Observation: The Eisenstein series $g_n = 2 \cdot G_n$ satisfy all these conditions.

At $p=2$, the story is not as clean ...

$$\pi_q \, \mathrm{fibre}(gl_1\, tmf \longrightarrow L_2\, gl_1\, tmf) = 0 \quad \text{if } q>3$$

More generally, if $R = L_n R$ and R is E_∞ ring spectrum, then

$$\pi_q \, \mathrm{fibre}(gl_1 R \longrightarrow L_n\, gl_1 R) = 0 \quad \text{if } q > n+1.$$

The key input is:

$$K(n)_* \, K(q, \mathbb{Z}/p) = 0 \quad \text{if } q > n.$$

17.10.03

$L_{K(2)}\, tmf$

Let R be an $\mathbb{F}_p$-algebra, C/R elliptic curve.

C is super-singular if $\hat{C}$ has strict height 2.

Example: $y^2 + y = x^3 / \mathbb{F}_2$

Classical: Over a field $R = k$, elliptic curves are classified by $j(C) \in k$ if $k = \bar{k}$ is algebraically closed.

Super-singular curves correspond to a finite set of j-values.

$\mathcal{M}_{ss}(R) =$ groupoid of super-singular curves over R

$=$ the locus where $v_1 = 0$

$$v_1 \in \Gamma(\mathcal{M}_{ell} \otimes \mathbb{F}_p, \omega^{\otimes p-1})$$

At primes ≥ 11, there is more than one iso class of s.s. elliptic curves, so then $L_{K(2)}tmf$ is a product of homotopy fixed point spectra.

Fact: for $p \geq 5$, $p \nmid \#G_i$.

Where is the elliptic curve?

Serre-Tate deformation theory for abelian varieties:

A ring, I ideal nilpotent, $p \in I$,

$A_0 = A/I$

$$\begin{tikzcd} \mathcal{M}_{ell}(A) \arrow[rr, "\text{reduction}"] \arrow[dr, "\simeq"'] & & \mathcal{M}_{ell}(A_0) \\ & \mathrm{Def}(A, A_0) & \end{tikzcd}$$

objects:
- C_0/A_0 elliptic curve
- p-divisible group G/A
- isomorphism $\varepsilon : G_0 \xrightarrow{\cong} C_0[p^\infty]$

Scheme-theoretic pullback

$$\begin{tikzcd} C[p^h] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C_0 \arrow[d, "{[p^h]}"] \\ \mathrm{Spec}(A_0) \arrow[r] & C_0 \end{tikzcd}$$

p-divisible group = inductive limit of such group schemes.

$$C_0[p^h]^1 \rightarrowtail C_0[p^h] \twoheadrightarrow C_0[p^h]^{et}$$

formal part

étale part (= 0 for strict height 2)

Thus if C_0 is super-singular, then all we need to get a curve on A is a lift of the formal group. Lubin-Tate theory does this, and so there is a unique lift.

$\Rightarrow$ "Near the locus of super-singular curves, the morphism $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is étale"

Haynes Miller
13.10.03

Formal groups and ring spectra

	S $\longrightarrow$	MU $\longrightarrow$	$H\mathbb{Z}$
H_*:	easy	OK	hard
π_*:	hard	OK	easy

Localize : $K_* \otimes_{MU_*} MU_*(X)$ (Todd genus $MU_* \to K_*$)

$$\begin{array}{c} K_* \otimes_{MU_*} MU_*(X) \\ \cong \big\downarrow \text{ Conner-Floyd isomorphism} \\ K_*(X) \end{array}$$

This leads to Landweber exactness, main topic of this lecture.

$KO \longrightarrow K$

Atiyah : $KO \simeq K^{hC_2}$

Try to construct interesting spectra by similar homotopy fixed point constructions.

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§ 1. Formal groups Quillen, Novikov, Mischenko.

ξ/X complex line bundle, want Euler class $e(\xi) \in E^2(X)$

E is complex orientable if it has Euler classes for complex line bundles.

(Today: Ring spectrum = commutative 1-monoid in stable homotopy category;

Formula for Euler class of tensor product

$$e(\xi \otimes \eta) = e(\xi) +_F e(\eta) = e(\xi) + e(\eta) + \dots \quad \text{formal group law.}$$

Preferable to re-locate Euler class into dimension 0.

Often there is a unit in $E^2(*)$, but this can be done more generally.

Definition: A Laurent ring is an evenly graded ring R_* such that $R_2 = \omega$ is a rank one projective R_0-module and such that

mult: $R_2 \otimes_R R_{-2} \xrightarrow{\cong} R$ is an isomorphism ($R = R_0$)

Thus $R_{-2} \cong \omega^{-1}$ and $R_{2n} \cong \omega^n$ for all $n \in \mathbb{Z}$, $R_* = R(\omega)$

Definition: A Laurent spectrum is a ring spectrum $\underline{\underline{R}}$ for which $R_* = \pi_* \underline{\underline{R}}$ is a Laurent ring.

Then $R^0 \mathbb{C}P^\infty$ is a formal group.

Definition: A formal curve $/R$ is a complete augmented R-algebra $\mathcal{O}_C$ which is formally smooth and 1-dimensional.

Formal smoothness $\Rightarrow$ $\omega = I/I^2$ is projective R-module ($I = \ker(\varepsilon)$)

1-dimensional $\underset{\text{Def}}{\Longleftrightarrow}$ ω is rank 1 projective.

Formally smooth: $gr\, \mathcal{O}_C \cong R[\omega] = \bigoplus_{n \geq 0} \omega^n$

This implies that $\mathcal{O}_C \xleftarrow{\cong} R[\![\omega]\!]$ un-canonically.

Definition A formal group $/R$ is a formal curve $G/_R$ together with a coproduct $\Delta : \mathcal{O}_G \longrightarrow \mathcal{O}_G \hat{\otimes} \mathcal{O}_G$

Example: For Laurent spectrum $\underline{\underline{R}}$, $R^0(\mathbb{C}P^\infty)$ is a formal group.

Notation: $d: \mathcal{O}_G \longrightarrow I \longrightarrow \omega = I/I^2$ "module of inv. differentials"

Assume ω is free. Then a coordinate on G is a $t \in \mathcal{O}_G$ such that $\omega = R\langle dt \rangle$, i.e. dt generates ω.

Then ω^{-1} has generator $\frac{\partial}{\partial t}$ (ω^{-1} = tangent space)

In the topological context

$$e = t\cdot\frac{\partial}{\partial t} \in R^2(\mathbb{CP}^\infty) \quad \text{is an Euler class}$$

(= complex orientation) determined by the coordinate.

$\mathcal{F}(R)$ = formal grps/R = pre-additive category

so have any map $[-]: \mathbb{Z} \longrightarrow \mathrm{End}_R(G)$ for any $G \in \mathcal{F}(R)$

For a formal grp laws, a homomorphism $\Theta: F \to G$ is a power series $\Theta(t) \in t\cdot R[[t]]$ such that $\Theta(x +_F y) = \Theta(x) +_G \Theta(y)$

For any homomorphism $f: R \to S$ (or $\mathrm{Spec}\, R \leftarrow \mathrm{Spec}\, S$)

can pull back formal grps $F \mapsto f^*F$.

A <u>formal grp</u> is a pair (R, F) where F is a formal grp over R.

A morphism in $\mathcal{F}$ is a pair $(f: R \to S,\ \Theta: f^*F \xrightarrow[\text{iso}]{\cong} G)$

of FG/ S

There exists an initial formal grp law $= F_u/L$

where L = Lazard ring, such that for any F/R, there exists a unique ring homomorphism $f: L \to R$ such that $f^*F_u = F$.

Define $\underline{L} = \underline{MP} = \bigvee_{n\in\mathbb{Z}} \Sigma^{2n} MU$, so that $\pi_0 \underline{L} = MU_* = L$

<u>Quillen's theorem</u>: the formal grp law associated to $\underline{MP}$ is universal.

§ <u>2: Formal groups in characteristic p</u>

Assume $pR = 0$. Frobenius $\phi: R \to R$, $x \mapsto x^p$

For formal grp law F/R, get formal grp law ϕF by

$$x^p +_{\phi F} y^p = (x +_F y)^p, \quad \text{i.e.}$$

$\varphi(t) = t^p$ is a homomorphism of fgls $\varphi: F \longrightarrow \phi F$.

Lemma: Let $\Theta : F \longrightarrow G$ be homomorphism of FGLs$/R$ such that $d\Theta = \Theta'(0) = 0$. Then there exists a unique homomorphism Θ_1 such that the following diagram commutes:

$$\begin{tikzcd} F \arrow[rr, "\Theta"] \arrow[dr, "\varphi"'] & & G \\ & \Phi F \arrow[ur, "\Theta_1"'] & \end{tikzcd}$$

i.e., $\Theta(t) = \Theta_1(t^p)$.

Definition $\Theta : F \longrightarrow G$ is of strict height n if

$$\Theta(t) = \Theta_n(t^{p^n}) \quad \text{for } \Theta_n'(0) \in R^{\times} \text{ is a unit.}$$

A formal grp law F has strict height n if $[p]_F$ has strict height n.

$\mathcal{F}_n(R)$ = full subcategory of $\mathcal{F}(R)$ of formal grps of strict height n, with isomorphisms as morphisms.

So for $F \in \mathcal{F}_n(R)$, we have

$$\begin{tikzcd} F \arrow[rr, "{[p]}"] \arrow[dr, "\varphi^n"'] & & F \\ & \Phi^n F \arrow[ur, "V_n"'] & \end{tikzcd}$$

for an iso of formal grps $V_n : \Phi^n F \longrightarrow F$

Lurie's formulation: natural isomorphism $V_n : \Phi^n \xrightarrow{\cong} \mathrm{id}$ on $\mathcal{F}_n$

Thus Φ is fully faithful.

Remark: The height of a formal grp is a complete isomorphism invariant if R is an algebraically closed field (Lazard)

Variant (Strickland): for $F, G \in \mathcal{F}_n(R)$, any ring R with $pR = 0$, then there exists a faithfully flat extension $f : R \to S$ such that

$$fF \cong fG \text{ over } S.$$

§ 3 Landwebers theorem

Consider homomorphism $\Theta : F \longrightarrow G$ over R with $pR = 0$.

$\Theta(t) = \sum_{i=1}^{\infty} a_{i-1} t^i$. Define ideals

$$I_n(\Theta) = (a_0, \dots, a_{p^{n-1}-1}) \subset R \quad \text{give a chain}$$

$$0 = I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \subseteq R.$$

By the lemma, $I_\bullet$ is a scale, i.e. a sequence of ideals such that for all n there exists $u_n \in R$ such that $I_{n+1} = I + Ru_n$.

A scaled Ring $(R, I_\bullet)$ acts regularly on an R-module M if u_n is monic on $M/I_n M$ for all n (this is independent of the choice of u_n).

Thm: If the height scale of $[p]_F$ is regular, then

$$X \longmapsto L_*(X) \otimes_{L_*} R_*$$ defines a homology theory.

14.10.03

Today: (1) Exact functor theorem
(2) Power operations
(3) Co-operation algebras

§ Comodules and Landwebers exact functor theorem

L = Lazard ring, $\mathrm{Ring}(L, R) = \{FGL/R\}$

$$\text{Iso of } fgl/R = \mathrm{Ring}(W, R)$$

where $W = L[b_0^{\pm 1}, b_1, b_2, \dots]$ and the universal isomorphism is $\Theta(t) = \sum_{i=1}^{\infty} b_{i-1} t^i$.

(L, W) is a Hopf-algebroid.

Comodules over (L,W): $M \in L\text{-mod}$, $\psi : M \to W \otimes_L M$

... counital, coassociative.

Introduce a grading by setting $M_{2n} = \{ x \in M \mid \psi x = b_0^n \otimes x + \dots \}$

Then $\mathrm{Comod}/(L,W) \cong$ Evenly graded MU_*MU-comodules.

$\Updownarrow$

sheaf over the stack of formal groups $\qquad \tilde{M}(F \text{ fgl}/R) = R \otimes_L M$

where R is an L-algebra via the classifying map for F.

any morphism of formal groups $F/R \longrightarrow G/S : (f : R \to S,\ \theta : f_*F \xrightarrow{\cong} G)$

$$R \otimes_L M \xrightarrow{1 \otimes \psi} R \otimes_L W \otimes_L M \xrightarrow{\text{"}f,\theta\text{"} \otimes 1} S \otimes_L M$$

"sheaf"-condition: $\tilde{M}(F/R)$ is an R-module & $\tilde{M}(F/R) \to \tilde{M}(G/S)$ is R-linear such that the extension

$$S \otimes_R \tilde{M}(F/R) \longrightarrow \tilde{M}(G/S) \text{ is iso.}$$

To get all (i.e. also odd degrees) MU_*MU-comodules, can work over the Hopf-algebroid (L, W^s), where

$$W^s = L[e_0^{\pm 1}, b_1, b_2, \dots] \longleftarrow W$$
$$e_0^2 \longleftarrow\!\shortmid\ b_0$$

There is a symmetric monoidal structure on (L,W^s)-comodules. These comodules are equivalent to sheaves on the stack of "spin formal groups".

Spin formal group $:= (F \text{ fgl}/R,\ \lambda \text{ line bundle},\ \lambda^{\otimes 2} \cong \omega)$

Thm (Landweber) Let F/R be a formal grp law.

$$(L,W)\text{-comod} \longrightarrow Ab$$

$$M \longmapsto R \otimes_L M$$

is exact if and only if the height scale is regular on R for all primes.

Recall: $[p]_F(t) = \sum a_{i-1} t^i$, then set

$I_n = (a_0, \ldots, a_{p^{n-1}-1})$; this is a scale in the sense that for all n $\exists\, u_n \in R$ such that $I_{n+1} = I_n + Ru_n$.

Regular on M: u_n is injective on M/I_nM.

We'll prove this assuming also that the height scales are all finite, i.e. $I_nR = R$ for some n, i.e. u_n is a unit for some n.

The proof is by a series of exercises:

Exercise: F/R is Landweber exact $\iff$ $\eta_R : L \to R\otimes_L W$ is flat.

Now let M be an R-module.

$$\begin{array}{ccccc} M & \xrightarrow{u_0} & M & \longrightarrow & M_0 = u_0^{-1}M \\ & & \downarrow & & \\ M/I_1 & \xrightarrow{u_1} & M/I_1 & \longrightarrow & M_1 = u_1^{-1} M/I_1 \\ & & \vdots & & \\ M/I_n & \xrightarrow{u_n} & M/I_n & \longrightarrow & M_n = u_n^{-1}(M/I_n) \end{array}$$

In the universal case, $L_n = u_n^{-1} L/I_n$ is a ring and any M_n as above is an L_n-module.

Exercise 1 If $I_\bullet$ is a regular scale, then for all n, the flat dimension $\operatorname{fdim}_R(R/I_n) \le n$.

Thus $\operatorname{fdim}_R(R_n) \le n$ $\quad (R_n = u_n^{-1} R/I_n)$

Exercise 2: If $I_\bullet$ is regular and M_n is flat$/R_n$ for all n,

then $\operatorname{fdim}_R M_n \leq n$

Hint: Spectral sequence.

Or: M_n/R_n flat $\Rightarrow$ $M_n/R/I_n$ flat $\Rightarrow$ $\operatorname{Tor}_j^R(-,M_n) \cong \operatorname{Tor}_j^R(-,R/I_n) \otimes_{R/I_n} M_n$

Exercise 3: If I acts regularly and finitely on M and $\operatorname{fdim}_R M_n \leq n$ for all n, then M is flat$/R$.

Claim Proof of Landweber EFT is one prime at a time.

Let p be a prime. Then L_n classifies formal gp laws over R of strict height n ($L_n = u_n^{-1} L/I_n$)

$$\mathrm{Ring}(L_n, R) = \left\{ \begin{array}{c} \text{fgl}/R \text{ of} \\ \text{strict height } n \end{array} \right\} = \mathcal{F}_n(R)$$

For any fgl of strict height n over R with $pR=0$, the map

$$\eta_R : L_n \longrightarrow R \otimes_L W \text{ is flat.}$$

One example: F_u universal FGL, $(L \to L_n)_* F_u / L_n$ is the universal fgl of strict height n.

$$\begin{tikzcd}
L_n \arrow[r, "\eta_R"] \arrow[dr, "\eta_L"'] & L_n \otimes_L W \arrow[d, "c"', "\cong\ \text{conjugation}"] \\
& L_n \otimes_L W
\end{tikzcd}$$

$$L_n \otimes_L W = L_n[b_0^{\pm 1}, b_1, \dots] \quad \text{flat by inspection}$$

Now use that after faithfully flat extension, any fgl of strict height n is isomorphic to one of this form (Lazard, Strickland) $\square$