The Bott cofiber sequence in deformation K-theory and simultaneous similarity in $\mathrm{U}(n)$

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(*Received*)

Abstract

We show that there is a homotopy cofiber sequence of spectra relating Carlsson's deformation K-theory of a group G to its "deformation representation ring," analogous to the Bott periodicity sequence relating connective K-theory to ordinary homology. We then apply this to study simultaneous similarity of unitary matrices.

The algebraic K-theory of a category uses the machinery of infinite loop space theory to associate spectra to symmetric monoidal categories. The homotopy groups of these spectra give information about the structure of the category itself. However, some symmetric monoidal categories arise with natural topologies on their objects and morphisms that give information about how objects in the category can behave in families.

For example, given a group G, we can consider the category of its finite-dimensional complex representations or unitary representations, each of which comes with a natural topology. Carlsson's "deformation K-theory," or the associated unitary variant, produces a K-theory spectrum which depends on both the symmetric monoidal structure and the behavior in families.

The purpose of this article is to identify the cofiber of the Bott map on unitary deformation K-theory ([**2**], [**8**]) of a finitely generated group G. For a finite group G, this cofiber can be identified with the Eilenberg-MacLane spectrum associated to the complex representation ring $R[G]$. More generally one obtains a "unitary deformation representation ring," also denoted by $R[G]$, which is a commutative $\mathbb{H}Z$-algebra spectrum. This deformation representation ring was considered in a previous paper [**7**]. Results of Park and Suh [**9**] will be applied to show that this deformation representation ring admits a cellular construction as an $\mathrm{H}\mathbb{Z}$-module spectrum.

There is a resulting first quadrant Atiyah-Hirzebruch style spectral sequence converging to the homotopy groups of deformation K-theory, as follows.

$$E_2^{p,q} = E_3^{p,q} = \pi_p(R[G]) \otimes \pi_q(ku) \Rightarrow \pi_{p+q}\mathcal{K}G.$$

As a side effect of this identification of $R[G]$ with the cofiber of the Bott map, we obtain results about the homotopy type of spaces parameterizing representations of the group G. In particular, when G is free, we obtain information about simultaneous similarity.

The spectral theorem in linear algebra implies that a unitary matrix A is determined, up to similarity, by its set of eigenvalues $\{z_1, \ldots, z_n\}$, counted with multiplicity. Taking the eigenvalues of a matrix gives a map from $\mathrm{U}(n)$ to the n-fold symmetric product $\mathrm{Sym}^n(S^1)$, inducing a bijection

$$\mathrm{U}(n)^{\mathrm{Ad}}/\mathrm{U}(n) \to \mathrm{Sym}^n(S^1).$$

In fact, both sides have natural topologies that make this map a homeomorphism.

The *simultaneous similarity* problem in $\mathrm{U}(n)$ is to classify the orbits of k-tuples of matrices $(A_1, \ldots, A_k)$ under unitary change of basis, or simultaneous conjugation. There is an analogous classification in $\mathrm{GL}(n)$ due to Friedland [**4**], which generalizes the Jordan canonical form but is much more involved.

The simplest invariant that can be extracted from this situation is the collection of eigenvalues. This gives a continuous *eigenvalue map*

$$\phi_{n,k}\colon\ X(n,k) = \left[\mathrm{U}(n)^{\mathrm{Ad}}\right]^k/\mathrm{U}(n) \to \left[\mathrm{Sym}^n(S^1)\right]^k.$$

In addition, there are stabilization maps $X(n,k) \hookrightarrow X(n+1,k)$, given by

$$(A_i) \mapsto \left(\begin{bmatrix} A_i & 0 \\ 0 & 1 \end{bmatrix}\right)$$

Define $X(\infty,k)$ to be the (homotopy) colimit of the $X(n,k)$. These stabilization maps commute with the stabilization maps $\mathrm{Sym}^n(S^1) \hookrightarrow \mathrm{Sym}^{n+1}(S^1)$, given by adding an extra copy of the basepoint 1.

We will show that the stable eigenvalue map

$$\phi_k\colon\ X(\infty,k) \to \left[\mathrm{Sym}^\infty(S^1)\right]^k$$

is a homotopy equivalence. The Dold-Thom theorem already implies that the map $S^1 \to \mathrm{Sym}^\infty(S^1)$ is an isomorphism on homotopy groups. Therefore, this result can be rephrased by saying that the map $(S^1)^k = X(1,k) \to X(\infty,k)$ is a homotopy equivalence.

At the end of this paper we will give two proofs of this result. The first proof presented here applies to more general spaces of representations and makes use of recently developed categories of module spectra, particularly of smash products over the connective K-theory spectrum. The author does not know general results about the stabilization of the homotopy groups of the spaces $X(n,k)$.

In section 6, we give an interpretation of the eigenvalue map in terms of simplicial spaces, using the simplicial decomposition of $\mathrm{U}(n)$ given in Harris [**6**]. We then establish a geometric proof by establishing contractibility for various spaces parameterizing multiple hyperplane arrangements in $\mathbb{C}^n$ for large n.

The geometric proof amounts to showing that the maps ϕ_k are quasifibrations. It should be noted that the eigenvalue maps $\phi_{n,k}$ are not quasifibrations, even for $n = 3$, $k \geq 2$. The fiber of the eigenvalue map at the basepoint is a single point; an exercise is to show that the fiber over a point of the form

$$(\{\zeta_1, \zeta_2, 1,\}, \{\zeta_1, \zeta_2, 1,\}, \{1,1,1\}, \{1,1,1\}, \ldots) \in \left[\mathrm{Sym}^3(S^1)\right]^k,$$

for $\zeta_1, \zeta_2, 1$ distinct elements of S^1, has the homotopy type of S^4.

1. *Definitions*

We briefly recall the definition and several properties of deformation K-theory from [**8**, Section 4] and the deformation representation ring functor from [**7**, Section 4].

Recall ([**13**], [**12**]) that a Γ-space M is a functor from finite based sets to based spaces such that $M(*) = *$. Associated to a levelwise finite simplicial set K, there is an associated based space $M(K)$ obtained by applying M levelwise and taking geometric realization. There is a natural assembly map $K \wedge M(L) \to M(K \wedge L)$, and so a Γ-space gives rise to a symmetric spectrum

$$\mathcal{S}p(M) = \{M(S^n)\}.$$

If X is a topological abelian monoid, we can define a Γ-space associated to X by

$$X(Z) = F(Z, X),$$

where F denotes the based mapping space, such that for $\alpha\colon Z \to Z'$,

$$\alpha_*(f)(z') = \sum_{\alpha(z)=z'} f(z).$$

Associated to a (topological) group G, we let $\mathrm{Rep}(G)$ be the space

$$\coprod_{n\in\mathbb{N}} \mathrm{Hom}\,(G, \mathrm{U}(n))/\mathrm{U}(n),$$

where the space of homomorphisms has the compact-open topology and $\mathrm{U}(n)$ acts by conjugation. This space parameterizes isomorphism classes of unitary representations of G.

The operations $\oplus$ and $\otimes$ give rise to the structure of a commutative topological semiring on $\mathrm{Rep}(G)$. In particular, the abelian addition operation $\oplus$ allows us to construct a spectrum

$$R[G] = \mathcal{S}p\left(\mathrm{Rep}(G)\right).$$

One can show that $R[G]$ is the spectrum obtained by iterated application of the the classifying space functor. This spectrum can be viewed as a homotopical group completion functor, generalizing the Grothendieck group construction of the ordinary representation ring.

The natural map

$$\mathrm{Rep}(G) \to \Omega^\infty R[G]$$

is a homotopy group completion map. The operation $\otimes$ gives rise to the structure of an E_∞-algebra over $\mathrm{H}\mathbb{Z}$ on $R[G]$.

The construction of $\mathrm{Rep}(G)$ has a K-theoretic analogue. Let $\mathcal{U}$ be a fixed countably infinite inner product space over $\mathbb{C}$. A *G-plane* in $\mathcal{U}$ is a pair (V, ρ), where V is a finite dimensional subspace of $\mathcal{U}$ and $\rho\colon G \to \mathrm{U}(V)$ is a group homomorphism. We define the deformation K-theory of G, $\mathcal{K}G$, to be the Γ-space given by

$$\mathcal{K}G(Z) = \left\{ (V_z, \rho_z)_{z\in Z} \;\middle|\; V_z \text{ a } G\text{-plane, } V_z \perp V_{z'} \text{ if } z \neq z',\ V_* = 0 \right\}.$$

For a morphism $\alpha\colon Z \to Z'$, we define

$$\alpha_*((V_z, \rho_z)_{z\in Z}) = \left(\bigoplus_{\alpha(z)=z'} V_z, \bigoplus_{\alpha(z)=z'} \rho_z \right)_{z'\in Z'}.$$

Taking isomorphism classes gives a map $\mathcal{K}G \to \mathrm{Rep}(G)$ of Γ-spaces. Therefore, there is a natural map of spectra $\mathcal{S}p(\mathcal{K}G) \to R[G]$.

The justification for the name K-theory arises as follows. We define the following spaces.

$$Ob(\mathcal{C}_G) = \coprod_{n\in\mathbb{N}} \mathrm{Hom}\,(G, \mathrm{U}(n))$$
$$Mor(\mathcal{C}_G) = \coprod_{n\in\mathbb{N}} \mathrm{Hom}\,(G, \mathrm{U}(n)) \times \mathrm{U}(n)$$

View a homomorphism $\rho \in Ob(\mathcal{C}_G)$ as a (unitary) representation of G, and a pair $(\rho, A) \in Mor(\mathcal{C})$ as an isometry of representations $A\colon\ \rho \to A\rho A^{-1}$. These form an internal category in spaces; the source, target, unit, and composition maps are all continuous. Further, the block sum $\oplus$ makes this into an internal symmetric monoidal category (in fact, a permutative category) in spaces. The spectrum $\mathcal{S}p(\mathcal{K}G)$ is homotopy equivalent to the associated K-theory object $K(\mathcal{C}_G)$.

Explicitly, we have a nerve

$$N(\mathcal{C}_G) \simeq \coprod_{n\in\mathbb{N}} \mathrm{Hom}\,(G, \mathrm{U}(n)) \times_{\mathrm{U}(n)} \mathrm{EU}(n).$$

The permutative category structure makes this into a topological monoid with an E_∞-H-space structure, and $K(\mathcal{C}_G)$ is the connective spectrum associated to $N(\mathcal{C}_G)$. If G is trivial, the associated spectrum is the connective K-theory spectrum.

2. *Filtrations of the representation ring*

We now provide a cellular construction of the topological monoid $\mathrm{Rep}(G)$ of the previous section.

If G is finitely generated and discrete, the space $\mathrm{Hom}\,(G, \mathrm{U}(n))$ is the set of real points of an algebraic variety, with $\mathrm{U}(n)$ acting algebraically by conjugation. In particular, by [**9**, Theorem 3.7], it admits the structure of a $\mathrm{U}(n)$-CW complex.

For any $N \in \mathbb{N}$, let $\mathrm{Rep}(G, N)$ be the submonoid of $\mathrm{Rep}(G)$ generated by the subspace $\coprod_{n\le N} \mathrm{Hom}\,(G, \mathrm{U}(n))$. This gives rise to a sequence of inclusions

$$* = \mathrm{Rep}(G, 0) \subset \mathrm{Rep}(G, 1) \subset \mathrm{Rep}(G, 2) \subset \cdots$$

A point of $\mathrm{Rep}(G)$ is an isomorphism class of unitary representations of G. In particular, any such representation admits a unique decomposition into irreducible subrepresentations. Let $\mathrm{Sum}(G, N) \subset \mathrm{Hom}\,(G, \mathrm{U}(N))$ be the subspace consisting of those representations which are reducible.

Equivalently, a representation $V \in \mathrm{Hom}\,(G, \mathrm{U}(N))$ is irreducible if and only if the stabilizer of it under the action of $\mathrm{U}(N)$ is the diagonal subgroup S^1, as follows. If $V \cong V' \oplus V''$, the stabilizer contains an action of $S^1 \times S^1$ acting individually on each factor. Conversely, Schur's lemma shows that the endomorphism ring of any irreducible object V is a finite dimensional division algebra over $\mathbb{C}$, and hence consists only of scalar maps. This shows that the map $\mathrm{Sum}(G, N) \hookrightarrow \mathrm{Hom}\,(G, \mathrm{U}(N))$ must be a $\mathrm{U}(N)$-CW inclusion.

This gives rise to the following diagram of spaces.

$$\begin{tikzcd}
\mathrm{Sum}(G,N)/\mathrm{U}(N) \arrow[r, hook] \arrow[d] & \mathrm{Hom}\,(G,\mathrm{U}(N))/\mathrm{U}(N) \arrow[d] \\
\mathrm{Rep}(G,N-1) \arrow[r] & \mathrm{Rep}(G,N)
\end{tikzcd}$$

Applying the free abelian topological monoid functor $\mathrm{Sym}^\infty((-)_+)$, which is left adjoint to the forgetful functor, to the top row gives a diagram of abelian topological monoids.

$$\begin{tikzcd}
\mathrm{Sym}^\infty(\mathrm{Sum}(G,N)/\mathrm{U}(N)_+) \arrow[r, hook] \arrow[d] & \mathrm{Sym}^\infty(\mathrm{Hom}\,(G,\mathrm{U}(N))/\mathrm{U}(N)_+) \arrow[d] \\
\mathrm{Rep}(G,N-1) \arrow[r] & \mathrm{Rep}(G,N)
\end{tikzcd}$$

The statement that any unitary representation of G is uniquely (up to isomorphism) a direct sum of irreducible subrepresentations implies that on the level of underlying abelian monoids, this diagram is a pushout diagram.

The monoids in the above diagram admit augmentations to the monoid $\mathbb{N}$, and are compact Hausdorff in each fiber. Therefore, the above diagram is a pushout diagram of topological abelian monoids. These pushout diagrams are preserved by cartesian products, and hence the associated diagram of classifying spaces is a pushout diagram.

We include a proof of the following for completeness.

Proposition 1. *Suppose $A \to B$ is a CW-inclusion and $\mathrm{Sym}^\infty(A_+) \to M$ is a map of topological abelian monoids. Let N be the pushout of the diagram*

$$M \leftarrow \mathrm{Sym}^\infty(A_+) \to \mathrm{Sym}^\infty(B_+)$$

of topological abelian monoids. Then the map $M \to N$ is a CW-inclusion, and the sequence of maps

$$M \to N \to \mathrm{Sym}^\infty(B/A)$$

induces a homotopy fibration sequence of spectra

$$\mathcal{S}p(M) \to \mathcal{S}p(N) \to \mathrm{H}\mathbb{Z} \wedge (B/A).$$

Proof. The pushout N is formed as a sequence of iterated CW attachments $N_i \to N_{i+1}$, where $N_0 = M$ and

$$N_{i+1} = N_i \bigcup_{(\cup B^j \times A \times B^{i-j})/\Sigma_{i+1}} B^{i+1}/\Sigma_{i+1}.$$

Weak equivalences are preserved by pushouts along cofibrations, so a weak equivalence $M' \to M$ of topological monoids induces a homotopy equivalence of pushouts. In particular, the natural weak equivalence

$$B(M, \mathrm{Sym}^\infty(A_+), \mathrm{Sym}^\infty(A_+)) \to M,$$

using the bar construction with respect to the monoid structure, induces a weak equivalence of topological abelian monoids

$$B(M, \mathrm{Sym}^\infty(A_+), \mathrm{Sym}^\infty(B_+)) \to N.$$

The classifying space functor commutes with products, and hence with the bar construction. Upon iterative application, we find that there is a natural weak equivalence of spectra

$$B(\mathcal{S}p(M), \mathcal{S}p(\mathrm{Sym}^\infty(A_+)), \mathcal{S}p(\mathrm{Sym}^\infty(B_+))) \to \mathcal{S}p(N).$$

The generalized Dold-Thom theorem implies that there is a natural weak equivalence

$$\mathrm{H}\mathbb{Z} \wedge \Sigma^\infty X_+ \to \mathcal{S}p(\mathrm{Sym}^\infty(X_+))$$

for spaces X of the homotopy type of a CW-complex. Therefore, there is a natural weak equivalence

$$B(\mathcal{S}p(M), \mathrm{H}\mathbb{Z} \wedge \Sigma^\infty A_+, \mathrm{H}\mathbb{Z} \wedge \Sigma^\infty B_+) \to \mathcal{S}p(N),$$

where the bar construction is taken with respect to coproduct (wedge) in spectra. Equivalently, there is a homotopy pushout diagram of spectra

$$\begin{tikzcd}
\mathrm{H}\mathbb{Z} \wedge \Sigma^\infty A_+ \arrow[r] \arrow[d] & \mathrm{H}\mathbb{Z} \wedge \Sigma^\infty B_+ \arrow[d] \\
\mathcal{S}p(M) \arrow[r] & \mathcal{S}p(N).
\end{tikzcd}$$

The result follows by considering the homotopy cofibers of the rows in this diagram. $\square$

Let $R_N = \mathrm{Hom}\,(G, \mathrm{U}(N))/\,\mathrm{Sum}(G, \mathrm{U}(N))$; it is a based $\mathrm{PU}(N)$-CW complex with free action away from the basepoint.

COROLLARY 2. *The spectrum $R[G]$ is the homotopy colimit of the spectra*

$$\mathcal{S}p(\mathrm{Rep}(G, N)).$$

There are fibration sequences of spectra for each $N \geq 1$

$$\mathcal{S}p(\mathrm{Rep}(G, N-1)) \to \mathcal{S}p(\mathrm{Rep}(G, N)) \to \mathrm{H}\mathbb{Z} \wedge (R_N/\mathrm{PU}(N)).$$

Proof. The inclusions $\mathrm{Rep}(G, N-1) \to \mathrm{Rep}(G, N)$ are CW-inclusions and induce pushout diagrams of spectra. The space $\mathrm{Rep}(G)$ is therefore the homotopy colimit of the subspaces $\mathrm{Rep}(G, N)$, and $\mathcal{S}p(\mathrm{Rep}(G))$ is the homotopy colimit of its subspectra $\mathcal{S}p(\mathrm{Rep}(G, N))$.

The existence of the fibration sequence is immediate from the proposition. $\square$

3. *Filtrations of unitary deformation K-theory*

We briefly recall the following results from [**8**]. From this point forward we abuse notation by writing $\mathcal{K}G$ to denote the spectrum associated to the Γ-space of section 1.

In the previous section we showed that that $\mathrm{Rep}(G)$ has a filtration by submonoids $\mathrm{Rep}(G, N)$ consisting of representations that are direct sums of irreducible subrepresentations of dimension N or smaller. There is an associated filtration of the symmetric monoidal category $\mathcal{C}_G$ by closed subcategories $\mathcal{C}_{G,N}$, and a diagram of maps of spectra as follows.

$$\begin{tikzcd}
K(\mathcal{C}_{G,1}) \arrow[r] \arrow[d] & K(\mathcal{C}_{G,2}) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{K}G \arrow[d] \\
\mathcal{S}p(\mathrm{Rep}(G,1)) \arrow[r] & \mathcal{S}p(\mathrm{Rep}(G,2)) \arrow[r] & \cdots \arrow[r] & R[G].
\end{tikzcd}$$

The map $\operatorname{hocolim} K(\mathcal{C}_{G,N}) \to \mathcal{K}G$ is a weak equivalence [**8**, Proposition 14]. Additionally, the objects $K(\mathcal{C}_{G,N})$ are module spectra over ku for all N [**8**, Proposition 30].

Recall that $R_N = \operatorname{Hom}(G, \mathrm{U}(N))/\operatorname{Sum}(G, \mathrm{U}(N))$. In [**8**, Section 4], for each N, a spectrum $ku^{\mathrm{PU}(N)}$ with a continuous action of $\mathrm{PU}(N)$ was constructed, with underlying spectrum homotopy equivalent to ku, so that there is a cofibration sequence up to homotopy

$$K(\mathcal{C}_{G,N-1}) \to K(\mathcal{C}_{G,N}) \to R_N \underset{\mathrm{PU}(N)}{\wedge} ku^{\mathrm{PU}(N)}.$$

([**8**, Corollary 19] and [**8**, Corollary 22].) Taking smash products over ku with $\mathrm{H}\mathbb{Z}$ gives a natural cofibration sequence

$$\mathrm{H}\mathbb{Z} \underset{ku}{\wedge} K(\mathcal{C}_{G,N-1}) \to \mathrm{H}\mathbb{Z} \underset{ku}{\wedge} K(\mathcal{C}_{G,N}) \to \mathrm{H}\mathbb{Z} \wedge (R_N/\mathrm{PU}(N)).$$

([**8**, Section 8].)

Theorem 3. *The map $K(\mathcal{C}_G) \to R[G]$ induces a weak equivalence*

$$\mathrm{H}\mathbb{Z} \underset{ku}{\wedge} K(\mathcal{C}_G) \to R[G].$$

Proof. The ku-module structure on $K(\mathcal{C}_{G,N})$ is induced by tensor product with trivial vector spaces. It coherently commutes with the abelian group structure on $\operatorname{Rep}(G, N)$ via the augmentation map sending a vector space to its dimension. Therefore, the map $N(\mathcal{C}_{G,N}) \to \operatorname{Rep}(G, N)$ induces a map of ku-modules.

The proof proceeds by proving inductively that the adjoint map of $\mathrm{H}\mathbb{Z}$-modules

$$\mathrm{H}\mathbb{Z} \underset{ku}{\wedge} K(\mathcal{C}_{G,N}) \to \mathcal{S}p(\operatorname{Rep}(G, N))$$

is a weak equivalence, and taking homotopy colimits. By the five-lemma, it suffices to show that the induced maps of homotopy cofibers are weak equivalences for all N. By corollary 2 and [**8**, Section 8], these homotopy cofibers are both weakly equivalent to

$$\mathrm{H}\mathbb{Z} \wedge (R_N/\mathrm{PU}(N)).$$

Therefore, it suffices to produce a map demonstrating that this map is a weak equivalence. There is a natural diagram of maps of spaces

$$\begin{tikzcd}
\operatorname{Sum}(G, \mathrm{U}(N)) \arrow[r] \arrow[d] & N(\mathcal{C}_{G,N-1}) \arrow[r] \arrow[d] & \operatorname{Rep}(G, N-1) \arrow[d] \\
\operatorname{Hom}(G, \mathrm{U}(N)) \arrow[r] & N(\mathcal{C}_{G,N}) \arrow[r] & \operatorname{Rep}(G, N).
\end{tikzcd}$$

Suspension is left adjoint to the forgetful functor to spaces, so there is an induced diagram of maps of symmetric spectra

$$\begin{tikzcd}
\Sigma^\infty \operatorname{Sum}(G, \mathrm{U}(N))_+ \arrow[r] \arrow[d] & K(\mathcal{C}_{G,N-1}) \arrow[r] \arrow[d] & \mathcal{S}p(\operatorname{Rep}(G, N-1)) \arrow[d] \\
\Sigma^\infty \operatorname{Hom}(G, \mathrm{U}(N))_+ \arrow[r] & K(\mathcal{C}_{G,N}) \arrow[r] & \mathcal{S}p(\operatorname{Rep}(G, N)).
\end{tikzcd}$$

Taking pushouts in columns gives maps

$$\Sigma^\infty R_N \to ku^{\mathrm{PU}(N)} \underset{\mathrm{PU}(N)}{\wedge} R_N \to \mathrm{H}\mathbb{Z} \wedge (R_N/\mathrm{PU}(N))$$

whose adjoint maps

$$\mathrm{H}\mathbb{Z} \underset{\mathrm{PU}(N)}{\wedge} R_N \to \mathrm{H}\mathbb{Z} \underset{ku}{\wedge} ku^{\mathrm{PU}(N)} \underset{\mathrm{PU}(N)}{\wedge} R_N \to \mathrm{H}\mathbb{Z} \wedge (R_N/\mathrm{PU}(N))$$

are equivalences. □

COROLLARY 4. *There is a homotopy cofiber sequence of ku-modules*

$$\Sigma^2 \mathcal{K}G \xrightarrow{\beta} \mathcal{K}G \to R[G],$$

where β is multiplication by the Bott element in $\pi_2(ku)$. There is a corresponding convergent "Atiyah-Hirzebruch" spectral sequence with E_2-term

$$\pi_p(R[G]) \otimes \pi_q(ku) \Rightarrow \pi_{p+q}\mathcal{K}G.$$

Proof. This follows by smashing the homotopy cofiber sequence

$$\Sigma^2 ku \xrightarrow{\beta} ku \to \mathrm{H}\mathbb{Z}$$

with the spectrum $\mathcal{K}G$ over ku, and using the theorem to identify the terms in the result. The spectral sequence follows by considering the tower of spectra

$$\cdots \to \Sigma^4 \mathcal{K}G \to \Sigma^2 \mathcal{K}G \to \mathcal{K}G,$$

whose filtration quotients are $\Sigma^{2k} R[G]$. □

4. *Example computations*

In this section, we analyze irreducible representations to compute the deformation ring spectrum $R[G]$, and then apply Corollary 4 to obtain information about the deformation K-theory groups of several groups.

For further examples relating deformation K-theory of surface groups to gauge theory, the reader should consult [**10**].

4·1. *Finitely generated abelian groups*

Let G be a finitely generated abelian group, with character group $G^* = \mathrm{Hom}(G, \mathrm{U}(1))$. Any irreducible representation is uniquely, up to isomorphism, a direct sum of characters. The topological monoid $\mathrm{Rep}(G)$ is the infinite symmetric product $\mathrm{Sym}^\infty(G^*)$, and so $R[G] \simeq \mathrm{H}\mathbb{Z} \wedge G^*$.

In particular, if $G \cong \mathbb{Z}^r \oplus A$ where A is finite, then $G^* \cong (S^1)^r \times A^*$, and so we obtain the following.

$$\pi_* R[G] \cong \bigoplus_{a \in A^*} H_*(S^1)^{\otimes r} \cong H_*(S^1)^{\otimes r} \otimes R[A]$$

In particular, it is free abelian in each degree. The E_2-term of the spectral sequence for deformation K-theory is therefore

$$H_*(S^1)^{\otimes r} \otimes R[A] \otimes \mathbb{Z}[\beta].$$

It remains to exclude the possibility of differentials in this spectral sequence. Either naturality in G or the results of [**8**] imply that the spectral sequence degenerates at the E_2-term.

4·2. *The integer Heisenberg group*

Let G be the integer Heisenberg group of upper triangular integer matrices with 1 on the diagonal. In [**7**], the deformation representation ring $R[G]$ was shown to satisfy

$$\pi_* R[G] \cong \begin{cases} \oplus\,\mathbb{Z} & \text{if } * = 0, \\ \oplus\,\mathbb{Z}^2 & \text{if } * = 1, \\ \oplus\,\mathbb{Z} & \text{if } * = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Here the direct sum ranges over roots of unity in $\mathbb{C}$; these index the irreducible representations via the "central character." The spectral sequence for deformation K-theory is therefore forced to degenerate at E_2, with no hidden extensions possible as all groups involved are free. Therefore, we find that

$$\pi_* \mathcal{K}(G) \cong \begin{cases} \oplus\,\mathbb{Z} & \text{if } * = 0, \\ \oplus\,\mathbb{Z}^2 & \text{if } * \geq 1. \end{cases}$$

4·3. $\mathbb{Z} \rtimes \mathbb{Z}/2$

Let G be the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}/2$, where $\mathbb{Z}/2$ acts by negation on $\mathbb{Z}$. It has an abelian subgroup $\mathbb{Z}$ of index 2, and hence any irreducible representation has dimension 1 or 2.

The commutator subgroup of G is $2\mathbb{Z}$, and there are four 1-dimensional representations $1, \sigma, \tau, \sigma\tau$. (Here we take σ to be the nontrivial representation factoring through the "obvious" quotient map $\mathbb{Z} \rtimes \mathbb{Z}/2 \to \mathbb{Z}/2$.) Therefore, $R_1/\mathrm{PU}(1) \cong \vee^4 S^0$.

For any $\alpha \in S^1$, there is a corresponding unitary character of $\mathbb{Z}$ (also denoted by α) which sends the generator to α. The induced representation $V_\alpha = \mathrm{Ind}_{\mathbb{Z}}^{G}(\alpha)$ is a two-dimensional unitary representation whose restriction to $\mathbb{Z}$ is isomorphic to $\alpha \oplus \alpha^{-1}$. One readily checks the following facts.

- $V_\alpha \cong V_\beta$ if and only if $\alpha = \beta^{\pm 1}$.
- V_α is irreducible if and only if $\alpha \neq \pm 1$.
- $V_1 \cong 1 \oplus \sigma$ and $V_{-1} \cong \tau \oplus \sigma\tau$.
- All 2-dimensional representations of G are either reducible or isomorphic to V_α for some α.

As a result, the space $R_2/\mathrm{PU}(2)$ of 2-dimensional representations modulo reducibles is homeomorphic to $[0,1]/\partial[0,1] \simeq S^1$. The cofiber sequences of corollary 2 degenerate to a single cofiber sequence

$$\mathrm{H}\mathbb{Z} \wedge (\vee^4 S^0) \to R[G] \to \mathrm{H}\mathbb{Z} \wedge ([0,1]/\partial[0,1]).$$

Therefore, $\pi_* R[G] = 0$ for $* > 0$, and there is a short exact sequence as follows.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}\sigma \oplus \mathbb{Z}\tau \oplus \mathbb{Z}\sigma\tau \longrightarrow \pi_0 R[G] \longrightarrow 0.$$

The left-hand map in this sequence is multiplication by $(1+\sigma) - (\tau + \sigma\tau)$.

As the homotopy of $R[G]$ is concentrated in degree zero, the spectral sequence for the deformation K-theory degenerates and we find

$$\pi_* \mathcal{K}(G) \cong \begin{cases} \mathbb{Z}^3 & \text{if } * \text{ is even}, * \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(The degeneration of the spectral sequence actually implies that the homotopy type of the spectrum is $\vee^3 ku$.)

We note that this group is isomorphic to the amalgamated product $\mathbb{Z}/2 * \mathbb{Z}/2$, and the main theorem of [**11**] recovers this result as part of a general formula for amalgamated products.

4·4. $\mathbb{Z}^2 \rtimes \mathbb{Z}/4$

We list one final example which is not known by methods of excision or product formulas.

Suppose G is the semidirect product $\mathbb{Z}^2 \rtimes \mathbb{Z}/4$, where the cyclic group of order 4 acts on $\mathbb{Z}^2$ by the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Choose generators x and y for $\mathbb{Z}^2$. The group G has an index 4 abelian subgroup, and so the irreducible representations have dimensions 1, 2, or 4.

More specifically, let $T = \mathrm{Hom}(\mathbb{Z}^2, \mathrm{U}(1))$ be the character group of $\mathbb{Z}^2$, with action of $\mathbb{Z}/4$ by precomposition. Elementary Frobenius reciprocity breaks the irreducible representations of G into the following types.

- Associated to each character in T fixed by $\mathbb{Z}/4$, there are 4 distinct extensions to irreducible 1-dimensional representations. These are acted on freely transitively by the character group of $\mathbb{Z}/4$.
  There are precisely 2 characters in T fixed by $\mathbb{Z}/4$, given by the trivial character and the character $x \mapsto -1, y \mapsto -1$. The group of characters of G is $(\mathbb{Z}/2 \times \mathbb{Z}/4)^*$.
- Associated to any $\mathbb{Z}/4$-orbit in T of order 2, each representative has 2 distinct extensions to 1-dimensional representations of $\mathbb{Z}^2 \rtimes \mathbb{Z}/2$. These induce to 2 distinct irreducible 2-dimensional representations of G determined only by the orbit. These are interchanged by the character group of $\mathbb{Z}/4$.
  There is precisely 1 orbit in T of size 2, with a representative given by the character $x \mapsto -1, y \mapsto 1$. There are then 2 irreducible representations of degree 2.
- Associated to each $\mathbb{Z}/4$-orbit in T of order 4, any representative in the orbit induces to an irreducible 4-dimensional representation of G. This is fixed by the character group of $\mathbb{Z}/4$.
  The space of isomorphism classes of representations of degree 4, modulo reducibles, is therefore the quotient of T by $\mathbb{Z}/4$ (homeomorphic to S^2), modulo the 3 points corresponding to orbits of size less than 4. We can give this space a cell structure with two 1-dimensional cells (attaching the three points reducing to the basepoint), together with a 2-cell attached via a map trivial in homology.

One can carry out analysis as in the previous example to show that the boundary map on the generators of the 1-cells in homology injects to a direct summand onto the previously attached 0-cells. One can then determine that the deformation representation ring $R[G]$ has homotopy as follows.

$$\pi_* R[G] \cong \begin{cases} \mathbb{Z}^8 & \text{if } * = 0, \\ \mathbb{Z} & \text{if } * = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The spectral sequence for the deformation K-theory then degenerates at the E_2-term, and we find

$$\pi_*\mathcal{K}(G) \cong \begin{cases} \mathbb{Z}^8 & \text{if } * = 0, \\ \mathbb{Z}^9 & \text{if } * \text{ is even, } * \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(The degeneration of the spectral sequence actually implies that the homotopy type of the spectrum is $(\vee^8 ku) \vee \Sigma^2 ku$.)

5. *Representation ring spectra of free groups*

Let F_k be the free group on k generators $x_1, \ldots, x_k$ with the discrete topology. A unitary representation of F_k consists of a choice of image of each generator. Therefore,

$$\mathrm{Rep}(F_k) = \coprod_{n\in\mathbb{N}} \left(\mathrm{U}(n)^{\mathrm{Ad}}\right)^k / \mathrm{U}(n) = \coprod_{n\in\mathbb{N}} X(n,k).$$

The direct sum maps $X(n,k) \times X(m,k) \to X(m+n,k)$ respect stabilization, and therefore give rise to the structure of an abelian topological monoid on $X(\infty,k)$. In particular, there is a map of abelian topological monoids

$$\mathrm{Rep}(F_k) \to \mathbb{Z} \times X(\infty,k).$$

The spaces $X(n,k)$ are connected and have abelian fundamental group, so the same holds for $X(\infty,k)$.

Due to classical results of Quillen [**5**, Appendix Q], the homotopy group completion map $\mathrm{Rep}(G) \to \Omega \mathrm{B}\, \mathrm{Rep}(G)$ is characterized as inducing a localization map on homology

$$H_*(\mathrm{Rep}(G)) \to (\pi_0\, \mathrm{Rep}(G))^{-1} H_*(\mathrm{Rep}(G)).$$

In particular, the map $\mathrm{Rep}(F_k) \to \mathbb{Z} \times X(\infty,k)$ is a homotopy group completion map. As a result, we find that

$$\pi_*(R[G]) \cong \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \pi_*(X(\infty,k)) & \text{if } * > 0 \end{cases}$$

Remark 5. *Ideally, one would like to prove a stability result at this point. This is not strictly necessary to show that the given map is a group completion; this is unnecessary for topological monoids which are homotopy commutative, since these admit a "calculus of fractions". See* [**5**, Appendix Q].

Proposition 6. *As a spectrum,*

$$R[F_k] \simeq \mathrm{H}\mathbb{Z} \vee \left(\vee^k \Sigma \mathrm{H}\mathbb{Z}\right).$$

The group $\pi_1(R[F_k])$ is isomorphic to $\mathrm{Hom}(F_k, \mathbb{Z})$*, naturally in maps of free groups.*

Proof. The deformation K-theory spectrum of F_k is the spectrum associated to the E_∞-H-space

$$\coprod_{n\in\mathbb{N}} \left[\mathrm{U}(n)^{\mathrm{Ad}}\right]^k \times_{\mathrm{U}(n)} \mathrm{EU}(n).$$

We briefly sketch an identification of the homotopy type of this spectrum; a more general decomposition of the homotopy type for free products can be found in [**11**].

Recall that for a group G, there is a natural weak equivalence

$$G^{\mathrm{Ad}} \times_G EG \simeq \Lambda BG,$$

the free loop space on BG [**1**, Proposition 2.6]. Taking the k-fold fiber product over BG, we find that there is a natural weak equivalence

$$\left[G^{\mathrm{Ad}}\right]^k \times_G EG \simeq \mathrm{Map}(BF_k, BG).$$

Naturality implies that the weak equivalence

$$\coprod_{n\in\mathbb{N}} \left[\mathrm{U}(n)^{\mathrm{Ad}}\right]^k \times_{\mathrm{U}(n)} \mathrm{EU}(n) \simeq \mathrm{Map}\left(BF_k, \coprod_{n\in\mathbb{N}} \mathrm{BU}(n)\right)$$

respects the E_∞-structure, where the E_∞-structure on the right-hand space is derived from the range of the mapping space.

The spectrum associated to $\coprod \mathrm{BU}(n)$ is the connective K-theory spectrum ku. Therefore, there are maps

$$\mathrm{Map}\left(BF_k, \coprod \mathrm{BU}(n)\right) \to \mathrm{Map}\left(BF_k, \Omega^\infty ku\right) \simeq \Omega^\infty F\left(\Sigma^\infty (BF_k)_+, ku\right).$$

This composite map is a homotopy group completion map by inspection. Therefore, the map

$$\mathcal{K}F_k \to F\left(\Sigma^\infty (BF_k)_+, ku\right)$$

is an isomorphism on homotopy groups in positive degrees, and the left-hand spectrum is connective. The function spectrum is equivalent to

$$ku \vee \left(\bigvee^k \Omega ku\right)$$

as a ku-module. The connective cover of this spectrum is

$$\mathcal{K}F_k \simeq ku \vee \left(\bigvee^k \Sigma ku\right)$$

by Bott periodicity.

By theorem 3, we then find that

$$R[F_k] \simeq \mathrm{H}\mathbb{Z} \vee \left(\bigvee^k \Sigma \mathrm{H}\mathbb{Z}\right).$$

The first homotopy group of $R[F_k]$ has a natural isomorphism to the (-1)'st homotopy group of $F((BF_k)_+, ku)$ by Bott periodicity, which gives rise to the natural isomorphism

$$\pi_1 R[F_k] \cong \mathrm{Hom}\left(F_k, \mathbb{Z}\right).$$

$\square$

COROLLARY 7. *The eigenvalue map $X(\infty, k) \to \left[\mathrm{Sym}^\infty(S^1)\right]^k$ is a weak equivalence; in particular, the higher homotopy groups of $X(\infty, k)$ vanish.*

Proof. This follows from identification of the identity component of $\Omega^\infty R[F_k]$ with $X(\infty, k)$, and the eigenvalue map with the product of the restriction maps

$$X(\infty, k) \to X(\infty, 1)^k,$$

which is an isomorphism on π_1. $\square$

6. *Simplicial interpretation of the eigenvalue map*

In [**6**], the spectral theorem was be reinterpreted as a simplicial decomposition of the conjugation action of $\mathrm{U}(n)$ on itself. We will now recall this construction.

For $\{n_i\}_{i=1}^p$ a sequence of integers with $\sum n_i \leq n$, define

$$\mathrm{Gr}(n_1, n_2, \cdots ; n) = \mathrm{U}(n) \Big/ [\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \cdots \times \mathrm{U}(n - \Sigma n_i)] \,.$$

This space is a Grassmannian parameterizing configurations $(V_1, V_2, \cdots)$ of orthogonal systems of subspaces in $\mathbb{C}^n$ with $\dim(V_i) = n_i$. It has a natural left action of $\mathrm{U}(n)$.

Define a simplicial space $X_.$ by

$$X_p = \coprod_{\sum_{i=1}^p n_i \leq n} \mathrm{Gr}(n_1, n_2, \cdots ; n).$$

Face maps are given as follows:

$$d_i(V_1, \cdots, V_p) = \begin{cases} (V_2, \cdots, V_p) & \text{if } i = 0, \\ (V_1, \cdots, V_i + V_{i+1}, \cdots, V_p) & \text{if } 0 < i < p, \\ (V_1, \cdots, V_{p-1}) & \text{if } i = p. \end{cases}$$

The degeneracy maps are insertion of a zero-dimensional subspace.

A point of the geometric realization $|X_.|$ consists of an arrangement $(V_1, \cdots, V_p)$ of orthogonal hyperplanes and a point of Δ^p, i.e. a sequence of numbers $0 \leq t_1 \leq \cdots \leq t_p \leq 1$. Define a map $f \colon |X_.| \to \mathrm{U}(n)$ by sending this point to the matrix A such that each space V_i is an eigenspace for A with eigenvalue $e^{2\pi i t_i}$, and the orthogonal complement of $\oplus V_i$ is acted on trivially by A. The map f is a homeomorphism of $\mathrm{U}(n)$-spaces.

There is a map of simplicial spaces

$$X_. = \left\{ \coprod_{\sum n_i \leq n} \mathrm{Gr}(n_1, \cdots ; n) \right\} \to \left\{ \coprod_{\sum n_i \leq n} * \right\}.$$

The right-hand space is obtained from $X_.$ by taking the quotient by the action of $\mathrm{U}(n)$. The right-hand simplicial set is $\mathrm{Sym}^n(S^1)$.

By taking k-fold products, we recover the map of spaces

$$\mathrm{U}(n)^k \cong \left| X_.^k \right| \to \left[\mathrm{Sym}^n(S^1) \right]^k .$$

By taking quotients by the conjugation action, we find that the map $X(n,k) \to \left[\mathrm{Sym}^n(S^1) \right]^k$ can be expressed as the geometric realization of the following map of simplicial spaces.

$$\left\{ \coprod_{n_{i,j}} \left[\mathrm{U}(n) \Big\backslash \prod_{i=1}^k \mathrm{Gr}(n_{i,1}, n_{i,2}, \cdots ; n) \right] \right\} \to \left\{ \coprod_{n_{i,j}} * \right\}$$

Therefore, one way to get estimates on the connectivity of the eigenvalue map would be to obtain increasing bounds on the connectivity of the spaces on the left.

Note that the space

$$\mathrm{U}(n) \Big\backslash \prod_{i=1}^k \mathrm{Gr}(n_{i,1}, n_{i,2}, \cdots ; n)$$

becomes fixed for $n \geq N = \sum_{i,j} n_{i,j}$; this follows because any configuration of hyperplanes of this type is contained within its span, which is of dimension less than or equal to N.

PROPOSITION 8. *The space*

$$Y_n = \mathrm{U}(n) \Big\backslash \prod_{i=1}^{k} \mathrm{Gr}(n_{i,1}, n_{i,2}, \cdots ; n)$$

is contractible for large n.

Proof. This sequence of spaces stabilizes for large n, so it suffices to show that the stabilization map

$$s\colon Y_n \to Y_{kn}$$

is null-homotopic.

Write $\mathbb{C}^{kn} \cong V^{\oplus k}$, where $V = C^n$. For $2 \leq i \leq k$ and $0 \leq \theta \leq \pi/4$, let $A_i(\theta)$ be the block element of $\mathrm{U}(kn)$

$$\begin{bmatrix} \cos\theta I & 0 & & \sin\theta I & \\ 0 & I & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \\ -\sin\theta I & 0 & \cdots & \cos\theta I & \cdots \\ & \vdots & & \vdots & \end{bmatrix},$$

which rotates the first copy of V to the ith copy, leaving the other copies fixed. (For simplicity, we define $A_1(\theta)$ to be the identity.)

One then checks that we have a well-defined homotopy

$$(H_1, H_2, \ldots, H_k, \theta) \mapsto (A_1(\theta)s(H_1), A_2(\theta)s(H_2), \ldots, A_k(\theta s(H_k)))$$

from the stabilization map s to the map

$$(H_1, H_2, \ldots, H_k) \mapsto (H_1 \oplus 0, 0 \oplus H_2 \oplus 0, \ldots, 0 \oplus H_k).$$

However, the right-hand side is constant after the quotient by the action of $\mathrm{U}(kn)$. □

We find that the stable eigenvalue map of the introduction is a homotopy equivalence from this proposition and the simplicial decomposition of the stable eigenvalue map.

Stability questions naturally give rise to the following question: How does the connectivity of these spaces of hyperplane arrangements depend on the $n_{i,j}$ and n?

One can obtain some partial answers to this question. For example,

$$\mathrm{U}(n) \Big\backslash [\mathrm{Gr}(n_1; n) \times \mathrm{Gr}(n_2; n)]$$

is always contractible. Given an n_1-dimensional plane V and an n_2-dimensional plane W in $\mathbb{C}^n$, let p be the orthogonal projection from W to V and $q = p^T$ the projection from V to W. The singular value decomposition in linear algebra shows that this configuration is determined up to isomorphism by the eigenvalues $\sigma_1^2 \geq \sigma_2^2 \geq \ldots$ of pq, which agree with those of qp up to additional zeros. (This method was indicated to us by Neil Strickland.)

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