§4. Power operations in periodic ring spectra

Ando, Ando–Hopkins–Strickland

Let A be a finite abelian p-group, acting on itself by translation, so it acts on $X^{\wedge A}$ $(= X^{\wedge \# A})$.

Form orbit spectrum

$$X^{\wedge A}/A \simeq EA \ltimes_A X^{\wedge A} = D_A X$$

Comes with maps $X^{\wedge A} \longrightarrow D_A X$

and for spaces X, a diagonal map $BA^+ \wedge X \longrightarrow D_A X$.

If R is an E_∞ ring spectrum, it comes with maps

$$\begin{tikzcd} D_A R \arrow[r, "\theta_A"] & R \\ R^{\wedge A} \arrow[u] \arrow[ur, "A\text{-fold product}"'] & \end{tikzcd}$$

(for us, "H_∞" is actually good enough)

Get external power operation

$$P^A : R^0(X) = [X, R] \xrightarrow{D_A} [D_A X, D_A R] \xrightarrow{\theta_{A*}} [D_A X, R]$$

Now suppose R is periodic with formal group G_R,

$\pi_0 R$ is complete local ring with residue characteristic $p > 0$.

height (G/residue field) $= n < \infty$.

Landweber: $R^0(BA)$ is free of rank $|A|^n$ over $\pi_0 R$.

Thus

If X is a space, then

$$\begin{tikzcd} R^0(BA^+ \wedge X) & R^0(BA^+) \otimes_{R^0} R^0(X) \arrow[l, "\cong"'] \\ R^0(D_A X) \arrow[u, "\text{diagonal}^*"] \arrow[ur] & \end{tikzcd}$$

$\mathcal{C}$ = category of pairs (S, J) where S is commutative ring and

$$S = \varprojlim S/J^n .$$

Example: $(R = \pi_0 \underline{R}, m)$

$$\mathcal{C}_R = (R, m)/\mathcal{C}$$

$$R^0(BA) \supseteq I = \ker\left(R^0(BA) \xrightarrow{\varepsilon} R \to R/m \right)$$

For $(S, J) \in \mathcal{C}_R$, we get the ~~formal~~ group

$$G_{\underline{R}}(J) = J \text{ with } +_{G_{\underline{R}}}$$ (works since S is complete wrt J)

Then $R^0(BA)$ is characterised by

$$\mathcal{C}_R\left(R^0(BA), (S, J) \right) = \mathrm{Hom}\left(A^*, G_{\underline{R}}(J)\right)$$

where $A^* = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ is dual group.

Universal example : $c : A^* \longrightarrow G(I)$

$$(\chi : A \to S^1) \longmapsto \quad R^0(BA) \leftarrow R^0(BS^1) = R^0 \mathbb{C}P^\infty$$

$$c(t) \hookleftarrow t$$

Now switch notation; the previous A is now A^*.
Given $(S, J) \in \mathcal{C}_R$ and an inclusion $A \hookrightarrow G_{\underline{R}}(J)$

$$\begin{tikzcd}
R^0(X) \arrow[r, "P^{A^*}"'] \arrow[drr, bend right, "\psi^A \text{ : Ring homomorphism}"'] & R^0(D_A X) \arrow[r] & R^0(BA) \otimes_R R^0(X) \arrow[d] \\
 & & S \otimes_R R^0(X)
\end{tikzcd}$$

Example: Take $X = *$ to get $\psi^A : R \longrightarrow S$ ring map

Take $X = \mathbb{C}P^\infty$ to get $\psi^A : R^0(\mathbb{C}P^\infty) \longrightarrow S \otimes_R R^0(\mathbb{C}P^\infty)$

$$t \mapsto \psi^A(t) \in S[\![t]\!]$$

Proposition: $\psi^A(t) : G_R \longrightarrow \psi^A G_R$ is a homomorphism of formal groups

Proposition: $\ker\left(G_R(J) \xrightarrow{\psi^A(t)} \psi^A G_R(J) \right)$
$$\cup\!|$$
$$A$$

The composite is an embedding, sometimes an iso
($\longrightarrow$ level structures)

15.10.03

P.S.: (to last lecture)

Thm (Landweber, Hovey-Strickland)

$$E : \begin{pmatrix}\text{regular formal} \\ \text{groups}\end{pmatrix} \xrightarrow{\text{Landweber}} \mathrm{Ho}(\text{Laurent spectra})$$

Hovey-Strickland solve the problem of phantom maps.

such that $\pi_0 E_{F/R} = R$ and $G_{E_{F/R}} = F$, $\pi_2 E_{F/G} = \omega_{G_F}$.

P.S.: $\chi_A : R^0(D_A X) \longrightarrow S \otimes_R R^0(X)$ is called the Hopkins-Kuhn-Ravenel character.

§5: Deformation theory

Consider formal grp law $\sum a_{ij} x^i y^j$ over K.

How can we vary/deform the coefficients a_{ij} and retain associativity?

We assume throughout that K is a perfect ring ~~of p≠0~~ of characteristic $p>0$, i.e. $\Phi: K \to K$, $x \mapsto x^p$ is bijective.

<u>Definition</u> A <u>p-adic ring</u> is a ring R which is (p)-adically complete and such that

1. R/p is perfect
2. p is not a zero divisor in R.

<u>Thm (Witt)</u>

$$(\text{p-adic rings}) \xrightarrow{\simeq \text{ red.}} (\text{perfect rings})$$

p-typical Witt vectors

is an equivalence of categories.

<u>Example:</u> $\mathbb{Z}_p[\omega]$, $\omega^{q-1}=1$ $\longleftrightarrow$ $\mathbb{F}_q$

<u>Lubin, Tate:</u>

<u>Def:</u> A ~~[illegible]~~ <u>universal deformation</u> is

1. $(R, I \text{ ideal})$ such that R is I-adically complete and R/I is perfect.
2. F is a formal grp over R whose height scale is regular and finite with $I_{p,n+1}=R$ and $I=I_{p,n}$.

In this situation, F reduces to a formal grp of strict height n.

<u>Thm (Lubin, Tate)</u>

$$(\text{universal deformations}) \xrightarrow[\text{reduce}]{\simeq} \left(\begin{array}{c}\text{formal grps of} \\ \text{finite strict height} \\ \text{over perfect rings}\end{array}\right)$$

Example: $W(\mathbb{F}_q)[\![u_1,\dots,u_{n-1}]\!]$ $\longleftrightarrow$ F ht n over $\mathbb{F}_q$
Lubin-Tate ring

By Landweber's theorem this gives a functor

$$\begin{tikzcd} \begin{pmatrix}\text{formal grps of} \\ \text{finite strict height} \\ \text{over perfect rings}\end{pmatrix} \arrow[r] & \mathrm{Ho}\begin{pmatrix}\text{Laurent} \\ \text{Spectra}\end{pmatrix} \\ E_\infty^{\text{Lubin-Tate}} \arrow[u] \arrow[r] & (E_\infty \text{ Laurent spectra}) \end{tikzcd}$$

<u>Thm (Hopkins-Miller)</u> This forgetful functor is an equivalence.
In particular, the mapping spaces are homotopically discrete.

The proof uses the obstruction they developed in Paul Goerss' talks.
We'll show why the obstruction grps vanish.

§6 Cooperation algebras and étaleness

Let $\underline{R}, \underline{S}$ be Laurent spectra (in the homotopy category)

Then $\quad \mathrm{Ring}(\underline{R},\underline{S}) \xrightarrow{\cong} S_0\text{-alg}(S_0\underline{R}, S_0)$ is an iso.

$$(f:\underline{R}\to\underline{S}) \mapsto \left(S_0R \xrightarrow{S_0(f)} S_0S \xrightarrow{\text{mult}} S_0\right)$$

Assume also that $\underline{R}$ and $\underline{S}$ are Landweber exact. Then

$$S_0R = S\otimes_{MP_0} MP_0(R) = S\otimes_{MP_0} MP_0MP\otimes_{MP_0} R$$

$$= S\otimes_L W\otimes_L R.$$

earlier notation

So $\quad S_0\text{-alg}(S_0\underline{R}, S_0) = S_0\text{-alg}(S\otimes_L W\otimes_L R, S) \cong$ morphism of formal grps $\mathbb{G}_{\underline{R}}/R \longrightarrow \mathbb{G}_{\underline{S}}/S$
$(=(f:R\to S,\ \theta: \mathbb{G}_R \cong \mathbb{G}_S))$

Next up is the cohomology $H^*_{Ass}(S_0 R, S)$ and $H^*_{Comm}(S_0 R, S)$.

<u>Key Lemma</u>: Let $\bar R$ and $\bar S$ be perfect rings. Consider formal groups $F_0/\bar R$ and $G_0/\bar S$ of strict height n.

Then $\bar R \otimes_L W \otimes_L \bar S$ is a perfect ring.

<u>Proof:</u>

$$\begin{tikzcd} \bar R \otimes_L W \otimes_L \bar S \arrow[rr, "\Phi"] \arrow[dr, "h"'] & & \bar R \otimes_L W \otimes_L \bar S \arrow[dl, dashed, "?(\exists!)"] \\ & T & \end{tikzcd}$$

test ring

h corresponds to $(f : \bar R \to T,\ g : \bar S \to T,\ \theta : fF_0 \xrightarrow{\cong} gG_0)$

$$\Phi(\tilde f : \bar R \xrightarrow{\Phi^{-1}} \bar R \xrightarrow{f} T) = f$$
$$\Phi(\tilde g : \bar S \xrightarrow{\Phi^{-1}} \bar S \xrightarrow{g} T) = g$$

We proved earlier that Φ is fully faithful on $\mathcal{F}_n$.
So there exists unique $\tilde\theta$ such that $\Phi\tilde\theta = \theta$. □

<u>Corollary</u>: $\mathrm{Der}(\bar R \otimes_L W \otimes_L \bar S, M) = 0$ for any module M
since $dy = d(x^p) = 0$, so $\Omega_{\bar R \otimes_L W \otimes_L \bar S} = 0$.

This implies that $H^*_{Com}(\bar R \otimes_L W \otimes_L \bar S / \bar S, -) = 0$.
With some tricks, this makes all the obstruction groups vanish.

If R, S are universal deformations, then $\eta_R : L \to S \otimes_L W$ is flat,
thus $\eta_R : L \to W \otimes_L S$ is also flat. Thus

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R \otimes_L W \otimes_L S \arrow[d] \\ \bar R \arrow[r] & \bar R \otimes_L W \otimes_L S = \bar R \otimes_L W \otimes_L \bar S \end{tikzcd}$$

is flat

By flat base change ~~$H^*(\tilde{R}\otimes_L W\otimes_L \tilde{S}/\tilde{R}, M)$~~

$$H^*(R\otimes_L W\otimes_L S/R, M) = 0 \quad \text{for } M \text{ any module over } \tilde{R}\otimes_L W\otimes_L \tilde{S}.$$

This can be promoted to vanish for all $R\otimes_L W\otimes_L S$-modules by completeness.

16.10.03
Haynes Miller
Question session.

Genus : $\begin{pmatrix}\text{manifolds} \\ \text{with structure}\end{pmatrix} \longrightarrow (\text{number})$

cobordism invariant, multiplicative

Todd genus : $td : MU_{2n} \longrightarrow \mathbb{Z}$

$\Longleftrightarrow$ multiplicative formal grp

MU complex structure on $\nu \; (M^{2n} \hookrightarrow \mathbb{R}^{2d+2n})$

exponential for multiplicative formal grp $\quad \exp : G_a \to G_m$

$\exp(x) = 1 - e^{-x}$

$$\frac{x}{1-e^{-x}} = \frac{\exp(\cancel{add}\, G_a)}{\exp(G_m)} = K(x)$$

Then $\prod_{i=1}^{d} \frac{x_i}{1-e^{x_i}}$ is symmetric function in x_i's so it can be expand as $\backslash\backslash\, F(c_1, \dots, c_d)$ in elementary symmetric functions c_i.

Then $td(M) = \langle F(c_1(M), \dots, c_d(M)), [M] \rangle \in \mathbb{Z}$

$\uparrow$ Chern classes.

Geometric/analytic expression for $td(M)$ if M is a complex manifold (Debeault complex)

$$0 \to C^\infty(M) \xrightarrow{\bar\partial} C^\infty(\Lambda T^*_{\mathbb{C}} M) \xrightarrow{\bar\partial} \cdots$$

$$\sum_{i=0}^{d} (-1)^i H^i(-) = \text{arithmetic genus} = td(M).$$

Hirzebruch?

Given a family of complex manifolds / fibration

$$\begin{tikzcd} M^{2d} \arrow[r] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

Then $\sum (-1)^i H^i(\text{fibres})$ is a virtual vector bundle, so it defines a class in $K(B)$.

This can be twisted with vector bundles on E, and then gives a map $K(E) \to K(B)$.

This map coincides with the homotopically defined pushforward:

$$\begin{tikzcd} E \arrow[r, hook, "i"] \arrow[d, "\pi"'] & B \times \mathbb{R}^{2d+2n} \arrow[dl, "\text{proj}"] \\ B & \end{tikzcd}$$

The complex structure on M^{2d} determines a complex structure on the normal bundle $\nu(i)$

Thom-Pontryagin construction gives

$$\Sigma^{2d+2n} B_+ = (B \times \mathbb{R}^{2d+2n})^+_\emptyset \longrightarrow Th(\nu(i))$$

$$\begin{tikzcd} K^0(\Sigma^{2d+2n} B_+) & K^0(Th(\nu(i))) \arrow[l] \\ \tilde{K}^0(B) \arrow[u, "\text{Bott periodicity}", "\cong"'] & \tilde{K}^0(E) \arrow[l] \arrow[u, "\cong", "\text{Thom-isomorphism}"'] \end{tikzcd}$$

Index theorem: pushforward coincides with above map.

The ring spectrum map $MU \to K$ is the topologist's version of the geometry behind families of complex manifolds.

$n=0$	$n=1$	$n=2$
$\mathbb{G}_a$	$\mathbb{G}_m$	$C/_R$ elliptic curve
$MV \longrightarrow H\mathbb{Q}$	$MV \overset{td}{\underset{\overline{td}}{\rightrightarrows}} K$	$MV \xrightarrow{t} E_{C/R}$
	Symmetry ± 1	Symmetries : M_{ell}
	$\downarrow$ equalise symmetry grp	$\downarrow$ equalise symmetries
$MSO \xrightarrow{\#} H\mathbb{Q}$	$MSpin \xrightarrow{\hat{A}} KO$	$MString \longrightarrow tmf$
$w_1 = 0$	$w_1 = w_2 = 0$	$w_1 = w_2 = \frac{p_1}{2} = 0$
	$+ \mathbb{Z}/2$ invariants in dim $= 1,2$ (8)	$+$ 2,3-torsion invariants

$$\begin{array}{c} BString \\ | \\ BSpin \\ | \\ BSO \\ | \\ BO \end{array}$$

Mathew: two orientations $\Rightarrow$ two Thom isomorphisms

$\Rightarrow$ the ratios of the isos in a well defined unit in the cohomology of the base

(Since $E^*(\text{Thom})$ is free module of rk. 1 over $E^*(\text{Base})$)

$\Rightarrow$ ratios of two orientations $MV \rightrightarrows E$ lies in

$$E^0(BU)^\times \iff bu \longrightarrow gl_1(E)$$

Neil Strickland: elliptic curve $C \hookrightarrow S$

proper, flat, relative dimension 1 over S,

all geometric fibres of genus 1.