Brauer groups and Galois cohomology of commutative ring spectra

David Gepner* and Tyler Lawson†

July 5, 2016

Contents

*Partially supported by NSF grant 1406529.
†Partially supported by NSF grant 1206008.

1 Introduction

The Brauer group of a field F, classifying central simple algebras over F, plays a critical role in class field theory. The definition was generalized by Auslander and Goldman [1] to the case of a commutative ring: the Brauer group of R consists of Morita equivalence classes of Azumaya algebras over R.

In recent years these concepts have been extended to derived algebraic geometry [2], to homotopy theory [3], to more general categorical frameworks [4], and generalized to the Morita theory of E_n-algebras [5]. Associated to a commutative ring spectrum R, there is a category of Azumaya algebras over R and a Brauer space $\mathrm{Br}(R)$ classifying Morita equivalence classes of such R. Joint work of Antieau with the first author gave an in-depth study of these Brauer spaces when R is connective [6], and in particular found that the set of Morita equivalence classes could be calculated cohomologically.

There are two important tools developed in [6] which make this cohomological identification possible. First, Azumaya algebras A over connective R are étale-locally trivial: there exist enough "π_*-étale" maps $R \to S$ such that $S \otimes_R A$ is Morita trivial. Second, generators descend: an R-linear category which is étale-locally a category of modules over an Azumaya algebra is a category of modules over a global Azumaya algebra. The goal of this paper is to calculate the Brauer group of *nonconnective* ring spectra R, and these tools are absent in the case when R is nonconnective. Moreover, the first outright fails: there exist Azumaya algebras which are not π_*-étale-locally trivial.

This should not necessarily be suprising: detecting étale extensions on the level of π_* is fundamentally not adequate for nonconnective ring spectra. For example, the homotopy pullback of the diagram of Eilenberg–Mac Lane spectra

$$\begin{array}{ccc} R & \dashrightarrow & \mathbb{C}[x, y^{\pm 1}] \\ \vdots & & \downarrow \\ \mathbb{C}[x^{\pm 1}, y] & \longrightarrow & \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \end{array}$$

has a map $\mathbb{C}[x, y] \to R$ which is not π_*-étale. On the level of module categories, however, R-modules are equivalent to $\mathbb{C}[x, y]$-modules supported away from the origin, and so this gives an "affine" but nonconnective model for the open immersion $\mathbb{A}^2 \setminus \{0\} \hookrightarrow \mathbb{A}^2$ [7, 2.4.4]. In this and other quasi-affine cases, the coefficient ring does not exhibit all of the useful properties of this map [8, Section 8].

Our first tool for calculations will be obstruction theory. We show that the homotopy category of those Azumaya algebras over R whose underlying graded coefficient ring is a projective module over $\pi_* R$ form a category equivalent to the category of Azumaya $\pi_* R$-algebras in the graded sense (a result of Baker–Richter–Szymik [3]). Moreover, we show that there exist natural exact sequences that calculate the homotopy groups of the space of automorphisms of such an Azumaya algebra. For example, the space of automorphisms of the matrix algebra $M_n(R)$ is an extension of a discrete group of "outer automorphisms" by a group which might be called $PGL_n(R)$. With an eye towards future applications, we have developed our obstruction theory so that one may extend from a $\mathbb{Z}$-grading to general families Γ of elements of the Picard groupoid of R.

Our second tool for calculations will be descent theory. For a Galois extension of ring spectra $R \to S$ with Galois group G in the sense of Rognes [9] we develop descent-theoretic methods for lifting Azumaya algebras and Morita equivalences from S to R. In particular, there are maps $B\operatorname{Pic}(S)^{hG} \to \operatorname{Br}(S)^{hG} \xrightarrow{\sim} \operatorname{Br}(R)$. The first map is an equivalence above degree zero and an injection on π_0, with image consisting of those Morita equivalence classes of R-algebras which become Morita trivial S-algebras. This allows us to use calculations with the homotopy fixed-point spectrum of the Picard spectrum $\mathfrak{pic}(S)$ from [10] to detect interesting Brauer classes, and employ an obstruction theory for cosimplicial spaces due to Bousfield [11] to lift Azumaya algebras. In order to carry this out we need to connect the space of autoequivalences of a module to the space of autoequivalences of its endomorphism algebra. We will make heavy use of the machinery of ∞-categories to make this possible.

In Section 7 we will collect these together and apply them to calculations. For even-periodic ring spectra E, we find that the algebraic Azumaya algebras are governed by the Brauer–Wall group [12] and are generated by three phenomena: ordinary Azumaya algebras over $\pi_0 E$, $\mathbb{Z}/2$-graded "quaternion" algebras over E, and (if 2 is invertible) associated 1-periodic ring spectra. In particular, all algebraic Azumaya algebras over KU are Morita trivial, and the algebraic Azumaya algebras over Lubin–Tate spectra have either 4 or 2 Morita equivalence classes depending on whether 2 is invertible or not.[1]

Finally, our most difficult calculation studies Azumaya KO-algebras which become Morita-trivial KU-algebras; we show that there exist exactly two Morita equivalence classes of these. The nontrivial Morita equivalence class is realized by an "exotic" KO-algebra lifting $M_2(KU)$ which we construct by finding a path through an obstruction theory spectral sequence. This requires a careful analysis of what happens near the bottom of the homotopy fixed-point spectral sequence for $B\operatorname{Pic}(KU)^{hC_2}$.

[1] We note that Angeltveit–Hopkins–Lurie have announced the existence of a host of "exotic" elements exhausting the $K(n)$-local Brauer group of a Lubin–Tate spectrum E, obtained as E-module Thom spectra on tori.

Acknowledgements

The authors would like to thank Benjamin Antieau, Andrew Baker, Clark Barwick, Saul Glasman, Akhil Mathew, Lennart Meier, Birgit Richter, Vesna Stojanoska, and Markus Szymik for discussions related to this paper.

2 Homological algebra

In this section we will recall some important results on categories of graded objects, their algebras, and their homological algebra.

2.1 Graded objects

Definition 2.1. A Picard groupoid Γ is a symmetric monoidal groupoid such that the monoidal operation makes $\pi_0(\Gamma)$ into a group.

Given a symmetric monoidal category $\mathcal{C}$, the Picard groupoid $\mathrm{Pic}(\mathcal{C})$ is the groupoid of objects in $\mathcal{C}$ which have an inverse under the monoidal product, with maps being isomorphisms between them.

We will abusively use the symbol $+$ to denote the symmetric monoidal structure on a Picard groupoid Γ, and write 0 for the unit object.

Definition 2.2. For an ordinary category $\mathcal{C}$, we define the category $\mathcal{C}_\Gamma$ of Γ-graded objects to be the category of contravariant functors $M_\star \colon \Gamma^{\mathrm{op}} \to \mathcal{C}$, and for $\gamma \in \Gamma$ we write M_γ for the image. In particular, Ab_Γ is the category of Γ-graded abelian groups.

Suppose $\mathcal{C}$ is cocomplete and symmetric monoidal under an operation $\otimes$ with unit $\mathbb{I}$. If $\otimes$ preserves colimits in each variable separately, then $\mathcal{C}_\Gamma$ has a symmetric monoidal closed structure given by the Day convolution product. Specifically, its values are given by

$$(M \otimes N)_\gamma = \mathrm{colim}_{\alpha+\beta\to\gamma} M_\alpha \otimes N_\beta,$$

and the unit is given by the functor $\gamma \mapsto \coprod_{\mathrm{Hom}(\gamma,0)} \mathbb{I}$. Making choices of representatives for all isomorphism classes $[\gamma] \in \pi_0\Gamma$ gives rise to a noncanonical isomorphism

$$(M \otimes N)_\gamma \cong \coprod_{\{([\alpha],[\beta]) \mid \alpha+\beta\cong\gamma\}} M_\alpha \otimes_{\mathrm{Aut}_\Gamma(0)} N_\beta.$$

Definition 2.3. A Γ-graded commutative ring $R_\star$ is a commutative monoid object in Ab_Γ. The unit of $R_\star$ is the induced map $\mathbb{Z}[\mathrm{Aut}_\Gamma(0)] \to R_0$.

Proposition 2.4. *The category* $\mathrm{Mod}_{R_\star}$ *of* Γ*-graded* $R_\star$*-modules is a symmetric monoidal closed abelian category, with tensor product* $\otimes_{R_\star}$*, internal* Hom *objects* $F_{R_\star}(-,-)$*, and arbitrary products and coproducts which are exact.*

Remark 2.5. Suppose that A and G are abelian groups and ε is a bilinear pairing $A \times A \to G$. Then ε determines the structure of a Picard groupoid on $\Gamma = A \times BG$. The monoidal structure is split, in the sense that it is the product of the abelian group structures on A and BG, but the natural symmetry isomorphism $\tau_{a,b}\colon\, a+b \to b+a$ is given by $\varepsilon_{a,b} \in \mathrm{Aut}(a+b)$. In particular, the splitting usually does not respect the symmetry isomorphism.

A Γ-graded commutative ring then consists of an A-indexed collection R_γ of abelian groups, multiplication maps $R_\alpha \otimes R_\beta \to R_{\alpha+\beta}$, and a homomorphism $G \to R_0^\times$. These are required to satisfy associativity and unitality conditions, and the commutativity condition takes the form $x \cdot y = \varepsilon_{\alpha,\beta}(y \cdot x)$ for $x \in R_\alpha$, $y \in R_\beta$. The category of graded R-modules then inherits a symmetric monoidal structure using ε to describe a "Koszul sign convention" for the tensor product. We thus recover the framework of [13, 14] without the assumption that R is concentrated in degree zero.

For $\gamma \in \Gamma$, write $\mathbb{Z}^\gamma$ for the Γ-graded abelian group obtained from the Γ-graded set $\mathrm{Hom}_\Gamma(-,\gamma)$ by taking the free group levelwise. We have natural isomorphisms $\mathbb{Z}^\alpha \otimes \mathbb{Z}^\beta \to \mathbb{Z}^{\alpha+\beta}$ that determine a functor $\Gamma \to \mathrm{Pic}(\mathrm{Ab}_\Gamma)$. Let the suspension operator Σ^γ be the tensor product with $\mathbb{Z}^\gamma$, an automorphism of the category of $R_\star$-modules. There is an isomorphism $M_\delta \cong (\Sigma^\gamma M)_{\gamma+\delta}$, and this extends to isomorphisms

$$M_\gamma \cong \mathrm{Hom}_{R_\star}(\Sigma^\gamma R_\star, M_\star).$$

In general, if I is a finite Γ-graded set, we will write ${R_\star}^I$ for the free Γ-graded $R_\star$-module on I, which can be constructed as the tensor product of $R_\star$ with the free Γ-graded abelian group on I.

Definition 2.6. Suppose $A_\star$ is an algebra in the category $\mathrm{Mod}_{R_\star}$. We call a right $A_\star$-module $P_\star$ a *graded generator* if $\{\Sigma^\gamma P_\star\}_{\gamma\in\Gamma}$ is a set of compact projective generators of $\mathrm{Mod}_{A_\star}$.

For example, $R_\star$ is always a graded generator of $\mathrm{Mod}_{R_\star}$. It is unlikely to be a generator of $\mathrm{Mod}_{R_\star}$ in the ordinary sense unless the Γ-graded ring $R_\star$ contains units in R_γ for each $\gamma \in \Gamma$.

Let $\theta\colon\, \Gamma \to \Gamma'$ be a homomorphism of Picard groupoids and let $R_\star$ be a Γ-graded commutative ring. The pullback functor θ^* from Γ'-graded modules to Γ-graded modules has a left adjoint $\theta_!$, given by left Kan extension along θ.

Proposition 2.7. *Suppose $\mathcal{C}$ is cocomplete and symmetric monoidal, and that the symmetric monoidal structure preserves colimits in each variable. Then the functor $\theta_!\colon\, \mathcal{C}_\Gamma \to \mathcal{C}'_\Gamma$ is symmetric monoidal.*

Proof. For M, N objects of $\mathcal{C}_\Gamma$, we consider the square

$$\begin{array}{ccc} \Gamma^{\mathrm{op}} \times \Gamma^{\mathrm{op}} & \xrightarrow{+} & \Gamma^{\mathrm{op}} \\ \downarrow & & \downarrow \\ (\Gamma')^{\mathrm{op}} \times (\Gamma')^{\mathrm{op}} & \xrightarrow[+]{} & (\Gamma')^{\mathrm{op}}. \end{array}$$

The object $\theta_!(M \otimes N)$ is obtained by starting with $M \otimes N \colon \Gamma^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \to \mathcal{C}$ and taking Kan extension along the two functors in the upper-right portion of the square. Because the tensor product preserves colimits in each variable, the composite Kan extension of $M \otimes N$ along the lower-left portion of the square is canonically isomorphic to $(\theta_! M) \otimes (\theta_! N)$. The natural isomorphism making the square commute determines a natural isomorphism between these two composites. Similar diagrams show that when θ preserves the unit and is compatible with the associativity, symmetry, and unit isomorphisms, $\theta_!$ does the same. ☐

In particular, the ring $R_\star$ gives rise to a Γ'-graded ring $(\theta_! R)_\star$ defined by the formula

$$(\theta_! R)_{\gamma'} = \operatorname{colim}_{\gamma' \to \theta(\gamma)} R_\gamma.$$

Moreover, an $R_\star$-module $M_\star$ determines an $(\theta_! R)_\star$-module $(\theta_! M)_\star$.

We also have the notion of a θ-graded ring map $R_\star \to {R_\star}'$, which is just a Γ'-graded ring map $\theta_! R_\star \to {R_\star}'$. Given a θ-graded ring map $R_\star \to {R_\star}'$, we obtain a functor

$$(-) \otimes_{R_\star} {R_\star}' \colon \operatorname{Mod}_{R_\star} \longrightarrow \operatorname{Mod}_{{R_\star}'}$$

which sends the $R_\star$-module $M_\star$ to the ${R_\star}'$-module $M'_\star := M_\star \otimes_{R_\star} {R_\star}'$ defined by

$$M' = (\theta_! M)_\star \otimes_{(\theta_! R)_\star} {R_\star}'.$$

Here the tensor product on the right is the usual base-change along a Γ'-graded ring map.

Proposition 2.8. *For a map $\theta \colon \Gamma \to \Gamma'$ and a θ-graded map $R_\star \to {R_\star}'$, the functor $(-) \otimes_{R_\star} {R_\star}' \colon \operatorname{Mod}_{R_\star} \longrightarrow \operatorname{Mod}_{{R_\star}'}$ is symmetric monoidal. In particular, it extends to a functor $(-) \otimes_{R_\star} {R_\star}' \colon \operatorname{Alg}_{R_\star} \longrightarrow \operatorname{Alg}_{{R_\star}'}$ between categories of algebra objects.*

Proposition 2.9. *Suppose $\mathcal{A}$ is an additive symmetric monoidal category with unit $\mathbb{I}$ such that the monoidal product is additive in each variable, and that we have a symmetric monoidal functor $\Gamma \to \operatorname{Pic}(\mathcal{A})$ given by $\gamma \mapsto A^\gamma$. Then there is a canonical additive, lax symmetric monoidal functor $\phi \colon \mathcal{A} \to \mathrm{Ab}_\Gamma$, sending M to the object $M_\star$ with*

$$M_\gamma = \operatorname{Hom}(A^\gamma, M).$$

In particular, $\mathbb{I}_\star$ is a Γ-graded commutative ring, and ϕ lifts to the category of $\mathbb{I}_\star$-modules.

Proof. Since $\mathcal{A}$ is additive, the set $\operatorname{Hom}(M, N)$ of maps from M to N admits an abelian group structure such that composition is bilinear. This determines the functor ϕ. It remains to show that ϕ is lax symmetric monoidal.

The lax monoidal structure map sends a pair $(A^\alpha \to M)$ in M_α and $(A^\beta \to N)$ in N_β to the composite determined by

$$A^{\alpha+\beta} \xleftarrow{\sim} A^\alpha \otimes A^\beta \to M \otimes N,$$

an element in $(M \otimes N)_{\alpha+\beta}$. The natural associativity and commutativity diagrams

$$\begin{array}{ccc} A^\alpha \otimes A^\beta & \xrightarrow{\sim} & A^\beta \otimes A^\alpha \\ \downarrow & & \downarrow \\ M \otimes N & \xrightarrow{\sim} & N \otimes M \end{array} \qquad \begin{array}{ccc} (A^\alpha \otimes A^\beta) \otimes A^\gamma & \xrightarrow{\sim} & A^\alpha \otimes (A^\beta \otimes A^\gamma) \\ \downarrow & & \downarrow \\ (M \otimes N) \otimes P & \xrightarrow{\sim} & M \otimes (N \otimes P) \end{array}$$

(together with a similar unitality diagram) reduce the proof that ϕ is a lax symmetric monoidal functor to the fact that $\Gamma \to \mathrm{Pic}(\mathcal{A})$ is symmetric monoidal. □

Definition 2.10. Suppose $\mathcal{A}$ is an additive symmetric monoidal category such that the monoidal product is additive in each variable, and that we have a symmetric monoidal functor $\Gamma \to \mathrm{Pic}(\mathcal{A})$ given by $\gamma \mapsto A^\gamma$. The shift operator $\Sigma^\gamma \colon \mathcal{A} \to \mathcal{A}$ is defined by

$$\Sigma^\gamma M = A^\gamma \otimes M.$$

We then define

$$\mathrm{Hom}(M, N)_\gamma := \mathrm{Hom}(\Sigma^\gamma M, N).$$

The notation is compatible with the shift notation for Γ-graded abelian groups, because there is a natural isomorphism $(\Sigma^\gamma M)_\star \cong \Sigma^\gamma(M_\star)$.

Proposition 2.11. *In the situation of the previous definition, the Γ-graded abelian groups $\mathrm{Hom}(-,-)_\star$ make $\mathcal{A}$ into a category enriched in $\mathbb{I}_\star$-modules. More, this enrichment is compatible with the symmetric monoidal structure.*

Proof. There are canonical isomorphisms $\Sigma^{\alpha+\beta} L \cong \Sigma^\alpha \Sigma^\beta L$. Using this, we may define composition of graded maps by

$$\begin{aligned} \mathrm{Hom}(\Sigma^\alpha M, N) \otimes \mathrm{Hom}(\Sigma^\beta L, M) &\longrightarrow \mathrm{Hom}(\Sigma^\alpha M, N) \otimes \mathrm{Hom}(\Sigma^\alpha \Sigma^\beta L, \Sigma^\alpha M) \\ &\longrightarrow \mathrm{Hom}(\Sigma^{\alpha+\beta} L, N). \end{aligned}$$

This composition is associative, and the unit $\mathbb{I}_\star \to \mathrm{Hom}(M, M)_\star$ sends $f \colon A^\gamma \to \mathbb{I}$ to $f \otimes \mathrm{id}_M$. □

Remark 2.12. In section 14 of [15], a group cohomology element in $H^3(\pi_0\Gamma; \pi_1\Gamma)$ is described which obstructs our ability to make Γ-grading monoidal, in the sense of the functor $\otimes$ inducing an associative exterior product $\otimes \colon \pi_\alpha(X) \otimes \pi_\beta(Y) \to \pi_{\alpha+\beta}(X \otimes Y)$. This group cohomology element is the unique k-invariant of the classifying space $B\Gamma$.

Since Γ is assumed symmetric monoidal, $B\Gamma$ admits an infinite delooping and one can calculate that this k-invariant must vanish. This removes the obstruction to $\otimes$ inducing a monoidal pairing. However, this becomes replaced by a spectrum k-invariant

$$\varepsilon \in H^2(H\pi_0\Gamma, \pi_1\Gamma) \cong \mathrm{Hom}(\pi_0\Gamma, \pi_1\Gamma)[2]$$

which classifies the "sign rule."

More specifically, the sign rule is equivalent to a bilinear pairing $\pi_0\Gamma \times \pi_0\Gamma \to \pi_1\Gamma$ sending $\alpha, \beta \in \pi_0\Gamma$ to the element $\varepsilon_{\alpha,\beta} \in \pi_1\Gamma$ such that for $x \in \pi_\alpha X$ and $y \in \pi_\beta Y$, $x \otimes y = \varepsilon_{\alpha,\beta}(y \otimes x)$. (The elements $\varepsilon_{\alpha,\beta}$ are not invariant under equivalence; the isomorphism with the group of 2-torsion homomorphisms indicates that such Picard groupoids are determined completely by the $\varepsilon_{\alpha,\alpha}$, together describing a 2-torsion homomorphism $\pi_0\Gamma \to \pi_1\Gamma$ [16].)

Remark 2.13. One needs to be extremely cautious with isomorphisms between Γ-graded objects due to the sign rule. For example, a casual expression like

$$F_{R_\star}(\Sigma^\alpha M_\star, \Sigma^\beta N_\star) = \Sigma^{\beta-\alpha} F_{R_\star}(M_\star, N_\star)$$

hides several implicit isomorphisms [17].

2.2 Graded Azumaya algebras

We continue to fix a Picard groupoid Γ and let $R_\star$ be a Γ-graded commutative ring with module category $\mathrm{Mod}_{R_\star}$.

Definition 2.14. If $A_\star$ is an algebra in $\mathrm{Mod}_{R_\star}$ with multiplication μ, the opposite algebra $A_\star^{\mathrm{op}}$ is the algebra with the same underlying object and unit, but with multiplication $\mu \circ \tau$ precomposed with the twist isomorphism τ.

Definition 2.15. A Γ-graded Azumaya $R_\star$-algebra is an associative algebra $A_\star$ in the category $\mathrm{Mod}_{R_\star}$ such that

- the underlying module $A_\star$ is a graded projective generator of the category $\mathrm{Mod}_{R_\star}$, and
- the natural map of algebras $A_\star \otimes_{R_\star} A_\star^{\mathrm{op}} \to \mathrm{End}_{R_\star}(A_\star)$, adjoint to the left action

$$(A_\star \otimes_{R_\star} A_\star^{\mathrm{op}}) \otimes_{R_\star} A_\star \xrightarrow{1\otimes\tau} A_\star \otimes_{R_\star} A_\star \otimes_{R_\star} A_\star^{\mathrm{op}} \xrightarrow{\mu(1\otimes\mu)} A_\star,$$

  is an isomorphism.

Proposition 2.16. *If $P_\star$ is a graded generator of the category $\mathrm{Mod}_{R_\star}$, then the endomorphism algebra $\mathrm{End}_{R_\star}(P_\star)$ is an Azumaya $R_\star$-algebra.*

Definition 2.17. Let $\mathrm{Cat}_{R_\star}$ be the 2-category of additive categories which are *left-tensored* over the monoidal category $\mathrm{Mod}_{R_\star}$: abelian categories $\mathcal{A}$ with a functor $\otimes\colon\ \mathrm{Mod}_{R_\star} \times \mathcal{A} \to \mathcal{A}$ which preserves colimits in each variable, together with a natural isomorphism

$$\mathbb{I} \otimes A \xrightarrow{\sim} A$$

and

$$(M \otimes N) \otimes A \xrightarrow{\sim} M \otimes (N \otimes A)$$

that respects the unit and pentagon axioms.

Morphisms in $\mathrm{Cat}_{R_\star}$ are $\mathrm{Mod}_{R_\star}$-linear: colimit-preserving functors $F\colon \mathcal{A} \to \mathcal{A}'$, together with natural isomorphisms $M \otimes F(A) \to F(M \otimes A)$ that respect associativity and the unit isomorphisms. The 2-morphisms in $\mathrm{Cat}_{R_\star}$ are natural isomorphisms of functors which commute with the tensor structure.

Remark 2.18. In particular, a left-tensored category $\mathcal{A}$ inherits suspension operators by defining $\Sigma^\gamma M = (\Sigma^\gamma R) \otimes M$ via the left action. This allows us to define graded function objects by

$$F_{\mathcal{A}}(M, N)_\gamma = \mathrm{Hom}_{\mathcal{A}}(\Sigma^\gamma M, N).$$

This definition makes a $\mathrm{Mod}_{R_\star}$-linear category into a category enriched in $R_\star$-modules in such a way that $\mathrm{Mod}_{R_\star}$-linear functors preserve this enrichment.

Definition 2.19. The functor

$$\mathrm{Mod}\colon\ \mathrm{Alg}_{R_\star} \to \mathrm{Cat}_{R_\star}$$

sends an $R_\star$-algebra $A_\star$ to the category $\mathrm{Mod}_{A_\star}$ of right $A_\star$-modules in $\mathrm{Mod}_{R_\star}$, viewed as left-tensored over $R_\star$ via the tensor product in the underlying category $\mathrm{Mod}_{R_\star}$. A map $A_\star \to B_\star$ is sent to the functor $\mathrm{Mod}_{A_\star} \to \mathrm{Mod}_{B_\star}$ given by extension of scalars. (Composite ring maps have natural isomorphisms of composite functors which satisfy a coherence condition: Mod is a pseudofunctor.)

The following theorems have proofs which are essentially identical to their classical counterparts; for example, see [14]. We will sketch the main points below.

Theorem 2.20 (Graded Eilenberg–Watts)**.** *The map sending an $A_\star$-$B_\star$-bimodule $L_\star$ to the functor*

$$N_\star \mapsto N_\star \otimes_{A_\star} L_\star$$

determines a canonical equivalence of categories from the category ${}_{A_\star}\mathrm{Mod}_{B_\star}$ of $A_\star$-$B_\star$-bimodules to the category of $\mathrm{Mod}_{R_\star}$-linear functors $\mathrm{Mod}_{A_\star} \to \mathrm{Mod}_{B_\star}$.

Proof. Functors of the form $(-) \otimes_{A_\star} L_\star$ are colimit-preserving and come with a natural associativity isomorphism

$$M_\star \otimes_{R_\star} (N_\star \otimes_{A_\star} L_\star) \to (M_\star \otimes_{R_\star} N_\star) \otimes_{A_\star} L_\star,$$

making them maps $\mathrm{Mod}_{A_\star} \to \mathrm{Mod}_{B_\star}$ in $\mathrm{Cat}_{R_\star}$. This produces the desired functor. Conversely, any $\mathrm{Mod}_{R_\star}$-linear functor $G\colon\ \mathrm{Mod}_{A_\star} \to \mathrm{Mod}_{B_\star}$ preserves the shift operators Σ^γ and extends to a Γ-graded functor. In particular, the action map

$$A_\star \otimes_{R_\star} G(A_\star) \to G(A_\star \otimes_{R_\star} A_\star) \to G(A_\star)$$

induced by the multiplication is adjoint to a ring map $A_\star \to F_{B_\star}(G(A_\star), G(A_\star))$ making $G(A_\star)$ into an $A_\star$-$B_\star$-bimodule. Given the canonical presentation

$$N_\star \cong \mathrm{colim}\left(\bigoplus_{\Sigma^\delta A_\star \to \Sigma^\gamma A_\star \to N_\star} \Sigma^\delta A_\star \rightrightarrows \bigoplus_{\Sigma^\gamma A_\star \to N_\star} \Sigma^\gamma A_\star \right),$$

the two colimit-preserving functors G and $(-)\otimes_{A_\star} G(A_\star)$ both give us naturally isomorphic presentations

$$G(N_\star) \cong \operatorname{colim}\left(\bigoplus_{\Sigma^\delta A_\star \to \Sigma^\gamma A_\star \to N_\star} \Sigma^\delta G(A_\star) \rightrightarrows \bigoplus_{A_\star \to N_\star} \Sigma^\gamma G(A_\star)\right).$$

Therefore, these two functors are canonically equivalent. $\square$

Theorem 2.21 (Graded Morita theory)**.** *Let $A_\star$ be an $R_\star$-algebra, and $\mathrm{Mod}^{\mathrm{cg}}_{A_\star}$ be the full subcategory of $\mathrm{Mod}_{A_\star}$ spanned by the graded generators $P_\star$. Then there are canonical pullback diagrams of categories:*

$$\begin{array}{ccccc}
\mathrm{Pic}({}_{A_\star}\mathrm{Mod}_{A_\star}) & \longrightarrow & (\mathrm{Mod}^{\mathrm{cg}}_{A_\star})^{\simeq} & \longrightarrow & \{\mathrm{Mod}_{A_\star}\} \\
\downarrow & & \downarrow & & \downarrow \\
\{A_\star\} & \longrightarrow & (\mathrm{Alg}_{R_\star})^{\simeq} & \longrightarrow & \mathrm{Cat}_{R_\star}\,.
\end{array}$$

More generally, the fiber of $(\mathrm{Mod}^{\mathrm{cg}}_{A_\star})^{\simeq} \to (\mathrm{Alg}_{R_\star})^{\simeq}$ over an algebra $B_\star$ is either empty or a principal torsor for the Picard group $\mathrm{Pic}({}_{A_\star}\mathrm{Mod}_{A_\star})$ of the category of bimodules.

Proof. We will first identify $\mathrm{Mod}^{\mathrm{cg}}_{A_\star}$ with the right-hand fiber product. The pullback of the diagram $\mathrm{Alg}_{R_\star} \to \mathrm{Cat}_{R_\star} \leftarrow \{\mathrm{Mod}_{A_\star}\}$ is the category of pairs $(B_\star, \phi)$, where $B_\star$ is an $R_\star$-algebra and ϕ is an equivalence $\mathrm{Mod}_{B_\star} \to \mathrm{Mod}_{A_\star}$ in $\mathrm{Cat}_{R_\star}$. Such a functor is colimit-preserving, so by the graded Eilenberg–Watts theorem such a functor is represented by a certain type of pair $(B_\star, P_\star)$. For this functor to be an equivalence, the graded generator $B_\star$ must map to a graded generator $P_\star$, and we must have $B_\star = \mathrm{End}_{A_\star}(P_\star)$. It remains to show that any such $P_\star$ determines an equivalence of categories.

Given a right $A_\star$-module $P_\star$ as in the statement, we obtain an $R_\star$-algebra $B_\star = F_{A_\star}(P_\star, P_\star)$ and a functor $(-)\otimes_{B_\star} P_\star \colon \mathrm{Mod}_{B_\star} \to \mathrm{Mod}_{A_\star}$. This functor is colimit-preserving. It also has a colimit-preserving right adjoint $F_{A_\star}(P_\star, -)$ because $P_\star$ is finitely generated projective.

The unit map

$$M_\star \to F_{A_\star}(P_\star, P_\star \otimes_{B_\star} M_\star)$$

is an isomorphism when $M_\star = \Sigma^\gamma B_\star$. Both sides preserve colimits, and so applying this unit to a resolution $F_1 \to F_0 \to M_\star \to 0$ where F_i are (graded) free modules shows that the unit is always an isomorphism.

The counit map

$$F_{A_\star}(P_\star, N_\star) \otimes_{B_\star} P_\star \to N_\star$$

is an isomorphism when $N_\star = P_\star$. Because the set of objects $\Sigma^\gamma P_\star$ is a set of generators there always exists a resolution $F_1 \to F_0 \to N_\star \to 0$ where F_i are direct sums of shifts of $P_\star$. Again, as the functors in question preserve colimits, the counit is always an isomorphism.

We now consider the left-hand square. As pullbacks can be calculated iteratively, the pullback of a diagram $B_\star \to \mathrm{Alg}_{R_\star} \leftarrow (\mathrm{Mod}^{\mathrm{cg}}_{A_\star})^{\simeq}$ is equivalent to the pullback of the diagram $\{\mathrm{Mod}_{B_\star}\} \to \mathrm{Cat}_{R_\star} \leftarrow \{\mathrm{Mod}_{A_\star}\}$. If these categories are inequivalent as $R_\star$-linear categories, this is empty. If these categories are equivalent, then this is instead isomorphic to the groupoid of self-equivalences of $\mathrm{Mod}_{A_\star}$. Such a functor is given up to unique isomorphism by tensoring with an $A_\star$-bimodule $P_\star$, and there must exist an inverse given by tensoring with an $A_\star$-bimodule $Q_\star$. For these to be inverse to each other, we must have an isomorphism of $A_\star$-bimodules

$$P_\star \otimes_{A_\star} Q_\star \cong Q_\star \otimes_{A_\star} P_\star \cong A_\star.$$

Such a $Q_\star$ exists if and only if $P_\star$ is an invertible element in the category of bimodules. □

Corollary 2.22 (Graded Rosenberg-Zelinsky exact sequence)**.** *For an Azumaya $R_\star$-algebra $A_\star$, there is an exact sequence of groups*

$$1 \to (R_0)^\times \to (A_0)^\times \to \mathrm{Aut}_{\mathrm{Alg}_{R_\star}}(A_\star) \to \pi_0 \operatorname{Pic}(\mathrm{Mod}_{R_\star}).$$

The group $\operatorname{Pic}(\mathrm{Mod}_{R_\star})$ acts on the set of isomorphism classes of compact generators of $\mathrm{Mod}_{A_\star}$ with quotient the set of isomorphism classes of Azumaya $R_\star$-algebras $B_\star$ such that $\mathrm{Mod}_{A_\star} \simeq \mathrm{Mod}_{B_\star}$. The stabilizer of $A_\star$, viewed as a right $A_\star$-module, is the image of the outer automorphism group in $\operatorname{Pic}(\mathrm{Mod}_{R_\star})$.

Proof. We consider the pullback diagram of categories

$$\begin{array}{ccc} \operatorname{Pic}({}_{A_\star}\mathrm{Mod}_{A_\star}) & \longrightarrow & (\mathrm{Mod}^{\mathrm{cg}}_{A_\star})^{\simeq} \\ \downarrow & & \downarrow \\ \{A_\star\} & \longrightarrow & (\mathrm{Alg}_{R_\star})^{\simeq} \end{array}$$

obtained from graded Morita theory. This is a homotopy pullback diagram of groupoids, and so we may take the nerve and obtain a long exact sequence in homotopy groups at the basepoint $A_\star$ of Pic. Put together, this gives an exact sequence

$$1 \to \mathrm{Aut}_{\operatorname{Pic}({}_{A_\star}\mathrm{Mod}_{A_\star})}(A_\star) \to \mathrm{Aut}_{\mathrm{Mod}_{A_\star}}(A_\star) \to \mathrm{Aut}_{\mathrm{Alg}_{R_\star}}(A_\star) \to \pi_0 \operatorname{Pic}({}_{A_\star}\mathrm{Mod}_{A_\star}).$$

Moreover, the category of $A_\star$-bimodules is equivalent to the category of modules over $A_\star \otimes_{R_\star} A_\star^{\mathrm{op}}$, which is Morita equivalent to $R_\star$. This gives us a symmetric monoidal equivalence of categories $\operatorname{Pic}({}_{A_\star}\mathrm{Mod}_{A_\star}) \simeq \operatorname{Pic}(R_\star)$ that carries $A_\star$ to $R_\star$. The desired description of this exact sequence follows by identifying $\mathrm{Aut}_{\mathrm{Mod}_{A_\star}}(A_\star)$ with $A_0^\times$ and $\mathrm{Aut}_{\operatorname{Pic}(R_\star)}(R_\star)$ with $R_0^\times$.

Similarly, the description of the action of Pic follows by identifying this fiber square with the principal fibration associated to the map $(\mathrm{Alg}_{R_\star})^{\simeq} \to (\mathrm{Cat}_{R_\star})^{\simeq}$. □

Remark 2.23. In the exact sequence above, suppose $v \in A_\gamma$ is a unit in the graded ring $A_\star$. Then conjugation by v determines an element in $\mathrm{Aut}_{\mathrm{Alg}_{R_\star}}(A_\star)$ whose image in $\operatorname{Pic}(\mathrm{Mod}_{R_\star})$ is $[\Sigma^\gamma A]$.

2.3 Matrix algebras over graded commutative rings

Definition 2.24. Let $R_\star$ be a Γ-graded commutative ring. An $R_\star$-algebra is a matrix $R_\star$-algebra if it is isomorphic to the endomorphism $R_\star$-algebra

$$\mathrm{End}_{R_\star}(M_\star) = F_{R_\star}(M_\star, M_\star)$$

of an $R_\star$-module of the form $M_\star \cong {R_\star}^I$ for some Γ-graded set I.

In general we write $\mathrm{Mat}_I(R_\star)$ for the Γ-graded matrix algebra $\mathrm{End}_{R_\star}({R_\star}^I)$ and $\mathrm{GL}_I(R_\star)$ for the group $[\mathrm{Aut}_{R_\star}({R_\star}^I)]_0^\times$ of automorphisms of the graded $R_\star$-module ${R_\star}^I$.

Proposition 2.25. *If $R_\gamma = 0$ for $\gamma \neq 0$ then there is an isomorphism of groups*

$$\mathrm{GL}_I(R) \cong \prod_{\gamma \in \Gamma} \mathrm{GL}_{I_\gamma}(R_0),$$

where the groups on the right are the usual general linear groups of the commutative ring R_0.

Proposition 2.26. *If I is a finite Γ-graded set, then I is a disjoint union of $\mathrm{Hom}(\gamma_i, -)$ for some I, and there is a canonical isomorphism of $R_\star$-modules*

$$\mathrm{End}_{R_\star}({R_\star}^I) \cong \bigoplus_{i,j \in I} \Sigma^{\gamma_j - \gamma_i} R_\star.$$

In particular, there is a natural Γ-graded set ∂I such that it is of the form ${R_\star}^{\partial I}$.

Proposition 2.27. *The formation of matrix algebras is compatible with base-change. That is, for any homomorphism $\theta\colon \Gamma \to \Gamma'$ of abelian groups, any θ-graded ring map $R_\star \to {R_\star}'$, and any finite Γ-graded set I, the canonical ${R_\star}'$-algebra map*

$$\mathrm{Mat}_I(R_\star) \otimes_{R_\star} {R_\star}' \longrightarrow \mathrm{Mat}_{\theta_! I}({R_\star}')$$

is an isomorphism.

Proof. Write ∂I for the Γ-graded set as in the previous proposition. First, let us assume that θ is the identity of Γ, so that $R_\star \to {R_\star}'$ is just a Γ-graded ring map. Then $\theta_! \partial I = \partial I$ and the map ${R_\star}^{\partial I} \otimes_{R_\star} {R_\star}' \to ({R_\star}')^{\partial I}$ is an equivalence between free ${R_\star}'$-modules on the same Γ-graded set.

Now suppose instead that θ is arbitrary and ${R_\star}' = \theta_! R_\star$. Then the desired map is a composite

$${R_\star}^{\partial I} \otimes_{R_\star} {R_\star}' \cong \theta_!({R_\star}^{\partial I}) \cong (\theta_! R_\star)^{\partial \theta_! I}.$$

Finally, an arbitrary θ-graded ring map $R \to R'$ is a composite of ring maps of the type treated above, so the result follows. □

2.4 Derivations and Hochschild cohomology

The following recalls some of Quillen's work on cohomology for associative rings [18].

In a symmetric monoidal abelian category where the monoidal operation $\otimes$ preserves colimits in each variable, any algebra A sits in a short exact sequence

$$0 \to \mathbb{L}_A \to A \otimes A^{\mathrm{op}} \to A \to 0$$

of A-bimodules, split (as left modules) by the unit. If A is the tensor algebra on a projective object P, then $\mathbb{L}_A$ can be identified with the projective bimodule $A \otimes P \otimes A^{\mathrm{op}}$. Moreover, for any A-bimodule M with associated square-zero extension $M \rtimes A \to A$ in Alg_A, there are canonical isomorphisms

$$\mathrm{Der}(A, M) = \mathrm{Hom}_{\mathrm{Alg}_A/A}(A, M \rtimes A) \cong \mathrm{Hom}_{{}_A\mathrm{Mod}_A}(\mathbb{L}_A, M).$$

This allows us to relate the derived functors of derivations, in the sense of [18], to Hochschild cohomology in this category. The André-Quillen cohomology groups of A with coefficients in M may be identified with the nonabelian derived functors $\mathrm{Der}^s(A, M)$. Applying the right derived functors of $\mathrm{Hom}_{{}_A\mathrm{Mod}_A}(-, M)$ to the exact sequence defining $\mathbb{L}_A$ gives us isomorphisms

$$\mathrm{Der}^s(A, M) \to HH^{s+1}(A, M)$$

for $s > 0$ and an exact sequence

$$0 \to HH^0(A, M) \to M \to \mathrm{Der}(A, M) \to HH^1(A, M) \to 0.$$

Proposition 2.28. *Suppose $A_\star$ is an Azumaya $R_\star$-algebra. For any $A_\star$-bimodule $M_\star$ in the category of Γ-graded $R_\star$-modules, we have a short exact sequence*

$$0 \to HH^0(A_\star, M_\star) \to M_\star \to \mathrm{Der}(A_\star, M_\star) \to 0.$$

Both the Hochschild cohomology groups $HH^s_{R_\star}(A_\star, M_\star)$ and the derived functors $\mathrm{Der}^s_{R_\star}(A_\star, M_\star)$ vanish for $s > 0$.

Proof. Consider the short exact sequence

$$0 \to \mathbb{L}_{A_\star} \to A_\star \otimes_{R_\star} A_\star^{\mathrm{op}} \to A_\star \to 0$$

of bimodules. The center bimodule is free, hence projective. Moreover, under the chain of Morita equivalences

$$\mathrm{Mod}_{R_\star} \simeq \mathrm{Mod}_{\mathrm{End}_{R_\star}(A_\star)} \simeq \mathrm{Mod}_{A_\star \otimes_{R_\star} A_\star^{\mathrm{op}}},$$

the image of the projective $R_\star$-module $R_\star$ is $A_\star$, and hence $A_\star$ is also projective. Therefore, the sequence splits and $\mathbb{L}_{A_\star}$ is projective too. ☐

3 Obstruction theory

3.1 Gradings for ring spectra

Definition 3.1. Let R be an $\mathbb{E}_\infty$-ring spectrum, with Γ_R the algebraic Picard groupoid of invertible R-modules and homotopy classes of equivalences; similarly let $\Gamma_{\mathbb{S}}$ be the Picard groupoid of the sphere spectrum. A *grading* for R is a Picard groupoid Γ together with a commutative diagram

$$\begin{array}{ccc} & \Gamma & \\ {\scriptstyle\nu}\nearrow & & \searrow \\ \Gamma_{\mathbb{S}} & \longrightarrow & \Gamma_R \end{array}$$

of Picard groupoids.

The *period* of the grading is the minimum of the set

$$\{n > 0 \mid \nu[S^n] = 0 \text{ in } \pi_0\Gamma\} \cup \{\infty\},$$

where $[S^n] \in \pi_0\Gamma_{\mathbb{S}}$ is the equivalence class of the n-sphere.

A grading provides a chosen lift of the suspension ΣR to Γ such that the twist on $\Sigma R \otimes \Sigma R$ lifts the automorphism $-1 \in (\pi_0 R)^\times$; it also provides an action $\Gamma_{\mathbb{S}} \times \Gamma \to \Gamma$, $(n, \gamma) \mapsto n + \gamma$, compatible with that on Γ_R. The minimal and maximal options are $\Gamma_{\mathbb{S}}$-grading (usually referred to as "$\mathbb{Z}$-grading") and Γ_R-grading (usually referred to as "Picard-grading"). If R is connective (and nontrivial) then $\pi_0\Gamma_{\mathbb{S}} \to \pi_0\Gamma_R$ is a monomorphism, and so R has period ∞ (usually referred to as "not being periodic").

Throughout this section we will assume that we have chosen a grading for R. This produces elements $R^\gamma \in \mathrm{Mod}_R$ for $\gamma \in \Gamma$ and gives the category of R-modules Γ-graded homotopy groups $\pi_\star M$ as in Section 5.1. These homotopy groups preserve coproducts and filtered colimits, as well as take cofiber sequences to long exact sequences. The fact that weak equivalences are detected on $\mathbb{Z}$-graded homotopy groups implies the following.

Proposition 3.2. *If R has a grading by Γ, a map $X \to Y$ of R-modules is an equivalence if and only if the map $\pi_\star X \to \pi_\star Y$ is an isomorphism of $\pi_\star R$-modules.*

Proposition 3.3. *If A is an R-algebra, the Γ-graded groups $\pi_\star A$ form a $\pi_\star R$-algebra. If A is a commutative R-algebra, $\pi_\star A$ is a graded-commutative $\pi_\star R$-algebra.*

3.2 Picard-graded model structures

In this section we describe model structures on categories of R-modules and R-algebras based on using elements of $\mathrm{Pic}(R)$ as basic cells. The structure of this section is based on Goerss–Hopkins' work on obstruction theory for algebras over an operad [19], which in turn is based on Bousfield's work [20]. We carry this out under the simplifying assumptions that we are not using an auxiliary homology theory, and that the operad in question is the associative operad. However, we will remove the assumption that the base category is the stable homotopy category, and allow ourselves the use of homotopy groups graded by a Picard groupoid Γ rather than integer-graded homotopy groups.

In this section we work in the flat stable model category structure on symmetric spectra (the S-model structure of [21]). Fix a commutative model for our $\mathbb{E}_\infty$-ring spectrum R, and let Mod_R be the category of R-modules. We also fix a grading Γ for R as in the previous section, giving any R-module M natural Γ-graded homotopy groups $\pi_\star M$.

According to [21, 2.6-2.7], the category Mod_R is a cofibrantly generated, proper, stable model category with generating sets of cofibrations and acyclic cofibrations with cofibrant source; it is also, compatibly, a simplicial model category (e.g. see [22] for references in this direction). The smash product $\wedge_R$ and function object $F_R(-,-)$ give Mod_R a symmetric monoidal closed structure under which Mod_R is a monoidal model category, and the category Alg_R of associative R-algebras is a cofibrantly generated simplicial model category with fibrations and weak equivalences detected in Mod_R [23]. We let $\mathbb{T}$ denote the monad taking M to the free R-algebra $\mathbb{T}(M) = \bigvee M^{\wedge_R n}$; algebras over $\mathbb{T}$ are associative R-algebras.

The following definitions are dual to those in Bousfield [20], taking the category Γ as generating a class $\mathcal{P}$ of cogroup objects.

Definition 3.4. Let $\mathcal{D}_R$ denote the homotopy category of Mod_R.

1. A map $p\colon X \to Y$ in $\mathcal{D}_R$ is Pic-*epi* if the map $\pi_\star X \to \pi_\star Y$ is surjective.
2. An object $A \in \mathcal{D}_R$ is Pic-*projective* if the map $p_*\colon [A,X] \to [A,Y]$ is surjective whenever $p\colon X \to Y$ is Pic-epi.
3. A morphism $A \to B$ in Mod_R is a Pic-*projective cofibration* if it has the left lifting property with respect to all Pic-epi fibrations in Mod_R.

Remark 3.5. Technically speaking, we should include the group Γ in the notation, but we do not.

Any object $P \in \mathrm{Pic}(R)$ with a lift to an element $\gamma \in \Gamma$ is is automatically Pic-projective, and the class of projective cofibrations is closed under coproducts, suspensions, and desuspensions. There are enough Pic-projective objects: to construct a Pic-projective P and a map $P \to X$ inducing a surjection $\pi_\star P \to \pi_\star X$, we can choose generators $\{x_\alpha \in \pi_{\gamma_\alpha} X\}$ of $\pi_\star X$ which are represented by a map $\bigvee_\alpha R^{\gamma_\alpha} \to X$. We can then describe a model structure on the category $s\,\mathrm{Mod}_R$ of simplicial R-modules.

Definition 3.6. Let $f\colon X_\bullet \to Y_\bullet$ be a map of simplicial R-modules.

1. The map f is a Pic-*equivalence* if the map $\pi_\gamma f\colon \pi_\gamma X_\bullet \to \pi_\gamma Y_\bullet$ is a weak equivalence of simplicial abelian groups for all $\gamma \in \Gamma$.

2. The map f is a Pic-*fibration* if it is a Reedy fibration and the map $\pi_\gamma f\colon \pi_\gamma X_\bullet \to \pi_\gamma Y_\bullet$ is a fibration of simplicial abelian groups for all $\gamma \in \Gamma$.

3. The map f is a Pic-*cofibration* if the latching maps

$$X_n \coprod_{L_n X} L_n Y \to Y_n$$

are Pic-projective cofibrations for $n \geq 0$.

Theorem 3.7 ([20]). *These definitions give the category $s\,\mathrm{Mod}_R$ of simplicial R-modules the structure of a simplicial model category, which we call the* Pic-resolution model structure. *This model structure is cofibrantly generated, and has generating sets of cofibrations and acyclic cofibrations with cofibrant source. The forgetful functor to simplicial R-modules (with the Reedy model structure) creates fibrations.*

As in [19, Section 3], for a simplicial R-module X and $\gamma \in \Gamma$ we have "natural" homotopy groups $\pi_n^\natural(X;\gamma)$. On geometric realization there is a homotopy spectral sequence with E_2-term

$$\pi_p \pi_\gamma(X) \Rightarrow \pi_{p+\gamma}|X|\,.$$

The E_2-term of this spectral sequence comes from an exact couple, the spiral exact sequence [19, Lemma 3.9]:

$$\cdots \to \pi_{n-1}^\natural(X;\gamma) \to \pi_n^\natural(X;\gamma) \to \pi_n\pi_\gamma(X) \to \pi_{n-2}^\natural(X;\gamma) \to \cdots$$

As applications of the Pic-resolution model structure, we obtain Pic-graded Künneth and universal coefficient spectral sequences.

Theorem 3.8. *For $X, Y \in \mathcal{D}_R$, there are spectral sequences of Γ-graded $R_\star$-modules:*

$$\mathrm{Tor}^{R_\star}_{p,\gamma}(\pi_\star X, \pi_\star Y) \Rightarrow \pi_{p+\gamma}(X \wedge_R Y)$$
$$\mathrm{Ext}^{s,\tau}_{R_\star}(\pi_\star X, \pi_\star Y) \Rightarrow \pi_{-s-\tau} F_R(X,Y)$$

Proof. Lift X and Y to Mod_R, cofibrant or fibrant as appropriate. Then choose a cofibrant replacement $P \to X$, where X is viewed as a constant simplicial object in the Pic-resolution model category structure. The result is a simplicial R-module, augmented over X, such that the map $|P| \to X$ is a weak equivalence and such that the associated simplicial object $\pi_\star P$ is levelwise projective as a Γ-graded $\pi_\star R$-module. The spectral sequences in question are associated to the geometric realization of $P \wedge_R Y$ and the totalization of $F_R(P,Y)$, which are equivalent to the derived smash $X \wedge_R Y$ and derived function object $F_R(X,Y)$, respectively. □

Corollary 3.9. *Suppose P is a cofibrant R-module such that $\pi_\star P$ is a projective $\pi_\star R$-module. Then $\pi_\star \mathbb{T}(P)$ is isomorphic to the free $\pi_\star R$-algebra on $\pi_\star P$.*

Proof. This follows by first observing that the Künneth formula degenerates to isomorphisms

$$\pi_\star(P \wedge_R \cdots \wedge_R P) \cong \pi_\star P \otimes_{\pi_\star R} \cdots \otimes_{\pi_\star R} \pi_\star P,$$

and then applying $\pi_\star$ to the identification

$$\mathbb{T}(P) \cong \bigvee_{k \geq 0} P^{\wedge_R k}$$

of R-modules. ☐

The Pic-resolution model structure on simplicial R-modules now lifts to R-algebras.

Theorem 3.10. *There is a simplicial model category structure on $s\,\mathrm{Alg}_R$ such that the forgetful functor $s\,\mathrm{Alg}_R \to s\,\mathrm{Mod}_R$ creates weak equivalences and fibrations. We call this the* Pic-resolution model structure *on simplicial R-algebras. This model structure is cofibrantly generated, and has generating sets of cofibrations and acyclic cofibrations with cofibrant source.*

For each $X \in s\,\mathrm{Alg}_R$, there is a Pic*-equivalence $Y \to X$ with the following properties:*

1. *The simplicial object Y is cofibrant in the* Pic*-resolution model category structure on $s\,\mathrm{Alg}_R$.*
2. *[19, 3.7] There are objects Z_n, which are wedges of cofibrant R-modules in* $\mathrm{Pic}(R)$*, such that the underlying degeneracy diagram of Y is of the form*

$$Y_n = \mathbb{T}\left(\coprod_{\phi:[n]\twoheadrightarrow[m]} Z_m\right).$$

Given this structure, we can use Goerss–Hopkins' moduli tower of Postnikov approximations to produce an obstruction theory. This both classifies objects and constructs a Bousfield-Kan spectral sequence for spaces of maps between R-algebras using Γ-graded homotopy groups. In order to describe the resulting obstruction theories, let $\mathrm{Der}^s_{\mathrm{Alg}_{\pi_\star R}}$ denote the derived functors of derivations in the category of Γ-graded $\pi_\star R$-algebras as in section 2.4.

Theorem 3.11. *1. There are successively defined obstructions to realizing an algebra $A_\star \in \mathrm{Alg}_{\pi_\star R}$ by an R-algebra A in the groups*

$$\mathrm{Der}^{s+2}_{\mathrm{Alg}_{\pi_\star R}}(A_\star, \Omega^s A_\star),$$

and obstructions to uniqueness in the groups

$$\mathrm{Der}^{s+1}_{\mathrm{Alg}_{\pi_\star R}}(A_\star, \Omega^s A_\star),$$

for $s \geq 1$.

2. *For R-algebras X and Y, there are successively defined obstructions to realizing a map $f \in \mathrm{Hom}_{\mathrm{Alg}_{\pi_\star R}}(\pi_\star X, \pi_\star Y)$ in the groups*

$$\mathrm{Der}^{s+1}_{\mathrm{Alg}_{\pi_\star R}}(\pi_\star X, \Omega^s \pi_\star Y),$$

and obstructions to uniqueness in the groups

$$\mathrm{Der}^{s}_{\mathrm{Alg}_{\pi_\star R}}(\pi_\star X, \Omega^s \pi_\star Y),$$

for $s \geq 1$.

3. *Let $\phi \in \mathrm{Map}_{\mathrm{Alg}_R}(X, Y)$ be a map of R-algebras. Then there is a fringed, second quadrant spectral sequence abutting to*

$$\pi_{t-s}(\mathrm{Map}_{\mathrm{Alg}_R}(X, Y), \phi),$$

with E_2-term given by

$$E_2^{0,0} = \mathrm{Hom}_{\mathrm{Alg}_{\pi_\star R}}(\pi_\star X, \pi_\star Y)$$

and

$$E_2^{s,t} = \mathrm{Der}^{s}_{\mathrm{Alg}_{\pi_\star R}}(\pi_\star X, \Omega^t \pi_\star Y) \text{ for } t > 0.$$

This theorem is obtained using simplicial resolutions. Given an R-algebra A we form a simplicial resolution of A by free R-algebras, which becomes a resolution of $\pi_\star A$ by free $\pi_\star R$-algebras by Corollary 3.9. We get the spectral sequences for mapping spaces from the associated homotopy spectral sequence (see [20]). The obstruction theory for the construction of such A, instead, relies on constructing *partial* resolutions $P_n A$ as simplicial free R-algebras whose homotopy spectral sequence degenerates in a specific way, and then identifying the obstruction to extending the construction of $P_n(A)$ to $P_{n+1}(A)$ as lying in an André–Quillen cohomology group.

3.3 Algebraic Azumaya algebras

We now apply the obstruction theory of the previous section to the algebraic case. We continue to let R be an $\mathbb{E}_\infty$-ring spectrum with a grading by Γ.

We recall that an algebra A is an Azumaya R-algebra if A is a compact generator of $\mathcal{D}_R$ and the left-right action map $A \wedge_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$ is an equivalence in $\mathcal{D}_R$ [3].

Proposition 3.12. *Suppose A is an R-algebra such that $\pi_\star A$ is a projective $\pi_\star R$-module. Then $\pi_\star A$ is an Azumaya $\pi_\star R$-algebra if and only if A is an Azumaya R-algebra.*

Proof. The projectivity of $\pi_\star A$ makes the Künneth and universal coefficient spectral sequences of Theorem 3.8 degenerate. We find that that the action map $A \wedge_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$ becomes, on $\pi_\star$, the map $\pi_\star A \otimes_{\pi_\star R} \pi_\star A^{\mathrm{op}} \to \mathrm{End}_{\pi_\star R}(\pi_\star A)$, and so the two conditions are equivalent. □

We have a similar result about Morita triviality.

Proposition 3.13. *Let M be an R-module whose Γ-graded homotopy groups $\pi_\star M$ form a finitely generated projective $\pi_\star R$-module. The function spectrum $\mathrm{End}_R(M)$ has homotopy groups given by the $\pi_\star R$-algebra $\mathrm{Hom}_{\pi_\star R}(\pi_\star M, \pi_\star M)$. The center of this algebra is the image of $\pi_\star R$, and if $\pi_\star M$ is a graded generator this algebra is Morita equivalent to $\pi_\star R$ in the category of Γ-graded $\pi_\star R$-algebras.*

Definition 3.14. An R-algebra is said to be an *algebraic Γ-graded Azumaya algebra* over R if the multiplication on $\pi_\star A$ makes it into an Azumaya $\pi_\star R$-algebra.

We may apply the Goerss–Hopkins obstruction theory to algebraic Azumaya R-algebras. Much of the following is originally due to Baker–Richter–Szymik [3, 6.1].

Theorem 3.15. *1. Any Azumaya $\pi_\star R$-algebra is isomorphic to $\pi_\star A$ for some Γ-graded algebraic Azumaya R-algebra A.*

2. Suppose A is a Γ-graded algebraic Azumaya R-algebra. For any R-algebra S (not necessarily Azumaya), the natural map

$$[A, S]_{\mathrm{Alg}_R} \xrightarrow{\pi_\star} \mathrm{Hom}_{\mathrm{Alg}_{\pi_\star R}}(\pi_\star A, \pi_\star S),$$

is an isomorphism. For any map $\phi\colon A \to S$ of R-algebras (making $\pi_\star S$ into a $\pi_\star A$-bimodule) and any $t > 0$, we have an isomorphism

$$\pi_t(\mathrm{Map}_{\mathrm{Alg}_R}(A, S), \phi) \cong (\pi_t S)/HH^0(\pi_\star A, \Omega^t \pi_\star S).$$

3. If A is a Γ-graded algebraic Azumaya R-algebra, the homotopy groups of the space $\mathrm{Aut}_{\mathrm{Alg}_R}(A)$ satisfy

$$\pi_t(\mathrm{Aut}_{\mathrm{Alg}_R}(A), \mathrm{id}) \cong \begin{cases} \mathrm{Aut}_{\mathrm{Alg}_{\pi_\star R}}(\pi_\star A) & \text{if } t = 0, \\ \pi_t A/\pi_t R & \text{if } t > 0. \end{cases}$$

4. If A is a Γ-graded algebraic Azumaya R-algebra, then for $t > 0$ the sequence

$$0 \to \pi_t \mathrm{GL}_1(R) \to \pi_t \mathrm{GL}_1(A) \to \pi_t \mathrm{Aut}_{\mathrm{Alg}_R}(A) \to 0$$

is exact, and there is an exact sequence of potentially nonabelian groups

$$1 \to \pi_0 \mathrm{GL}_1(R) \to \pi_0 \mathrm{GL}_1(A) \to \pi_0 \mathrm{Aut}_{\mathrm{Alg}_R}(A) \to \pi_0 \mathrm{Pic}(R).$$

The image in $\pi_0 \mathrm{Pic}(R)$ of the last map is the group of outer automorphisms of $\pi_\star A$ as a $\pi_\star R$-algebra.

Proof. The Goerss–Hopkins obstruction groups $\mathrm{Der}^s_{\mathrm{Alg}_{\pi_\star R}}(\pi_\star A, M)$ appearing in Theorem 3.15 vanish identically for $s > 0$ by Proposition 2.28. In particular, the obstructions to existence and uniqueness vanish, so every Azumaya $\pi_\star R$-algebra lifts to an Azumaya R-algebra. Moreover, the obstructions to existence and uniqueness for lifting maps also vanish, and so every map of Azumaya $\pi_\star R$-algebras lifts uniquely to a map of R-algebras.

We then apply the long exact sequence on homotopy groups to Corollary 5.17. The previous theorem implies that this decomposes into short exact sequences on π_t for $t > 1$ and the stated results on π_1 and π_0 once due caution is exercised regarding basepoints. □

Corollary 3.16. *The functor $\pi_\star$ restricts to an equivalence from the homotopy category of algebraic Γ-graded Azumaya R-algebras to the category of Azumaya $\pi_\star R$-algebras.*

Remark 3.17. There are two very common sources of non-algebraic Azumaya R-algebras. First, any compact generator M of Mod_R produces an Azumaya R-algebra $\mathrm{End}_R(M)$ regardless of whether $\pi_\star M$ is projective or not (for example, the derived endomorphism ring of $\mathbb{Z} \oplus \mathbb{Z}/p$ is a non-algebraic derived Azumaya algebra over $\mathbb{Z}$). Second, the property of being algebraic also depends on the grading. If P is an element in $\mathrm{Pic}(R)$ which is not a suspension of R, then $\mathrm{End}_R(R \oplus P)$ is likely to be exotic for $\mathbb{Z}$-grading but is definitely not exotic for Pic-grading.

4 Presentable symmetric monoidal ∞-categories

From this section forward, we will switch to an ∞-categorical point of view on categories of Azumaya algebras and their module categories so that we can make use of the results of [24] and [25]. Finding strict model-categorical versions of many of these constructions we will use seems extremely difficult. For example, it is hard to find point-set constructions that simultaneously give a construction of $GL_n(R)$ as a group, $M_n(R)$ as an R-algebra, an action of $GL_n(R)$ on $M_n(R)$ by conjugation, and a diagonal embedding $GL_1(R) \to GL_n(R)$ which acts trivially. If we also want these to be homotopically sensible then it becomes harder still.

Making this switch implicitly requires a translation process, which we will briefly sketch. Given a commutative symmetric ring spectrum R, its image $\bar{R}$ in the ∞-category Sp of spectra is a commutative algebra object in the sense of [24, 2.1.3.1].

- [24, 4.1.3.10] Associated to Mod_R there is a stable presentable symmetric monoidal ∞-category $\mathrm{N}^\otimes(\mathrm{Mod}_R^\circ)$, the operadic nerve of the category $(\mathrm{Mod}_R^\circ) \subset \mathrm{Mod}_R$ of cofibrant-fibrant R-modules.
- [24, 4.3.3.17] This ∞-category is equivalent to the ∞-category of modules over the associated commutative algebra object $\bar{R}$ in Sp.

- [24, 4.1.4.4] The model category of associative algebra objects Alg_R has ∞-category equivalent to the ∞-category of associative algebra objects of $\mathrm{N}^{\otimes}(\mathrm{Mod}_R^{\circ})$ in the sense of [24, 4.1.1.6].

- [24, 4.3.3.17] For such R-algebras, the model categories of left A-modules, right A-modules, or A-B bimodules in Mod_R have associated ∞-categories equivalent to the left modules, right modules, or bimodules over the corresponding associative algebra objects in $\mathrm{N}^{\otimes}(\mathrm{Mod}_R^{\circ})$.

Definition 4.1. Let $\mathrm{Ring} := \mathrm{CAlg}(\mathrm{Sp})$ denote the ∞-category of $\mathbb{E}_\infty$-ring spectra, or, equivalently, commutative algebra objects in Sp.

4.1 Closed symmetric monoidal ∞-categories

Definition 4.2 ([24, 4.1.1.7])**.** A monoidal ∞-category $\mathcal{C}^{\otimes}$ is *closed* if, for each object A of $\mathcal{C}$, the functors $A \otimes (-)\colon \mathcal{C} \to \mathcal{C}$ and $(-) \otimes A\colon \mathcal{C} \to \mathcal{C}$ admit right adjoints. A symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is *closed* if the underlying monoidal ∞-category is closed.

Recall [24, 4.8] that the ∞-category of $\mathcal{P}r^{\mathrm{L}}$ of presentable ∞-categories and colimit-preserving functors [26, 5.5.3.1] admits a symmetric monoidal structure with unit the ∞-category $\mathcal{S}$ of spaces. We refer to (commutative) algebra objects in this ∞-category as *presentable (symmetric) monoidal ∞-categories.*

Proposition 4.3 ([24, 4.2.1.33])**.** *A presentable monoidal ∞-category is closed.*

Proof. Let $\mathcal{C}^{\otimes}$ be a presentable monoidal ∞-category. Then, by definition, the underlying ∞-category $\mathcal{C}$ is presentable, and for each object A of $\mathcal{C}$ the functors $A \otimes (-)$ and $(-) \otimes A$ commute with colimits. It follows from the adjoint functor theorem [26, 5.5.2.2] that both of these functors admit right adjoints. □

Note that this implies that (the underlying ∞-category of) a presentable symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is canonically enriched, tensored and cotensored over itself. If $\mathcal{C}^{\otimes}$ is stable, then $\mathcal{C}$ is enriched, tensored and cotensored over Sp, the ∞-category of spectra. We will not normally notationally distinguish between the internal mapping object and the mapping spectrum, which should always be clear from the context.

Proposition 4.4. *A symmetric monoidal ∞-category $\mathcal{R}$ is stable and presentable (as a symmetric monoidal ∞-category) if and only if the underlying ∞-category is stable and presentable and (any choice of) the tensor bifunctor $\mathcal{R} \times \mathcal{R} \to \mathcal{R}$ preserves colimits in each variable. In particular, a closed symmetric monoidal ∞-category $\mathcal{R}$ is stable and presentable if and only if the underlying ∞-category is stable and presentable.*

There is also the following multiplicative version of Morita theory.

Proposition 4.5 ([24, 7.1.2.7], [6, 3.1])**.** *The functor*

$$\mathrm{Mod}\colon\ \mathrm{CAlg}(\mathrm{Sp}) \longrightarrow \mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}),$$

sending R to the (symmetric monoidal, presentable, stable) ∞-category of R-modules, is a fully faithful embedding.

4.2 Structured fibrations

We will write $\mathrm{Cat}_\infty^\wedge$ for the very large ∞-category of large ∞-categories.

Definition 4.6. Given a (possibly large) ∞-category $\mathcal{C}$ and a functor $\mathcal{C} \to \mathrm{Cat}_\infty^\wedge$, we will say that a coCartesian fibration $X \to S$ admits a $\mathcal{C}$-structure if its classifying functor $X \to \mathrm{Cat}_\infty^\wedge$ factors through $\mathcal{C} \to \mathrm{Cat}_\infty^\wedge$.

We have a coCartesian fibration $\mathrm{Mod} \to \mathrm{Ring}$ [24, 4.5.3.6] whose fiber over the $\mathbb{E}_\infty$-ring spectrum R is the (large) ∞-category Mod_R of R-modules.

Proposition 4.7 ([24, 4.5.3.1, 4.5.3.2])**.** *The coCartesian fibration* $\mathrm{Mod} \to \mathrm{Ring}$ *admits a canonical symmetric monoidal structure: there is a coCartesian family of ∞-operads*

$$\mathrm{Mod}^\otimes \to \mathrm{Ring} \times \mathrm{Comm}^\otimes$$

classifying a functor $R \mapsto \mathrm{Mod}_R\colon\ \mathrm{Ring} \to \mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$ from $\mathbb{E}_\infty$-ring spectra to presentable stable symmetric monoidal ∞-categories.

We next consider algebra objects. By applying [24, 4.8.3.13], we similarly find that we have a coCartesian fibration $\mathrm{Alg} \to \mathrm{Ring}$ whose fiber over the ring R is the (large) ∞-category Alg_R of R-algebras.

Proposition 4.8. *The coCartesian fibration* $\mathrm{Alg} \to \mathrm{Ring}$ *admits a canonical symmetric monoidal structure such that the forgetful functor from algebras to modules induces a morphism of symmetric monoidal coCartesian fibrations*

$$\begin{array}{ccc} \mathrm{Alg} & \longrightarrow & \mathrm{Mod} \\ & \searrow \quad \swarrow & \\ & \mathrm{Ring} & \end{array}$$

over Ring*.*

Proof. As in [24, 5.3.1.20], the coCartesian family of ∞-operads $\mathrm{Mod}^\otimes \to \mathrm{Ring} \times \mathrm{Comm}^\otimes$ classifies a functor $\mathrm{Ring} \to (\mathrm{Op}_\infty)^{/\,\mathrm{Comm}^\otimes}$, taking R to the coCartesian fibration $\mathrm{Mod}_R^\otimes \to \mathrm{Comm}^\otimes$. Applying [24, 3.4.2.1], we obtain a functor $\mathrm{Alg}\colon\ \mathrm{Ring} \to (\mathrm{Op}_\infty)^{/\,\mathrm{Comm}^\otimes}$, taking R to a coCartesian fibration $\mathrm{Alg}_R^\otimes \to \mathrm{Comm}^\otimes$ with a forgetful map

$$\begin{array}{ccc} \mathrm{Alg}_R^\otimes & \longrightarrow & \mathrm{Mod}_R^\otimes \\ & \searrow \quad \swarrow & \\ & \mathrm{Comm}^\otimes & \end{array}$$

that preserves coCartesian arrows [24, 3.2.4.3]. Converting this back, we obtain a diagram

$$\begin{array}{ccc} \mathrm{Alg}^{\otimes} & \xrightarrow{\qquad\qquad} & \mathrm{Mod}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathrm{Ring} \times \mathrm{Comm}^{\otimes} & \end{array}$$

of coCartesian Ring-families of symmetric monoidal ∞-operads, lifting the underlying map Alg $\to$ Mod to one compatible with the symmetric monoidal structure. ☐

Restricting the coCartesian fibration Mod $\to$ Ring to the subcategory of coCartesian arrows between compact modules, we obtain a left fibration

$$\mathrm{Mod}^{\omega} \longrightarrow \mathrm{Ring}$$

whose fiber over R is the ∞-groupoid Mod_R^{ω} of compact (or perfect [24, 7.2.5.2]) R-modules and equivalences thereof. More specifically, an arrow $(R, M) \to (R', M')$ of Mod^{ω} is an arrow $(R, M) \to (R', M')$ of Mod such that M is compact (as an R-module) and the map $M \otimes_R R' \to M'$ is an equivalence. Note that Mod_R^{ω} should not be confused with the full subcategory of Mod_R spanned by the compact objects; rather, it is the full subgroupoid of Mod_R spanned by the compact objects.

Lastly, let $\mathrm{Alg}^{\mathrm{prop}} \to \mathrm{Ring}$ denote the left fibration whose source ∞-category is the subcategory $\mathrm{Alg}^{\mathrm{prop}}$ of proper algebras defined by the pullback

$$\begin{array}{ccc} \mathrm{Alg}^{\mathrm{prop}} & \longrightarrow & \mathrm{Alg} \\ \downarrow & & \downarrow \\ \mathrm{Mod}^{\omega} & \longrightarrow & \mathrm{Mod}\,. \end{array}$$

This time, however, $\mathrm{Alg}_R^{\mathrm{prop}}$ is not the full subgroupoid of Alg_R on the compact R-algebras, but rather the full subgroupoid of Alg_R consisting of the R-algebras A whose underlying R-module is compact.

Proposition 4.9. *The morphism of symmetric monoidal coCartesian fibrations* Alg $\longrightarrow$ Mod *over* Ring *restricts to a morphism of symmetric monoidal left fibrations* $\mathrm{Alg}^{\mathrm{prop}} \to \mathrm{Mod}^{\omega}$ *over* Ring.

Proof. Tensors of compact modules are compact [24, 5.3.1.17]. ☐

4.3 Functoriality of endomorphisms

In order to construct the endomorphism algebra as a functor, we need to extend the results of [24, 4.7.2]. In this, Lurie considers the category of tuples $(A, M, \phi\colon A \otimes M \to M)$, which has a forgetful functor p given by $p(A, M, \phi) = M$. He extends it in such a way as to give this functor p monoidal fibers; this gives the terminal object $\mathrm{End}(M)$ in the fiber over M a canonical monoid structure. For the reader's convenience, we will first review some details of Lurie's construction.

Let $\mathcal{LM}^{\otimes}$ denote the ∞-operad parametrizing pairs of an algebra and a left module [24, 4.2.1.7]. A coCartesian fibration $\mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}$ of ∞-operads determines a monoidal ∞-category $\mathcal{C}$ and an ∞-category $\mathcal{M}$ such that $\mathcal{M}$ is *left-tensored* over $\mathcal{C}$ [24, 4.2.1.19]: in particular, there exist objects $A \otimes M$ for $A \in \mathcal{C}$ and $M \in \mathcal{M}$. Associated to this there is a category $\mathrm{LMod}(\mathcal{M})$ of left module objects in $\mathcal{M}$ [24, 4.2.1.13]: such an object is determined by an algebra $A \in \mathcal{C}$ and a left A-module $M \in \mathcal{M}$. There is a forgetful map $\mathrm{LMod}(\mathcal{M}) \to \mathcal{M}$ which is a categorical fibration.

Proposition 4.10. *Let* $\mathrm{Act}(\mathcal{M})$ *be the fiber product* $\mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \mathcal{M}^{\simeq}$*. The natural map* $\mathrm{Act}(\mathcal{M}) \to \mathcal{M}^{\simeq}$ *is a coCartesian fibration.*

Proof. The map $\mathrm{Act}(\mathcal{M}) \to \mathcal{M}$ is a categorical fibration to a Kan complex, and so by [26, 2.4.1.5, 2.4.6.5] it is a coCartesian fibration. □

Definition 4.11 ([24, 4.2.1.28])**.** Suppose that $\mathcal{M}$ is left-tensored over the monoidal ∞-category $\mathcal{C}$. A *morphism object* for M and N is an object $F_{\mathcal{M}}(M,N)$ of $\mathcal{C}$ equipped with a map $F_{\mathcal{M}}(M,N) \otimes M \to N$ such that the resulting natural homotopy class of map

$$\mathrm{Map}_{\mathcal{C}}(C, F_{\mathcal{M}}(M,N)) \to \mathrm{Map}_{\mathcal{M}}(C \otimes M, N)$$

is a homotopy equivalence for all $C \in \mathcal{C}$. If morphism objects exist for all M and N, we say that the left-tensor structure gives $\mathcal{M}$ a $\mathcal{C}$-*enrichment*.

Proposition 4.12 ([24, 4.7.2.40])**.** *Suppose that* $\mathcal{M}$ *is left-tensored over* $\mathcal{C}$*, giving it a* $\mathcal{C}$*-enrichment. For any* $M \in \mathcal{M}$*, the fiber* $\mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}$ *has a final object* $\mathrm{End}(M)$ *whose image under the composite* $\mathrm{LMod}(\mathcal{M}) \to \mathrm{Alg}_{\mathcal{C}} \to \mathcal{C}$ *is* $F_{\mathcal{M}}(M,M)$*.*

Corollary 4.13. *Under these assumptions, there exists a functor* $\mathrm{End}\colon \mathcal{M}^{\simeq} \to \mathrm{Alg}_{\mathcal{C}}$ *sending* M *to* $\mathrm{End}(M)$*.*

Proof. By [26, 2.4.4.9], the full subcategory of $\mathrm{Act}(\mathcal{M})$ spanned by the final objects determines a trivial Kan fibration $\mathrm{End}(\mathcal{M}) \to \mathcal{M}^{\simeq}$. Choosing a section of this map, we obtain a composite functor

$$\mathcal{M}^{\simeq} \to \mathrm{LMod}(\mathcal{M}) \to \mathrm{Alg}_{\mathcal{C}}$$

with the desired properties. □

Remark 4.14. It should be sufficient to assume that $\mathcal{C}$ is monoidal and $\mathcal{M}$ merely $\mathcal{C}$-enriched, rather than including the stronger assumption that $\mathcal{M}$ is left-tensored over $\mathcal{C}$. However, we require this assumption in order to make use of the results from [24, 4.7.2].

5 Picard and Brauer spectra

5.1 Picard spectra

If $\mathcal{C}$ is a small ∞-category, we write $\pi_0\mathcal{C}$ for the set of equivalence classes of objects of $\mathcal{C}$. Note that, by definition, $\pi_0\mathcal{C}$ is an invariant of the underlying ∞-groupoid $\mathcal{C}^{\simeq}$ of $\mathcal{C}$ (the ∞-groupoid obtained by discarding the noninvertible arrows).

Definition 5.1. A symmetric monoidal ∞-category $\mathcal{C}$ is grouplike if the monoid $\pi_0\mathcal{C}$ is a group.

A symmetric monoidal ∞-category $\mathcal{C}$ has a unique maximal grouplike symmetric monoidal subgroupoid $\mathcal{C}^{\times}$, the subcategory $\mathcal{C}^{\times} \subset \mathcal{C}$ consisting of the invertible objects and equivalences thereof. That this is actually a symmetric monoidal subcategory in the ∞-categorical sense follows from the fact that invertibility and equivalence are both detected upon passage to the symmetric monoidal homotopy category, and the grouplike condition is guaranteed by considering only the invertible objects.

Let $\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}} \subset \mathcal{P}r^{\mathrm{L}}$ denote the ∞-category of *stable* presentable ∞-categories and colimit-preserving functors; by [24, 4.8.2.18] this is the category $\mathrm{Mod}_{\mathrm{Sp}}$ of left modules over the ∞-category of spectra. We have the ∞-category $\mathrm{CAlg}(\mathrm{Mod}_{\mathrm{Sp}})$ of commutative ring objects in $\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}}$; these are the same as commutative Sp-algebras or presentable symmetric monoidal stable ∞-categories.

Definition 5.2. Let $\mathcal{R}$ be a commutative Sp-algebra. The Picard ∞-groupoid $\mathrm{Pic}(\mathcal{R})$ of $\mathcal{R}$ is $\mathcal{R}^{\times}$, the maximal subgroupoid of the underlying ∞-category of $\mathcal{R}$ spanned by the invertible objects.

By [27, 8.9] $\mathrm{Pic}(\mathcal{R})$ is equivalent to a small space, and by [27, 8.10] the functor Pic commutes with limits.

We have a symmetric monoidal cocartesian fibration

$$\mathrm{Mod}(\mathrm{Mod}_{\mathrm{Sp}}) \longrightarrow \mathrm{CAlg}(\mathrm{Mod}_{\mathrm{Sp}})$$

whose fiber over a commutative Sp-algebra $\mathcal{R}$ is the symmetric monoidal ∞-category $\mathrm{Cat}_{\mathcal{R}}$ of $\mathcal{R}$-linear ∞-categories. Writing

$$\mathrm{Mod}_{\mathrm{Sp},\omega} \subset \mathrm{Mod}_{\mathrm{Sp}}$$

for the symmetric monoidal subcategory consisting of the compactly-generated Sp-modules and compact object preserving functors, this restricts to a symmetric monoidal cocartesian fibration

$$\mathrm{Mod}(\mathrm{Mod}_{\mathrm{Sp},\omega}) \longrightarrow \mathrm{CAlg}(\mathrm{Mod}_{\mathrm{Sp},\omega})$$

over the subcategory $\mathrm{CAlg}(\mathrm{Mod}_{\mathrm{Sp},\omega}) \subset \mathrm{CAlg}(\mathrm{Mod}_{\mathrm{Sp}})$ of commutative algebra objects in $\mathrm{Mod}_{\mathrm{Sp},\omega} \subset \mathrm{Mod}_{\mathrm{Sp}}$. For a commutative algebra object $\mathcal{R} \in \mathrm{CAlg}(\mathrm{Mod}_{\mathrm{Sp},\omega})$, a.k.a. a compactly generated commutative Sp-algebra, we write $\mathrm{Cat}^{\simeq}_{\mathcal{R},\omega}$ for the full subgroupoid of the fiber $\mathrm{Cat}_{\mathcal{R},\omega}$ over $\mathcal{R}$, the symmetric monoidal ∞-category of compactly generated $\mathcal{R}$-linear ∞-categories in the sense of [26, 5.3.5], and note that the map $\mathcal{R} \mapsto \mathrm{Cat}^{\simeq}_{\mathcal{R},\omega}$ defines a left fibration $\mathrm{Mod}^{\simeq}(\mathrm{Mod}_{\mathrm{Sp},\omega}) \to \mathrm{CAlg}(\mathrm{Mod}_{\mathrm{Sp},\omega})$.

Proposition 5.3 (cf. [10, 2.1.3])**.** *Let $\mathcal{R}$ be a compactly-generated commutative* Sp*-algebra. Then any invertible object of $\mathcal{R}$ is compact.*

Proof. Let I be any invertible object of $\mathcal{R}$ and let $\{M_\alpha\}$ be a filtered system of objects of $\mathcal{R}$. Then there are natural equivalences

$$\begin{aligned}\mathrm{Map}(I, \operatorname{colim} M_\alpha) &\simeq \mathrm{Map}(\mathbf{1}, \operatorname{colim} I^{-1} \otimes M_\alpha)\\ &\simeq \operatorname{colim} \mathrm{Map}(\mathbf{1}, I^{-1} \otimes M_\alpha)\\ &\simeq \operatorname{colim} \mathrm{Map}(I, M_\alpha).\end{aligned}$$

The first equivalence follows because $\otimes$ commutes with colimits. The second follows because the monoidal unit $\mathbf{1}$ (the image of the sphere spectrum under the map $\mathrm{Sp} \to \mathcal{R}$) is compact by definition. □

Because $\mathrm{Pic}(\mathcal{R})$ is closed under the symmetric monoidal product on $\mathcal{R}$, it is a grouplike symmetric monoidal ∞-groupoid, so by the recognition principle for infinite loop spaces we may regard $\mathrm{Pic}(\mathcal{R})$ as having an associated (connective) spectrum $\mathfrak{pic}(\mathcal{R}) = K(\mathrm{Pic}(\mathcal{R}))$ [28, 1.4]. Let $\Gamma_{\mathcal{R}}$ be the algebraic Picard groupoid of $\mathcal{R}$: the homotopy category of $\mathrm{Pic}(\mathcal{R})$, which is the 1-truncation of $\mathrm{Pic}(\mathcal{R})$. If $\mathcal{R}$ is unambiguous, we drop it and simply write Γ. We will notationally distinguish between an object $\gamma \in \Gamma$ and the associated invertible object $R^\gamma \in \mathcal{R}$.

Proposition 5.4. *The homotopy category of $\mathcal{R}$ is canonically enriched in the symmetric monoidal category of Γ-graded abelian groups.*

Proof. Since $\mathcal{R}$ is stable, the set $\pi_0 \mathrm{Map}(M, N)$ of homotopy classes of maps from M to N admits an abelian group structure which is natural in the variables M and N of $\mathcal{R}$, and composition is bilinear. The result then follows from Proposition 2.11, defining $\pi_\star \mathrm{Map}(M, N)$ by the rule

$$\pi_\gamma \mathrm{Map}(M, N) := \pi_0 \mathrm{Map}(\Sigma^\gamma M, N).$$

□

If R is an $\mathbb{E}_\infty$-ring spectrum, then we will typically write $\mathrm{Pic}(R)$ in place of $\mathrm{Pic}(\mathrm{Mod}_R)$ and Γ_R in place of Γ_{Mod_R}.

5.2 Brauer spectra

Definition 5.5. Let $\mathcal{R}$ be a compactly generated Sp-algebra. The Brauer ∞-groupoid $\mathrm{Br}(\mathcal{R})$ of $\mathcal{R}$ is the subgroupoid $\mathrm{Pic}(\mathrm{Cat}_{\mathcal{R},\omega})$ of $\mathrm{Cat}_{\mathcal{R},\omega}$ on the invertible $\mathcal{R}$-linear categories which admit a compact generator.

Remark 5.6. If $\mathcal{C}$ is a presentable ∞-category, then $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$ is the κ-filtered colimit completion of the full subcategory $\mathcal{C}^\kappa \subset \mathcal{C}$ on the κ-compact objects for some sufficiently large cardinal κ. If κ can be taken to be countable, then $\mathcal{C}$ is said to be compactly generated, and if there exists a compact object $P \in \mathcal{C}$ such that $\mathcal{C} \simeq \mathrm{Mod}_{\mathrm{End}(P)}$ as Sp-modules, then $\mathcal{C}$ is said to admit a compact generator. Note that an Sp-module $\mathcal{C}$ admits a compact generator P if and only if the smallest thick subcategory of $\mathcal{C}^\omega$ containing P is $\mathcal{C}^\omega$ itself, in which case $\mathcal{C}^\omega \simeq \mathrm{Mod}^\omega_{\mathrm{End}(P)}$. Also observe that there is a distinction between these objects—$\mathcal{R}$-linear ∞-categories with a compact generator—and the compact objects in $\mathrm{Cat}_{\mathcal{R}}$.

Because $\mathrm{Br}(\mathcal{R})$ is closed under the symmetric monoidal product on $\mathrm{Cat}_{\mathcal{R}}$, it is a grouplike symmetric monoidal ∞-groupoid, so we may associate to it a connective spectrum $\mathfrak{br}(\mathcal{R})$.

Proposition 5.7. *Let $\mathcal{R}$ be a compactly-generated stable symmetric monoidal ∞-category. Then there is a canonical equivalence* $\mathrm{Pic}(\mathcal{R}) \to \Omega\,\mathrm{Br}(\mathcal{R})$.

Proof. We must show that $\Omega\,\mathrm{Pic}(\mathrm{Cat}_{\mathcal{R},\omega}) \simeq \mathrm{Pic}(\mathcal{R})$. To see this, first note that if X is a pointed ∞-groupoid, then $\Omega X \simeq \mathrm{Aut}_X(*)$ is the space of automorphisms of the distinguished object $*$ of X. Hence $\Omega\,\mathrm{Pic}(\mathrm{Mod}^\omega_{\mathcal{R}}) \simeq \mathrm{Aut}_{\mathcal{R}}(\mathcal{R}) \simeq \mathrm{Pic}(\mathcal{R})$, where the last equivalence follows from the fact that invertible $\mathcal{R}$-module endomorphisms of $\mathcal{R}$ correspond to invertible objects of $\mathcal{R}$ under the equivalence $\mathrm{End}_{\mathcal{R}}(\mathcal{R}) \simeq \mathcal{R}$ [24, 4.8.4]. □

If R is an $\mathbb{E}_\infty$-ring spectrum, we will typically write Br_R for the ∞-groupoid $\mathrm{Br}(\mathrm{Mod}_R)$.

5.3 Azumaya algebras

Definition 5.8 ([3], [6, 3.1.3], [2])**.** Let R be an $\mathbb{E}_\infty$-ring spectrum. An Azumaya R-algebra is an R-algebra A such that

- the underlying R-module of A is a compact generator of Mod_R in the sense of [26, 5.5.8.23], and
- the "left-and-right" multiplication map $A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$ is an R-algebra equivalence.

Remark 5.9. In [6, 3.15] it is shown that such A are characterized by the fact that Mod_A is invertible in the cateory $\mathrm{Cat}_{R,\omega}$ of compactly generated Mod_R-linear ∞-categories.

Proposition 5.10. *If A is an Azumaya R-algebra and $R \to R'$ is a ring map, then $A \otimes_R R'$ is an Azumaya R'-algebra.*

We write Az for the full subcategory of Alg determined by pairs (R, A) such that A is an Azumaya R-algebra. Because Azumaya algebras are stable under base-change, we have a morphism of left fibrations

$$\begin{array}{ccc} \mathrm{Az} & \longrightarrow & \mathrm{Alg} \\ & \searrow \quad \swarrow & \\ & \mathrm{Ring} & \end{array}$$

over Ring.

Proposition 5.11. *Let A be an Azumaya R-algebra. Then the category of right A-modules Mod_A is an invertible Mod_R-module with inverse $\mathrm{Mod}_{A^{\mathrm{op}}}$.*

Proof. We must show that $\mathrm{Mod}_A \otimes \mathrm{Mod}_{A^{\mathrm{op}}} \simeq \mathrm{Mod}_R$, where the tensor is taken in the category of left Mod_R-linear categories. Since Mod is symmetric monoidal [24, 4.8.5.16], we have an equivalence $\mathrm{Mod}_A \otimes_{\mathrm{Mod}_R} \mathrm{Mod}_{A^{\mathrm{op}}} \simeq \mathrm{Mod}_{A \otimes_R A^{\mathrm{op}}}$, and as "left-and-right" multiplication $A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$ is an equivalence of R-algebras we see that $\mathrm{Mod}_A \otimes \mathrm{Mod}_{A^{\mathrm{op}}} \simeq \mathrm{Mod}_{\mathrm{End}_R(A)}$. Finally, because A is a compact generator of Mod_R, Morita theory gives an equivalence $\mathrm{Mod}_R \simeq \mathrm{Mod}_{\mathrm{End}_R(A)}$ [24, 8.1.2.1], and the result follows. $\square$

Remark 5.12. We can instead show that the functor Mod itself is symmetric monoidal using the results of [29]. There it is shown that that the category of stable ∞-categories is the symmetric monoidal localization of the category of spectral ∞-categories, obtained by inverting the Morita equivalences. In particular, regarding ring spectra A and B as one-object spectral ∞-categories, it follows that $\mathrm{Mod}_A \otimes \mathrm{Mod}_B \simeq \mathrm{Mod}_{A\otimes B}$. The relative tensors are computed as the geometric realization of two-sided bar constructions $B(A, R, B)$ and $B(\mathrm{Mod}_A, \mathrm{Mod}_R, \mathrm{Mod}_B)$ [24, 4.4.2.8]; the localization functor preserves geometric realization due to being a left adjoint.

Proposition 5.13. *The map of ∞-groupoids $\mathrm{Az}_R \to \mathrm{Br}_R$ is essentially surjective. Moreover, if A and B are Azumaya algebras such that the images of A and B become equal in $\pi_0 \mathrm{Br}_R$, then A and B are Morita equivalent.*

Proof. Let $\mathcal{R} = \mathrm{Mod}_R$ and let $\mathcal{I}$ be an invertible object of $\mathrm{Cat}_{\mathcal{R},\omega}$. Then $\mathcal{I}$ has a compact generator, so $\mathcal{I} \simeq \mathrm{Mod}_A$ for some R-algebra A ([30], [24, 7.1.2.1]), and invertibility implies that A is a compact generator of Mod_R. It follows that $\mathrm{End}_R(A)$ is Morita equivalent to R, and thus that the R-algebra map $A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$ is an equivalence. $\square$

We remark that we can identify the homotopy types of the fibers of the various left fibrations over Ring.

Proposition 5.14. *Let R be an $\mathbb{E}_\infty$-ring spectrum. Then*

$$\mathrm{Mod}_R^\omega \simeq \coprod_{[M]\in\pi_0 \mathrm{Mod}_R^\omega} B\,\mathrm{Aut}_R(M)$$

and

$$\mathrm{Az}_R \simeq \coprod_{[A]\in\pi_0 \mathrm{Az}_R} B\,\mathrm{Aut}_{\mathrm{Alg}_R}(A).$$

5.4 The conjugation action on endomorphisms

Let $\mathcal{R}$ be a symmetric monoidal presentable stable ∞-category with unit $\mathbf{1}$, which is therefore enriched over itself (see [24, 4.2.1.33] or [25, 7.4.10])), and let M be an object of $\mathcal{R}$. In this section we analyze the fiber of the map

$$\mathrm{Aut}_{\mathcal{R}}(M) \longrightarrow \mathrm{Aut}_{\mathrm{Alg}_{\mathcal{R}}}(\mathrm{End}_{\mathcal{R}}(M)),$$

which roughly sends an automorphism α of M to the conjugation automorphism $\alpha^{-1} \circ (-) \circ \alpha$ of the endomorphism algebra $\mathrm{End}(M)$. This map arises from the map $\mathrm{End}_{\mathcal{R}} \colon \mathcal{R}^{\simeq} \to \mathrm{Alg}_{\mathcal{R}}$ of Corollary 4.13.

Proposition 5.15. *Let R be an $\mathbb{E}_\infty$ ring spectrum, A an Azumaya R-algebra, and $\mathrm{Mod}_A^{\mathrm{cg}}$ denote the ∞-category of compact generators of Mod_A. Then there are canonical pullback diagrams of ∞-categories*

$$\begin{array}{ccccc} \mathrm{Pic}(R) & \longrightarrow & \mathrm{Mod}_A^{\mathrm{cg}} & \longrightarrow & \{\mathrm{Mod}_A\} \\ \downarrow & & \downarrow & & \downarrow \\ \{A\} & \longrightarrow & \mathrm{Az}_R & \longrightarrow & \mathrm{Br}_R\,. \end{array}$$

More generally, the fiber of $\mathrm{Mod}_A^{\mathrm{cg}} \to \mathrm{Alg}_R$ over an R-algebra B is either empty or a principal torsor for $\mathrm{Pic}(R)$.

Remark 5.16. Note that $\mathrm{Mod}_A^{\mathrm{cg}}$ is not to be confused with the larger subcategory Mod_A^{ω} of compact objects.

Proof. The pullback of the right-hand square is the ∞-category of R-algebras equipped with a Morita equivalence to Mod_A. In [24, 4.8.4] it is shown that the category of functors $\mathrm{Mod}_A \to \mathrm{Mod}_B$ is equivalent to a category of bimodules, and so this pullback category of Morita equivalences is equivalent to the category of compact generators of Mod_A via the map $M \mapsto \mathrm{End}_A(M)$.

The pullback of the left-hand square is the ∞-category of A-bimodules inducing Mod_R-linear Morita self-equivalences of Mod_A. These are, in particular, invertible modules over $A \otimes_R A^{op} \simeq \mathrm{End}_R(A)$, and Morita theory implies that the map $I \mapsto I \otimes_R A$ makes this equivalent to $\mathrm{Pic}(R)$. □

Taking preimages of the unit component, we obtain the following.

Corollary 5.17. *For an $\mathbb{E}_\infty$-ring R, there is a fiber sequence*

$$\coprod_{[M] \in \pi_0 \mathrm{Mod}_R^{\mathrm{cg}}} B\,\mathrm{Aut}_R(M) \to \coprod_{[A] \in \pi_0 (\mathrm{Az}_R)^{triv}} B\,\mathrm{Aut}_{\mathrm{Alg}_R}(A) \to B\,\mathrm{Pic}(R),$$

where the middle coproduct is over Azumaya R-algebras Morita equivalent to R.

In particular, this implies that the map $\mathrm{Aut}_R(M) \to \mathrm{Aut}_{\mathrm{Alg}_R}(\mathrm{End}_R(M))$ factors through a quotient by $\mathrm{GL}_1(R)$.

Corollary 5.18 (cf. 2.22)**.** *For any Azumaya R-algebra A, there is a long exact sequence of groups*

$$\cdots \to (\pi_n R)^\times \to (\pi_n A)^\times \to \pi_n(\mathrm{Aut}_{\mathrm{Alg}_R}(A)) \to \cdots$$
$$\to (\pi_0 R)^\times \to (\pi_0 A)^\times \to \pi_0(\mathrm{Aut}_{\mathrm{Alg}_R}(A)) \to \pi_0 \operatorname{Pic}(R).$$

Moreover, the group $\pi_0 \operatorname{Pic}(R)$ acts on the set of isomorphism classes of compact generators of Mod_A. The quotient is the set of isomorphism classes of Azumaya algebras A Morita equivalent to R, and the stabilizer of A is the image of the group of outer automorphisms of $\pi_\star A$ as a $\pi_\star R$-algebra.

This long exact sequence generalizes the short exact sequences of Theorem 3.15 for Γ-graded algebraic Azumaya R-algebras.

6 Galois cohomology

6.1 Galois extensions and descent

In this section we will review definitions of Galois extensions of ring spectra, due to Rognes [9]. Let R be an $\mathbb{E}_\infty$-ring spectrum and let G be an R-dualizable ∞-group: $G \simeq \Omega BG$ is the space of automorphisms of an object in some pointed, connected ∞-groupoid BG, and the associated group ring $R[G] := R \otimes_{\mathbb{S}} \Sigma^\infty_+ G$ is dualizable as an R-module.

Definition 6.1. A Galois extension of R by G is a functor $f\colon BG \to \mathrm{Ring}_{R/}$, sending the basepoint to a commutative R-algebra S with G-action, such that

- the unit map $R \to S^{hG} = \lim f$ is an equivalence, and
- the map $S \otimes_R S \to S \otimes_R D_R R[G] \simeq D_S S[G]$, induced by the action $R[G] \otimes_R S \to S$, is an equivalence.

A G-Galois extension $R \to S$ is *faithful* if S is a faithful R-module.

We will usually just write $f\colon R \to S$ for the Galois extension without explicitly mentioning the G-action. All of the Galois extensions that we consider in this paper will be assumed to be faithful.

We have the following important result.

Proposition 6.2 ([9, 6.2.1])**.** *Let $R \to S$ be a G-Galois extension. Then the underlying R-module of S is dualizable.*

In other words, S is a *proper* R-algebra in the sense of [24, 4.6.4.2]. Using this, Mathew has deduced several important consequences.

Proposition 6.3 ([31, 3.36])**.** *Let $R \to S$ be a faithful G-Galois extension with G a finite group. Then $R \to S$* admits descent *in the sense of [31, 3.17].*

Proposition 6.4. *Let $R \to S$ be a faithful G-Galois extension with G a finite group, and M an R-module. Then several properties of S-modules descend:*

- *A map $M \to N$ of R-modules is an equivalence if and only if $S \otimes_R M \to S \otimes_R N$ is an equivalence.*

- *M is a faithful R-module if and only if $S \otimes_R M$ is a faithful S-module.*

- *M is a perfect R-module if and only if $S \otimes_R M$ is a perfect S-module.*

- *M is an invertible R-module if and only if $S \otimes_R M$ is an invertible S-module.*

Proof. The first statement is equivalent to the statement that N/M is trivial if and only if $S \otimes_R N/M$ is, which is the definition of faithfulness. The second statement follows from the tensor associativity equivalence $N \otimes_S (S \otimes_R M) \simeq N \otimes_R M$. The third statement is [31, 3.27] and the fourth is [31, 3.29]. $\square$

Associated to a commutative R-algebra S, there is the associated Amitsur complex, a cosimplicial commutative R-algebra:

$$S^{\otimes_R} := \left\{ S \rightrightarrows S \otimes_R S \overrightarrow{\rightrightarrows} S \otimes_R S \otimes_R S \overrightarrow{\overrightarrow{\rightrightarrows}} \cdots \right\}$$

In degree n this is the $(n+1)$-fold tensor power of S over R. More explicitly, the Amitsur complex is the left Kan extension of the map $\{[0]\} \to \mathrm{Ring}_{R/}$ classifying S along the inclusion $\{[0]\} \hookrightarrow \Delta$.

Composing with the functor $\mathrm{Mod}\colon\ \mathrm{CAlg} \to \mathrm{Cat}_\infty^\wedge$, we obtain a cosimplicial object

$$\mathrm{Mod}_{S^{\otimes_R}} := \left\{ \mathrm{Mod}_S \rightrightarrows \mathrm{Mod}_{S \otimes_R S} \overrightarrow{\rightrightarrows} \mathrm{Mod}_{S \otimes_R S \otimes_R S} \overrightarrow{\overrightarrow{\rightrightarrows}} \cdots \right\}$$

in Mod_R-linear ∞-categories; we also refer to this as the Amitsur complex.

Proposition 6.5 (cf. [32, 6.15, 6.18], [31, 3.21])**.** *Suppose S is a proper commutative R-algebra and A is an R-algebra. Then the natural map*

$$\theta\colon\ \mathrm{Mod}_A \to \lim \mathrm{Mod}_{(S^{\otimes_R}) \otimes_R A}$$

has fully faithful left and right adjoints. If S is faithful as an R-module, then θ is an equivalence.

Proof. We will prove this result by verifying the two criteria of [24, 4.7.6.3] (a consequence of the ∞-categorical Barr–Beck theorem) for both this cosimplicial diagram of categories and the corresponding diagram of opposite categories.

The first criterion asks that colimits of simplicial objects exist in Mod_A and that the extension-of-scalars functor $S \otimes_R (-)\colon\ \mathrm{Mod}_A \to \mathrm{Mod}_{S \otimes_R A}$ preserve them. However, both categories are cocomplete and the given functor is left adjoint to the forgetful functor, hence preserves all colimits. The same condition on the opposite category asks that $S \otimes_R (-)$ preserve totalizations of certain cosimplicial objects, but since S is R-dualizable there is a natural equivalence

$$S \otimes_R M \simeq F_R(D_R S, M).$$

This equivalent functor has a left adjoint, given by $N \mapsto D_R S \otimes_S M$, and so preserves all limits.

The second criterion is a "Beck–Chevalley" condition, as follows. For any $\alpha\colon [m] \to [n]$ in Δ, consider the induced diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Mod}_{S^{\otimes_R m} \otimes_R A} & \xrightarrow{d^0} & \mathrm{Mod}_{S^{\otimes_R (1+m)} \otimes_R A} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{S^{\otimes_R n} \otimes_R A} & \xrightarrow{d^0} & \mathrm{Mod}_{S^{\otimes_R (1+n)} \otimes_R A} . \end{array}$$

Then we ask that these diagrams are left adjointable and right adjointable [24, 4.7.5.13]: the horizontal arrows admit left and right adjoints, and that the resulting natural transformation between the composites is an equivalence. In our case, this diagram is generically of the following form:

$$\begin{array}{ccc} \mathrm{Mod}_{B} & \longrightarrow & \mathrm{Mod}_{S \otimes_R B} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{B'} & \longrightarrow & \mathrm{Mod}_{S \otimes_R B'} \end{array}$$

Here the horizontal arrows are extension of scalars to S, while the vertical arrows are extensions of scalars induced by a map of R-algebras $B \to B'$. The natural transformation between composed left adjoints is the natural equivalence

$$(D_R S \otimes_S M) \otimes_B B' \to D_R S \otimes_S (M \otimes_B B'),$$

and the one between composed right adjoints is the natural equivalence

$$N \otimes_{S \otimes_R B} (S \otimes_R B') \to N \otimes_B B',$$

verifying the Beck–Chevalley condition and its opposite.

Therefore, the map from Mod_A to the limit category has fully faithful left and right adjoints. If S is faithful, then the functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S \otimes_R A}$ is conservative and [24, 4.7.6.3] additionally verifies that Mod_A is equivalent to the limit, making Mod_A monadic and comonadic over $\mathrm{Mod}_{S \otimes_R A}$. □

Remark 6.6. This construction has a stricter lift. If we lift R and S to strictly commutative ring objects in a model category and G is an honest group acting on S, the operation of tensoring with the right R-module S implements a left Quillen functor between the category of R-modules and the category of modules over the twisted group algebra $S\langle G \rangle \simeq \mathrm{End}_R(S)$.

6.2 Group actions

Let $G \simeq \Omega BG$ be the space of automorphisms of a connected Kan complex BG with basepoint $i\colon \Delta^0 \to BG$. For an ∞-category $\mathcal{C}$, the category of G-objects in $\mathcal{C}$ is the functor category $\mathcal{C}^{BG} = \mathrm{Fun}(BG, \mathcal{C})$. Evaluation at the basepoint determines a functor $i^*\colon \mathcal{C}^{BG} \to \mathcal{C}$.

If $\mathcal{C}$ is complete and cocomplete, the functor i^* admits left and right adjoints $i_!$ and i_* respectively, given by left and right Kan extension. These are naturally described by the colimit and limit of the constant diagram on G with value X, or equivalently the tensor and cotensor of X with G:

$$i_! X \simeq G \otimes X, \qquad\qquad i_* X \simeq X^G.$$

Proposition 6.7. *If $\mathcal{C}$ is complete and cocomplete, the forgetful functor i^* makes $\mathcal{C}^{BG}$ monadic and comonadic over $\mathcal{C}$ in the sense of [24, 4.7.4.4].*

Suppose G is equivalent to a finite discrete group and let $p\colon \mathcal{C} \to \mathcal{D}$ be a functor between complete and cocomplete ∞-categories which preserves finite products, with induced map $p_*\colon \mathcal{C}^{BG} \to \mathcal{D}^{BG}$. If $\mathbb{T}^{\mathcal{C}}$ and $\mathbb{T}^{\mathcal{D}}$ are the induced comonads on $\mathcal{C}$ and $\mathcal{D}$, then the resulting natural transformation $p \circ \mathbb{T}^{\mathcal{C}} \to \mathbb{T}^{\mathcal{D}} \circ p$ between comonads is an equivalence.

Proof. For the first statement it suffices, by [24, 4.7.4.5] and its dual, to observe that i^* is conservative and preserves all limits and colimits, being both a left and right adjoint.

For the second statement, the natural map is provided by the adjunction in the form of a composite

$$p i^* i_* \cong i^* p_* i_* \xrightarrow{i^*(\eta)} i^* i_* i^* p_* i_* \cong i^* i_* p i^* i_* \xrightarrow{i^* i_* p(\varepsilon)} i^* i_* p.$$

For $X \in \mathcal{C}$, this takes the form of the natural limit transformation $p(X^G) \to p(X)^G$, which is an equivalence by assumption. □

Corollary 6.8. *If G is a finite group, associated to a ring $S \in (\mathrm{Ring}_{R/})^{BG}$ there is a cosimplicial commutative R-algebra*

$$\mathbb{T}^\bullet(S) = \left\{ i^* S \rightrightarrows \mathbb{T}(i^*S) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathbb{T}(\mathbb{T}(i^*S)) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right\}$$

induced by the comonad $\mathbb{T}$, whose underlying cosimplicial R-module is the fixed-point construction

$$S^{hG} = \left\{ S \rightrightarrows S^G \substack{\rightarrow\\\rightarrow\\\rightarrow} S^{G\times G} \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right\}.$$

6.3 Descent

Proposition 6.9. *For a Galois extension $R \to S$, there is an equivalence of cosimplicial R-algebras between the Amitsur complex $S^{\otimes}_R$ and the fixed-point construction $\mathbb{T}^\bullet(S)$ of Corollary 6.8.*

Proof. The universal property of the left Kan extension implies that the identity map $S \simeq \mathbb{T}^0(S)$ extends to a map of cosimplicial objects $S^{\otimes_R} \to S^{hG}$, unique up to contractible choice. It suffices to verify that this induces equivalences $S^{\otimes_R (n+1)} \to S^{G^n}$, which follows by induction from the case $n = 1$. □

Now suppose that $R \to S$ is a faithful Galois extension of R by the stably dualizable group G. Write $f\colon BG \to \mathrm{Ring}_{R/}$ for the functor classifying S as a (naive) G-equivariant commutative R-algebra, so that $S \simeq f(*)$ and $R \simeq \lim f$, and write

$$(\mathrm{Mod}_S)^{hG} := \lim \mathrm{Mod}_f$$

for the "fixed-points" of the ∞-category Mod_f, the ∞-category of G-semilinear S-modules. Lastly, we write N^{hG} for the limit of a G-semilinear S-module N, and view it as an $R \simeq S^{hG}$-module.

Theorem 6.10. *Let $R \to S$ be a faithful G-Galois extension with G finite, and $A \in \mathrm{Alg}_R$. Then the canonical map*

$$\mathrm{Mod}_A \longrightarrow (\mathrm{Mod}_{S \otimes A})^{hG}$$

is an equivalence of ∞-categories.

Proof. Tensoring the equivalence of Proposition 6.9 with A, we obtain maps of cosimplicial objects

$$(S^{\otimes_R}) \otimes_R A \xrightarrow{\sim} \mathbb{T}^\bullet(S) \otimes_R A \to \mathbb{T}^\bullet(A).$$

The natural map $\mathbb{T}(X) \otimes_R Y \to \mathbb{T}(X \otimes_R Y)$ is equivalent to the map $X^G \otimes_R Y \to (X \otimes_R Y)^G$ and is therefore an equivalence, because both sides are a $|G|$-fold coproduct of copies of $X \otimes_R Y$.

Since S is faithful and dualizable as an R-module, Proposition 6.5 shows that there is an equivalence

$$\mathrm{Mod}_A \simeq \lim(\mathrm{Mod}_{S^{\otimes_R} \otimes_R A}).$$

The equivalence of cosimplicial rings shows that this extends to an equivalence

$$\mathrm{Mod}_A \simeq \lim \mathrm{Mod}_{\mathbb{T}^\bullet A} \simeq (\mathrm{Mod}_{S \otimes_R A})^{hG}. \qquad \square$$

Corollary 6.11. *Let $R \to S$ be a faithful G-Galois extension with G finite, associated to a functor $f\colon BG \to \mathrm{Ring}_{R/}$, and consider the diagram*

$$\begin{array}{ccc} & & \mathrm{Mod} \\ & \nearrow & \downarrow \\ BG & \xrightarrow{f} & \mathrm{Ring}\,. \end{array}$$

Then the map

$$\mathrm{Mod}_R \longrightarrow \mathrm{Fun}_{/\,\mathrm{Ring}}(BG, \mathrm{Mod}),$$

which sends the R-module M to the G-Galois module $S \otimes_R M$, is an equivalence.

Proof. The ∞-category of sections from BG to the pullback of $\mathrm{Mod} \to \mathrm{Ring}$ is equivalent to the limit of the functor $\mathrm{Mod}_f\colon BG \to \mathrm{Cat}_\infty^\wedge$ it classifies [26, 3.3.3.2], which in turn is equivalent to Mod_R by Theorem 6.10. $\square$

Lemma 6.12. *For an ∞-operad $\mathcal{O}$, the ∞-category of $\mathcal{O}$-monoidal ∞-categories has limits which are computed in* Cat_∞.

Proof. In [24, 2.4.2.6] it is shown that there is an equivalence between $\mathcal{O}$-monoidal ∞-categories and $\mathcal{O}$-algebra objects in Cat_∞, and so [24, 3.2.2.1] shows that limits of the underlying ∞-categories lift uniquely to limits of $\mathcal{O}$-monoidal ∞-categories. The same proof applies within the category of large ∞-categories. □

Corollary 6.13. *Let $f\colon I \to \mathrm{Cat}_\infty^{\mathcal{O}}$ be a diagram of $\mathcal{O}$-monoidal ∞-categories and $\mathcal{O}$-monoidal functors. Then the canonical map*

$$\mathrm{Alg}_{/\mathcal{O}}(\lim f) \to \lim(\mathrm{Alg}_{/\mathcal{O}} \circ f)$$

is an equivalence.

Proof. The ∞-category $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}^\otimes)$ of $\mathcal{O}$-algebra objects in an $\mathcal{O}$-monoidal ∞-category $p\colon \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is the ∞-category of functors $\mathrm{Fun}_{/\mathcal{O}^\otimes}(\mathcal{O}^\otimes, \mathcal{C}^\otimes)$, and $\mathrm{Fun}_{/\mathcal{O}^\otimes}(\mathcal{O}^\otimes, -)\colon\ \mathrm{Cat}_\infty^{\mathcal{O}} \to \mathrm{Cat}_\infty$ evidently preserves limits in the target. □

Proposition 6.14. *Let $R \to S$ be the G-Galois extension associated to a functor $f\colon\ BG \to \mathrm{Ring}_{R/}$, and consider the diagram*

$$\begin{array}{ccc} & & \mathrm{Alg} \\ & \nearrow & \downarrow \\ BG & \xrightarrow{f} & \mathrm{Ring}\,. \end{array}$$

Then the map $\mathrm{Alg}_R \longrightarrow \mathrm{Fun}_{/\,\mathrm{Ring}}(BG, \mathrm{Alg})$, which sends the R-algebra A to the G-equivariant S-algebra $S \otimes_R A$, is an equivalence.

Proof. This follows from the corresponding statement for modules, by noting that f is comes from a diagram $BG \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ of symmetric monoidal ∞-categories and symmetric monoidal functors, together with Corollary 6.13. □

We now consider the diagram of ∞-categories

$$\begin{array}{ccccccc} & & \mathrm{Pic} & \longrightarrow & \mathrm{Mod}^{\mathrm{cg}} & \longrightarrow & \mathrm{Az} \longrightarrow \mathrm{Br} \\ & & & \searrow & \downarrow & \swarrow & \\ BG^{\triangleleft} & & \longrightarrow & & \mathrm{Ring}, & & \end{array}$$

where the bottom map describes R as the limit of the G-action on S. For each of the vertical maps we may take spaces of sections over the cone point or over BG, recovering fixed-point objects for the action of G on Pic_S, $\mathrm{Mod}_S^{\mathrm{cg}}$, Az_S, and Br_S respectively.

Theorem 6.15. *Let $R \to S$ be the G-Galois extension with G finite. There is a commutative diagram of symmetric monoidal ∞-categories*

$$\begin{array}{ccccccc}
\mathrm{Pic}_R & \longrightarrow & \mathrm{Mod}_R^{\mathrm{cg}} & \longrightarrow & \mathrm{Az}_R & \longrightarrow & \mathrm{Cat}_{R,\omega} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\mathrm{Pic}_S)^{hG} & \longrightarrow & (\mathrm{Mod}_S^{\mathrm{cg}})^{hG} & \longrightarrow & (\mathrm{Az}_S)^{hG} & \longrightarrow & (\mathrm{Cat}_{S,\omega})^{hG}
\end{array}$$

where the left three vertical arrows are equivalences. On the full subcategory of $\mathrm{Cat}_{R,\omega}$ spanned by categories of the form Mod_A for A an R-algebra, the right-hand map is fully faithful.

Proof. We already have equivalences $\mathrm{Mod}_R \simeq (\mathrm{Mod}_S)^{hG}$ and $\mathrm{Alg}_R \simeq (\mathrm{Alg}_S)^{hG}$, so for the left three arrows it suffices to identify the essential images of the ∞-categories Pic_R, $\mathrm{Mod}_R^{\mathrm{cg}}$, and Az_R of invertible modules, compact generators, and Azumaya algebras.

Proposition 6.4 implies that the property of being invertible descends, as do subcategories of equivalences, so that the essential image of Pic_R is the subcategory $(\mathrm{Pic}_S)^{hG}$ of $(\mathrm{Mod}_S)^{hG}$. Similarly, Proposition 6.4 implies that the properties of being dualizable and faithful descend, and that for a dualizable R-algebra A the map $A \otimes_R A^{op} \to \mathrm{End}_R(A) \simeq D_R A \otimes_R A$ is an equivalence if and only if the same is true for the S-algebra $S \otimes_R A$. Therefore, the essential image of Az_R is the subcategory $(\mathrm{Az}_S)^{hG}$ of $(\mathrm{Alg}_S)^{hG}$.

Compact generators are taken to compact generators, and so the second vertical arrow is defined. Further, if M is an R-module whose image in Mod_S is a compact generator, then M is compact and $\mathrm{End}_R(M)$ is an R-algebra whose image $\mathrm{End}_S(S \otimes_R M)$ is an Azumaya S-algebra, as already shown. Therefore $\mathrm{End}_R(M)$ is an Azumaya R-algebra, implying that M is a generator.

It remains to verify full faithfulness of the right-hand map. We have a commutative diagram

$$\begin{array}{ccc}
(\mathrm{Mod}^{\simeq}_{A \otimes_R B^{\mathrm{op}}}) & \longrightarrow & (\mathrm{Mod}^{\simeq}_{(S \otimes_R A) \otimes_S (S \otimes_R B)^{\mathrm{op}}})^{hG} \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathrm{Cat}_R}(\mathrm{Mod}_A, \mathrm{Mod}_B) & \longrightarrow & \mathrm{Map}_{\mathrm{Cat}_S}(\mathrm{Mod}_{S \otimes_R A}, \mathrm{Mod}_{S \otimes_R B})^{hG}.
\end{array}$$

The vertical maps are equivalences by the results of [24, 4.8.4], and the top map is an equivalence by Theorem 6.10 using the equivalence $S \otimes_R (A \otimes B^{\mathrm{op}}) \simeq (S \otimes_R A) \otimes_S (S \otimes_R B)^{\mathrm{op}}$. □

In particular, this gives a descent criterion for Morita equivalence.

Corollary 6.16. *The group $\pi_0(B\,\mathrm{Pic}_S^{hG})$ has, as a subgroup, the group of Morita equivalence classes of R-algebras A such that there exists an S-module M and an equivalence of S-algebras $S \otimes_R A \simeq \mathrm{End}_S(M)$.*

Proof. The ∞-category of such R-algebras is the preimage of the component of Mod_S in Cat_S, and all such algebras are Azumaya R-algebras by the previous result; this component is $B\,\mathrm{Pic}_S$ by Proposition 5.15. The maximal subgroupoid in Cat_R spanned by objects in this preimage is therefore equivalent to $(B\,\mathrm{Pic}_S)^{hG}$, as taking maximal subgroupoids preserves all limits and colimits. $\square$

6.4 Spectral sequence tools

For an object X in an ∞-category $\mathcal{C}$, we write $B\,\mathrm{Aut}_{\mathcal{C}}(X)$ for the subgroupoid of $\mathcal{C}^{\simeq}$ spanned by objects equivalent to X and $\mathrm{Aut}_{\mathcal{C}}(X)$ for the space of self-equivalences.

Definition 6.17. Let G be a group and $f\colon BG \to \mathrm{Cat}_\infty$ a functor classifying the action of G on an ∞-category $\mathcal{C}$. Write $\mathcal{C}_{hG} \to BG$ for the associated fibration (the colimit) and $\mathcal{C}^{hG}$ for the limit.

Restricting gives us a Kan fibration $(\mathcal{C}_{hG})^{\simeq} \to BG$ of Kan complexes whose space of sections is $(\mathcal{C}^{hG})^{\simeq}$ [26, 3.3.3.2]. The descent diagram in Theorem 6.15 will now allow us to carry out computations using the Bousfield–Kan spectral sequence for spaces of sections. In our cases of interest there will be obstruction groups that are annihilated by late differentials, and so we need to use the more sophisticated obstruction theory due to Bousfield [11]. We will review this obstruction theory now.

For a cosimplicial object $\mathcal{D}^\bullet\colon \Delta \to \mathrm{Cat}_\infty$, the limits

$$\mathrm{Tot}^n(\mathcal{D}) = \lim_{\Delta_{\le n}} \mathcal{D}^n$$

give a tower of ∞-categories whose limit is the limit of $\mathcal{D}^\bullet$.

Proposition 6.18. *Let $f\colon \mathcal{C}_{hG} \to BG$ be a Kan fibration classifying the action of G on a Kan complex $\mathcal{C}$, viewed as an ∞-groupoid. Then there is a tower*

$$\cdots \to \mathrm{Tot}^2 \to \mathrm{Tot}^1 \to \mathrm{Tot}^0 = \mathcal{C}$$

of Kan fibrations whose limit is $\mathcal{C}^{hG}$. Given an object $X \in \mathcal{C}$, there is an obstruction theory for existence and uniqueness of lifts of X to an object of $\mathcal{C}^{hG}$, natural in X and $\mathcal{C}$.

1. *An object $X \in \mathcal{C}$ is in the essential image of Tot^1 if and only if the equivalence class $[X] \in \pi_0\mathcal{C}$ is fixed by the action of the group G. Equivalently, this is true if and only if the map*

$$\pi_0\,\mathrm{Aut}_{\mathcal{C}_{hG}}(X) \to \pi_0\,\mathrm{Aut}_{BG}(*) = G$$

is surjective.

2. *Given an object $X \in \mathcal{C}$ with a lift $Y \in \mathrm{Tot}^1$, consider the surjection of groups*

$$\pi_0\,\mathrm{Aut}_{\mathcal{C}_{hG}}(X) \to G$$

as above. The obstruction to X being in the essential image of Tot^2 is whether this map of groups splits, and the obstruction to uniqueness of lift to Tot^2 is parametrized by the choice of splitting.

3. *[11, 2.4] If X lifts to $Y \in \mathrm{Tot}^n$ for $n \geq 2$, we have a fringed spectral sequence (starting at E_1) with E_2-term*

$$H^s(G; \pi_t(B\,\mathrm{Aut}_{\mathcal{C}}\, X)),$$

defined for $t > 1$ or for $0 \leq s \leq t \leq 1$. Further pages $E_r^{s,t}$ only exist for $2r - 2 \leq n$, and the E_r-page depends on a choice of lift of X to Tot^{r-1}. For $r \geq 2$ the E_r-page is defined on the region

$$\{(s,t) \mid s \geq 0, t - s \geq 0\} \cup \{(s,t) \mid s \geq 0, t - r \geq \tfrac{r-2}{r-1}(s - r)\}.$$

4. *[11, 5.2] If $r \geq 1$ and Y is a lift of X to Tot^r which admits a further lift to Tot^{2r}, then there is an obstruction class*

$$\theta_{2r+1} \in E_{r+1}^{2r+1,2r}$$

which is zero if and only if Y can be lifted to Tot^{2r+1}.

5. *[11, 5.2] If $r \geq 2$ and Y is a lift of X to Tot^r which admits a further lift to Tot^{2r-1}, then there is an obstruction class*

$$\theta_{2r} \in E_r^{2r,2r-1}$$

which is zero if and only if Y can be lifted to Tot^{2r}.

6. *If $Y \in \mathcal{C}^{hG}$, the above spectral sequences converge (in the region $t - s > 0$) to $\pi_{t-s}(B\,\mathrm{Aut}_{\mathcal{C}^{hG}}\, Y)$.*

7. *If $\mathcal{C} \simeq \Omega\mathcal{D}$ for a Kan complex $\mathcal{D}$ with compatible G-action, the spectral sequences for $\mathcal{C}$ and $\mathcal{D}$ are compatible. In particular, if the map $BG \to \mathrm{Sp}$ representing the G-action on $\mathcal{C}$ lifts to a functor from BG to the category of E_∞-spaces, we can construct an associated K-theory spectrum $K(\mathcal{C})$ such that the spectral sequence above extends to the homotopy fixed-point spectral sequence for the action of G on $K(\mathcal{C})$.*

Remark 6.19. The beginning of the obstruction theory may be described as follows. In order for X to lift to the limit $\mathcal{C}^{hG}$, a lift to Tot^1 is determined by choosing equivalences $\phi_g \colon {}^gX \to X$ for all $g \in G$. A lift to Tot^2 is then determined by witnesses for the cocycle condition, in the form of homotopies from ϕ_{gh} to $\phi_g \circ {}^g(\phi_h)$.

Remark 6.20. The user (particularly if they are used to stable work) may benefit from being explicitly reminded of some of the dangers of the "fringe effect." While the splittings in the second obstruction can be parametrized by $H^1(G; \pi_0\,\mathrm{Aut}_{\mathcal{C}}(X))$, this does not occur until an initial splitting is chosen (indeed, otherwise the action of G on $\pi_*\,\mathrm{Aut}_{\mathcal{C}}(X)$ is not even defined). The structure of the spectral sequence, at arbitrarily large pages, may also depend strongly on the choices of lift Y.

Because we will be interested in understanding different lifts, it will be useful to be more systematic about the obstructions to this.

Definition 6.21. For an ∞-category $\mathcal{C}$ and objects X and Y in $\mathcal{C}$, let $\mathrm{Equiv}_{\mathcal{C}}(X,Y)$ be the pullback in the diagram

$$\begin{array}{ccc} \mathrm{Equiv}_{\mathcal{C}}(X,Y) & \longrightarrow & \mathrm{Map}(\Delta^1, \mathcal{C}^{\simeq}) \\ \downarrow & & \downarrow \\ \{(X,Y)\} & \longrightarrow & \mathrm{Map}(\partial\Delta^1, \mathcal{C}^{\simeq}). \end{array}$$

Proposition 6.22. *The space* $\mathrm{Equiv}_{\mathcal{C}}(X,Y)$ *is a Kan complex, and composition of functions gives a left action of the group* $\mathrm{Aut}_{\mathcal{C}}(Y)$ *on* $\mathrm{Equiv}_{\mathcal{C}}(X,Y)$*. If* $\mathrm{Equiv}_{\mathcal{C}}(X,Y)$ *is nonempty, any choice of point* $f \in \mathrm{Equiv}_{\mathcal{C}}(X,Y)$ *produces an equivalence* $f^*\colon \mathrm{Aut}_{\mathcal{C}}(Y) \to \mathrm{Equiv}_{\mathcal{C}}(X,Y)$*.*

Proposition 6.23. *Let G be a group acting on an ∞-category $\mathcal{C}$, let $p\colon \mathcal{C}^{hG} \to \mathcal{C}$ be the limit, and suppose and X and Y are objects in $\mathcal{C}^{hG}$. Then the map*

$$\mathrm{Equiv}_{\mathcal{C}^{hG}}(X,Y) \to \mathrm{Equiv}_{\mathcal{C}}(p(X),p(Y))^{hG}.$$

is an equivalence of Kan complexes.

Proof. The fixed-point construction, as a limit, commutes with taking maximal subgroupoids, mapping objects, and pullbacks. ☐

We may therefore apply the tower of Tot-objects to both $\mathrm{Aut}_{\mathcal{C}}(Y)$ and $\mathrm{Equiv}_{\mathcal{C}}(X,Y)$ to obtain the following.

Proposition 6.24. *Let $f\colon \mathcal{C}_{hG} \to BG$ be a Kan fibration classifying the action of G on a Kan complex $\mathcal{C}$, $p\colon \mathcal{C}^{hG} \to \mathcal{C}$ the limit, and X and Y objects in $\mathcal{C}^{hG}$.*

1. *There are towers of Kan fibrations:*

$$\cdots \to \mathrm{Aut}^2(Y) \to \mathrm{Aut}^1(Y) \to \mathrm{Aut}^0(Y) = \mathrm{Aut}_{\mathcal{C}}(p(Y))$$
$$\cdots \to \mathrm{Equiv}^2(X,Y) \to \mathrm{Equiv}^1(X,Y) \to \mathrm{Equiv}^0(X,Y) = \mathrm{Equiv}_{\mathcal{C}}(p(X),p(Y))$$

   The limits are $\mathrm{Aut}_{\mathcal{C}^{hG}}(Y)$ *and* $\mathrm{Equiv}_{\mathcal{C}^{hG}}(X,Y)$ *respectively.*

2. *The spaces* $\mathrm{Aut}^n(Y)$ *are ∞-groups which act on the spaces* $\mathrm{Equiv}^n(X,Y)$*.*

3. *If* $\mathrm{Equiv}^n(X,Y)$ *is nonempty, any choice of point produces an equivalence of partial towers* $\mathrm{Aut}^{\leq n}(Y) \to \mathrm{Equiv}^{\leq n}(X,Y)$*.*

4. *[11, 5.2] If* $\mathrm{Equiv}^n(X,Y)$ *is nonempty, there is an obstruction class*

$$\theta_{n+1} \in E_r^{n+1,n+1}$$

   in the spectral sequence calculating $\pi_* B\,\mathrm{Aut}_{\mathcal{C}^{hG}}(Y)$, defined for $2r \leq n+1$, which is zero if and only if $\mathrm{Equiv}^{n+1}(X,Y)$ *is nonempty.*

7 Calculations

7.1 Algebraic Brauer groups of even-periodic ring spectra

In this section we assume that E is an even-periodic $\mathbb{E}_\infty$-ring spectrum: there is a unit in $\pi_2 E$, and $\pi_1 E$ is trivial.

We can describe specific Azumaya algebras for these groups using Theorem 3.15 and the algebras described in the proof of [12, 7.10].

Example 7.1. Let $u \in \pi_2 E$ be a unit and $\pi_0 E \to R$ a quadratic Galois extension with Galois automorphism σ. There is an Azumaya E-algebra whose coefficient ring is the graded quaternion algebra

$$R\langle S\rangle/(S^2 - u, Sr - {}^\sigma r S),$$

where S is in degree 1 and R is concentrated in degree zero.

Example 7.2. Suppose 2 is a unit in $\pi_0 E$ and $u \in \pi_2 E$ is a unit. There is an Azumaya E-algebra whose coefficient ring is (perhaps unexpectedly) the 1-periodic graded ring

$$(\pi_* E)[x]/(x^2 - u) \cong (\pi_0 E)[x^{\pm 1}],$$

which is of rank two over $\pi_* E$. If A and B are two such algebras determined by units u and v, then $A \wedge_E B$ is equivalent to a quaternion algebra from Example 7.1 determined by the unit $u \in \pi_2(E)$ and the quadratic Galois extension $\pi_0(E) \to \pi_0(E)[y]/(y^2 + uv^{-1})$.

If E is even periodic and we fix a unit $u \in \pi_2 E$, the category of E-modules has $\mathbb{Z}/2$-graded homotopy groups in the classical sense. Therefore, the set of Morita equivalence classes of algebraic Azumaya algebras over E is the same as the set of Morita equivalence classes of $\mathbb{Z}/2$-graded Azumaya algebras over $\pi_0(E)$: the *Brauer–Wall* group $BW(\pi_0 E)$. This $\mathbb{Z}/2$-graded Brauer group of a commutative ring has been largely determined (generalizing work of Wall over a field). In order to state the result, we will need to recall the definition of the group of $\mathbb{Z}/2$-graded quadratic extensions of a ring R.

Definition 7.3. Suppose R is a commutative ring, viewed as $\mathbb{Z}/2$-graded and concentrated in degree 0. Then $Q_2(R)$ is the set of isomorphism classes of quadratic graded R-algebras: $\mathbb{Z}/2$-graded R-algebras whose underlying ungraded R-algebra is commutative, separable, and projective of rank two.

In the ungraded case the corresponding set is identified with the étale cohomology group $H^1_{et}(\mathrm{Spec}(R), \mathbb{Z}/2)$; similarly $Q_2(R)$ admits a natural group structure. If $\mathrm{Spec}(R)$ is connected, then there are two possible types of element in $Q_2(R)$. In a quadratic graded R-algebra $L = (L_0, L_1)$, either L_1 has rank 0 and we have an ungraded quadratic extension $R \to L_0$, or L_1 has rank 1 and L is of the form (R, L_1) for some rank 1 projective R-module L_1. In the latter case, the multiplication map $L_1 \otimes_R L_1 \to R$ must be an isomorphism. Carrying this analysis further yields the following result.

Proposition 7.4. *When* $\mathrm{Spec}(R)$ *is connected, there is a short exact sequence*

$$0 \to H^1_{et}(R, \mathbb{Z}/2) \to Q_2(R) \to \mathbb{Z}/2.$$

Here the étale cohomology group $H^1_{et}(R, \mathbb{Z}/2)$ *parametrizes ungraded* $\mathbb{Z}/2$*-Galois extensions of* R*, and the map* $Q_2(R) \to \mathbb{Z}/2$ *sends a* $\mathbb{Z}/2$*-graded quadratic* R*-algebra* (L_0, L_1) *to the rank of* L_1*. The image of* $Q_2(R)$ *in* $\mathbb{Z}/2$ *is nontrivial if and only if* 2 *is a unit in* R*.*

Theorem 7.5 ([12])**.** *Suppose that* R *possesses no idempotents. Then the Brauer–Wall group* $BW(R)$ *is contained in a short exact sequence*

$$0 \to \mathrm{Br}(R) \to BW(R) \to Q_2(R) \to 0,$$

where the subgroup is generated by Azumaya algebras concentrated in even degree.

Corollary 7.6. *Suppose that* E *is even-periodic and that* $\pi_0 E$ *possesses no idempotents. Then the subgroup of the Brauer group of* E *generated by algebraic Azumaya algebras is contained in a short exact sequence*

$$0 \to \mathrm{Br}(\pi_0 E) \to \pi_0 \mathrm{Br}(E)^{\mathrm{alg}} \to Q_2(\pi_0 E) \to 0,$$

where the subgroup is generated by algebraic Azumaya algebras with homotopy concentrated in even degrees. In $Q_2(\pi_0 E)$*, the elements of* $H^1_{et}(\pi_0 E, \mathbb{Z}/2)$ *detect the algebras of Example 7.1, while the map to* $\mathbb{Z}/2$ *detects any of the "half-quaternion" algebras of Example 7.2.*

Example 7.7. In the case where E is the complex K-theory spectrum KU, with coefficient ring $\mathbb{Z}[\beta^{\pm 1}]$, the relevant Brauer–Wall group $BW(\mathbb{Z})$ is trivial and all $\mathbb{Z}/2$-graded algebraic Azumaya algebras are Morita equivalent. Therefore, there are no $\mathbb{Z}$-graded algebraic Azumaya algebras over KU other than those of the form $\mathrm{End}_{KU}(M)$ for M a coproduct of suspensions of KU.

Example 7.8. Suppose that $\pi_0 E$ is a Henselian local ring with residue field k. Then extension of scalars determines isomorphisms $H^1_{et}(\pi_0 E, \mathbb{Z}/2) \to H^1_{et}(k, \mathbb{Z}/2)$ and $\mathrm{Br}(\pi_0 E) \to \mathrm{Br}(k)$ ([33, 5], [34, 6.1]), and hence an isomorphism $BW(\pi_0 E) \to BW(k)$. If k is finite (for example, when E is a Lubin–Tate spectrum associated to a formal group law over a finite field) the group $\mathrm{Br}(k)$ is trivial and the Galois cohomology group is $\mathbb{Z}/2$, so we find that the Brauer–Wall group of k is $\mathbb{Z}/2$ if k has characteristic 2 and is of order 4 if k has odd characteristic. The algebraic $\mathbb{Z}/2$-graded Azumaya E-algebras are generated (up to Morita equivalence) by those of Examples 7.1 and 7.2.

Example 7.9. If we form the localized ring $KU[1/2]$, we may use global class field theory to analyze the result. The ordinary Brauer group is $\mathbb{Z}/2$, generated by the Hamilton quaternions over $\mathbb{Z}[1/2]$, and this algebra lifts to an Azumaya algebra as originally shown in [3, 6.3]. The étale cohomology group is $\mathbb{Z}/2 \times \mathbb{Z}/2$, with nonzero elements corresponding to the quadratic extensions obtained by adjoining i, $\sqrt{2}$, or $\sqrt{-2}$. Finally, $KU[1/2]$ also has Azumaya algebras given by its 1-periodifications, generating the quotient $\mathbb{Z}/2$ of the Brauer–Wall group $BW(\mathbb{Z}[1/2])$. The full group has order 16, and one can show that it is isomorphic to $\mathbb{Z}/8\times\mathbb{Z}/2$. These can be given specific generators: the $\mathbb{Z}/8$-factor is generated by an algebra with coefficient ring $\mathbb{Z}[\beta^{\pm(1/2)}, 1/2]$ as an algebra over KU_*, while the $\mathbb{Z}/2$-factor is generated by an algebra with coefficient ring

$$KU_*\left[\sqrt{2}, 1/2\right]\langle S\rangle/(S^2-\beta, S\sqrt{2}+\sqrt{2}S).$$

Remark 7.10. The short exact sequence of Theorem 7.5 is generalized in [13, Section 4] for many more groups, and by applying their results one can compute the Brauer–Wall group classifying algebraic Azumaya algebras for an overwhelming abundance of examples. For the 4-periodic localization $KO[1/2]$ we may show that the Brauer–Wall group has 16 elements, combining the order-2 Brauer group of $\mathbb{Z}[1/2]$ with the order-8 collection of Galois extensions of $\mathbb{Z}[1/2]$ with cyclic Galois group of order four. For the p-complete Adams summand L_p at an odd prime p, the Brauer–Wall group has $(p-1)$ elements if $p \equiv 1 \mod 4$ and $2(p-1)$ elements if $p \equiv 3 \mod 4$. By contrast, p-*local* spectra such as $K_{(p)}$, $KO_{(p)}$, or $L_{(p)}$ tend to have much larger Brauer groups because $\mathbb{Z}_{(p)}$ and its finite extensions have infinite Brauer groups.

7.2 Homotopy fixed-points of $\mathrm{Pic}(KU)$

In this section we study the Galois extension $KO \to KU$. Most of the structure of the homotopy fixed-point spectral sequence for $\mathrm{Pic}(KU)$ has been determined in depth by Mathew and Stojanoska using tools they developed for comparing with the homotopy fixed-point spectral sequence for KU [10, 7.1]. However, for our purposes we will require information about the behavior of the spectral sequence in small, negative degrees.

We recall the following about the category of naive G-spectra.

Proposition 7.11. *For a G-equivariant spectrum X such that $\pi_i(X) = 0$ for $n < i < m$, the d_{n-m+1}-differential*

$$H^s(G; \pi_n(X)) \to H^{s+m-n+1}(G; \pi_m(X))$$

in the homotopy fixed-point spectral sequence for X^{hG} is given by an equivariant k-invariant

$$k^G \in \pi_{n-m-1}F_{\mathbb{S}[G]}(H\pi_n X, H\pi_m X),$$

which determines a cohomology operation of degree $(m-n+1)$ on Borel equivariant cohomology. The forgetful map

$$\pi_{n-m-1}F_{\mathbb{S}[G]}(H\pi_n X, H\pi_m X) \to \pi_{n-m-1}F_{\mathbb{S}}(H\pi_n X, H\pi_m X)$$

sends k^G to the underlying k-invariant of X.

Using the adjunction

$$F_{\mathbb{S}[G]}(X,Y) \simeq F_{\mathbb{S}[G]}(\mathbb{S}, F_{\mathbb{S}}(X,Y)) = F_{\mathbb{S}}(X,Y)^{hG},$$

we recover the following computational tool.

Proposition 7.12. *For functors $BG \to \mathrm{Sp}$ representing spectra X and Y with G-action, there exists a spectral sequence with E_2-term*

$$E_2^{s,t} = H^s(G; \pi_t F_{\mathbb{S}}(X,Y)) \Rightarrow \pi_{t-s} F_{\mathbb{S}[G]}(X,Y).$$

Furthermore, the edge morphism in this spectral sequence recovers the natural map to $\pi_* F_{\mathbb{S}}(X,Y)$.

We may then apply this to calculate the possible first two C_2-equivariant k-invariants of $\mathfrak{pic}(KU)$, both between degrees 0 and 1 and between degrees 1 and 3.

Proposition 7.13. *Let $\mathbb{Z}^-$ be $\mathbb{Z}$ with the sign action of C_2, and*

$$\beta^- \colon H^*(C_2; \mathbb{Z}/2) \to H^{*+1}(C_2; \mathbb{Z}^-)$$

the Bockstein map associated to the short exact sequence

$$0 \to \mathbb{Z}^- \to \mathbb{Z}^- \to \mathbb{Z}/2 \to 0.$$

Let $x \in H^1(G; \mathbb{Z}/2)$ denote the generator. We have

$$\pi_{-2} F_{\mathbb{S}[C_2]}(H\mathbb{Z}/2, H\mathbb{Z}/2) \cong (\mathbb{Z}/2)^3,$$

generated by the operations $\mathrm{Sq}^2(-)$, $x \cdot \mathrm{Sq}^1(-)$, and $x^2 \cdot (-)$. We also have

$$\pi_{-3} F_{\mathbb{S}[C_2]}(H\mathbb{Z}/2, H\mathbb{Z}^-) \cong (\mathbb{Z}/2)^2,$$

generated by the operations $\beta^- \circ \mathrm{Sq}^2(-)$ and $\beta^-(x^2 \cdot (-))$.

The restriction to the group of nonequivariant operations sends the generators involving x to zero.

Proof. Proposition 7.12 gives us two spectral sequences, pictured in Figure 7.1:

$$H^s(C_2; \pi_t F_{\mathbb{S}}(H\mathbb{Z}/2, H\mathbb{Z}/2)) \Rightarrow \pi_{t-s} F_{\mathbb{S}[C_2]}(H\mathbb{Z}/2, H\mathbb{Z}/2)$$
$$H^s(C_2; \pi_{t-s} F_{\mathbb{S}}(H\mathbb{Z}/2, H\mathbb{Z})) \Rightarrow \pi_{t-s} F_{\mathbb{S}[C_2]}(H\mathbb{Z}/2, H\mathbb{Z}^-)$$

There is an isomorphism $\pi_{-*} F_{\mathbb{S}}(H\mathbb{Z}/2, H\mathbb{Z}/2) \cong A^*$, where A^* is the mod-2 Steenrod algebra; this group is isomorphic to $\mathbb{Z}/2$ for $-2 \le * \le 0$ and is trivial for all other $* \ge -2$. Similarly, there is an isomorphism $\pi_{-*} F_{\mathbb{S}}(H\mathbb{Z}/2, H\mathbb{Z}) \cong \mathrm{Sq}^1 \cdot A^* \subset A^*$; this group is isomorphic to $\mathbb{Z}/2$ for $* = -1, -3$ and is trivial for all other $* \ge -3$. The associated spectral sequences appear in Figure 7.1. These spectral sequences place an upper bound of 8 on the size of the group $\pi_{-2} F_{\mathbb{S}[C_2]}(H\mathbb{Z}/2, H\mathbb{Z}/2)$ and of 4 on the size of the group $\pi_{-3} F_{\mathbb{S}[C_2]}(H\mathbb{Z}/2, H\mathbb{Z}^-)$.

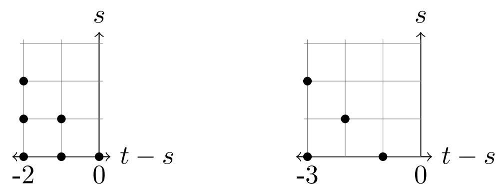

Figure 7.1: Spectral sequences for equivariant k-invariants

However, these cohomology operations we have described in these groups are linearly independent over $\mathbb{Z}/2$, as can be checked by applying them to elements in the group

$$\pi_* F_{\mathbb{S}[C_2]}(\Sigma^\infty_+ EC_2, H\mathbb{Z}/2) \cong H^*(BC_2; \mathbb{Z}/2).$$

(These represent elements in different cohomological filtration in this spectral sequence.) □

Proposition 7.14. *The first two C_2-equivariant k-invariants of $\mathfrak{pic}(KU)$ are $\mathrm{Sq}^2 + x\,\mathrm{Sq}^1$ and $\beta^- \mathrm{Sq}^2$.*

Proof. The underlying nonequivariant k-invariants must be the first two k-invariants of $\mathfrak{pic}(KU)$. These are Sq^2 and $\beta\,\mathrm{Sq}^2$, where β is the nonequivariant Bockstein.

Moreover, the generating elements in $\pi_0\,\mathfrak{pic}(KU)$ and $\pi_1\,\mathfrak{pic}(KU)$ are the images of the classes $[\Sigma KO]$ and -1 from $\mathfrak{pic}(KO)$ respectively, and hence must survive the homotopy fixed-point spectral sequence. These classes would support a nontrivial d_2 or d_3 differential if the cohomology operation involved a nonzero multiple of x^2 or $\beta^- x^2$ respectively. This shows that the second k-invariant can only be $\beta^- \mathrm{Sq}^2$, and the first k-invariant can only be Sq^2 or $\mathrm{Sq}^2 + x\,\mathrm{Sq}^1$.

Suppose that the second k-invariant were Sq^2. This k-invariant is in the image of the map

$$\pi_{-2} F_{\mathbb{S}}(H\mathbb{Z}/2, H\mathbb{Z}/2) \to \pi_{-2} F_{\mathbb{S}[C_2]}(H\mathbb{Z}/2, H\mathbb{Z}/2)$$

induced by the ring map $\mathbb{S}[C_2] \to \mathbb{S}$, and so the resulting C_2-equivariant Postnikov stage $\tau_{\le 1}\,\mathfrak{pic}(KU)$ would be equivalent to one with the trivial C_2-action. We would then have the equivalence

$$(\tau_{\le 1}\,\mathfrak{pic}(KU))^{hC_2} \simeq F((BC_2)_+, \tau_{\le 1}\,\mathfrak{pic}(KU)).$$

This splits off a copy of $\tau_{\le 1}\,\mathfrak{pic}(KU)$ so there could be no hidden extensions from $H^0(C_2; \pi_0\,\mathfrak{pic}(KU)$ to $H^1(C_2; \pi_1\,\mathfrak{pic}(KU))$ in the homotopy fixed-point spectral sequence. However, there is a hidden extension: the class $[\Sigma KO] \in \pi_0\,\mathfrak{pic}(KO)$ has nontrivial image in $H^0(C_2; \pi_0\,\mathfrak{pic}(KU))$ and twice it is $[\Sigma^2 KO]$, which has nontrivial image in $H^1(C_2; (\pi_0 KU)^\times)$. □

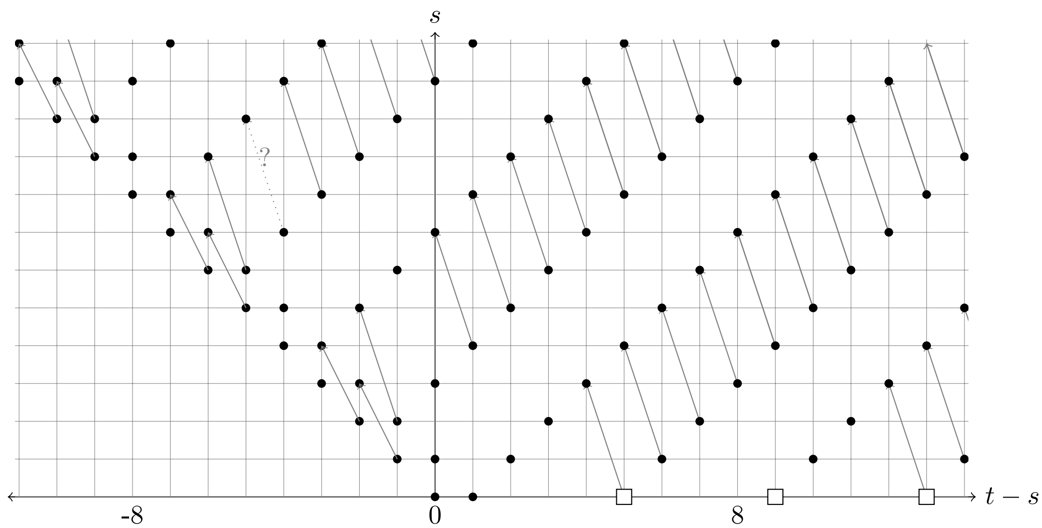

Figure 7.2: Fixed-point spectral sequence for $\mathrm{Pic}(KU)$ up to E_3

Proposition 7.15. *The homotopy fixed-point space $B\,\mathrm{Pic}(KU)^{hC_2}$ has homotopy groups*

$$\pi_n B\,\mathrm{Pic}(KU)^{hC_2} = \begin{cases} \pi_{n-2} GL_1(KO) & \text{if } n \geq 2 \\ \mathbb{Z}/8 & \text{if } n = 1 \\ \mathbb{Z}/2 & \text{if } n = 0. \end{cases}$$

Proof. The homotopy fixed-point spectral sequence

$$H^s(C_2; \pi_t\,\mathfrak{pic}(KU)) \Rightarrow \pi_{t-s}\,\mathfrak{pic}(KU)^{hC_2}$$

is pictured in Figure 7.2; we refer to [10] for the portion with $t > 3$, obtained by comparison with the homotopy fixed point spectral sequence for KU. The differentials supported on $t = 0$ and $t = 1$ are the stable cohomology operations we just determined. The inclusion of $\mathbb{Z}/8$ into $\pi_0\,\mathfrak{pic}(KO) \cong \pi_0\,\mathfrak{pic}(KU)^{hC_2}$ forces the hidden extension in degree 0. □

This recovers the calculation of the Picard group of KO by [35].

There are potential further differentials in negative degrees in the homotopy fixed-point spectral sequence which we have not addressed here. There are potential sources for a d_4-differential when $t = 0$, $s \equiv 3 \mod 4$. There are also potential targets for a d_3- or d_5- or d_6-differential when $t = 5$, $s \equiv 2 \mod 4$, though these latter would be impossible if the Postnikov stage $\mathfrak{pic}(KU) \to \tau_{\leq 3}\,\mathfrak{pic}(KU)$ split off equivariantly. It seems likely that a precise formulation of the periodic structure in this spectral sequence would be able to address these questions.

7.3 Lifting from KU to KO

In this section we examine those Azumaya KO-algebras whose extension to KU are algebraic.

By Example 7.7, we have the following.

Proposition 7.16. *Any algebraic Azumaya KU-algebra is of the form $\mathrm{End}_{KU} N$, where N is a finite coproduct of suspensions of KU.*

Therefore, by Proposition 7.15 and Corollary 6.16, there are at most two Morita equivalence classes of Azumaya KO-algebras whose extensions to KU are algebraic.

The following shows that the nontrivial Morita equivalence class is realizable.

Proposition 7.17. *There exists a unique equivalence class of quaternion algebra Q over KO such that*

- *$KU \otimes_{KO} Q \simeq M_2(KU)$, and*
- *there is no KO-module M such that $Q \not\simeq \mathrm{End}_{KO}(M)$ as KO-algebras.*

This algebra has homotopy groups isomorphic, as a KO_*-algebra, to the homotopy groups of a twisted group algebra:

$$\pi_* Q \cong \pi_* KU\langle C_2\rangle \cong \pi_* \mathrm{End}_{KO} KU.$$

Proof. The KO-algebras A such that $KU \otimes_{KO} A \simeq M_2(KU)$ are parametrized by the preimage of the component $B\,\mathrm{Aut}_{\mathrm{Alg}_{KU}} M_2(KU) \subset \mathrm{Az}_{KU}$. We may therefore apply the obstruction theory of Section 6.4. We know that there is a chain of equivalences

$$\begin{aligned} KU \otimes_{KO} \mathrm{End}_{KO}(KU) &\simeq \mathrm{End}_{KU}(KU \otimes_{KO} KU) \\ &\simeq \mathrm{End}_{KU}(KU \oplus KU) \simeq M_2(KU), \end{aligned}$$

and so we may use $\mathrm{End}_{KO}(KU)$ as a basepoint for the purposes of calculations. The obstruction theory then takes place in a fringed spectral sequence with E_2-term

$$H^s(C_2; \pi_t B\,\mathrm{Aut}_{\mathrm{Alg}_{KU}}(M_2(KU))).$$

By Corollary 5.18, we have a long exact sequence

$$\begin{aligned} \cdots \to (\pi_n KU)^\times \to (\pi_n M_2(KU))^\times \to \pi_n(\mathrm{Aut}_{\mathrm{Alg}_{KU}}(M_2 KU)) \to \cdots \\ \to (\pi_0 KU)^\times \to (\pi_0 M_2(KU))^\times \to \pi_0(\mathrm{Aut}_{\mathrm{Alg}_{KU}}(M_2(KU))) \to \pi_0 \mathrm{Pic}(KU). \end{aligned}$$

Since $\pi_* M_2(KU) \cong M_2(\pi_*(KU))$, we find that $\mathrm{Aut}_{\mathrm{Alg}_{KU}}(M_2 KU)$ has trivial homotopy groups in odd degrees, and that for $k > 0$ there are short exact sequences

$$0 \to \pi_{2k} KU \to \pi_{2k} M_2(KU) \to \pi_{2k} \mathrm{Aut}_{\mathrm{Alg}_{KU}}(M_2 KU) \to 0.$$

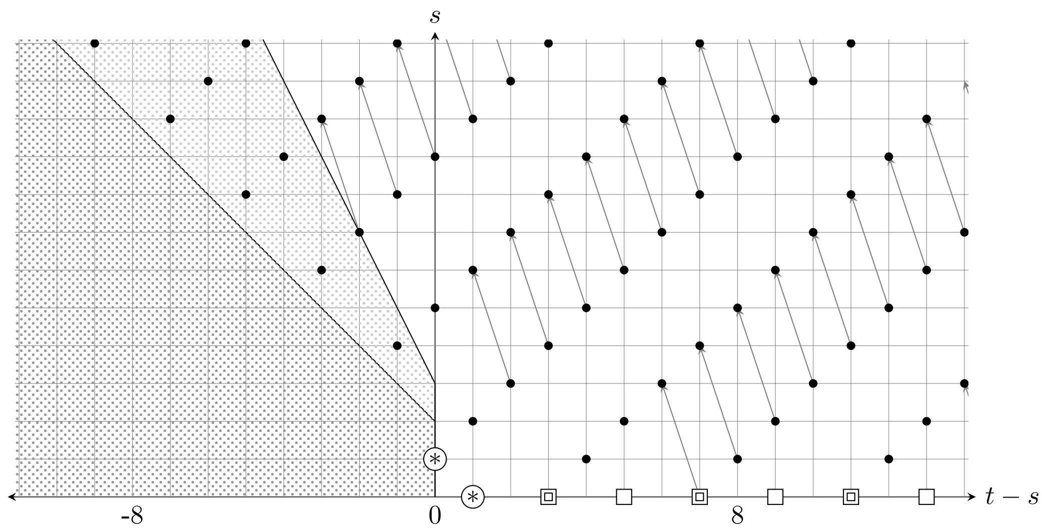

Figure 7.3: Fixed-point spectral sequence for $B\,\mathrm{Aut}(KU \otimes_{KO} \mathrm{End}_{KO}(KU))$ up to E_3

Moveover, the C_2-action on $\pi_{2k}(KU \otimes_{KO} \mathrm{End}_{KO}(KU)) \cong M_2(KU_{2k})$ is given in matrix form by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (-1)^k \begin{bmatrix} d & c \\ b & a \end{bmatrix}.$$

We may now use this to calculate group cohomology. We find that for $s, t > 0$, the cohomology $H^s(C_2; \pi_t M_2(KU))$ vanishes with this action and we have isomorphisms

$$H^s(C_2; \pi_t \,\mathrm{Aut}_{\mathrm{Alg}_{KU}}(M_2 KU)) \to H^{s+1}(C_2; \pi_t \,\mathrm{Pic}(KU)),$$

realized by the natural map $B\,\mathrm{Aut}_{\mathrm{Alg}_{KU}}(M_2 KU) \to B\,\mathrm{Pic}(KU)$. We display the spectral sequence for calculating lifts of $M_2(KU)$ in Figure 7.3 through the E_3-term. The regions where the spectral sequence is undefined at E_2 or E_3 are blocked out, and the nonabelian cohomology $H^s(C_2; \mathrm{PGL}_2(\mathbb{Z}))$ is indicated with $\circledast$. The first detail we note about this spectral sequence is that for $t - s \geq -1$, the E_4-page vanishes entirely for $s \geq 5$. There are potential obstructions to lifting in the column $t - s = -1$ and to uniqueness in the column $t - s = 0$; we will now discuss these obstruction groups using the machinery of Section 6.4.

Because the groups $E_r^{s,s-1}$ and $E_r^{s,s}$ become trivial at E_4 for $s > 5$, there are no obstructions to existence or uniqueness of lifting algebras beyond Tot^5: any Azumaya KU-algebra equivalent to $M_2(KU)$ with a lift to Tot^5 has an essentially unique further lift to an Azumaya KO-algebra.

The group $E_2^{4,3}$ is $\mathbb{Z}/2$, and this group is a potential home for obstructions for a point in Tot^2 which lifts to Tot^3 to also lift to Tot^4 (see Remark 7.18 for further elaboration). Since we have already chosen a lift of $M_2(KU)$ to the algebra $\mathrm{End}_{KO}(KU)$ in the homotopy limit to govern the obstruction theory, the obstruction must be zero at this basepoint.

The group $E_3^{5,5}$ parametrizes differences between lifts from Tot^4 to Tot^5. This group is $\mathbb{Z}/2$, and contains only permanent cycles due to the fact that the spectral sequence has a vanishing region at E_4. Therefore, there are two distinct lifts of $KU \otimes_{KO} \mathrm{End}_{KO}(KU)$ from Tot^2 to Tot^5, representing two inequivalent KO-algebras which become equivalent to $M_2(KU)$ after extending scalars. One of these is $\mathrm{End}_{KO}(KU)$; we will refer to the other algebra as Q.

Moreover, the map $B\,\mathrm{Aut}_{\mathrm{Alg}_{KU}}(M_2 KU) \to B\,\mathrm{Pic}(KU)$ induces an isomorphism on homotopy fixed-point spectral sequences in the relevant degree. The generator of $E_3^{5,5}$ representing Q therefore maps to the nontrivial element of $\pi_0(B\,\mathrm{Pic}(KU))^{hC_2} \subset \pi_0\,\mathrm{Br}(KO)$, and so any points of the fixed-point category with distinct lifts to Tot^5 are Morita inequivalent.

Hence, there exists precisely one other KO-algebra, Q, whose image in Tot^4 is the same as the image of $\mathrm{End}_{KO}(KU)$, and Q is Morita inequivalent to any endomorphism algebra. □

Remark 7.18. The obstruction group $E_2^{4,3}$ deserves some mention. There is an element in $H^1(C_2; \pi_1 B\,\mathrm{Aut}_{\mathrm{Alg}_{KU}}(M_2 KU))$ whose image in $H^2(C_2; \pi_2 B\,\mathrm{Pic}(KU))$ is nontrivial. More explicitly, $\pi_1 B\,\mathrm{Aut}_{\mathrm{Alg}_{KU}}(M_2 KU)$ contains $PGL_2(\mathbb{Z})$ and this H^1-class is represented by the alternative action

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (-1)^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

of C_2 on $\pi_{2t} M_2(KU)$. One might hope that there is a KO-algebra A such that $KU \otimes_{KO} A$ is $M_2(KU)$ with this alternative C_2-action on the coefficient ring.

For example, we might imagine finding a self-map $\phi\colon KO \to KO$ representing multiplication by $-1 \in \pi_0(KO)$, and using it to produce an action of C_2 on $KU \otimes_{KO} M_2(KO)$ such that the generator acts on the KU factor by complex conjugation and on the $M_2(KO)$-factor by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 \\ \phi & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & \phi \\ 1 & 0 \end{bmatrix}.$$

In the classical setup, one encounters a sequence of difficulties with carrying this program out. The spectrum KO cannot be a fibrant-cofibrant KO-module if KO is strictly commutative, so we require a replacement in order for ϕ to be defined. Then this replacement is not strictly the unit for the smash product and so we cannot move ϕ across a smash product without an intervening homotopy. In order to make this a ring homomorphism one either wants ϕ^2 to be the identity, or one wants to replace ϕ by an automorphism so that we can genuinely replace this with a conjugation action. And so on. One is left with the feeling that these are technical details and the tools are just barely inadequate for the job, but this is not the case: this H^1-class cannot be realized by an algebra *at all* because the image in $H^2(C_2; \pi_2 B\,\mathrm{Pic}(KU))$ supports a d_3-differential (see Figure 7.2). These seemingly mild details are *fundamental* to the situation.

References

[1] M. Auslander and O. Goldman, "The Brauer group of a commutative ring," *Trans. Amer. Math. Soc.*, vol. 97, pp. 367–409, 1960.

[2] B. Toën, "Derived Azumaya algebras and generators for twisted derived categories," *Invent. Math.*, vol. 189, no. 3, pp. 581–652, 2012.

[3] A. Baker, B. Richter, and M. Szymik, "Brauer groups for commutative S-algebras," *J. Pure Appl. Algebra*, vol. 216, no. 11, pp. 2361–2376, 2012.

[4] N. Johnson, "Azumaya objects in triangulated bicategories," *J. Homotopy Relat. Struct.*, vol. 9, no. 2, pp. 465–493, 2014.

[5] R. Haugseng, "The higher Morita category of E_n-algebras," 2014.

[6] B. Antieau and D. Gepner, "Brauer groups and étale cohomology in derived algebraic geometry," *Geom. Topol.*, vol. 18, no. 2, pp. 1149–1244, 2014.

[7] J. Lurie, "Derived Algebraic Geometry VIII: Quasi-coherent sheaves and Tannaka duality theorems." http://www.math.harvard.edu/~lurie/papers/DAG-VIII.pdf, 2011.

[8] A. Mathew, "Residue fields for a class of rational $\mathbf{E}_\infty$-rings and applications," 2014.

[9] J. Rognes, "Galois extensions of structured ring spectra. Stably dualizable groups," *Mem. Amer. Math. Soc.*, vol. 192, no. 898, pp. viii+137, 2008.

[10] A. Mathew and V. Stojanoska, "The Picard group of topological modular forms via descent theory," 2014.

[11] A. K. Bousfield, "Homotopy spectral sequences and obstructions," *Israel J. Math.*, vol. 66, no. 1-3, pp. 54–104, 1989.

[12] C. Small, "The Brauer-Wall group of a commutative ring," *Trans. Amer. Math. Soc.*, vol. 156, pp. 455–491, 1971.

[13] L. N. Childs, G. Garfinkel, and M. Orzech, "The Brauer group of graded Azumaya algebras," *Trans. Amer. Math. Soc.*, vol. 175, pp. 299–326, 1973.

[14] H. Ikai, "Azumaya algebras with general gradings and their automorphisms," *Tsukuba J. Math.*, vol. 23, no. 2, pp. 293–320, 1999.

[15] M. Hovey and N. P. Strickland, "Morava K-theories and localisation," *Mem. Amer. Math. Soc.*, vol. 139, no. 666, pp. viii+100, 1999.

[16] N. Johnson and A. M. Osorno, "Modeling stable one-types," *Theory Appl. Categ.*, vol. 26, pp. No. 20, 520–537, 2012.

[17] J. F. Adams, "Prerequisites (on equivariant stable homotopy) for Carlsson's lecture," in *Algebraic topology, Aarhus 1982 (Aarhus, 1982)*, vol. 1051 of *Lecture Notes in Math.*, pp. 483–532, Springer, Berlin, 1984.

[18] D. Quillen, "On the (co-) homology of commutative rings," in *Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968)*, pp. 65–87, Providence, R.I.: Amer. Math. Soc., 1970.

[19] P. G. Goerss and M. J. Hopkins, "Moduli spaces of commutative ring spectra," in *Structured ring spectra*, vol. 315 of *London Math. Soc. Lecture Note Ser.*, pp. 151–200, Cambridge: Cambridge Univ. Press, 2004.

[20] A. K. Bousfield, "Cosimplicial resolutions and homotopy spectral sequences in model categories," *Geom. Topol.*, vol. 7, pp. 1001–1053 (electronic), 2003.

[21] B. Shipley, "A convenient model category for commutative ring spectra," in *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, vol. 346 of *Contemp. Math.*, pp. 473–483, Providence, RI: Amer. Math. Soc., 2004.

[22] D. G. Davis and T. Lawson, "Commutative ring objects in pro-categories and generalized Moore spectra," *Geom. Topol.*, vol. 18, no. 1, pp. 103–140, 2014.

[23] S. Schwede and B. E. Shipley, "Algebras and modules in monoidal model categories," *Proc. London Math. Soc. (3)*, vol. 80, no. 2, pp. 491–511, 2000.

[24] J. Lurie, "Higher Algebra." http://www.math.harvard.edu/~lurie/papers/higheralgebra.pdf, 2014.

[25] D. Gepner and R. Haugseng, "Enriched ∞-categories via non-symmetric ∞-operads," *Adv. Math.*, vol. 279, pp. 575–716, 2015.

[26] J. Lurie, *Higher Topos Theory*, vol. 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[27] M. Ando, A. J. Blumberg, and D. Gepner, "Parametrized spectra, multiplicative Thom spectra, and the twisted Umkehr map," 2011.

[28] D. Clausen, "p-adic J-homomorphisms and a product formula," 2011.

[29] A. J. Blumberg, D. Gepner, and G. Tabuada, "Uniqueness of the multiplicative cyclotomic trace," *Adv. Math.*, vol. 260, pp. 191–233, 2014.

[30] S. Schwede and B. Shipley, "Stable model categories are categories of modules," *Topology*, vol. 42, no. 1, pp. 103–153, 2003.

[31] A. Mathew, "The Galois group of a stable homotopy theory," *Adv. Math.*, vol. 291, pp. 403–541, 2016.

[32] J. Lurie, "Derived Algebraic Geometry VII: Spectral Schemes." http://www.math.harvard.edu/~lurie/papers/DAG-VII.pdf, 2011.

[33] G. Azumaya, "On maximally central algebras," *Nagoya Math. J.*, vol. 2, pp. 119–150, 1951.

[34] A. Grothendieck, "Le groupe de Brauer. I. Algèbres d'Azumaya et interprétations diverses," in *Dix Exposés sur la Cohomologie des Schémas*, pp. 46–66, North-Holland, Amsterdam; Masson, Paris, 1968.

[35] M. J. Hopkins, M. Mahowald, and H. Sadofsky, "Constructions of elements in Picard groups," in *Topology and representation theory (Evanston, IL, 1992)*, vol. 158 of *Contemp. Math.*, pp. 89–126, Providence, RI: Amer. Math. Soc., 1994.