9/10/03

Analytic theory (of modular forms).

Doubly periodic analytic functions w.r.t. lattice $\Lambda \subset \mathbb{C}$,

$$f(z+\lambda) = f(z), \qquad \lambda \in \Lambda .$$

If f is also holomorphic, then f is constant. Must have at least 2 poles in a fundamental domain for Λ. First guess

$$\sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^2}$$

but this does not converge. Correct to

$$\wp(z,\Lambda) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right),$$

which converges absolutely. This does not obviously satisfy $\wp(z+\lambda,\Lambda) = \wp(z)$. However,

$$\wp'(z+\lambda,\Lambda) = \wp'(z) \quad \left(= -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3} \right),$$

so $\wp(z+\lambda,\Lambda) - \wp(z)$ is a constant c.

Take $z = -\frac{\lambda}{2}$ to get $\wp(\frac{\lambda}{2}, \Lambda) = \wp(-\frac{\lambda}{2}, \Lambda) + c$
But $\wp(-, \Lambda)$ is even, so $c = 0$. The field of doubly periodic analytic fcts is generated by $\wp$ and $\wp'$, but $\wp$ and $\wp'$ are not algebraically independent.
Look at Taylor expansion

$$\wp(z, \Lambda) = \frac{1}{z^2} + \sum_{n \geq 1} (n+1) G_{n+2} z^n$$

$$G_k = \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^k} \quad \text{Eisenstein series}$$

$$G_k = 0 \qquad k \text{ odd.}$$

so starts out

$$\wp(z, \Lambda) = \frac{1}{z^2} + 3 G_4 z^2 + 5 G_6 z^4$$

$$\wp'(z, \Lambda) = -\frac{2}{z^3} + 6 G_4 z + 20 G_6 z^3$$

This gives the equation

$$(\wp')^2 = 4 \wp^3 + 60 G_4 \wp + 140 G_6 .$$

Emb. of $\mathbb{C}/\Lambda$ as cubic plane curve C

$$\mathbb{C}/\Lambda \longrightarrow \mathbb{P}^2$$

$$[\wp : \wp' : 1]$$

with equation

$$y^2 = 4x^3 + g_4 x + g_6$$

Genus $C = 1$:

$\dim H^0(C, \Omega^1) = 1$ on $\mathbb{C}/\Lambda$

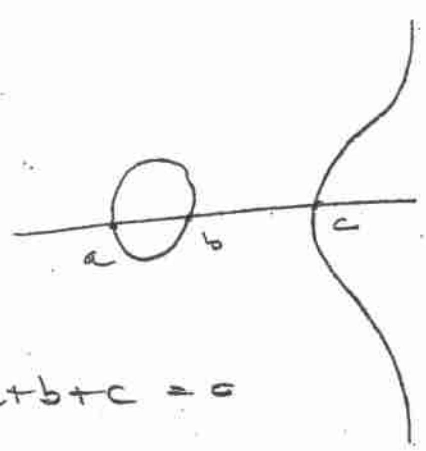
$\frac{dx}{y}$ $\quad x = x(z)$ $\quad dz$

$\quad y = y(z)$

$$\frac{dx}{y} = dz$$

$$y = x'.$$

Group structure



point at $\infty = 0$.

$a+b+c = 0$

$$\begin{matrix}\text{moduli space of} \\ \text{lattices } \Lambda \subset \mathbb{C}\end{matrix} \Big/ \ \Lambda \sim t\Lambda \quad 0 \neq t \in \mathbb{C}$$

$$\begin{matrix}\text{moduli space of} \\ \text{plane cubic curves}\end{matrix} \Big/ \sim$$

Want not only to study invariant fcts. ($f(t\Lambda) = f(\Lambda)$.) but fcts. that satisfy

$$f(t\Lambda) = t^{-k} f(\Lambda).$$

For example, $G_k(t\Lambda) = t^{-k} G_k(\Lambda)$. These will correspond to sections of tensor powers of a line bdl.

$$\langle \omega_1, \omega_2 \rangle = \text{lattice spanned by } \omega_1, \omega_2$$

$$\sim \langle 1, \tau \rangle \quad , \quad \operatorname{im}(\tau) > 0$$

$$\langle c\tau + d, a\tau + b \rangle \sim \langle 1, \frac{a\tau+b}{c\tau+d} \rangle, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

so equation $f(t\Lambda) = t^{-k} f(\Lambda)$ becomes

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Space of lattices

$= \mathcal{H} / SL_2(\mathbb{Z})$

$\mathcal{H} = \{ z \in \mathbb{C} \mid \mathrm{im}(z) > 0 \}$.

Def An analytic modular form of wt k is a holomorphic fct. f on $\mathcal{H}$ s.t.

(i) $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

(ii) $\lim_{t\to\infty} f(it) < \infty$. "

Now consider moduli space of plane cubic curve. Plane cubic

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

over $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. Transf.:

$$\mathrm{I} \begin{cases} x \longmapsto x + r \\ y \longmapsto y + sx + t \end{cases}$$

$$\mathrm{II} \begin{cases} x \longmapsto \lambda^{-2} x \\ y \longmapsto \lambda^{-3} y \end{cases}$$

Again do not only want to consider the functions ($f \in A$) invariant w.r.t. these transf.

Def An algebraic modular form of wt k (over $\mathbb{Z}$) is an elem. $f \in A$ s.t. f is invariant under (I) and transf. as $f \mapsto \lambda^k f$ under (II). //

Make A into graded ring $|a_i| = 2i$, then $f \mapsto \lambda^k f$ if and only if $|f| = 2k$.
Compute the ring of alg. modular forms following Deligne's paper in Antwerp Proceedings.

First invert 6. Then

$$y \mapsto y - \tfrac{1}{2}(a_1 x + a_3)$$

transforms the gen. equation to

$$y^2 = x^3 + \frac{b_2}{4} x^2 + \frac{b_4}{2} + \frac{b_6}{4}, \quad b_i \in A.$$

Also

$$x \longmapsto x - \frac{b_2}{12}$$

transf. this eq. to

$$y^2 = x^3 + \frac{c_4}{48} + \frac{c_6}{864} \quad , \quad c_4, c_6 \in A .$$

It turns out that

$$\frac{c_4^3 - c_6^2}{1728} = \Delta$$

and that this is the discriminant of the curve C. Moreover, the last form of the equation is unique. So we have produced a map of graded rings

$$M = \mathbb{Z}[c_4, c_6, \Delta] / (c_4^3 - c_6^2 - 1728\Delta)$$

$$\longrightarrow \text{invariants in } A$$

which is an is after inverting 6.

Prop (Tate) This map is an iso.

To prove this, it suffices to check this after localizing at 2 and 3. By Nakayama's lemma, it suffices to show mono modulo 2 and 3. Modulo 3, consider

$$M \otimes \mathbb{Z}/3 \to \mathbb{Z}/3\,[b_2, b_4, b_6] \hookrightarrow b_2^{-1}\,\mathbb{Z}/3\,[b_2, b_4, b_6]$$

$$\to b_2^{-1}\,\mathbb{Z}/3\mathbb{Z}\,[b_2, b_6] \qquad (b_4 \mapsto 0).$$

Calc.

$$c_4 \mapsto b_2^2 - 24 b_4 \equiv b_2^2 \quad (\mathrm{mod}\ 3)$$

$$c_6 \mapsto b_2^3 \quad (\mathrm{mod}\ 3)$$

$$\Delta \mapsto b_2^3\, b_6 \quad (\mathrm{mod}\ 3)$$

so easy to check this is mono. Modulo 2, invert a_1 and set $a_3 = a_4 = 0$

$$M \otimes \mathbb{Z}/2 \longrightarrow a_1^{-1}\,\mathbb{Z}/2\,[a_1, a_2, a_6]$$

$$c_4 \mapsto a_1^2$$

$$c_4 \mapsto a_1^3$$

$$\Delta \mapsto a_1^6\, a_6.$$

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9/15/03

Investigate notion of modular forms in a more geometric manner.

$$\begin{array}{c} E \\ e \curvearrowleft \big\downarrow \; \text{family of curves} \\ S \end{array}$$

$\omega = e^* \Omega^1_{E/S}$

line bdl. of inv. 1-forms along the fibers.

then

modular form of wt. k

= section of $\omega^{\otimes k}$ compatible with maps of families of curves.

ex $\mathbb{C}/\Lambda$ ω - trivialize using dz;

section of $\omega^{\otimes k}$: $f(\Lambda)\,dz^k$; compatible with all maps : $f(t\Lambda)(d\,tz)^k = f(\Lambda)\,dz^k$,

or $f(t\Lambda) = t^{-k} f(\Lambda)$. //

- reproduces

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$$

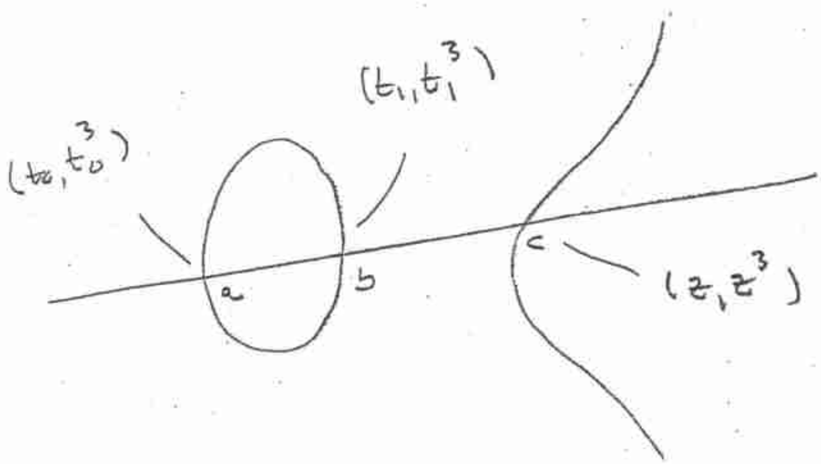
- gives eigenvectors for scaling in purely algebraic case.

Modular forms of wt k

$$= \varprojlim_{E/S} H^0(S, \omega^{\otimes k})$$

where the limit is over families of plane cubic curve. There are higher limits in this case which will appear in the topological theory later. //

Express group law in local parameter near e = point at infinity.



ex $y^2 = x^3$ or $s = t^3$ with $s = \frac{1}{y}$, $t = \frac{x}{y}$.

$s = t = 0$ origin, t = parameter.

$$\frac{\dfrac{z^3 - t_0^3}{z - t_0} - \dfrac{z^3 - t_1^3}{z - t_1}}{z - t_1} = 0$$