrow [0, 1] \times S^1$ so that there is some $\epsilon = \epsilon(\phi) > 0$ and some $\psi_0, \psi_1 \in \mathrm{Diff}'(S^1)$ so that $\phi|_{[0, 1] \times B_\epsilon(1)} = \mathrm{Id}$, and $\phi(p, q) = (p, \psi_0(q))$ for all $p \in [0, \epsilon)$, and $\phi(p, q) = (p, \psi_1(q))$ for all $p \in (1 - \epsilon, 1]$. (That is, ϕ is the identity near 1 and is invariant in the $[0, 1]$ -direction near the boundary.)

(Compare [LLSb, Definitions 2.46 and 2.47].)

Definition 5.12. Let $\odot = S^1 \times (-1, 1)$ denote the annulus. The *divided cobordism category of the annulus*, $\mathrm{Cob}_d(\odot)$, is defined as follows:

- An object of $\mathrm{Cob}_d(\odot)$ consists of:
 - A smooth, closed 1-manifold Z embedded in \odot .

– A collection of disjoint, closed arcs $A \subset Z$ such that $I = Z \setminus A$ is also a union of disjoint arcs. We call components of A *active arcs* and components of I *inactive arcs*.

We declare two objects (Z, A) and (Z', A') to be equivalent if there is a $\phi \in \text{Diff}'(S^1)$ so that $(\phi \times \text{Id}_{(-1,1)})(Z, A) = (Z', A')$.

- A morphism from (Z, A) to (Z', A') is an equivalence class of pairs (Σ, Γ) where
 - Σ is a smoothly embedded cobordism in $[0, 1] \times \odot$ from Z to Z' which is vertical (invariant in the $[0, 1]$ -direction) near $\{0, 1\} \times \odot$ and $[0, 1] \times \{1\} \times (-1, 1)$.
 - $\Gamma \subset \Sigma$ is a collection of properly embedded arcs in Σ , also vertical near $\{0, 1\} \times \odot$, with $(\partial A \cup \partial A') = \partial \Gamma$, and so that every component of $\Sigma \setminus \Gamma$ has one of the following forms:
 - (I) A rectangle, with two sides components of Γ and two sides components of $A \cup A'$.
 - (II) A $(2n + 2)$ -gon, with $(n + 1)$ sides components of Γ , one side an arc component of I' , and the other n sides arc components of I . (The integer n is allowed to be zero.)

We call the components of Γ *divides*.

The pairs (Σ, Γ) and (Σ', Γ') are equivalent if there is a $\phi \in \text{Diff}'([0, 1] \times S^1)$ with $(\phi \times \text{Id}_{(-1,1)})(\Sigma) = \Sigma'$, and $(\phi \times \text{Id}_{(-1,1)})(\Gamma) = \Gamma'$.

- There is a unique 2-morphism from (Σ, Γ) to (Σ', Γ') whenever (some representative of the equivalence class of) (Σ, Γ) is isotopic to (some representative of the equivalence class of) (Σ', Γ') rel boundary and $[0, 1] \times \{1\} \times (-1, 1)$.
- Composition of divided cobordisms is defined as follows. Given $(\Sigma, \Gamma): (Z, A) \rightarrow (Z', A')$ and $(\Sigma', \Gamma'): (Z', A') \rightarrow (Z'', A'')$, choose a representative of the equivalence class of (Z', A') and representatives of the equivalence classes (Σ, Γ) and (Σ', Γ') which end / start at this representative of (Z', A') . Define $(\Sigma', \Gamma') \circ (\Sigma, \Gamma)$ to be $(\Sigma' \circ \Sigma, \tilde{\Gamma}' \circ \Gamma)$. The same proof from [LLSb] shows that composition is well-defined.

Recall that a multicategory \mathcal{C} has a canonical groupoid enrichment [LLSb, §2.4.1]. In the case that \mathcal{C} is an ordinary category—the case of interest in this section—the canonical groupoid enrichment \mathcal{C}' has the same objects as \mathcal{C} , 1-morphisms $\mathcal{C}'(x, y)$ the set of finite sequences of morphisms $x \xrightarrow{f_1} z_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} z_k \xrightarrow{f_k} y$, and a unique 2-morphism $(f_1, \dots, f_k) \rightarrow (g_1, \dots, g_\ell)$ whenever $f_k \circ \cdots \circ f_1 = g_\ell \circ \cdots \circ g_1$. The relevance of this enrichment is that there is a (strict) 2-functor $V_{HKK}: \text{Cob}'_d \rightarrow \mathcal{B}$, the Burnside category [LLSb, §3.4]. The definition of V_{HKK} extends immediately to a functor $V_{HKK}: \text{Cob}_d(\odot)' \rightarrow \mathcal{B}$.

Recall that to define a functor $\tilde{\mathcal{T}}_{m;n} \rightarrow \text{Cob}_d$ induced by a tangle T , we introduce some extra decorations on T :

Definition 5.13. [LLSb, §3.3] A *poxed tangle* is a tangle diagram T together with a collection of marked points (*pox*) on the arcs in T so that for every resolution T_v of T , there is at least one pox on each closed circle of T_v .

Call a poxed tangle T *sufficiently poxed* if for every resolution T_v of T there is at least one pox on each closed circle of the annular closure $[T_v]$.

Given a poxed (n, n) -tangle T , there is an induced functor $\underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{2n;2n} \rightarrow \text{Cob}_d$ which restricts to a functor $\underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{n;n}^{\text{lk};\text{lk}} \rightarrow \text{Cob}_d$ [LLSb, §3.4]. Further, the composition $\underline{2}^N \tilde{\times} \tilde{\mathcal{T}}_{2n;2n} \rightarrow \text{Cob}_d \rightarrow \mathcal{B}$ is independent of the choice of pox. If T is sufficiently poxed, there is also an induced functor $\underline{2}^N \rightarrow$

$\text{Cob}_d(\odot)$ coming from the annular closure $[T]$ of T . Again, the composition $\underline{2}^N \rightarrow \text{Cob}_d(\odot) \rightarrow \mathcal{B}$ is independent of the choice of pox .

Proof of Theorem 8. It suffices to show that there is a map of spectra

$$\text{THH}(\mathcal{X}_T^{\text{lk};\text{lk}}) \rightarrow \mathcal{A}\mathcal{X}_{[T]}^{n-2k}\{n-2k\}$$

so that the induced map on Hochschild complexes agrees with the map from Definition 5.5.

Fix a collection of pox on T making T sufficiently poxed . When talking about \overline{T} , below, add a pox in the middle of each horizontal strand added to T .

Let Δ_{inj} be the subcategory of the simplex category generated by the face maps. That is, Δ_{inj} has objects the positive integers and $\text{Hom}(p, q)$ the order-preserving injections $\{0, \dots, p-1\} \hookrightarrow \{0, \dots, q-1\}$. Then the topological Hochschild homology $\text{THH}(\mathcal{A}_n^{\text{lk}}; \mathcal{X}_T^{\text{lk};\text{lk}})$ is the homotopy colimit of a diagram $\Delta_{inj}^{\text{op}} \rightarrow \mathbb{Z}^{\mathcal{S}}$.

We will reformulate this homotopy colimit over a larger diagram, but first we need some more notation from the guts of the construction. Recall that $\underline{2}_+^N$ is the result of adding one object $*$ to $\underline{2}^N$ and a morphism $v \rightarrow *$ for each vertex v except $\bar{1}$. Given a functor $G: \underline{2}^N \rightarrow \mathbb{Z}^{\mathcal{S}}$, we can extend G to a functor $G_+: \underline{2}_+^N \rightarrow \mathbb{Z}^{\mathcal{S}}$ by declaring that $G(*) = \{\text{pt}\}$. Then $\text{hocolim } G_+$ is the iterated mapping cone of G . Consider the functor

$$\overline{(K \circ \Psi_T^{\text{lk};\text{lk}})}_+: \underline{2}_+^N \tilde{\times} \mathcal{T}_{n;n}^{\text{lk};\text{lk}} \rightarrow \mathbb{Z}^{\mathcal{S}}$$

where K is Elmendorf-Mandell's K -theory, $\Psi_T^{\text{lk};\text{lk}}$ is as in Section 4, and the bar denotes applying Elmendorf-Mandell's rectification construction, to obtain an honest functor. To shorten notation, let

$$\overline{\mathcal{X}}_{T_v}^{\text{lk};\text{lk}}(a_1, a_2) = \overline{(K \circ \Psi_T^{\text{lk};\text{lk}})}_+(v, a_1, T, a_2).$$

(The spectrum $\overline{\mathcal{X}}_{T_v}^{\text{lk};\text{lk}}(a_1, a_2)$ is weakly equivalent to $\mathcal{X}_{T_v}^{\text{lk};\text{lk}}(a_1, a_2)\{-|v| - N_+ + 2N_-\}$; the difference arises based on when the rectification construction was applied.)

Now, let \mathcal{C} be the category with one object for every finite sequence $a_1, \dots, a_\alpha \in \mathbf{B}_n^{\text{lk}}$, $\alpha \geq 1$, and with a unique morphism $(a_1, \dots, a_\alpha) \rightarrow (a_1, \dots, \widehat{a}_i, \dots, a_\alpha)$ for each i , composing in the obvious way. Define a functor $F: \underline{2}_+^N \times \mathcal{C} \rightarrow \mathbb{Z}^{\mathcal{S}}$ (where $\underline{2}_+^N$ is viewed as a 1-category) by declaring that:

- For $v \in \text{Ob}(\underline{2}_+^N)$ and $a_1, \dots, a_\alpha \in \mathbf{B}_n^{\text{lk}}$,

$$F(v, a_1, \dots, a_\alpha) = \overline{\mathcal{X}}_{T_v}^{\text{lk};\text{lk}}(a_\alpha, a_1) \wedge \mathcal{A}_n^{\text{lk}}(a_1, a_2) \wedge \dots \wedge \mathcal{A}_n^{\text{lk}}(a_{\alpha-1}, a_\alpha).$$

- $F(*, a_1, \dots, a_\alpha) = \{\text{pt}\}$.
- F sends a morphism in $v \rightarrow w$ in $\underline{2}_+^N$ to the map from $\overline{\mathcal{X}}_{T_v}^{\text{lk};\text{lk}}(a_\alpha, a_1)$ to $\overline{\mathcal{X}}_{T_w}^{\text{lk};\text{lk}}(a_\alpha, a_1)$ from the functor $\overline{(K \circ \Psi_T^{\text{lk};\text{lk}})}_+$, smashed with the identity map in the other factors.
- F sends the morphism $(v, a_1, \dots, a_\alpha) \rightarrow (v, a_1, \dots, \widehat{a}_i, \dots, a_\alpha)$ to the multiplication map

$$\mathcal{A}_n^{\text{lk}}(a_{i-1}, a_i) \wedge \mathcal{A}_n^{\text{lk}}(a_i, a_{i+1}) \rightarrow \mathcal{A}_n^{\text{lk}}(a_{i-1}, a_{i+1}) \quad \text{if } 1 < i < \alpha$$

or maps from the functor $\overline{(K \circ \Psi_T^{\text{lk};\text{lk}})}_+$

$$\begin{aligned} \overline{\mathcal{X}}_{T_v}^{\text{lk};\text{lk}}(a_\alpha, a_1) \wedge \mathcal{A}_n^{\text{lk}}(a_1, a_2) &\rightarrow \overline{\mathcal{X}}_{T_v}^{\text{lk};\text{lk}}(a_\alpha, a_2) && \text{if } i = 1 \\ \mathcal{A}_\alpha^{\text{lk}}(a_{\alpha-1}, a_\alpha) \wedge \overline{\mathcal{X}}_{T_v}^{\text{lk};\text{lk}}(a_\alpha, a_1) &\rightarrow \overline{\mathcal{X}}_{T_v}^{\text{lk};\text{lk}}(a_{\alpha-1}, a_1) && \text{if } i = \alpha, \end{aligned}$$

smashed with the identity map in the remaining factors.

The homotopy colimit of F , shifted down by N_+ , is clearly equivalent to the topological Hochschild homology $\text{THH}(\mathcal{A}_n^{\text{lk}}, \overline{\mathcal{X}}_T^{\text{lk};\text{lk}})$.

The advantage of the reformulation in terms of F is that F factors through the divided cobordism category of the annulus. Specifically, there is a functor $G: (\underline{2}^N \times \mathcal{C})' \rightarrow \text{Cob}_d(\odot)'$ which sends an object $(v, a_1, \dots, a_\alpha)$ to the 1-manifold

$$a_\alpha \overline{T}_v \widehat{a}_1 \amalg a_1 \widehat{a}_2 \amalg \cdots \amalg a_{\alpha-1} \widehat{a}_\alpha,$$

embedded in \odot so that the middle of \overline{T}_v is on the line $\{1\} \times (-1, 1)$ and the disjoint unions are in the cyclic order shown. The set A is the union of:

- a small closed neighborhood of the crossings labeled 0 in v ,
- a small closed neighborhood of each pox in T , and
- the complement in each a_i (respectively \widehat{a}_i) of a neighborhood of the boundary.

Each morphism in $(\underline{2}^N \times \mathcal{C})'$ is sent to a composition of saddle cobordisms (cf. [LLSb, §3.3–3.4]).

Consider the composition $V_{HKK} \circ G: (\underline{2}^N \times \mathcal{C})' \rightarrow \mathcal{B}$. Define a functor $L: (\underline{2}^N \times \mathcal{C})' \rightarrow \mathcal{B}$ by declaring that for an object $(v, a_1, \dots, a_\alpha)$, $L(v, a_1, \dots, a_\alpha) = \emptyset$ if there is a type III circle in any of $a_\alpha \overline{T}_v \widehat{a}_1, a_1 \widehat{a}_2, \dots, a_{\alpha-1} \widehat{a}_\alpha$. Otherwise, after shifting quantum grading by $n - |v| - N_+ + 2N_-$, $L(v, a_1, \dots, a_\alpha)$ is the set of elements $y \in V_{HKK}(G(v, a_1, \dots, a_\alpha))$ which label each type II circle by 1. On morphisms, L is obtained by restricting $V_{HKK} \circ G$. It follows from Lemma 4.2 that this defines a 2-functor.

Clearly, composing the Elmendorf-Mandell K -theory functor with L , rectifying, and adding a basepoint $*$ to $\underline{2}^N$, gives a diagram equivalent to F . In particular, $\text{hocolim}(\overline{(K \circ L)}_+) \simeq \text{hocolim } F$, desuspended N_+ times, is the topological Hochschild homology.

Let \mathcal{C}^T be the result of adding a terminal object T to \mathcal{C} . Extend G to a functor $G^T: (\underline{2}^N \times \mathcal{C}^T)' \rightarrow \text{Cob}_d(\odot)'$ by declaring that $G^T(v, T) = [T]$, and G^T sends each morphism to the corresponding saddle cobordism. Define $L^T: \mathcal{C}^T \rightarrow \mathcal{B}$ as follows. On $(\underline{2}^N \times \mathcal{C})'$, $L^T|_{\underline{2}^N \times \mathcal{C}} = L$. After shifting quantum grading by $-|v| - N_+ + 2N_-$, define $L^T(v, T) \subset V_{HKK}(G^T(v, T))$ to be those elements which

- have annular filtration 0,
- label every lower horizontal circle X , and
- label every upper horizontal circle 1.

On morphisms, define L^T to be the restriction of $V_{HKK} \circ G^T$. It follows from the proof of Lemma 5.4 that this defines a 2-functor.

Consider the functor $K \circ L^T: (\underline{2}^N \times \mathcal{C}^T)' \rightarrow \mathbb{Z}^{\mathcal{S}}$, and let $\overline{K \circ L^T}$ be its rectification. By definition, $\text{hocolim}(\overline{(K \circ L^T)}_{\underline{2}^N \times \{T\}})_+$, desuspended N_+ times, is $\mathcal{A}\mathcal{X}_{[T]}^{n-2k}\{n-2k\}$. This leads to

a cofibration sequence

$$\Sigma^{N+} \mathcal{A} \mathcal{X}_{[T]}^{n-2k} \{n-2k\} \rightarrow \text{hocolim}(\overline{K \circ L^T})_+ \rightarrow \Sigma \text{hocolim}(\overline{K \circ L})_+ \simeq \Sigma \text{hocolim} F.$$

Further, by construction the map $H_*(\Sigma \text{hocolim} F) \rightarrow H_*(\Sigma^{N+1} \mathcal{A} \mathcal{X}_{[T]}^{n-2k} \{n-2k\})$ induced by the Puppe construction is the map Ξ_* from Theorem 7. Thus, the Puppe map is a weak equivalence $\text{hocolim} F \simeq \Sigma^{N+} \mathcal{A} \mathcal{X}_{[T]}^{n-2k} \{n-2k\}$. Since $\text{hocolim} F \simeq \Sigma^{N+} \text{THH}(\mathcal{A}_n^k; \mathcal{X}_T^{k;k})$, this proves the result. \square

Corollary 5.14. *The action of the Hochschild cohomology of \mathcal{A}_n^k on the annular Khovanov homology $AKh([T]; n-2k)\{n-2k\} \cong \text{HH}_*(\mathcal{A}_n^k; \mathcal{C}_T^{k;k})$ satisfies a Cartan formula with respect to the action by Steenrod operations. For example, with mod-2 coefficients, for $a \in \text{HH}^*(\mathcal{A}_n^k)$ and $\beta \in \text{HH}_*(\mathcal{A}_n^k; \mathcal{C}_T^{k;k})$,*

$$\text{Sq}^n(a \cdot \beta) = \sum_{i+j=n} \text{Sq}^i(a) \cdot \text{Sq}^j(\beta).$$

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