Dec 1 TMF

A map $\begin{array}{c} \operatorname{Spec} R \\ \downarrow J \\ \mathcal{M}_{Ell} \end{array}$ corresponds to an elliptic curve J over R.

Map between elliptic curves

$$\begin{tikzcd} \operatorname{Spec} S \arrow[r] \arrow[rr, bend left] \arrow[dr] & \operatorname{Spec} R_{12} \arrow[r] \arrow[d] & \operatorname{Spec} R_2 \arrow[d, "J_2"] \\ & \operatorname{Spec} R_1 \arrow[r, "J_1"'] & \mathcal{M}_{Ell} \end{tikzcd}$$

to give a map $R_{12} \to S$ is the same as $R_1 \xrightarrow{f_1} S \xleftarrow{f_2} R_2$

$$\begin{array}{c} \Updownarrow \\ \operatorname{Spec} S \to \operatorname{Spec} R_{12} \end{array} \qquad f_1^* J_1 \overset{t}{\simeq} f_2^* J_2$$

$\mathcal{A}ff/\mathcal{M}_{Ell}$: ob $\operatorname{Spec} R_1 \to \mathcal{M}_{Ell}$

map
$$\begin{tikzcd} \operatorname{Spec} R_{12} \arrow[r] \arrow[d, shift left] & \operatorname{Spec} R_2 \arrow[d] \\ \operatorname{Spec} R_1 \arrow[r] \arrow[u, shift left] & \mathcal{M}_{Ell} \end{tikzcd}$$

This is true for any stack (don't use property about elliptic curves)

CONCRETE WAY TO DESCRIBE THIS CATEGORY

i.e. ob (R, J) J = ell. curve over R

map $(R_2, J_2) \to (R_1, J_1)$
$R_2 \xrightarrow{f} R_1$ + iso $f^* J_2 \overset{t}{\simeq} J_1$

Fact:

- $\mathcal{M}_{Ell} \longrightarrow \mathcal{M}_{FG}$ is flat
- $\mathcal{M}_{Weier} \longrightarrow \mathcal{M}_{FG}$ is not flat

$y^3 = x^8$ is the bad point

But removing it:

- $\mathcal{M}_{Weier} \setminus \{y^3 = x^2\} \longrightarrow \mathcal{M}_{FG}$ is flat

this $\Rightarrow$

$$\begin{tikzcd} \mathrm{Spec} \arrow[r, "\text{flat}"] \arrow[dr, "\therefore\ \text{flat}"'] & \mathcal{M}_{Ell} \arrow[d] \\ & \mathcal{M}_{FG} \end{tikzcd}$$

$\rightsquigarrow$ Landweber exact homology theories

Let C^{SS}_{Top} (super singular; topological) the category with

objects:

A_∞, even, periodic E together with an elliptic curve

$$\begin{tikzcd} \mathrm{Spec}\, \pi_0 E \arrow[r, "J"] \arrow[dr, "G_E"'] & (\mathcal{M}_{Ell})^{\wedge}_{SS} \arrow[d] \\ & \mathcal{M}_{FG} \end{tikzcd}$$

étale

maps:

$A_\infty : E_2 \xrightarrow{f} E_1$
+ an iso $(\pi_0 f)^* J_2 \sim J_1$ of elliptic curves.

there is a forgetful functor

$$\mathcal{C}^{ss}_{Top} \xrightarrow{\operatorname{Spec} \pi_0} \left(Aff \Big/ \left(\mathcal{M}_{Ell}\right)^{\wedge}_{ss} \right)_{et} =: \mathcal{C}_{alg}$$

<u>Prop</u>: this is a weak equivalence of categories.

i.e.,
every etale $\operatorname{Spec} R \longrightarrow \left(\mathcal{M}_{Ell}\right)^{\wedge}_{ss}$ is $\operatorname{Spec} \pi_0 E$ for some A_∞ E.

And

$$\pi_i \, \mathcal{C}^{ss}_{Top}(E_2, E_1) = 0 \quad , \; i > 0$$

$$\pi_0 \, \mathcal{C}^{ss}_{Top}(E_2, E_1) \longrightarrow \mathcal{C}_{alg}(\_\,,\_) \text{ is iso.}$$

<u>Pf</u>:

$$\begin{tikzcd}
\operatorname{Spec} R \arrow[r] \arrow[dr, "{\substack{\therefore \\ \text{mod } p, \\ \text{relative Frobenius} \\ \text{is iso}}}"'] & \left(\mathcal{M}_{Ell}\right)^{\wedge}_{ss} \arrow[d, "{\substack{\text{relative Frobenius, after} \\ \text{reducing mod } p, \text{ is iso} \\ \text{(last time)}}}"] \\
& \mathcal{M}_{FG}
\end{tikzcd}$$

since etale

By our obstruction theory $\exists$ unique even periodic A_∞ E with correct

$$\begin{array}{c} \operatorname{Spec} \pi_0 E = R \\ \downarrow \mathcal{F} \\ \mathcal{M}_{FG} \end{array}$$

By looking at the resolution as we did earlier there is a spectral sequence for A_∞ maps

① choose $MP_* E_2 \longrightarrow MP_* E_1$ map of $(\pi_0 MP = L \,,\; \Gamma = \pi_0 MP \wedge MP)$ co-module algebras

③ $\mathrm{Der}^s_{(L,\Gamma)\text{ co-module}} (HP_* E_2, \Omega^t HP_* E_1) \Longrightarrow \pi_{t-s} A_\infty(E_2, E_1)_f$

In our case, the Der^s gps are all zero,

by rel Frobenius $\Longrightarrow$ $\pi_n A_\infty(E_2, E_1) = 0$, $n > 0$

$\pi_0 A_\infty(E_2, E_1) = (L,\Gamma)$ co-module maps

$\pi_0 HP \wedge E_2 \to \pi_0 HP \wedge E_1$

(L,Γ) co-mod maps = maps in $\mathrm{Aff}/\mathcal{M}_{FG}$

$$\begin{tikzcd} \text{from } R_1 \arrow[dr] & & R_2 \arrow[dl] \\ & \mathcal{M}_{FG} & \end{tikzcd}, \quad R_i = \mathrm{Spec}\ \pi_0 E_i$$

i.e.

$R_2 \xrightarrow{f} R_1$ + an iso of FG $f^* \hat{J}_2 \to \hat{J}_1$.

E_∞ & A_∞

A commutative ring R is an associative ring for which $R \otimes R \xrightarrow{\mu} R$ is a ring homomorphism

Suppose we have

$\mathrm{Spec}\ R \longrightarrow \mathcal{M}_{FG}$

flat, p-complete,

$\mathrm{Spec}\ R/p \longrightarrow \mathcal{M}_{FG} \otimes \mathbb{F}_p$ $\Longrightarrow$ unique A_∞ E.

relative Frobenius is iso

call this E "perfect".

Consider $A_\infty(E\wedge E, E)$ (want to understand A_∞ maps)

<u>Claim:</u>

E perfect $\Rightarrow$ $E\wedge\ldots\wedge E$ is perfect

More generally:

E_1, E_2 perfect $\Rightarrow$ $E_1\wedge E_2$ perfect

<u>Pf:</u>

$$\begin{tikzcd}
\operatorname{Spec}\pi_0 E_1\wedge E_2 \arrow[r] \arrow[d, "\text{rel Frob is iso}"'] \arrow[dr, phantom, "p.b"] & \operatorname{Spec}\pi_0 E_2 \arrow[d, "\text{rel Frob is iso}"] \\
\operatorname{Spec}\pi_0 E_1 \arrow[r, "\text{rel Frob is iso}"'] & \mathcal{M}_{FG}
\end{tikzcd}$$

reduce mod p

(stack theoretical p.b.)

$\Rightarrow$ $\operatorname{Spec}\pi_0 E_1\wedge E_2 \longrightarrow \mathcal{M}_{FG}$ is perfect (rel Frob is iso) //

multiplication $E\wedge E \longrightarrow E$

$\Updownarrow$

$G_E \to G_E \to G_E$

$$\begin{tikzcd}
\operatorname{Spec}\pi_0 E\wedge E\wedge E \arrow[r] \arrow[d] & \operatorname{Spec}\pi_0 E \arrow[d, "\text{classifies some formal gp } G_E"] \\
\operatorname{Spec}\pi_0 E\wedge E \arrow[r] & \mathcal{M}_{FG}
\end{tikzcd}$$

$\operatorname{iso}(G_E, G_E)$

want the map of co-modules corresponding to $E\wedge E \to E$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \operatorname{Spec}\pi_0 E \arrow[d] \arrow[dl, dotted] \\
\operatorname{Spec}\pi_0 E\wedge E \arrow[r] & \cdot
\end{tikzcd}$$

need a section $\boxed{}$

$(G_E \xrightarrow{id} G_E \xrightarrow{id} G_E) \leftarrow G_E$

gives a section of top map

to make X into an E_∞ ring, need to find contractible spaces C_n

$$C_n \longrightarrow \mathcal{J}(X^{\wedge n}, X) = \mathcal{E}_n$$

(forming a map of "operads")

$$X^{\wedge j_1} \longrightarrow X$$
$$\vdots$$
$$X^{\wedge j_n} \longrightarrow X \qquad\qquad X^{\wedge n} \longrightarrow X$$

$$X^{\wedge j_1} \wedge \dots \wedge X^{\wedge j_n} \longrightarrow \underbrace{X \wedge \dots \wedge X}_{n} \longrightarrow X$$

$$C_n \times C_{j_1} \times \dots \times C_{j_n} \longrightarrow C_{j_1 + \dots + j_n}$$
$$\mathcal{E}_n \times \mathcal{E}_{j_1} \times \dots \times \mathcal{E}_{j_n} \longrightarrow \mathcal{E}_{j_1 + \dots + j_n}$$

satisfying an evident associativity axiom.

take C_n = component of $A_\infty(E^{\wedge n}, E) \overset{(*)}{\subset} \mathcal{J}(E^{\wedge n}, E)$ containing the iterated multiplication

$\Rightarrow$ E_∞ structure on E

PROBLEM ABOUT (*)

but $E^{\wedge n}$ is not cofibrant even if E is.

$\Rightarrow$ the inclusion is not quite true.

But we assume

E has a E_∞ structure.

Def: $(tmf)^{\wedge}_{ss} := \underleftarrow{\text{holim}}\ \mathcal{C}^{ss}_{top}$.

supersingular completion of tmf

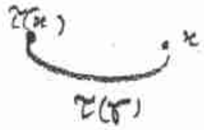
completion of tmf at the ss locus

About homotopy limit: $\underleftarrow{\text{holim}}$

suppose we have a space X with a $\mathbb{Z}/2$-action

$\underleftarrow{\lim}_{\mathbb{Z}_2} X = \{x \mid \tau(x) = x\}$

Suppose instead we have a **path** 

this gives another path 

... iterate

homotopy fixed point

$\mathbb{Z}/2$-equivariant map $S^{\infty} \to X$

p prime ≥ 5

$\pi_*(tmf)^{\wedge}_{ss} = \mathbb{Z}_p[c_4, c_6]^{\wedge}_I$ $\qquad I = (p, H(c_4, c_6))$

$H =$ coeff. of $x^{p-1}y^{p-1}$ in $\left(y^2 - (x^3 + c_4 x + c_6)\right)^{p-1}$

$p = 3$

look at $y^2 = x^3 - x$

$y^2 = x^3 + bx^2 - x$ gives $\operatorname{Spec} \mathbb{Z}_p[[b]]$

$$\operatorname{Spec} \mathbb{Z}_p[[b]] \xrightarrow{\text{étale}} (\mathcal{M}_{Ell})^{\wedge}_{ss}$$

over $\mathbb{F}_9$

$x \longmapsto x+1$ ... order 3

$x \longmapsto -x$
$y \longmapsto iy$ .. aut of order 4 —

they generate.

Aut gp $= \mathbb{Z}/3 \rtimes \mathbb{Z}/4 = G$

$\Rightarrow$ action of G on $\mathbb{Z}_p[[b]]$

Get ss

$$H^*(G; \mathbb{Z}_p[[b]][u^{\pm 1}]) \Rightarrow \pi_*(tmf)^{\wedge}_{ss}$$

$= 2$

have to look at $y^2 + y = x^3$

universal deformation now given by $y^2 + axy + y = x^3$

Aut $= \pm 1, \pm i, \pm j, \pm k, \dfrac{\pm 1 \pm i \pm j \pm k}{2}$.

11/26/03

E, F even periodic

$$\begin{array}{ccc} \operatorname{Spec} \pi_0 E\wedge F & \longrightarrow & \operatorname{Spec} \pi_0 E \\ \downarrow & & \downarrow \\ \operatorname{Spec} \pi_0 F & \longrightarrow & \mathcal{M}_{FG} \end{array}$$

If $\operatorname{Spec} \pi_0 E \to \mathcal{M}_{FG}$ is flat, then this square is a (homotopy) pull-back of stacks.

Proof $MP = MU[u^{\pm 1}]$, $\deg u = 2$, so $\pi_0 MP$ is the Lazard ring.

Case 1: $F = MP$ + can choose a coord. on G_E; let $\pi_0 MP \to \pi_0 E$ correspond to the coord.

$$E_*(X) = \pi_0 E \otimes_{\pi_0 MP} MP_*(X).$$

Take $X = MP$; then

$$\pi_* E\wedge MP = \pi_0 E \otimes_{\pi_0 MP} MP_* MP$$

and MP_*MP represents pairs of formal grps plus and iso. between them. This $\Rightarrow$ Case 1.

Case 2: General F

<u>lemma</u> $L = \pi_0 MP \to \pi_0 MP \wedge E$ is flat.

pf

$$\begin{tikzcd}
\operatorname{Spec} \pi_0 MP \wedge E \arrow[r] \arrow[d, "\text{flat}"'] & \operatorname{Spec} \pi_0 E \arrow[d, "\text{flat}"] \arrow[dl, phantom, "\Leftarrow"] \\
\operatorname{Spec} \pi_0 MP \arrow[r] & \mathcal{M}_{FG}
\end{tikzcd}$$

//

The lemma implies that

$$\pi_* E \wedge F = MP_* F \otimes_{MP_0} \pi_0 E \wedge MP .$$

The following diagram is a pull-back

$$\begin{tikzcd}
\operatorname{Spec} (\pi_0 MP \wedge E \otimes_{\pi_0 MP} \pi_0 MP \wedge F) \arrow[r] \arrow[d] & \operatorname{Spec} \pi_0 MP \wedge E \arrow[r] \arrow[d] & \operatorname{Spec} \pi_0 E \arrow[d] \\
\operatorname{Spec} \pi_0 F \arrow[r] & \operatorname{Spec} \pi_0 MP \arrow[r] & \mathcal{M}_{FG}
\end{tikzcd}$$

but the equality above shows that the upper left-hand term is

$\operatorname{Spec} \pi_* E \wedge F$. //

Cor If $\operatorname{Spec} \pi_0 E \to \mathcal{M}_{FG}$ is flat then $\pi_0 E \to \pi_0 E \wedge E$ is flat so $E_*(X)$ takes values in $E_* E$ - comodules.

pf
$$\begin{tikzcd} \operatorname{Spec} \pi_0 E \wedge E \arrow[r] \arrow[d, "\text{flat}"] \arrow[dr, phantom, "\Leftarrow"] & \operatorname{Spec} \pi_0 E \arrow[d, "\text{flat}"] \\ \operatorname{Spec} \pi_0 E \arrow[r] & \mathcal{M}_{FG} \end{tikzcd}$$

Rem Since a groupoid is equal to the 1-skeleton of its nerve, the homotopy pull-back of stacks is associative up to canonical isomorphism.

$$\begin{tikzcd} \operatorname{Spec} \Gamma \arrow[r] \arrow[d] & \operatorname{Spec} L \arrow[d] \\ \operatorname{Spec} L \arrow[r] & \mathcal{M}_{FG} \end{tikzcd}$$

$$\begin{tikzcd} \operatorname{Spec} R_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \operatorname{Spec} R_1 \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} \Gamma \arrow[r, shift left] \arrow[r, shift right] & \operatorname{Spec} L \arrow[r] & \mathcal{M}_{FG} \end{tikzcd}$$

or in the language of coh th

$$E_*(X) \to (MP \wedge E)_*(X) \rightrightarrows (MP \wedge MP \wedge E)_*(X).$$

So if we cannot choose a coord. on G_E, we can work with $MP \wedge E$ and $MP \wedge MP \wedge E$ and deduce the result for E.

Problem: Suppose $\operatorname{Spec} R \xrightarrow{G} \mathcal{M}_{FG}$ is flat ($\Leftrightarrow$ Landweber exact). Determine all A_∞-ring spectra E with $\pi_0 E = R$ and $G_E = G$.

Equivalent problem: Consider the pull-back

$$\begin{array}{ccc} \operatorname{Spec} R_1 & \longrightarrow & \operatorname{Spec} R \\ \downarrow \text{flat} & \Leftarrow & \downarrow \text{flat} \\ \operatorname{Spec} L & \longrightarrow & \mathcal{M}_{FG} \end{array}$$

Then R_1 is an alg. in (L,Γ)-comodules (comes from descent-data of pull-back). Determine all A_∞-ring spectra E s.t.

$$\pi_0 MP \wedge E \simeq R_1$$

as an alg. in (L,Γ)-comodules.

By adding the (L,Γ)-comod str. we get $G = G_E$ automatically. The comod. str. can be handled by the obstruction theory we have developed. The obstructions to existence are in

$$\mathrm{Der}^{n+2}_{(L,\Gamma)\text{-comod}}(R_{1*}, \Omega^n R_{1*}).$$

If we can choose a coord. on G, we can calc. R_1 as

$$\begin{tikzcd}
\mathrm{Spec}\, R_1 \arrow[r] \arrow[d] & \mathrm{Spec}\, R \arrow[d, "\text{coord.}"] \arrow[dd, bend left=60, "G"] \\
\mathrm{Spec}\, \Gamma \arrow[r] \arrow[d] & \mathrm{Spec}\, L \arrow[d] \\
\mathrm{Spec}\, L \arrow[r] & \mathcal{M}_{FG}
\end{tikzcd}$$

Hence, $R_1 = \Gamma \otimes_L R$ is an extended or co-free (L,Γ)-comodule, i.e.

$$\mathrm{Comod}_{(L,\Gamma)}(M, R_1) = \mathrm{Mod}_L(M, R).$$

Hence,

$$\mathrm{Der}^{n+2}_{\mathrm{comod}}(R_{1*}, \Omega^n R_{1*})$$

$$\approx \mathrm{Der}^{n+2}_{\mathrm{mod}}(R_{1*}, \Omega^n R_*).$$

<u>Criterion</u> Suppose R is p-complete and the relative Frobenius on $R_1 \otimes \mathbb{Z}/p \,/\, L \otimes \mathbb{Z}/p$ is an iso. Then

$$\mathrm{Der}^{n+2}_{\mathrm{mod}}(R_{1*}, \Omega^n R_*) = 0.$$

<u>Cor</u> If R is p-complete, and if the rel. Frob. on $R_1 \otimes \mathbb{Z}/p \,/\, L \otimes \mathbb{Z}/p$ is an iso., then there is a unique A_∞-ring spectrum E s.t. $\pi_0 MP \wedge E = R_1$ as an algebra in (L, Γ)-comodules. //

$$\begin{tikzcd}
\mathrm{Spec}\, R_1/p \arrow[r] \arrow[d] & \mathrm{Spec}\, R/p \arrow[d] \\
\mathrm{Spec}\, L/p \arrow[r] & \mathcal{M}_{FG}/p
\end{tikzcd}$$

(to get $\Rightarrow$ use descent)

rel. Frob. is iso. $\Longleftrightarrow$ rel. Frob. is iso.

Cor Let $\operatorname{Spec} R \xrightarrow{G} \mathcal{M}_{FG}$ be flat and suppose that the rel. Frobenius w.r.t.

$$\operatorname{Spec} R \otimes \mathbb{Z}/p \to \mathcal{M}_{FG} \otimes \mathbb{Z}/p$$

is an iso. Then there is a unique A_∞-ring spectrum E with $\pi_0 E = R$ and $G_E \approx G$. //

Recall def. of rel Frob.:

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd, "\varphi"'] & & B \arrow[dl] \arrow[dd, "\varphi"] \\
& A \otimes_A B \arrow[dr, "\varphi_{B/A}"] \arrow[ul, phantom, "\text{cocart.}"] & \\
A \arrow[rr] \arrow[ur] & & B
\end{tikzcd}$$

Lemma If $A \to B$ is étale, then the rel. Frob. $\varphi_{B/A}$ is an iso.

pf The map $\varphi^* B \to B$ is an iso if and only if after a faithfully flat base-change along $A \to A'$, the map $\varphi^* B' \to B'$ is an iso. But $A \to B$ is étale if and